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Research article

On accelerated fixed point convergence of generalized Bianchini mappings through hybrid Krasnosel'skii iteration

Maryam Iqbal[1,*] **and Afshan Batool**[2]

[1] Academy of Computer Science and Software Engineering, University of Johannesburg, Johannesburg 2006, South Africa; miqbal@uj.ac.za

[2] Department of Mathematical Sciences, Fatima Jinnah Women University, The Mall, Rawalpindi, Pakistan; afshan.batool@fjwu.edu.pk

* **Correspondence:** Email: miqbal@uj.ac.za.

Abstract: This paper studies the approximation of fixed points of generalized enriched Bianchini mappings in Banach spaces by a two-step Krasnosel'skii scheme, which we call the hybrid Krasnosel'skii iterative method (HKIM). For this class of mappings, we prove the existence and the uniqueness of the fixed point together with the convergence of the scheme. We then compare the scheme with the Krasnosel'skii iterative method (KIM) in quantitative terms: Under a common residual stopping rule, the guaranteed decay factor of the residual of the two-step scheme carries an additional factor at every step, and this gives an explicit bound on the reduction of the number of outer iterations. Numerical tests on the real line, on $\mathbb{R}^3$, and on the space $C[0,1]$ agree with the predicted bounds. Applications to variational inequality problems and to split feasibility problems are presented together with numerical examples.

Keywords: fixed point; hybrid Krasnosel'skii iterative method; generalized enriched Bianchini mapping; asymptotic regularity; Banach space; variational inequality problem; split feasibility problem
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1. Introduction

In nonlinear functional analysis, the study of fixed points of mappings in Banach spaces has proved to be a productive area of research. Let $(\mathbb{B}, \|\cdot\|)$ be a Banach space, and let $\mathbb{C}$ be a nonempty closed convex subset of $\mathbb{B}$. For a mapping $T : \mathbb{C} \to \mathbb{C}$, we write $Fix(T) = \{x \in \mathbb{C} : Tx = x\}$. The averaged

operator associated with T is

$$T_\chi x = (1-\chi)x + \chi T x, \qquad 0 < \chi \leq 1. \tag{1.1}$$

Since $T_\chi x - x = \chi(Tx - x)$, one has $Fix(T) = Fix(T_\chi)$ for every admissible χ, while T_χ often has better analytic properties than T itself [12, 24].

The Krasnosel'skii iteration and the Mann iteration [21, 23] provide an effective way of approximating fixed points of nonexpansive mappings [11, 13, 22]. The technique has been extended to several classes of operators, among them the enriched contractions in Banach spaces [1,3]. Working with the averaged operator T_χ, the convergence of the Krasnosel'skii iteration can be established under suitable assumptions. In a Hilbert space, if T is nonexpansive and $Fix(T) \neq \emptyset$, the sequence generated from an arbitrary x_0 converges weakly to a fixed point of T, and for enriched generalized contractive mappings, the same framework produces strong convergence [2, 3, 5].

The notion of asymptotic regularity goes back to Browder and Petryshyn [8]. A mapping $T : \mathbb{C} \to \mathbb{C}$ is asymptotically regular on $\mathbb{C}$ if

$$\|T^{n+1}x - T^n x\| \to 0 \qquad \text{as } n \to \infty$$

for every $x \in \mathbb{C}$. Ishikawa [18] proved that if T is nonexpansive and the orbit $\{T_\chi^n x\}$ is bounded, then T_χ is asymptotically regular for $0 < \chi < 1$; this is the case, in particular, when $\mathbb{C}$ is bounded. The notion has since been used for wider classes of mappings in order to obtain stronger convergence statements [2, 3, 14].

Berinde and Păcurar introduced the class of enriched contraction mappings [3] and subsequently the subclasses of enriched Kannan mappings and enriched Bianchini mappings [4]. These classes were designed with the split feasibility problem (SFP) and the variational inequality problem (VIP) in view, and they combine features of Banach contractions, nonexpansive mappings, Kannan mappings [20], and Bianchini mappings [6] inside one framework [5]. Related material on iterative procedures and their applications may be found in [16, 17, 19]: Fixed point theorems in S-metric spaces with an application to navigation and control systems are given in [16], Wardowski-type contractive mappings are treated in [17], the stability of iterative procedures for generalized α-nonexpansive mappings is studied in [15], and an application of fixed point methods to delay integral equations is given in [19].

Bisht [7] studied the Krasnosel'skii iterative method (KIM) for generalized Bianchini mappings in Banach spaces, established the existence, the uniqueness and the convergence of the scheme, and applied the results to variational inequality problems and to split feasibility problems. The scheme used there is

$$z_{n+1} = T_\chi z_n, \qquad n \geq 0. \tag{1.2}$$

Although the Krasnosel'skii iteration is widely used, its convergence can be slow. The present paper studies a two-step variant that carries one relaxation parameter and reduces the number of outer iterations required to meet a prescribed residual tolerance.

The contribution of the paper is organized as follows.

- Section 2.1 states three formal definitions, namely the classical, the enriched, and the generalized enriched Bianchini conditions, and describes the exact relations among them.
- Section 2.2 gives the closed formula of the HKIM, an implementable algorithm, and a residual stopping rule.

- Section 3 contains the existence, uniqueness, and convergence theorems.
- Section 4 contains a quantitative comparison with the KIM, together with a bound on the number of outer iterations and a statement of what the comparison does not assert.
- Section 5 reports numerical tests on $\mathbb{R}$, on $\mathbb{R}^3$, and on $C[0,1]$.
- Sections 6 and 7 treat the VIP and the SFP, each with a numerical example.

2. Preliminaries

2.1. Bianchini-type mappings

Throughout this subsection $\mathbb{C}$ is a nonempty closed convex subset of a Banach space $(\mathbb{B}, \|\cdot\|)$ and $T : \mathbb{C} \to \mathbb{C}$.

Definition 2.1. *The mapping T is a* Bianchini mapping *if there is a constant $h \in [0,1)$ such that*

$$\|Tx - Ty\| \le h \max\{\|x - Tx\|, \|y - Ty\|\}, \qquad x, y \in \mathbb{C}. \tag{2.1}$$

Definition 2.2. *Let $\kappa \in [0,\infty)$ and $h \in [0,\infty)$. The mapping T is a* (κ, h)-enriched Bianchini mapping *if*

$$\|\kappa(x - y) + Tx - Ty\| \le h \max\{\|x - Tx\|, \|y - Ty\|\}, \qquad x, y \in \mathbb{C}. \tag{2.2}$$

For a fixed $\kappa \ge 0$, let Φ_κ denote the set of mappings $\omega : [0,\infty) \to [0,\infty)$ such that

(i) ω is upper semicontinuous, that is, $t_n \to t \ge 0$ implies $\limsup_{n\to\infty} \omega(t_n) \le \omega(t)$; and
(ii) $\omega(t) < (\kappa + 1)\, t$ for every $t > 0$.

Let $\mathcal{F}$ denote the family of mappings $f : [0,\infty)^2 \to [0,\infty)$ such that

(i) $f(\lambda u, \lambda v) = \lambda f(u, v)$ for all $\lambda, u, v \ge 0$;
(ii) $f(0,0) = 0$; and
(iii) f is continuous at $(0,0)$.

Definition 2.3. *Let $\kappa \ge 0$, $\omega \in \Phi_\kappa$, and $f \in \mathcal{F}$. The mapping T is a* generalized (κ, ω, f)-enriched Bianchini mapping *if*

$$\|\kappa(x - y) + Tx - Ty\| \le \omega(\|x - y\|) + f(\|Tx - x\|, \|Ty - y\|), \qquad x, y \in \mathbb{C}. \tag{2.3}$$

Remark 2.1. *The three conditions are related as follows.*

(a) Condition (2.1) *is the particular case $\kappa = 0$ of condition* (2.2).
(b) Condition (2.2) *is the particular case $\omega \equiv 0$ and $f(u, v) = h \max\{u, v\}$ of condition* (2.3)*; indeed, $\omega \equiv 0$ belongs to Φ_κ and $(u, v) \mapsto h \max\{u, v\}$ belongs to $\mathcal{F}$.*
(c) Condition (2.3) *is therefore not obtained from* (2.2) *by permitting $h \ge 1$. It permits a separate comparison term that depends on $\|x-y\|$ and a positively homogeneous residual term that depends on $\|Tx - x\|$ and $\|Ty - y\|$, and the two terms play different roles in the proofs of Section 3.*

For $\kappa \ge 0$, set

$$\chi = \frac{1}{\kappa + 1} \in (0, 1], \qquad S := T_\chi = (1 - \chi)I + \chi T. \tag{2.4}$$

Because $\mathbb{C}$ is convex, S maps $\mathbb{C}$ into $\mathbb{C}$, and $Fix(S) = Fix(T)$.

Proposition 2.1. *Let $\kappa \geq 0$, χ and S be as in* (2.4)*, and $x, y \in \mathbb{C}$. Then*

$$S x - S y = \chi[\kappa(x - y) + T x - T y], \qquad S x - x = \chi\,(T x - x). \tag{2.5}$$

Consequently,

(a) if T satisfies (2.2)*, then*

$$\|S x - S y\| \leq h\, \max\{\|S x - x\|,\, \|S y - y\|\}, \qquad x, y \in \mathbb{C}; \tag{2.6}$$

(b) if T satisfies (2.3)*, then*

$$\|S x - S y\| \leq \chi\, \omega(\|x - y\|) + f(\|S x - x\|,\, \|S y - y\|), \qquad x, y \in \mathbb{C}, \tag{2.7}$$

and the comparison function of the averaged operator satisfies

$$\chi\, \omega(t) < t \qquad \text{for every } t > 0. \tag{2.8}$$

Proof. Since $\chi = 1/(\kappa + 1)$, we have $(1 - \chi)/\chi = \kappa$, hence

$$S x - S y = (1 - \chi)(x - y) + \chi(T x - T y) = \chi\Big[\tfrac{1-\chi}{\chi}(x - y) + T x - T y\Big] = \chi[\kappa(x - y) + T x - T y],$$

and $S x - x = \chi(T x - x)$ is immediate from (1.1). This proves (2.5).

(a) By (2.5) and (2.2),

$$\begin{aligned} \|S x - S y\| &= \chi\|\kappa(x - y) + T x - T y\| \leq \chi\, h\, \max\{\|T x - x\|, \|T y - y\|\} \\ &= \chi\, h\, \chi^{-1}\, \max\{\|S x - x\|, \|S y - y\|\}, \end{aligned}$$

which is (2.6).

(b) By (2.5) and (2.3),

$$\|S x - S y\| \leq \chi\, \omega(\|x - y\|) + \chi\, f(\|T x - x\|, \|T y - y\|).$$

Using $\|T x - x\| = \chi^{-1}\|S x - x\|$ and the positive homogeneity of f,

$$\chi\, f(\|T x - x\|, \|T y - y\|) = \chi\, f(\chi^{-1}\|S x - x\|, \chi^{-1}\|S y - y\|) = f(\|S x - x\|, \|S y - y\|),$$

which gives (2.7). Finally, $\omega \in \Phi_\kappa$ gives $\chi\, \omega(t) < \chi(\kappa + 1)t = t$ for $t > 0$, which is (2.8). □

Remark 2.2. *Proposition 2.1 shows that ω is not itself the comparison function of the averaged operator; the comparison function of S is $\chi\omega$, and* (2.8) *is the property actually used in Section 3. Throughout the paper, the symbol T_χ denotes the averaged operator* (1.1) *and is not a power or a fractional iterate of T.*

2.2. The hybrid Krasnosel'skii iterative method

Let $x_0 \in \mathbb{C}$, $\kappa \geq 0$, $\chi = 1/(\kappa+1)$, and $\beta \in (0,1)$, and let $S = T_\chi$ be as in (2.4). The HKIM generates two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathbb{C}$ by

$$\begin{cases} y_n = (1-\beta)x_n + \beta S x_n, \\ x_{n+1} = S y_n = (1-\chi)y_n + \chi T y_n, \end{cases} \qquad n \geq 0. \tag{2.9}$$

Both y_n and x_{n+1} lie in $\mathbb{C}$ because $\mathbb{C}$ is convex and $S(\mathbb{C}) \subseteq \mathbb{C}$. One step of (2.9) is the composition

$$x_{n+1} = \mathcal{H} x_n, \qquad \mathcal{H} := S \circ [(1-\beta)I + \beta S], \tag{2.10}$$

so that HKIM is the Picard iteration of the operator $\mathcal{H}$. The benchmark against which HKIM is measured is the KIM

$$z_{n+1} = S z_n = (1-\chi)z_n + \chi T z_n, \qquad z_0 = x_0. \tag{2.11}$$

For $\beta = 0$, the first line of (2.9) gives $y_n = x_n$, and (2.9) becomes (2.11), while for $\beta = 1$ one step of (2.9) coincides with two steps of (2.11). The parameter β therefore interpolates between these two schemes. The scheme is used with $\beta \in (0,1)$, while the statements of Section 3 are given for $\beta \in [0,1)$, so that they cover the benchmark scheme as a particular case.

Given a tolerance $\varepsilon > 0$ and a maximum iteration number $N_{\max}$, both schemes are stopped at the first index $n \geq 0$ for which the fixed point residual is at most ε, that is,

$$r_n := \|S x_n - x_n\| \leq \varepsilon \quad \text{(HKIM)}, \qquad \widehat{r_n} := \|S z_n - z_n\| \leq \varepsilon \quad \text{(KIM)}. \tag{2.12}$$

If (2.12) is not met, the run is terminated at $N_{\max}$ and reported as not convergent within the prescribed budget.

Algorithm 1 (HKIM for a generalized enriched Bianchini mapping).

Input: $T : \mathbb{C} \to \mathbb{C}$, $\kappa \geq 0$, $\beta \in (0,1)$, $x_0 \in \mathbb{C}$, $\varepsilon > 0$, $N_{\max} \in \mathbb{N}$.

Step 0. Set $\chi = 1/(\kappa+1)$ and $n = 0$.
Step 1. Compute $u_n = T x_n$ and $S x_n = (1-\chi)x_n + \chi u_n$.
Step 2. If $\|S x_n - x_n\| \leq \varepsilon$ or $n = N_{\max}$, stop and return x_n.
Step 3. Set $y_n = (1-\beta)x_n + \beta S x_n$.
Step 4. Compute $v_n = T y_n$ and $x_{n+1} = (1-\chi)y_n + \chi v_n$.
Step 5. Set $n \leftarrow n+1$ and go to Step 1.

Output: x_n and the residual $\|S x_n - x_n\|$.

Each pass through Steps 1 to 4 uses two evaluations of T, namely u_n and v_n, whereas one step of (2.11) uses one evaluation of T. This accounting is used in Sections 4 and 5.

3. Results

In this section, we establish the existence, the uniqueness, and the convergence of the scheme (2.9). Throughout, $(\mathbb{B}, \|\cdot\|)$ is a Banach space, $\mathbb{C} \subseteq \mathbb{B}$ is nonempty, closed and convex, $T : \mathbb{C} \to \mathbb{C}$, $\kappa \geq 0$, $\chi = 1/(\kappa+1)$, $S = T_\chi$, $\beta \in [0,1)$, $x_0 \in \mathbb{C}$, and $\{x_n\}$, $\{y_n\}$ are generated by (2.9). We write

$$r_n = \|S x_n - x_n\|, \qquad q_n = \|S y_n - y_n\| = \|x_{n+1} - y_n\|. \tag{3.1}$$

Condition 3.1. *The sequences generated by* (2.9) *from the initial point* x_0 *and with the parameter* β *satisfy*

$$\lim_{n\to\infty} r_n = 0 \qquad \text{and} \qquad \lim_{n\to\infty} q_n = 0.$$

Condition 3.1 is the counterpart, for the two-step scheme (2.9), of the asymptotic regularity of S in the sense of Browder and Petryshyn [8]. Since $\|y_n - x_n\| = \beta r_n$ and $\|x_{n+1} - x_n\| \le \|x_{n+1} - y_n\| + \|y_n - x_n\| = q_n + \beta r_n$, Condition 3.1 implies $\|x_{n+1} - x_n\| \to 0$, that is, the asymptotic regularity of the orbit of the operator $\mathcal{H}$ of (2.10) that starts at x_0. Theorem 4.1 shows that Condition 3.1 holds automatically whenever $h < 1$ in (2.2) and $\beta \in (h, 1)$, so that in the situation of Section 4, it is not an extra assumption.

Theorem 3.1. *Let* $T : \mathbb{C} \to \mathbb{C}$ *be a* (κ, h)*-enriched Bianchini mapping with* $\kappa \ge 0$ *and* $h \in [0, \infty)$*, and let* T *be continuous. Let* $x_0 \in \mathbb{C}$ *and* $\beta \in [0, 1)$ *be such that the sequences generated by* (2.9) *satisfy Condition 3.1. Then* T *has exactly one fixed point* $p \in \mathbb{C}$*, and* $x_n \to p$ *as* $n \to \infty$*.*

Proof. By Proposition 2.1(a), the averaged operator S satisfies (2.6) with the same constant h.

Step 1: $\{x_n\}$ *is a Cauchy sequence.* Let $m > n \ge 1$. By (2.9), we have $x_n = S y_{n-1}$ and $x_m = S y_{m-1}$, so (2.6) applied to the pair (y_{n-1}, y_{m-1}) gives

$$\begin{aligned} \|x_n - x_m\| = \|S y_{n-1} - S y_{m-1}\| &\le h \max\{\|S y_{n-1} - y_{n-1}\|, \|S y_{m-1} - y_{m-1}\|\} \\ &= h \max\{q_{n-1}, q_{m-1}\} \le h \sup_{k \ge n-1} q_k. \end{aligned}$$

By Condition 3.1, the right-hand side tends to 0 as $n \to \infty$, uniformly in $m > n$. Hence $\{x_n\}$ is a Cauchy sequence in the closed subset $\mathbb{C}$ of the Banach space $\mathbb{B}$, and there is $p \in \mathbb{C}$ with $x_n \to p$.

Step 2: p *is a fixed point.* Since T is continuous, so is S. Therefore $\|S p - p\| = \lim_{n\to\infty} \|S x_n - x_n\| = \lim_{n\to\infty} r_n = 0$ by Condition 3.1, so $S p = p$ and, by $Fix(S) = Fix(T)$, also $T p = p$.

Step 3: uniqueness. Let p and p' be fixed points of T. Then $S p = p$ and $S p' = p'$, and (2.6) gives

$$\|p - p'\| = \|S p - S p'\| \le h \max\{\|S p - p\|, \|S p' - p'\|\} = 0,$$

so $p = p'$. This holds for every $h \in [0, \infty)$. □

The next two examples describe the reach of Definition 2.2.

Example 3.1. *Let* $\mathbb{C} = \mathbb{B} = \mathbb{R}$ *with* $\|x\| = |x|$*, and let* $Tx = 1 - \frac{x}{2}$*. For* $\kappa = 1$*,*

$$|\kappa(x - y) + Tx - Ty| = \left|(x - y) - \tfrac{1}{2}(x - y)\right| = \tfrac{1}{2}|x - y|, \qquad |Tx - x| = \left|1 - \tfrac{3x}{2}\right| = \tfrac{3}{2}\left|x - \tfrac{2}{3}\right|.$$

Since $|x - y| \le \left|x - \frac{2}{3}\right| + \left|y - \frac{2}{3}\right| \le 2\max\{\left|x - \frac{2}{3}\right|, \left|y - \frac{2}{3}\right|\}$*, we obtain*

$$\tfrac{1}{2}|x - y| \le \max\left\{\left|x - \tfrac{2}{3}\right|, \left|y - \tfrac{2}{3}\right|\right\} = \tfrac{2}{3} \cdot \tfrac{3}{2} \max\left\{\left|x - \tfrac{2}{3}\right|, \left|y - \tfrac{2}{3}\right|\right\} = \tfrac{2}{3}\max\{|Tx - x|, |Ty - y|\}.$$

Hence T *is a* $(1, \frac{2}{3})$*-enriched Bianchini mapping, and the choice* $x = \frac{2}{3} - t$*,* $y = \frac{2}{3} + t$ *with* $t > 0$ *shows that* $h = \frac{2}{3}$ *is the smallest admissible constant for* $\kappa = 1$*. The unique fixed point is* $p = \frac{2}{3}$*, and the averaged operator is* $Sx = \frac{x}{4} + \frac{1}{2}$*.*

Example 3.2. *Let* $\mathbb{C} = \mathbb{B} = \mathbb{R}$ *and* $Tx = x^2$. *Then* $Fix(T) = \{0, 1\}$. *Taking* $x = 0$ *and* $y = 1$ *in* (2.2) *gives*

$$\|\kappa(x - y) + Tx - Ty\| = \kappa + 1 > 0, \qquad \max\{|Tx - x|, |Ty - y|\} = 0,$$

so no constant $h \in [0, \infty)$ *can satisfy* (2.2) *for any* $\kappa \geq 0$. *Thus* T *possesses fixed points without belonging to the class of Definition 2.2, which shows that* (2.2) *is sufficient but not necessary for the existence of a fixed point. The example is consistent with Theorem 3.1, whose conclusion includes uniqueness.*

Replacing the constant h by a comparison function enlarges the class of admissible mappings; this is the setting of Definition 2.3, for which we have the following result.

Theorem 3.2. *Let* $T : \mathbb{C} \to \mathbb{C}$ *be a generalized* (κ, ω, f)*-enriched Bianchini mapping with* $\kappa \geq 0$, $\omega \in \Phi_\kappa$, *and* $f \in \mathcal{F}$, *and let* T *be continuous. Let* $x_0 \in \mathbb{C}$, *and* $\beta \in [0, 1)$, *be such that the sequences generated by* (2.9) *satisfy Condition 3.1. Then* T *has exactly one fixed point* $p \in \mathbb{C}$, *and* $x_n \to p$ *as* $n \to \infty$.

Proof. By Proposition 2.1(b), S satisfies (2.7) and (2.8). Put $d_n = \|x_{n+1} - x_n\|$. From (2.9), $\|y_n - x_n\| = \beta r_n$ and $\|x_{n+1} - y_n\| = q_n$, so

$$d_n \leq q_n + \beta r_n \longrightarrow 0 \tag{3.2}$$

by Condition 3.1.

Step 1: $\{x_n\}$ *is a Cauchy sequence.* Suppose it is not. Then there exist $\epsilon > 0$ and integers $m_i > n_i \geq i$ such that

$$\|x_{n_i} - x_{m_i}\| \geq \epsilon, \qquad \|x_{n_i} - x_{m_i - 1}\| < \epsilon, \tag{3.3}$$

where m_i is chosen as the smallest index larger than n_i for which the first inequality holds. By the triangle inequality and (3.3),

$$\epsilon \leq \|x_{n_i} - x_{m_i}\| \leq \|x_{n_i} - x_{m_i - 1}\| + d_{m_i - 1} < \epsilon + d_{m_i - 1},$$

and (3.2) yields

$$\lim_{i \to \infty} \|x_{n_i} - x_{m_i}\| = \epsilon. \tag{3.4}$$

Since $\|y_k - x_k\| = \beta r_k \to 0$, the same limit holds for the auxiliary sequence,

$$\lim_{i \to \infty} \|y_{n_i} - y_{m_i}\| = \epsilon. \tag{3.5}$$

Applying (2.7) to the pair (y_{n_i}, y_{m_i}) and using $x_{k+1} = Sy_k$,

$$\|x_{n_i + 1} - x_{m_i + 1}\| \leq \chi\, \omega(\|y_{n_i} - y_{m_i}\|) + f(q_{n_i}, q_{m_i}). \tag{3.6}$$

The left-hand side of (3.6) is at least $\|x_{n_i} - x_{m_i}\| - d_{n_i} - d_{m_i}$, which tends to ϵ by (3.2) and (3.4). On the right-hand side, ω is upper semicontinuous, and (3.5) holds, so $\limsup_i \omega(\|y_{n_i} - y_{m_i}\|) \leq \omega(\epsilon)$; moreover $q_{n_i} \to 0$ and $q_{m_i} \to 0$ by Condition 3.1, and f is continuous at $(0, 0)$ with $f(0, 0) = 0$, so $f(q_{n_i}, q_{m_i}) \to 0$. Passing to the upper limit in (3.6) gives

$$\epsilon \leq \chi\, \omega(\epsilon) < \epsilon$$

by (2.8), a contradiction. Hence $\{x_n\}$ is a Cauchy sequence and converges to some $p \in \mathbb{C}$.

Step 2: p is a fixed point. As in Theorem 3.1, S is continuous and $r_n \to 0$, so $\|Sp - p\| = \lim_n r_n = 0$, that is, $Tp = p$.

Step 3: uniqueness. If $Tp = p$, $Tp' = p'$, and $p \neq p'$, then (2.7) and $f(0,0) = 0$ give

$$\|p - p'\| = \|Sp - Sp'\| \leq \chi\,\omega(\|p - p'\|) + f(0,0) = \chi\,\omega(\|p - p'\|) < \|p - p'\|$$

by (2.8), a contradiction. □

Corollary 3.1. *Let $\kappa \geq 0$, $\omega \in \Phi_\kappa$, and $c_1, c_2 \in [0, \infty)$, and let $T : \mathbb{C} \to \mathbb{C}$ be continuous and satisfy*

$$\|\kappa(x - y) + Tx - Ty\| \leq \omega(\|x - y\|) + c_1\|Tx - x\| + c_2\|Ty - y\|, \qquad x, y \in \mathbb{C}. \tag{3.7}$$

If $x_0 \in \mathbb{C}$ and $\beta \in [0, 1)$ are such that Condition 3.1 holds, then the conclusion of Theorem 3.2 is valid.

Proof. The mapping $f(u, v) = c_1u + c_2v$ satisfies $f(\lambda u, \lambda v) = \lambda f(u, v)$, $f(0, 0) = 0$ and is continuous, so $f \in \mathcal{F}$, and (3.7) is a particular case of (2.3). Theorem 3.2 applies. □

Remark 3.1. *Corollary 3.1 with $\omega \equiv 0$ and $c_1 = c_2 = h$ covers the enriched Kannan-type mappings of [4]. For $\beta = 0$, the scheme* (2.9) *is the Krasnosel'skii scheme* (2.11) *studied in [7], and both proofs above remain valid in that case, since then $y_n = x_n$ and $q_n = r_n$. Theorems 3.1 and 3.2 therefore contain the corresponding statements for that scheme.*

4. Rate of convergence and comparison with KIM

The results of Section 3 give convergence but no information on the speed of the two schemes. In this section, we compare (2.9) with (2.11) under a common stopping rule and quantify the difference. Throughout the section, $x_0 = z_0$, so that $r_0 = \widehat{r_0}$ and r_n, $\widehat{r_n}$, q_n are as in (3.1) and (2.12).

Lemma 4.1. *Let $S : \mathbb{C} \to \mathbb{C}$ satisfy* (2.6) *with $h \in [0, 1)$. Then*

$$\|S(Sv) - Sv\| \leq h\,\|Sv - v\| \qquad \text{for every } v \in \mathbb{C}.$$

Proof. Put $a = \|S(Sv) - Sv\|$ and $b = \|Sv - v\|$, and apply (2.6) to the pair (Sv, v) to obtain $a \leq h\max\{a, b\}$. If $\max\{a, b\} = a$, then $a \leq ha$ with $h < 1$ forces $a = 0 \leq hb$. If $\max\{a, b\} = b$, then $a \leq hb$ directly. □

Theorem 4.1. *Let $T : \mathbb{C} \to \mathbb{C}$ be a (κ, h)-enriched Bianchini mapping with $h \in [0, 1)$, let T be continuous, and let $\beta \in (h, 1)$. Put*

$$\theta := 1 - \beta + h \in (h, 1). \tag{4.1}$$

Then, for every $n \geq 0$,

(i) $q_n \leq \theta\, r_n$;
(ii) $r_{n+1} \leq h\,\theta\, r_n$, *and hence* $r_n \leq (h\theta)^n r_0$;
(iii) $\widehat{r_{n+1}} \leq h\widehat{r_n}$, *and hence* $\widehat{r_n} \leq h^n\widehat{r_0}$;
(iv) Condition 3.1 is satisfied, so Theorem 3.1 applies, and T has exactly one fixed point $p \in \mathbb{C}$; moreover

$$\|x_n - p\| \leq (1 + h)\, r_n \leq (1 + h)(h\theta)^n r_0, \qquad \|z_n - p\| \leq (1 + h)\widehat{r_n} \leq (1 + h)h^n\widehat{r_0}.$$

Since $\theta < 1$, the guaranteed contraction factor $h\theta$ for the residual of (2.9) *is strictly smaller than the corresponding factor h for* (2.11).

Proof. By Proposition 2.1(a), S satisfies (2.6) with the same h.

(i) From (2.9), $Sx_n - y_n = Sx_n - (1-\beta)x_n - \beta Sx_n = (1-\beta)(Sx_n - x_n)$, so $\|Sx_n - y_n\| = (1-\beta)r_n$. Hence

$$q_n = \|Sy_n - y_n\| \le \|Sy_n - Sx_n\| + \|Sx_n - y_n\| \le h\max\{q_n, r_n\} + (1-\beta)r_n.$$

If $q_n \le r_n$, this gives $q_n \le (h + 1 - \beta)r_n = \theta r_n$. If $q_n > r_n$, it gives $q_n(1-h) \le (1-\beta)r_n$, that is, $q_n \le \frac{1-\beta}{1-h} r_n$; since

$$\theta - \frac{1-\beta}{1-h} = (1 - \beta + h) - \frac{1-\beta}{1-h} = \frac{h(\beta - h)}{1-h} \ge 0$$

for $\beta \ge h$ and $h < 1$, we again obtain $q_n \le \theta r_n$.

(ii) By (2.9), $x_{n+1} = Sy_n$, so Lemma 4.1 with $v = y_n$ gives $r_{n+1} = \|S(Sy_n) - Sy_n\| \le h\|Sy_n - y_n\| = h\,q_n \le h\theta r_n$, and the geometric bound follows by induction.

(iii) By (2.11), $z_{n+1} = Sz_n$, so Lemma 4.1 with $v = z_n$ gives $\widehat{r}_{n+1} \le h\widehat{r}_n$.

(iv) Since $h\theta < 1$, part (ii) gives $r_n \to 0$, and part (i) then gives $q_n \to 0$; this is Condition 3.1. Theorem 3.1 therefore yields a unique fixed point $p \in \mathbb{C}$ with $x_n \to p$. Finally, using (2.6) with the pair (x_n, p) and $Sp = p$,

$$\|x_n - p\| \le \|x_n - Sx_n\| + \|Sx_n - Sp\| \le r_n + h\max\{r_n, 0\} = (1+h)r_n,$$

and the same computation with z_n in place of x_n gives the second bound. □

Corollary 4.1. *Assume the hypotheses of Theorem 4.1 with $h \in (0,1)$, and let $0 < \varepsilon < r_0 = \widehat{r}_0$. Then the stopping rule* (2.12) *is met after at most*

$$N_{\mathrm{HKIM}} = \left\lceil \frac{\ln(r_0/\varepsilon)}{\ln(1/(h\theta))} \right\rceil \quad \text{steps of (2.9)}, \qquad N_{\mathrm{KIM}} = \left\lceil \frac{\ln(\widehat{r}_0/\varepsilon)}{\ln(1/h)} \right\rceil \quad \text{steps of (2.11)}, \tag{4.2}$$

and the two bounds are related by

$$\frac{\ln(r_0/\varepsilon)}{\ln(1/(h\theta))} \Big/ \frac{\ln(\widehat{r}_0/\varepsilon)}{\ln(1/h)} = \frac{\ln(1/h)}{\ln(1/h) + \ln(1/\theta)} < 1. \tag{4.3}$$

The reduction factor in (4.3) *depends only on h and β; it is independent of the tolerance ε and of the initial point.*

Proof. By Theorem 4.1(ii), $r_n \le (h\theta)^n r_0 \le \varepsilon$ as soon as $n\ln(1/(h\theta)) \ge \ln(r_0/\varepsilon)$, which is the first bound in (4.2); the second is obtained in the same way from Theorem 4.1(iii). Dividing the two expressions and using $\ln(1/(h\theta)) = \ln(1/h) + \ln(1/\theta)$ together with $\theta < 1$ gives (4.3). □

The next statement covers the situation in which the averaged operator is a contraction, which is the case in most of the examples of Sections 5 to 7. It removes the restriction $\beta > h$ and holds for every $\beta \in (0,1)$.

Corollary 4.2. *Let $S = T_\chi$ satisfy $\|Sx - Sy\| \le \delta\|x - y\|$ for all $x, y \in \mathbb{C}$ and some $\delta \in [0, 1)$, and let $\beta \in (0, 1)$. This holds, in particular, when T satisfies* (2.3) *with $\omega(t) = at$, $a \in [0, \kappa + 1)$, and $f \equiv 0$, in which case $\delta = a/(\kappa + 1)$. Put*

$$\theta_\delta := 1 - \beta(1 - \delta) \in (\delta, 1). \tag{4.4}$$

Then T has exactly one fixed point $p \in \mathbb{C}$ and, for every $n \ge 0$,

$$\|x_{n+1} - p\| \le \delta\,\theta_\delta \|x_n - p\|, \qquad r_{n+1} \le \delta\,\theta_\delta\, r_n, \qquad \|z_{n+1} - p\| \le \delta\|z_n - p\|, \qquad \widehat{r}_{n+1} \le \delta\widehat{r}_n.$$

Consequently, if $\delta > 0$ and $0 < \varepsilon < r_0 = \widehat{r}_0$, the stopping rule (2.12) *is met after at most $\lceil \ln(r_0/\varepsilon)/\ln(1/(\delta\theta_\delta))\rceil$ steps of* (2.9) *and at most $\lceil \ln(\widehat{r}_0/\varepsilon)/\ln(1/\delta)\rceil$ steps of* (2.11).

Proof. The Banach contraction principle applied to S on the closed set $\mathbb{C}$ gives a unique p with $Sp = p$, and $Fix(S) = Fix(T)$. If T satisfies (2.3) with $\omega(t) = at$ and $f \equiv 0$, then (2.7) reads $\|Sx - Sy\| \le \chi a\|x - y\|$ with $\chi a = a/(\kappa + 1) < 1$.

From (2.9) and $Sp = p$,

$$\|y_n - p\| \le (1 - \beta)\|x_n - p\| + \beta\|Sx_n - Sp\| \le [(1 - \beta) + \beta\delta]\|x_n - p\| = \theta_\delta\|x_n - p\|,$$

and therefore $\|x_{n+1} - p\| = \|Sy_n - Sp\| \le \delta\|y_n - p\| \le \delta\theta_\delta\|x_n - p\|$. For the residual, the identity $Sx_n - y_n = (1 - \beta)(Sx_n - x_n)$ gives

$$q_n \le \|Sy_n - Sx_n\| + \|Sx_n - y_n\| \le \delta\|y_n - x_n\| + (1 - \beta)r_n = [\delta\beta + 1 - \beta]r_n = \theta_\delta r_n,$$

and $r_{n+1} = \|S(Sy_n) - Sy_n\| \le \delta\|Sy_n - y_n\| = \delta q_n \le \delta\theta_\delta r_n$. The two statements for (2.11) are immediate from $\|Sz_n - Sp\| \le \delta\|z_n - p\|$ and $\widehat{r}_{n+1} = \|S(Sz_n) - Sz_n\| \le \delta\widehat{r}_n$. The iteration counts follow as in Corollary 4.1. □

Remark 4.1. *Theorem 4.1 and Corollary 4.2 compare the two schemes per outer iteration. One step of* (2.9) *uses two evaluations of T, and one step of* (2.11) *uses one, so on the scale of evaluations of T, the guaranteed reduction factors are $(h\theta)^{1/2}$ and h, respectively, and $\theta > h$ gives $(h\theta)^{1/2} > h$. The statements above therefore assert a reduction of the number of outer iterations under a common stopping rule, and they do not assert a reduction of the number of evaluations of T. The distinction is relevant whenever the cost of an outer step is not dominated by the evaluation of T, for instance when each step carries a fixed overhead such as assembly, synchronisation, or data transfer, or when the quantity of interest is the number of outer steps itself. The numerical results of Section 5 report both counts.*

5. Numerical experiments

All comparisons use the same mapping T, the same κ and $\chi = 1/(\kappa+1)$, the same initial point $x_0 = z_0$, the same tolerance, and the same arithmetic. Computations are carried out in IEEE 754 binary 64 arithmetic with

$$\varepsilon = 10^{-6}, \qquad N_{\max} = 1000, \tag{5.1}$$

and both schemes are stopped by the residual rule (2.12). Two counts are reported for each test: the number of outer iterations and the number N_T of evaluations of the mapping T, where one step of (2.9)

contributes two evaluations and one step of (2.11) contributes one. The evaluation of S used in the diagnostic residual test is not included in N_T. In every example below, the defining inequality (2.3) is verified explicitly, and the parameters κ, χ, and β, the domain, the initial point, and the fixed point are stated.

Example 5.1. *Let $\mathbb{C} = \mathbb{B} = \mathbb{R}$ with the absolute value norm, $\kappa = 1$, $\chi = \frac{1}{2}$, $\beta = \frac{1}{2}$, and*

$$Tx = \frac{x}{3} + \frac{2}{3}, \qquad \omega(t) = \frac{t}{2}, \qquad f(u, v) = 2(u + v).$$

Then $\omega \in \Phi_1$, because $\frac{t}{2} < 2t$ for $t > 0$, and $f \in \mathcal{F}$. For $x, y \in \mathbb{R}$,

$$|\kappa(x - y) + Tx - Ty| = \frac{4}{3}|x - y|, \qquad |Tx - x| = \frac{2}{3}|1 - x|,$$

so that

$$\omega(|x - y|) + f(|Tx - x|, |Ty - y|) = \frac{1}{2}|x - y| + \frac{4}{3}(|1 - x| + |1 - y|) \geq \frac{1}{2}|x - y| + \frac{4}{3}|x - y| \geq \frac{4}{3}|x - y|,$$

and (2.3) *holds. The averaged operator is $Sx = \frac{2}{3}x + \frac{1}{3}$, a contraction with $\delta = \frac{2}{3}$, and the unique fixed point is $p = 1$. We take $x_0 = z_0 = 0.1$.*

Example 5.2. *Let $\mathbb{C} = \mathbb{B} = \mathbb{R}$, $\kappa = 1$, $\chi = \frac{1}{2}$, $\beta = 0.8$, and*

$$Tx = \frac{x + 6}{2}, \qquad \omega(t) = \frac{t}{2}, \qquad f(u, v) = 2(u + v).$$

The equation $x = (x + 6)/2$ gives the unique fixed point $p = 6$. Here,

$$|\kappa(x - y) + Tx - Ty| = \frac{3}{2}|x - y|, \qquad |Tx - x| = \frac{1}{2}|6 - x|,$$

so

$$\omega(|x - y|) + f(|Tx - x|, |Ty - y|) = \frac{1}{2}|x - y| + (|6 - x| + |6 - y|) \geq \frac{1}{2}|x - y| + |x - y| = \frac{3}{2}|x - y|,$$

and (2.3) *holds. The averaged operator is $Sx = \frac{3}{4}x + \frac{3}{2}$, with $\delta = \frac{3}{4}$. We take $x_0 = z_0 = 1$.*

Example 5.3. *Let $\mathbb{C} = [\frac{1}{2}, 2] \subset \mathbb{R} = \mathbb{B}$ and $Tx = \frac{1}{x}$, with $\kappa = 1$, $\chi = \frac{1}{2}$, $\beta = 0.8$, $\omega(t) = \frac{t}{2}$, and $f(u, v) = 2(u + v)$. The set $\mathbb{C}$ is closed and convex and $T(\mathbb{C}) \subseteq \mathbb{C}$. For $x, y \in \mathbb{C}$, we have $xy \in [\frac{1}{4}, 4]$ and therefore*

$$|\kappa(x - y) + Tx - Ty| = |x - y|\left|1 - \frac{1}{xy}\right| \leq 3|x - y|.$$

Moreover $|Tx - x| = |x - 1|\frac{x+1}{x} \geq \frac{3}{2}|x - 1|$ on $\mathbb{C}$, so

$$|x - y| \leq |x - 1| + |y - 1| \leq \frac{2}{3}(|Tx - x| + |Ty - y|),$$

which gives

$$|\kappa(x - y) + Tx - Ty| \leq 2(|Tx - x| + |Ty - y|) \leq \omega(|x - y|) + f(|Tx - x|, |Ty - y|).$$

The unique fixed point in $\mathbb{C}$ *is* $p = 1$*, and we take* $x_0 = z_0 = \frac{1}{2}$.

The mapping T *also satisfies the enriched Bianchini condition* (2.2) *on* $\mathbb{C}$ *with* $\kappa = 1$ *and* $h = \frac{2}{3}$. *Put* $\psi(z) = z + \frac{1}{z}$ *and* $\phi(z) = z - \frac{1}{z}$*, so that* $\kappa(x - y) + Tx - Ty = \psi(x) - \psi(y)$ *and* $Tz - z = -\phi(z)$. *Since*

$$\psi(x) - \psi(y) = (x - y)\frac{xy - 1}{xy}, \qquad \phi(x) - \phi(y) = (x - y)\frac{xy + 1}{xy},$$

we obtain, for $x \neq y$,

$$\psi(x) - \psi(y) = \frac{xy - 1}{xy + 1}(\phi(x) - \phi(y)). \tag{5.2}$$

If x *and* y *lie on the same side of* 1, *then* $\phi(x)$ *and* $\phi(y)$ *have the same sign, so* $|\phi(x) - \phi(y)| \leq \max\{|\phi(x)|, |\phi(y)|\}$, *while* $xy \in [\frac{1}{4}, 4]$ *gives* $\left|\frac{xy-1}{xy+1}\right| \leq \frac{3}{5}$. *If* x *and* y *lie on opposite sides of* 1, *then* $|\phi(x) - \phi(y)| = |\phi(x)| + |\phi(y)| \leq 2\max\{|\phi(x)|, |\phi(y)|\}$, *while now* xy *lies between* $\frac{1}{2}$ *and* 2, *which gives* $\left|\frac{xy-1}{xy+1}\right| \leq \frac{1}{3}$. *In both cases,* (5.2) *yields*

$$|\kappa(x - y) + Tx - Ty| \leq \tfrac{2}{3}\max\{|Tx - x|, |Ty - y|\}, \qquad x, y \in \mathbb{C}.$$

Since $\beta = 0.8 \in (\frac{2}{3}, 1)$, *Theorem 4.1 applies, and in particular Condition 3.1 is satisfied. The averaged operator* $Sx = \frac{1}{2}(x + \frac{1}{x})$ *is not a contraction on* $\mathbb{C}$, *since* $|S'(\frac{1}{2})| = \frac{3}{2}$, *so Corollary 4.2 does not apply, and this example is omitted from Table 2.*

Example 5.4. *Let* $\mathbb{B} = \mathbb{C} = \mathbb{R}^3$ *with the Euclidean norm,* $\kappa = 1$, $\chi = \frac{1}{2}$, $\beta = 0.75$, *and*

$$Tx = -x + Mx + b, \qquad M = \begin{pmatrix} 0.30 & 0.20 & -0.10 \\ 0.10 & 0.40 & 0.20 \\ -0.20 & 0.10 & 0.30 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}.$$

Then $\kappa(x - y) + Tx - Ty = M(x - y)$, *so* (2.3) *holds with* $\omega(t) = \|M\|_2\, t$ *and* $f \equiv 0$, *where* $\|M\|_2 = 0.528814 < 2 = \kappa + 1$, *so that* $\omega \in \Phi_1$. *The averaged operator is* $Sx = \frac{1}{2}(Mx + b)$, *a contraction with* $\delta = \frac{1}{2}\|M\|_2 = 0.264407$, *and the unique fixed point is the solution of* $(2I - M)p = b$, *namely*

$$p = (0.370615,\ -1.019193,\ 1.661152)^\top \quad \textit{to six decimal places.}$$

We take $x_0 = z_0 = (5, -5, 5)^\top$.

Example 5.5. *Let* $\mathbb{B} = \mathbb{C} = C[0, 1]$ *with the supremum norm* $\|u\|_\infty = \max_{0\leq s\leq 1} |u(s)|$, *and let*

$$(\mathcal{K}u)(s) = \int_0^1 \frac{u(t)}{1 + s + t}\, dt, \qquad g(s) = s^2, \qquad Tu = -u + \mathcal{K}u + g,$$

with $\kappa = 1$, $\chi = \frac{1}{2}$, *and* $\beta = 0.75$. *Then* $\kappa(u - v) + Tu - Tv = \mathcal{K}(u - v)$ *and*

$$\|\mathcal{K}\| = \max_{0\leq s\leq 1} \int_0^1 \frac{dt}{1 + s + t} = \max_{0\leq s\leq 1} \ln\frac{2 + s}{1 + s} = \ln 2,$$

so (2.3) *holds with* $\omega(t) = (\ln 2)\, t$ *and* $f \equiv 0$; *since* $\ln 2 < 2 = \kappa + 1$, *we have* $\omega \in \Phi_1$. *The averaged operator is* $Su = \frac{1}{2}(\mathcal{K}u + g)$, *a contraction with* $\delta = \frac{1}{2}\ln 2 = 0.346574$, *and its unique fixed point is the solution* u^* *of the Fredholm integral equation of the second kind*

$$u(s) - \frac{1}{2}\int_0^1 \frac{u(t)}{1 + s + t}\, dt = \frac{s^2}{2}, \qquad 0 \leq s \leq 1.$$

The iterations are carried out on a uniform grid of 801 *nodes with the composite trapezoidal rule, and norms are computed as maxima over the grid. The computed solution satisfies* $u^*(0) = 0.066771$, $u^*(1/2) = 0.175863$, *and* $u^*(1) = 0.541154$, *with* $\|u^*\|_\infty = 0.541154$. *We take* $u_0 = 0$.

Results

Table 1 lists the outcome of the protocol (5.1) for Examples 5.1 to 5.5 and for the two application examples of Sections 6 and 7. Figure 1 shows the corresponding residual histories on a logarithmic scale.

Table 1. Outer iterations and evaluations of T required to meet the residual rule (2.12) with $\varepsilon = 10^{-6}$ and $N_{\max} = 1000$. Both schemes start from the same initial point.

					outer iterations		evaluations N_T	
Example	Space	κ	χ	β	HKIM	KIM	HKIM	KIM
5.1	$\mathbb{R}$	1	1/2	0.5	22	32	44	32
5.2	$\mathbb{R}$	1	1/2	0.8	28	49	56	49
5.3	$[\frac{1}{2}, 2]$	1	1/2	0.8	2	4	4	4
5.4	$\mathbb{R}^3$	1	1/2	0.75	7	11	14	11
5.5	$C[0, 1]$	1	1/2	0.75	6	10	12	10
6.1	$\mathbb{R}^2$	0	1	0.7	10	15	20	15
7.1	$\mathbb{R}^2$	0	1	0.7	12	20	24	20

Table 2 compares the observed counts with the bounds of Corollary 4.2. In every case, the observed count is at most the predicted bound, and in Examples 5.1, 5.2, and 7.1, the two agree exactly. Example 5.3 is omitted from Table 2 because its averaged operator is not a contraction, so that it is covered by Theorem 4.1 rather than by Corollary 4.2.

Table 2. Contraction factors and the iteration bounds of Corollary 4.2 against the observed counts of Table 1.

				HKIM		KIM	
Example	δ	θ_δ	$\delta\theta_\delta$	bound	observed	bound	observed
5.1	0.666667	0.833333	0.555556	22	22	32	32
5.2	0.750000	0.800000	0.600000	28	28	49	49
5.4	0.264407	0.448305	0.118535	8	7	12	11
5.5	0.346574	0.509930	0.176728	8	6	13	10
6.1	0.523607	0.666525	0.348997	14	10	22	15
7.1	0.500000	0.650000	0.325000	12	12	20	20

The tables support two statements. First, under the common stopping rule (2.12), the scheme (2.9) needs fewer outer iterations than (2.11) in each of the reported tests, and the observed reduction is consistent with the factor predicted by Corollaries 4.1 and 4.2. Second, the columns headed N_T show that the reduction of the outer iteration count is not accompanied by a reduction of the number of evaluations of T, in agreement with Remark 4.1. The abstract and the conclusion are phrased accordingly.

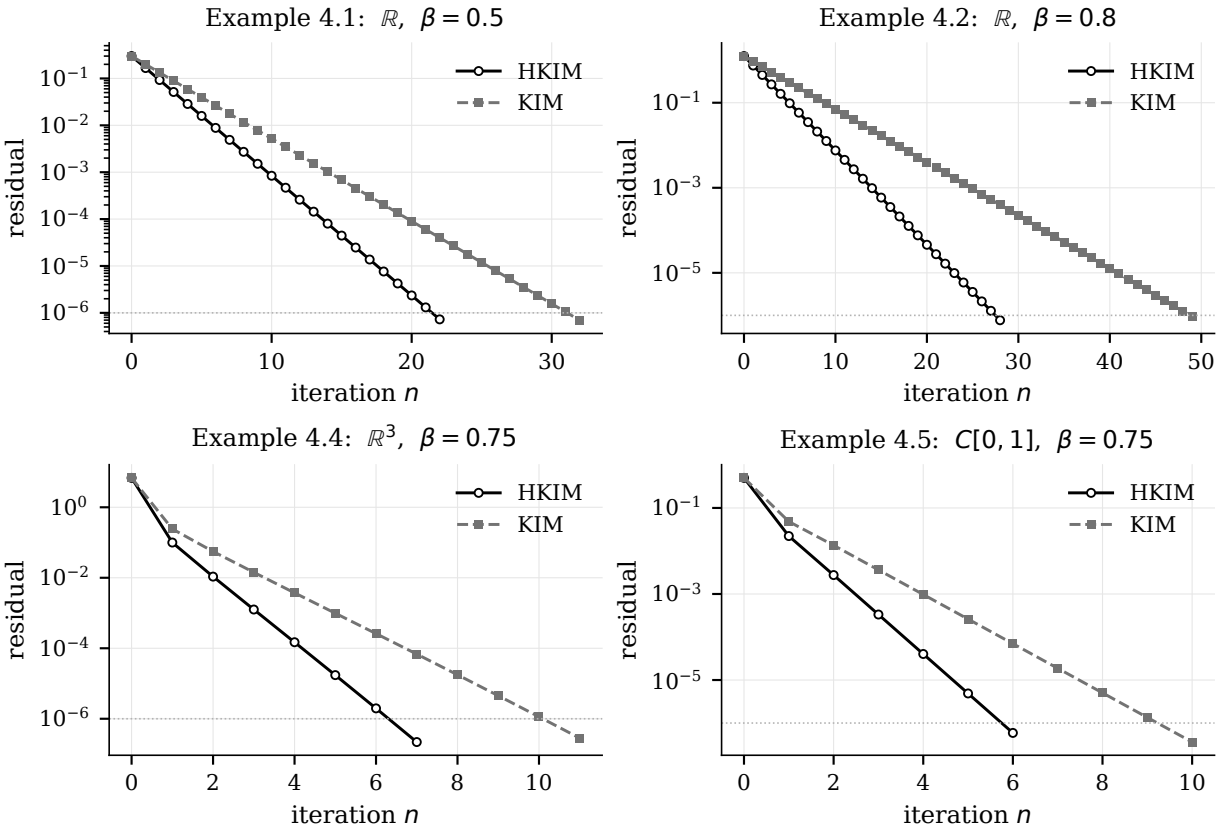


Figure 1. Residual $\|S\,w_n - w_n\|$ against the outer iteration index for Examples 5.1–5.5. The dotted line marks the tolerance $\varepsilon = 10^{-6}$.

6. Application to the variational inequality problem

Let $\mathbb{H}$ be a real Hilbert space, let $\mathbb{C} \subseteq \mathbb{H}$ be nonempty, closed, and convex, and let $G : \mathbb{H} \to \mathbb{H}$. The VIP is to find $x^* \in \mathbb{C}$ such that

$$\langle Gx^*,\, x - x^* \rangle \geq 0 \qquad \text{for every } x \in \mathbb{C}. \tag{6.1}$$

For $\gamma > 0$, define

$$F_\gamma := P_{\mathbb{C}}\,(I - \gamma G), \tag{6.2}$$

where $P_{\mathbb{C}}$ is the metric projection onto $\mathbb{C}$. The projection characterisation of (6.1) gives

$$x^* \text{ solves (6.1)} \iff x^* = F_\gamma x^*, \tag{6.3}$$

so that (6.1) is a fixed point problem for F_γ. The next statement identifies an interval of admissible γ for which F_γ belongs to the class of Definition 2.3.

Proposition 6.1. *Let G be η-strongly monotone and L-Lipschitz on $\mathbb{H}$, that is, $\langle Gx - Gy, x - y\rangle \geq \eta\|x-y\|^2$ and $\|Gx - Gy\| \leq L\|x - y\|$ with $\eta > 0$, and let*

$$0 < \gamma < \frac{2\eta}{L^2}. \tag{6.4}$$

Then F_γ is a contraction on $\mathbb{C}$ with constant $\delta = \sqrt{1-2\gamma\eta+\gamma^2L^2} < 1$. In particular, F_γ is a generalized $(0,\omega,f)$-enriched Bianchini mapping with $\omega(t) = \delta t$ and $f \equiv 0$, and (6.1) *has exactly one solution.*

Proof. For $x, y \in \mathbb{H}$,

$$\begin{aligned}\|(I-\gamma G)x-(I-\gamma G)y\|^2 &= \|x-y\|^2 - 2\gamma\langle Gx-Gy, x-y\rangle + \gamma^2\|Gx-Gy\|^2\\ &\le (1-2\gamma\eta+\gamma^2L^2)\|x-y\|^2,\end{aligned}$$

and (6.4) gives $1-2\gamma\eta+\gamma^2L^2 < 1$. Since $P_{\mathbb{C}}$ is nonexpansive, F_γ is a δ-contraction. Taking $\kappa = 0$ in (2.3), so that $\chi = 1$, the inequality $\|F_\gamma x - F_\gamma y\| \le \delta\|x-y\| = \omega(\|x-y\|)$ is exactly (2.3) with $f \equiv 0$, and $\omega \in \Phi_0$ because $\delta < 1$. Existence and uniqueness of the solution follow from (6.3) and the Banach contraction principle. □

Proposition 6.1 makes the scheme (2.9) applicable with $T = F_\gamma$, $\kappa = 0$, and $\chi = 1$, so that $S_\gamma := (1-\chi)I + \chi F_\gamma = F_\gamma$. Starting from $x_0 \in \mathbb{C}$, we compute

$$\begin{cases} y_n = (1-\beta)x_n + \beta F_\gamma x_n,\\ x_{n+1} = F_\gamma y_n, \end{cases} \qquad \beta \in (0,1) \tag{6.5}$$

and stop at the first n with $\|F_\gamma x_n - x_n\| \le \varepsilon$.

Theorem 6.1. *Under the hypotheses of Proposition 6.1, the sequence $\{x_n\}$ generated by* (6.5) *converges to the unique solution x^* of* (6.1) *for every $x_0 \in \mathbb{C}$ and every $\beta \in (0,1)$, and, with $\theta_\delta = 1-\beta(1-\delta)$,*

$$\|x_n - x^*\| \le (\delta\theta_\delta)^n\|x_0 - x^*\|, \qquad \|F_\gamma x_n - x_n\| \le (\delta\theta_\delta)^n\|F_\gamma x_0 - x_0\|.$$

The corresponding bounds for (2.11) *carry the factor δ in place of $\delta\theta_\delta$.*

Proof. Combine Proposition 6.1 with Theorem 3.2 and Corollary 4.2, applied with $\kappa = 0$, $\chi = 1$, and $S = F_\gamma$. □

Example 6.1. *Let $\mathbb{H} = \mathbb{R}^2$, $\mathbb{C} = [0,1]^2$, and $Gx = Mx - b$ with*

$$M = \begin{pmatrix} 4 & 1\\ 1 & 3\end{pmatrix}, \qquad b = \begin{pmatrix} 6\\ 2\end{pmatrix}.$$

The matrix M is symmetric positive definite with eigenvalues $\lambda_{\min} = \frac{7-\sqrt{5}}{2} = 2.381966$ and $\lambda_{\max} = \frac{7+\sqrt{5}}{2} = 4.618034$, so G is η-strongly monotone and L-Lipschitz with $\eta = \lambda_{\min}$ and $L = \lambda_{\max}$, and (6.4) *reads $0 < \gamma < 2\lambda_{\min}/\lambda_{\max}^2 = 0.223$. We take $\gamma = 0.2$. Since M is symmetric, the sharp contraction constant of $I - \gamma M$ is*

$$\delta = \|I-\gamma M\|_2 = \max\{|1-\gamma\lambda_{\min}|, |1-\gamma\lambda_{\max}|\} = 0.523607.$$

The unique solution of (6.1) *is $x^* = (1, \frac{1}{3})^\top$. Indeed, $Gx^* = (-\frac{5}{3}, 0)^\top$ and, for every $x \in [0,1]^2$,*

$$\langle Gx^*, x - x^*\rangle = -\tfrac{5}{3}(x_1 - 1) \ge 0$$

because $x_1 \le 1$. With $x_0 = (0,1)^\top$, $\beta = 0.7$, and the protocol (5.1)*, the scheme* (6.5) *meets the stopping rule after* 10 *outer iterations, whereas the Krasnosel'skii scheme needs* 15*; the corresponding evaluation counts are $N_T = 20$ and $N_T = 15$. The residual histories are shown in the left panel of Figure 2.*

7. Application to the split feasibility problem

Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be real Hilbert spaces, let $\mathbb{C} \subseteq \mathbb{H}_1$ and $\mathbb{Q} \subseteq \mathbb{H}_2$ be nonempty, closed, and convex, and let $A : \mathbb{H}_1 \to \mathbb{H}_2$ be a bounded linear operator with $A \neq 0$. The SFP is to find

$$x^* \in \mathbb{C} \qquad \text{such that} \qquad Ax^* \in \mathbb{Q}. \tag{7.1}$$

Problem (7.1) was introduced by Censor and Elfving [10] and studied by Byrne [9] and by Reich et al., [25]. For

$$0 < \gamma < \frac{2}{\|A\|^2}, \tag{7.2}$$

define

$$F_\gamma x := P_{\mathbb{C}}(x - \gamma A^*(I - P_{\mathbb{Q}})Ax). \tag{7.3}$$

Every solution of (7.1) is a fixed point of F_γ and, conversely, if (7.1) admits at least one solution, then every fixed point of F_γ solves (7.1). We call (7.1) consistent when it admits at least one solution; for consistent data, (7.1) is therefore a fixed point problem for F_γ. The following statement gives a class of data for which F_γ belongs to the class of Definition 2.3.

Proposition 7.1. *Let $\mathbb{Q} = q + V$ be an affine subset of $\mathbb{H}_2$, where $q \in \mathbb{H}_2$ and $V \subseteq \mathbb{H}_2$ is a closed subspace, and put $B := (I - P_V)A$. Assume that B is bounded below, that is, $\|Bx\| \geq \sigma\|x\|$ for some $\sigma > 0$ and all $x \in \mathbb{H}_1$. Then, for every γ satisfying* (7.2)*, the operator F_γ of* (7.3) *is a contraction on $\mathbb{C}$ with constant*

$$\delta = \|I - \gamma B^* B\| \leq \max\{|1 - \gamma\sigma^2|,\ |1 - \gamma\|B\|^2|\} < 1,$$

and therefore a generalized $(0, \omega, f)$-enriched Bianchini mapping with $\omega(t) = \delta t$ and $f \equiv 0$.

Proof. Since $\mathbb{Q}$ is affine, $P_{\mathbb{Q}}y = q + P_V(y - q)$, hence $(I - P_{\mathbb{Q}})y = (I - P_V)(y - q)$. Using $(I - P_V)^* = I - P_V = (I - P_V)^2$, we obtain $A^*(I - P_V)A = B^*B$, so

$$x - \gamma A^*(I - P_{\mathbb{Q}})Ax = (I - \gamma B^* B)x + \gamma A^*(I - P_V)q,$$

an affine map with linear part $I - \gamma B^* B$. The spectrum of the self-adjoint operator B^*B is contained in $[\sigma^2, \|B\|^2]$, so $\|I - \gamma B^* B\| \leq \max\{|1 - \gamma\sigma^2|, |1 - \gamma\|B\|^2|\}$. This quantity is smaller than 1 because $\sigma > 0$ and, by $\|B\| \leq \|A\|$ and (7.2), $\gamma\|B\|^2 < 2$. Composing with the nonexpansive projection $P_{\mathbb{C}}$ preserves the contraction constant. The last assertion follows as in Proposition 6.1. □

With $T = F_\gamma$, $\kappa = 0$, and $\chi = 1$, the scheme (2.9) reads

$$\begin{cases} y_n = (1 - \beta)x_n + \beta F_\gamma x_n, \\ x_{n+1} = F_\gamma y_n, \end{cases} \qquad \beta \in (0, 1), \tag{7.4}$$

and admissible stopping quantities are the fixed point residual $\|F_\gamma x_n - x_n\|$ and the feasibility residual $\operatorname{dist}(x_n, \mathbb{C}) + \operatorname{dist}(Ax_n, \mathbb{Q})$.

Theorem 7.1. *Let the hypotheses of Proposition 7.1 hold, and let* (7.1) *be consistent. Then* (7.1) *has exactly one solution x^*, and the sequence generated by* (7.4) *converges to x^* for every $x_0 \in \mathbb{C}$ and every $\beta \in (0, 1)$, with*

$$\|x_n - x^*\| \leq (\delta\theta_\delta)^n \|x_0 - x^*\|, \qquad \theta_\delta = 1 - \beta(1 - \delta).$$

Proof. By Proposition 7.1, the operator F_γ is a δ-contraction of the closed set $\mathbb{C}$ into itself, so it has exactly one fixed point x^*. Consistency of (7.1) turns this fixed point into the unique solution of (7.1). The convergence and the rate follow from Theorem 3.2 and Corollary 4.2 applied with $\kappa = 0$, $\chi = 1$, and $S = F_\gamma$. □

Example 7.1. *Let* $\mathbb{H}_1 = \mathbb{R}^2$, $\mathbb{H}_2 = \mathbb{R}^3$,

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{pmatrix}, \qquad \mathbb{C} = [0,2]^2, \qquad \mathbb{Q} = \{y \in \mathbb{R}^3 : y_1 = 2,\ y_2 = 0\}.$$

The set $\mathbb{Q}$ *is the affine line* $q + V$ *with* $q = (2,0,0)^\top$ *and* $V = \operatorname{span}\{e_3\}$, *and* $P_{\mathbb{Q}} y = (2,0,y_3)^\top$. *Here,* $A^\top A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$ *has eigenvalues* 2 *and* 7, *so* $\|A\|^2 = 7$ *and* (7.2) *reads* $0 < \gamma < \frac{2}{7} = 0.285714$*; we take* $\gamma = 0.25$. *Moreover* $B = (I - P_V)A$ *has* $B^\top B = 2I$, *so* $\sigma^2 = \|B\|^2 = 2$, *and Proposition 7.1 gives*

$$\delta = \|I - \gamma B^\top B\| = |1 - 2\gamma| = 0.5.$$

The system $Ax \in \mathbb{Q}$ *reads* $x_1 + x_2 = 2$ *and* $x_1 - x_2 = 0$, *so the problem is consistent, and its unique solution is* $x^* = (1,1)^\top \in \mathbb{C}$, *with* $Ax^* = (2,0,3)^\top \in \mathbb{Q}$. *With* $x_0 = (2,0)^\top$, $\beta = 0.7$, *and the protocol* (5.1), *the scheme* (7.4) *meets the stopping rule after* 12 *outer iterations against* 20 *for the Krasnosel'skii scheme; the evaluation counts are* $N_T = 24$ *and* $N_T = 20$. *The residual histories are shown in the right panel of Figure 2.*

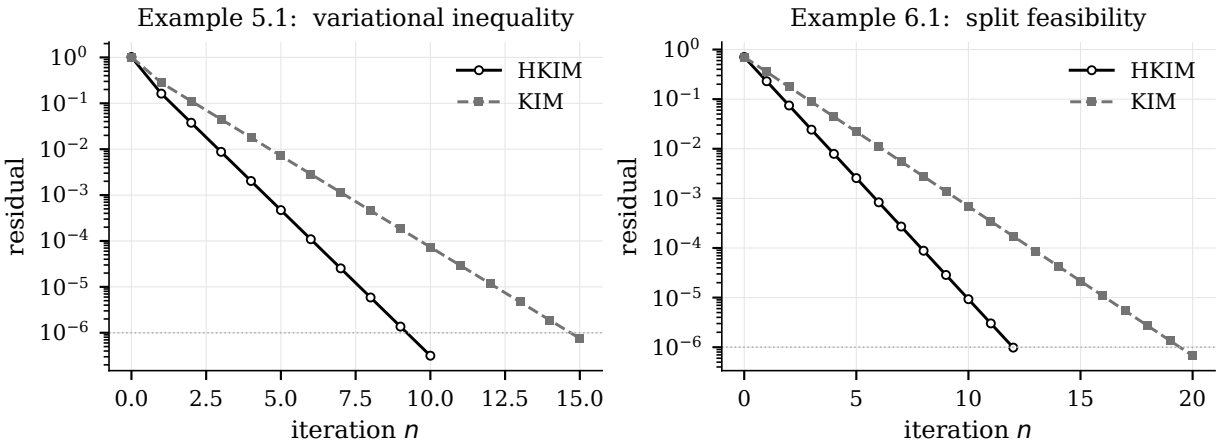


Figure 2. Residual histories for the VIP of Example 6.1 (left) and the SFP of Example 7.1 (right). The dotted line marks the tolerance $\varepsilon = 10^{-6}$.

8. Conclusions

We studied a two-step Krasnosel'skii scheme, the HKIM (2.9), for generalized enriched Bianchini mappings in Banach spaces. Working with the averaged operator T_χ, $\chi = 1/(\kappa + 1)$, we proved the existence and the uniqueness of the fixed point and the convergence of the scheme, first for the enriched

Bianchini condition (2.2) and then for the generalized condition (2.3). We also obtained a quantitative comparison with the KIM: The residual of the two-step scheme is reduced by the guaranteed factor $h\theta$ at every step, where $\theta = 1 - \beta + h < 1$, against the factor h of the one-step scheme, and this gives the explicit bound (4.3) on the reduction of the number of outer iterations. The numerical tests of Section 5, carried out on $\mathbb{R}$, on $\mathbb{R}^3$, and on $C[0, 1]$, agree with these bounds and show fewer outer iterations for the two-step scheme in each of the reported examples. As Remark 4.1 and the columns N_T of Table 1 record, the reduction concerns the number of outer iterations under a common stopping rule and not the number of evaluations of T.

The applicability of the results was illustrated on the VIP and on the SFP through their fixed point reformulations, in each case with an explicit interval of admissible step sizes, an implementable algorithm, and a worked numerical example. A description of classes of split feasibility data beyond the affine case of Proposition 7.1 for which the operator (7.3) satisfies the generalized enriched Bianchini condition would be a natural continuation of this work.

Author contributions

Maryam Iqbal: Conceptualization, methodology, formal analysis, investigation, software, visualization, writing original draft, and writing review and editing; Afshan Batool: Conceptualization, methodology, validation; supervision, writing review and editing, and project administration. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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