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Research article

New results on the geometric behavior of a normalized five-parameter Mittag–Leffler function

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Abstract: Our focus of this study was to investigate geometric properties of a normalized five-parameter Mittag–Leffler function defined in the open unit disk. By employing inequalities for the gamma and digamma functions together with monotonicity techniques for coefficient sequences, we derived sufficient conditions for starlikeness, convexity, close-to-convexity, and uniform convexity. The sufficient conditions were expressed in terms of explicit inequalities involving the parameters and gamma functions, together with parameter-dependent monotonicity conditions obtained from digamma-function estimates. Several results extended earlier studies of the classical two-parameter Mittag–Leffler function. We also discussed special cases that reduced to previously known Mittag–Leffler functions and provided examples to illustrate the applicability of the main results. The findings contribute to the growing literature on geometric function theory associated with special functions and multi-parameter Mittag–Leffler-type functions.

Keywords: Mittag–Leffler function; five-parameter Mittag–Leffler function; starlike and convex functions; convexity and close-to-convexity; uniform convexity
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1. Introduction

Let $\mathbb{D}$ be the open unit disk defined as

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}. \tag{1.1}$$

Let $\mathcal{H}$ represent the class of analytic functions (AFs) in $\mathbb{D}$. If $\mathcal{A}$ is the subclass of $\mathcal{H}$, then

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.2}$$

are functions that are analytic in $\mathbb{D}$ and fulfill $f(0) = 0$ and $f'(0) = 1$. Additionally, let $\mathcal{S}$ represent the class of functions in $\mathcal{A}$ that are univalent (UFs) in $\mathbb{D}$.

Definition 1.1. [1] A function $f \in \mathcal{A}$ is starlike if it maps $\mathbb{D}$ onto a starlike domain (with respect to the origin). This is equivalent to

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D}.$$

We denote the class of starlike functions by $\mathcal{S}^*$.

Definition 1.2. [1] A function $f \in \mathcal{A}$ is convex if it maps $\mathbb{D}$ onto a convex domain, characterized by

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D}.$$

This class is denoted by $\mathcal{K}$. A well-known inclusion relation is $\mathcal{K} \subset \mathcal{S}^*$.

Definition 1.3. [2] A function $f \in \mathcal{A}$ is said to be parabolic starlike if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|, \qquad z \in \mathbb{D}.$$

The class of parabolic starlike functions is denoted by $\mathcal{S}_p$.

Definition 1.4. A function $f \in \mathcal{A}$ is close-to-convex if it satisfies

$$\Re\left(\frac{f'(z)}{g'(z)}\right) > 0, \quad z \in \mathbb{D},$$

for some convex function $g \in \mathcal{K}$ (equivalently, $\Re(zf'(z)/h(z)) > 0$ for the starlike function $h(z) = zg'(z) \in \mathcal{S}^*$). In this case, we say that f is close-to-convex with respect to g. The class of close-to-convex functions, denoted C, is a subset of univalent functions in $\mathbb{D}$. In Section 3.4, the comparison functions are $g(z) = -\log(1 - z)$ and $g(z) = \frac{1}{2}\log\frac{1+z}{1-z}$; both are convex in $\mathbb{D}$, since $1 + zg''(z)/g'(z)$ equals $\frac{1}{1-z}$ and $\frac{1+z^2}{1-z^2}$, respectively, and both have a positive real part in $\mathbb{D}$. Thus, close-to-convexity with respect to these functions means $\Re((1 - z)f'(z)) > 0$ and $\Re((1 - z^2)f'(z)) > 0$ in $\mathbb{D}$, respectively.

Definition 1.5. [3, 4] A function $f \in \mathcal{A}$ is uniformly starlike ($\mathcal{UST}$) if

$$\Re\left(\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)}\right) > 0, \quad z, \zeta \in \mathbb{D}.$$

Definition 1.6. [3, 4] Let $f \in \mathcal{A}$ is uniformly convex ($\mathcal{UCV}$) if

$$\Re\left(1 + \frac{(z-\zeta)f''(z)}{f'(z)}\right) > 0, \quad z, \zeta \in \mathbb{D}.$$

References for uniformly starlike and uniformly convex functions include [3,4]. Uniform convexity is a stronger property than convexity and starlikeness, and the inclusion relations

$$\mathcal{UCV} \subset \mathcal{K} \subset \mathcal{S}^*$$

are well known.

The investigation of geometric properties such as univalence, starlikeness, and convexity for AFs defined in $\mathbb{D}$ has remained an active topic of research for many years. In particular, AFs associated with special functions have received considerable attention. Bansal and Prajapat [5] and Noreen et al. [8, 9] studied geometric properties of normalized Mittag–Leffler functions, while Mehrez [6] and Mehrez et al. [7] investigated functions related to the Fox–Wright functions and products of modified Bessel functions, respectively. Among special functions, the Mittag–Leffler (ML) function occupies a prominent position. It arises naturally in fractional calculus (FC), approximation theory (AT), and in diverse areas of science and engineering. Mittag–Leffler functions frequently arise in the solutions of fractional differential (FD) and integral equations, where they serve as natural generalizations of the exponential function while their importance in fractional modeling was highlighted in [10]. From an application-oriented viewpoint, the geometric properties obtained in this paper help describe the mapping and univalent behavior of normalized Mittag–Leffler functions arising in fractional differential and integral models. In particular, starlikeness, convexity, and close-to-convexity provide useful conditions for controlled geometric behavior in the unit disk. Therefore, the parameter conditions established here may be useful in fractional modeling and related analytical problems.

The classical ML function was introduced by Mittag-Leffler [11, 12] in 1903, which is defined as:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}, \qquad z \in \mathbb{C}, \alpha > 0. \tag{1.3}$$

A well-known two parameter generalization is given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta+\alpha k)}, \qquad z \in \mathbb{C}, \alpha, \beta > 0. \tag{1.4}$$

Several authors have examined geometric characteristics of the normalized ML function in the unit disk (see [5, 8, 9]). The normalized form is typically written as

$$\mathbb{E}_{\alpha,\beta}(z) = \Gamma(\beta)\, z\, E_{\alpha,\beta}(z) = z + \sum_{k=1}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha k+\beta)} z^{k+1}. \tag{1.5}$$

Al-Bassam and Luchko [13] suggested an additional extension by introducing the multi-index (vector-index). ML function with parameters of $2m$:

$$E^{(m)}_{(\alpha,\beta)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\prod_{i=1}^{m} \Gamma(\alpha_i k+\beta_i)}, \qquad \alpha_i, \beta_i > 0,\ m \in \mathbb{N}. \tag{1.6}$$

This function was employed to obtain explicit solutions of Cauchy-type problems for FD equations. The growing importance of multi-parameter ML functions, together with their strong connections

to fractional models, naturally motivates the study of their geometric characteristics. In this work, we continue this line of research by investigating geometric characteristics of normalized five-parameter ML functions. The study of geometric characteristics of AFs associated with special functions has developed into an active research area in recent years. Many classical and modern special functions have been investigated in the perspective of geometric function theory, including Bessel functions, Wright functions, and hypergeometric-type functions (see, for example, [14–16]). In particular, attention has been devoted to determining conditions under which such functions are univalent, starlike, convex, or close-to-convex in $\mathbb{D}$.

Mittag–Leffler functions constitute one of the great significant families of special functions in this direction. Since their introduction by Mittag-Leffler [11, 12], these functions have found widespread applications in FC and FD equations. Researchers have also investigated geometric properties of analytic functions associated with fractional differential operators and Mittag–Leffler functions [17,18]. The role of ML functions in fractional modeling and applied problems is highlighted in [10]. Further developments involving Le Roy-type ML functions have opened new directions for research. Analytic properties of these functions, such as complete monotonicity and integral representations, were studied in [19–21], while their geometric properties were investigated by Mehrez and Das [22]. Moreover, developments in GFT have highlighted the importance of special functions and q-calculus in the study of AFs. For instance, the application of q-difference operators to harmonic univalent functions was investigated in [23], where various structural properties were established. In a similar direction, coefficient bounds and Hankel determinants for subclasses of q-starlike functions were examined in [24], providing sharp analytical estimates. The role of special polynomials has also been explored, as seen in [25], where geometric properties of normalized Euler polynomials were studied. Furthermore, general criteria for starlikeness, convexity, and close-to-convexity under fixed initial coefficients were obtained in [26]. Moreover, generalized subclasses of uniformly starlike and uniformly convex functions related with Bessel and Gaussian hypergeometric functions were introduced and analyzed in [27]. These contributions collectively demonstrate the growing interplay between special functions and geometric properties of analytic functions. These generalizations enable greater flexibility in modeling and naturally lead to new questions concerning their geometric structure. Motivated by these developments, researchers aim to extend geometric investigations to broader classes of Mittag-Leffler-type functions with five parameters.

Definition 1.7. Let $d_1, d_2, d_3, d_4, \mu \in \mathbb{C}$ be five parameters satisfying

$$\Re(d_1 + d_2) > 0.$$

The five-parameter Mittag–Leffler function is defined by

$$MF^{\mu}_{d_1,d_2;d_3,d_4}(z) = \sum_{k=0}^{\infty} \frac{(\mu)_k}{\Gamma(d_1k + d_3)\,\Gamma(d_2k + d_4)} \frac{z^k}{k!}, \qquad z \in \mathbb{C},$$

where $(\mu)_k$ denotes the Pochhammer symbol given by

$$(\mu)_k = \frac{\Gamma(\mu + k)}{\Gamma(\mu)}.$$

The function may be defined under the more general complex-parameter setting, subject to the appropriate convergence restrictions. In this paper, however, we restrict ourselves to

$$d_1, d_3, d_4, \mu > 0, \qquad d_2 \geq 0, \qquad z \in \mathbb{D}.$$

Remark 1.1. Under the special choice $d_2 = 0$ and $\mu = 1$, the function $MF^{\mu}_{d_1,d_2;d_3,d_4}$ reduces to the classical two-parameter Mittag–Leffler function.

Definition 1.8. Since $MF^{\mu}_{d_1,d_2;d_3,d_4} \notin \mathcal{A}$, we consider the normalization

$$F^{\mu}_{d_1,d_2;d_3,d_4}(z) = z\,\Gamma(d_3)\Gamma(d_4)\,MF^{\mu}_{d_1,d_2;d_3,d_4}(z). \tag{1.7}$$

Then

$$F^{\mu}_{d_1,d_2;d_3,d_4}(z) = z + \sum_{k=1}^{\infty} \frac{(\mu)_k\,\Gamma(d_3)\Gamma(d_4)}{k!\,\Gamma(d_1k+d_3)\Gamma(d_2k+d_4)} z^{k+1} = z + \sum_{k=1}^{\infty} a_k(d_1,d_2,d_3,d_4,\mu)\,z^{k+1},$$

where

$$a_k(d_1,d_2,d_3,d_4,\mu) = \frac{(\mu)_k\,\Gamma(d_3)\Gamma(d_4)}{k!\,\Gamma(d_1k+d_3)\Gamma(d_2k+d_4)}. \tag{1.8}$$

Although (1.8) holds for $d_1, d_2, d_3, d_4, \mu \in \mathbb{C}$ and also $z \in \mathbb{C}$. However, in this article, we reduce our focus to $z \in \mathbb{D}$, $d_1, d_3, d_4, \mu > 0$ and $d_2 \geq 0$.

Remark 1.2. Under the special choice $d_2 = 0$ and $\mu = 1$, the function $MF^{\mu}_{d_1,d_2;d_3,d_4}$ reduces to the classical two-parameter Mittag–Leffler function. Indeed, since $(1)_k = k!$ and $\Gamma(d_2k + d_4) = \Gamma(d_4)$, we obtain

$$MF^{1}_{d_1,0;d_3,d_4}(z) = \frac{1}{\Gamma(d_4)} \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(d_1k+d_3)} = \frac{1}{\Gamma(d_4)} E_{d_1,d_3}(z),$$

where

$$E_{d_1,d_3}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(d_1k+d_3)}$$

is the classical two-parameter Mittag–Leffler function. Consequently, the normalized function in Definition 1.8 satisfies

$$F^{1}_{d_1,0;d_3,d_4}(z) = z\Gamma(d_3)E_{d_1,d_3}(z).$$

Thus, the normalized five-parameter Mittag–Leffler function contains a normalized form of the classical two-parameter Mittag–Leffler function as a special case.

The main novelty of this work is not only the introduction of additional parameters, but the development of a unified geometric framework for the normalized five-parameter Mittag–Leffler function. Moreover, as the lower-parameter and Le Roy-type Mittag–Leffler functions have already been studied, the analysis here simultaneously incorporates the two denominator parameters d_1, d_2, the two shift parameters d_3, d_4, and the Pochhammer parameter μ. This parameter structure leads to new coefficient sequences and requires parameter-dependent monotonicity conditions involving gamma and digamma functions. Within this framework, Theorems 3.1–3.7 establish sufficient conditions for starlikeness, convexity, parabolic starlikeness, uniform convexity, and close-to-convexity, including results in the unit disk and the half-unit disk. Moreover, the reductions to the two-parameter Mittag–Leffler function prove the consistency with known results while the general five-parameter conditions remain applicable beyond those reductions.

2. Preliminaries

To establish our main results, we will rely on the following lemmas:

Lemma 2.1. *[28] For any $s \in \mathbb{R}^+$, the digamma (psi) function*

$$\Psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$$

hold the following inequality

$$\log(s) - \frac{1}{s} < \Psi(s) < \log(s) - \frac{1}{2s}. \tag{2.1}$$

Lemma 2.2. *[29] Let $f \in \mathcal{A}$. If*

$$|f'(z) - 1| < \frac{2}{\sqrt{5}} \quad (z \in \mathbb{D}),$$

then f is starlike in $\mathbb{D}$.

Lemma 2.3. *[30] Let $f \in \mathcal{A}$. If*

$$\left|\frac{f(z)}{z} - 1\right| < 1 \quad (z \in \mathbb{D}),$$

then f is univalent and starlike in

$$\mathbb{D}_{1/2} = \{z : |z| < 1/2\}.$$

Lemma 2.4. *[31] Let $f \in \mathcal{A}$. If*

$$|f'(z) - 1| < 1 \quad (z \in \mathbb{D}),$$

then f is convex in $\mathbb{D}_{1/2}$.

Lemma 2.5. *[32] Let $f \in \mathcal{A}$. (i) If*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{1}{2},$$

then $f \in \mathcal{UCV}$. (ii) If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{1}{2},$$

then $f \in \mathcal{S}_p$.

Lemma 2.6. *[33] Let*

$$f(z) = z + \sum_{k=2}^{\infty} A_k z^k.$$

If either

$$1 \leq 2A_2 \leq \cdots \leq nA_n \leq (n+1)A_{n+1} \leq \cdots \leq 2,$$

or

$$1 \geq 2A_2 \geq \cdots \geq nA_n \geq (n+1)A_{n+1} \geq \cdots \geq 0,$$

then f is close-to-convex with respect to $-\log(1-z)$.

Lemma 2.7. *[33] Let*

$$f(z) = z + \sum_{k=2}^{\infty} A_{2k-1} z^{2k-1}$$

be analytic in $\mathbb{D}$*. If either*

$$1 \geq 3A_3 \geq \cdots \geq (2k-1)A_{2k-1} \geq \cdots \geq 0,$$

or

$$1 \leq 3A_3 \leq \cdots \leq (2k-1)A_{2k-1} \leq \cdots \leq 2,$$

then f *is close-to-convex with respect to the convex function* $\frac{1}{2}\log\frac{1+z}{1-z}$*, and in particular univalent in* $\mathbb{D}$*.*

Remark 2.1. In Lemma 2.6, the chain starts with the inequality between 1 and $2A_2$, and in Lemma 2.7, the chain starts with the inequality between 1 and $3A_3$. Since the coefficient a_k in (1.8) multiplies z^{k+1}, applying Lemma 2.6 to $F^{\mu}_{d_1,d_2;d_3,d_4}$ requires $A_{k+1} = a_k$; that is, the sequence $\{(k+1)a_k\}_{k\geq 1}$ must be monotone and the initial condition $2a_1 \leq 1$ must hold.

3. Main results

3.1. Starlikeness of the normalized five-parameter Mittag–Leffler function

Theorem 3.1. *Let* $d_1, d_3, d_4, \mu > 0$*, and* $d_2 \geq 0$ *such that*

$$d_1 \geq 1, \qquad d_1^2\mu \geq d_3,$$

and the relation

$$d_1\log(d_1+d_3) + d_2\log(d_2+d_4) - \log(\mu+1) - \frac{2\mu+1}{2(\mu+1)} - \frac{d_1}{d_1+d_3} - \frac{d_2}{d_2+d_4} > 0 \tag{3.1}$$

holds. (i) If

$$2\mu(e-1)\Gamma(d_3)\Gamma(d_4) < \Gamma(d_1+d_3)\Gamma(d_2+d_4), \tag{3.2}$$

then $F^{\mu}_{d_1,d_2;d_3,d_4}(z)$ *is starlike in* $\mathbb{D}$*.*

(ii) If

$$3\mu(e-1)\Gamma(d_3)\Gamma(d_4) < \Gamma(d_1+d_3)\Gamma(d_2+d_4),$$

then $F^{\mu}_{d_1,d_2;d_3,d_4} \in \mathcal{S}_p$*.*

Proof. (i) To prove starlikeness, it suffices to show

$$\left|\frac{z(F^{\mu}_{d_1,d_2;d_3,d_4})'(z)}{F^{\mu}_{d_1,d_2;d_3,d_4}(z)} - 1\right| < 1.$$

We have

$$(F^{\mu}_{d_1,d_2;d_3,d_4})' - \frac{F^{\mu}_{d_1,d_2;d_3,d_4}}{z} = \sum_{k=1}^{\infty} \frac{k(\mu)_k\Gamma(d_3)\Gamma(d_4)}{k!\Gamma(d_1k+d_3)\Gamma(d_2k+d_4)} z^k = \sum_{k=1}^{\infty} \frac{b_k z^k}{k!}, \tag{3.3}$$

where

$$b_k = \frac{k(\mu)_k \Gamma(d_3)\Gamma(d_4)}{\Gamma(d_1 k + d_3)\Gamma(d_2 k + d_4)}, \quad k \geq 1. \tag{3.4}$$

Define

$$f_1(s) = \frac{s(\mu)_s \Gamma(d_3)\Gamma(d_4)}{\Gamma(d_1 s + d_3)\Gamma(d_2 s + d_4)}, \quad s \geq 1. \tag{3.5}$$

Then

$$f_1'(s) = f_1(s) f_2(s), \tag{3.6}$$

where

$$\begin{aligned} f_2(s) &= \frac{1}{s} + \frac{\Gamma'(s+\mu)}{\Gamma(s+\mu)} - d_1 \frac{\Gamma'(d_1 s + d_3)}{\Gamma(d_1 s + d_3)} - d_2 \frac{\Gamma'(d_2 s + d_4)}{\Gamma(d_2 s + d_4)} \\ &= \frac{1}{s} + \Psi(s+\mu) - d_1 \Psi(d_1 s + d_3) - d_2 \Psi(d_2 s + d_4). \end{aligned} \tag{3.7}$$

Using Lemma 2.1, we get

$$f_2(s) < f_3(s), \tag{3.8}$$

where

$$\begin{aligned} f_3(s) =& \frac{1}{s} + \log(s+\mu) - \frac{1}{2(s+\mu)} \\ & - d_1 \log(d_1 s + d_3) + \frac{d_1}{d_1 s + d_3} \\ & - d_2 \log(d_2 s + d_4) + \frac{d_2}{d_2 s + d_4}. \end{aligned}$$

Differentiation yields

$$\begin{aligned} f_3'(s) =& -\frac{1}{s^2} + \frac{1}{s+\mu} + \frac{1}{2(s+\mu)^2} - \frac{d_1^2}{d_1 s + d_3} - \frac{d_1^2}{(d_1 s + d_3)^2} - \frac{d_2^2}{d_2 s + d_4} - \frac{d_2^2}{(d_2 s + d_4)^2} \\ =& \frac{d_1(1-d_1)s + (d_3 - d_1^2\mu)}{(\mu + s)(d_1 s + d_3)} - \frac{s^2 + 4\mu s + 2\mu^2}{2s^2(\mu + s)^2} - \frac{d_1^2}{(d_1 s + d_3)^2} - \frac{d_2^2(d_2 s + d_4 + 1)}{(d_2 s + d_4)^2} < 0, \end{aligned} \tag{3.9}$$

under the conditions $d_1 \geq 1$ and $d_1^2 \mu \geq d_3$, the numerator of the first fraction is non-positive and all remaining terms are negative. Hence, $f_3(s)$ is decreasing on $[1, \infty)$. Moreover,

$$f_3(1) = -\Big[d_1 \log(d_1 + d_3) + d_2 \log(d_2 + d_4) - \log(\mu + 1) - \frac{2\mu + 1}{2(\mu + 1)} - \frac{d_1}{d_1 + d_3} - \frac{d_2}{d_2 + d_4}\Big] < 0$$

by (3.1). Consequently $f_3(s) \leq f_3(1) < 0$ for all $s \geq 1$. Thus, $f_2(s) < 0$ and $f_1(s)$ is decreasing. Therefore, (b_k) is decreasing.

From (3.3), we have

$$\left| (F^{\mu}_{d_1,d_2;d_3,d_4})'(z) - \frac{F^{\mu}_{d_1,d_2;d_3,d_4}(z)}{z} \right| < b_1(e-1). \tag{3.10}$$

Additionally,

$$\left| \frac{F^{\mu}_{d_1,d_2;d_3,d_4}(z)}{z} \right| > 1 - \sum_{k=1}^{\infty} \frac{c_k}{k!}, \tag{3.11}$$

where

$$c_k = \frac{(\mu)_k \Gamma(d_3)\Gamma(d_4)}{\Gamma(d_1 k + d_3)\Gamma(d_2 k + d_4)}. \tag{3.12}$$

Since (c_k) is decreasing (indeed $c_k = b_k/k$ with (b_k) positive and decreasing),

$$\left| \frac{F^{\mu}_{d_1,d_2;d_3,d_4}(z)}{z} \right| > 1 - c_1(e-1). \tag{3.13}$$

Combining,

$$\left| \frac{z(F^{\mu}_{d_1,d_2;d_3,d_4})'(z)}{F^{\mu}_{d_1,d_2;d_3,d_4}(z)} - 1 \right| < \frac{b_1(e-1)}{1 - c_1(e-1)} < 1, \tag{3.14}$$

if

$$2\mu(e-1)\Gamma(d_3)\Gamma(d_4) < \Gamma(d_1 + d_3)\Gamma(d_2 + d_4).$$

Hence,

$$\Re\left(\frac{z(F^{\mu}_{d_1,d_2;d_3,d_4})'(z)}{F^{\mu}_{d_1,d_2;d_3,d_4}(z)} \right) > 0.$$

Thus, $F^{\mu}_{d_1,d_2;d_3,d_4} \in \mathcal{S}^*$.
(ii) Part (ii) follows similar steps and using Lemma 2.5 of part (ii). □

Corollary 3.1. *Let*

$$d_1 = d_2 = 2, \qquad d_4 = 2, \qquad \mu = 2.$$

If

$$\frac{-1 + \sqrt{1 + \frac{8(e-1)}{3}}}{2} < d_3 \leq 8,$$

then $F^2_{2,2;\,d_3,2}(z) \in \mathcal{S}^*$ *in* $\mathbb{D}$.

Proof. Let

$$d_1 = d_2 = 2, \quad d_4 = 2, \quad \mu = 2.$$

Then, (3.1) and (3.2) give us

$$f(d_3) = 2\log(2 + d_3) + 2\log 4 - \log 3 - \frac{4}{3} - \frac{2}{2 + d_3}$$

and

$$d_3 > \frac{-1 + \sqrt{1 + \frac{8(e-1)}{3}}}{2}.$$

Then $f(d_3) > 0$ is required.
Differentiating,

$$f'(d_3) = \frac{2}{2 + d_3} + \frac{2}{(2 + d_3)^2} > 0,$$

so f is increasing on $(0, \infty)$.

A direct computation gives

$$f\left(\frac{-1+\sqrt{1+\frac{8(e-1)}{3}}}{2}\right) = f(0.68) \approx 1.57 > 0,$$

and since the lower bound is ≈ 0.68, we conclude $f(d_3) > 0$ for all admissible d_3.

Theorem 3.1 requires

$$d_3 \leq d_1^2\mu = 8.$$

Combining all conditions,

$$\frac{-1+\sqrt{1+\frac{8(e-1)}{3}}}{2} < d_3 \leq 8,$$

and the proof is complete. □

Corollary 3.2. *Let*

$$d_1 = d_2 = 2, \qquad d_4 = 2, \qquad \mu = 1.$$

If

$$\frac{-1+\sqrt{1+\frac{4(e-1)}{3}}}{2} < d_3 \leq 4,$$

then $F^1_{2,2;\,d_3,2}(z) \in \mathcal{S}^*$ *in* $\mathbb{D}$.

Corollary 3.3. *Let*

$$d_1 = d_2 = 3, \qquad d_4 = 2, \qquad \mu = 1.$$

If

$$\frac{-1+\sqrt{1+\frac{4(e-1)}{3}}}{2} < d_3 \leq 9,$$

then $F^1_{3,3;\,d_3,2}(z) \in \mathcal{S}^*$ *in* $\mathbb{D}$.

Proof. Here, $d_1^2\mu = 9$. Condition (3.2) reads $2(e-1)\Gamma(d_3) < 24\,d_3(d_3+1)(d_3+2)\Gamma(d_3)$, i.e., $d_3(d_3+1)(d_3+2) > \frac{e-1}{12}$. If $d_3(d_3+1) > \frac{e-1}{3}$, then $d_3(d_3+1)(d_3+2) > \frac{2(e-1)}{3} > \frac{e-1}{12}$, so the stated lower bound is sufficient. Finally, (3.1) becomes $\varphi(d_3) = 3\log(3+d_3) + 3\log 5 - \log 2 - \frac{3}{4} - \frac{3}{3+d_3} - \frac{3}{5} > 0$; φ is increasing and $\varphi(0) \approx 5.08 > 0$, so (3.1) holds for every $d_3 > 0$. □

Putting $d_2 = 0$ and $\mu = 1$, we get the following known result.

Corollary 3.4. *[22] Let* $d_1 \geq 1$ *and* $d_3 > 0$ *such that*

$$d_1^2 \geq d_3,$$

and

$$d_1\log(d_1+d_3) - \log 2 - \frac{3}{4} - \frac{d_1}{d_1+d_3} > 0.$$

If

$$2(e-1)\Gamma(d_3) < \Gamma(d_1+d_3),$$

then $F^1_{d_1,0;d_3,d_4}(z) = z\,\Gamma(d_3)E_{d_1,d_3}(z) \in \mathcal{S}^*$ *in* $\mathbb{D}$. *Consequently,* $F^1_{d_1,0;d_3,d_4}(z) \in \mathcal{S}^*$ *in* $\mathbb{D}$.
(ii) If $3(e-1)\Gamma(d_3) < \Gamma(d_1+d_3)$, *then*

$$F^1_{d_1,0;d_3,d_4}(z) \in \mathcal{S}_p.$$

Example 3.1. Let $d_1 = 3$ and $d_3 = \frac{3}{2}$. Since

$$2(e-1)\Gamma\left(\tfrac{3}{2}\right) < \Gamma\left(\tfrac{9}{2}\right)$$

and

$$3(e-1)\Gamma\left(\tfrac{3}{2}\right) < \Gamma\left(\tfrac{9}{2}\right),$$

the function $F_{3,\frac{3}{2}}(z) \in \mathcal{S}_p$.

Remark 3.1. It is noteworthy that Theorem 3.1 also permits admissible values of d_3 in the interval $(0, 1)$ for suitable choices of d_1. Thus, our results also cover parameter ranges not typically considered in many earlier works that assume $0 < d_3 < 1$. See the following example.

Example 3.2. The conditions of Theorem 3.1 are satisfied for $d_1 = 4$, $d_2 = 3$, $d_3 = \frac{1}{2}$, $d_4 = 2$, and $\mu = 1$. Hence, the function $F^1_{4,3;\frac{1}{2},2}(z)$ is starlike in $\mathbb{D}$ and belongs to S_p.

Theorem 3.2. *Let* $d_1, d_3, d_4, \mu > 0$, *and* $d_2 \geq 0$, *such that* $d_1 \geq 1$ *and*

$$d_3 \leq \min(d_1^2\mu,\ \sqrt{2}\mu d_1)$$

and

$$\mu(e-1)\Gamma(d_3)\Gamma(d_4) < \Gamma(d_1+d_3)\Gamma(d_2+d_4). \tag{3.15}$$

Additionally, if

$$\frac{1}{2(1+\mu)} - \log(1+\mu) + d_1\log(d_1+d_3) - \frac{d_1}{d_1+d_3} + d_2\log(d_2+d_4) - \frac{d_2}{d_2+d_4} > 0, \tag{3.16}$$

then the function $F^{\mu}_{d_1,d_2;d_3,d_4} \in \mathcal{S}^*$ *in* $\mathbb{D}_{1/2}$.

Proof. After some simple calculations, we obtain:

$$\left|\frac{F^{\mu}_{d_1,d_2;d_3,d_4}(z)}{z} - 1\right| < \sum_{k=1}^{\infty} \frac{c_k(d_1,d_2,d_3,d_4,\mu)}{k!}, \quad z \in \mathbb{D}, \tag{3.17}$$

where $(c_k)_{k\geq 1}$ is defined in (3.12). Define

$$g_1(s) = \frac{(\mu)_s\Gamma(d_3)\Gamma(d_4)}{\Gamma(d_1s+d_3)\Gamma(d_2s+d_4)}, \quad s \geq 1. \tag{3.18}$$

Equivalently,

$$g_1(s) = \frac{\Gamma(\mu+s)\Gamma(d_3)\Gamma(d_4)}{\Gamma(\mu)\Gamma(d_1s+d_3)\Gamma(d_2s+d_4)}.$$

Then

$$g_1'(s) = g_1(s)g_2(s), \tag{3.19}$$

where

$$g_2(s) = \Psi(s+\mu) - d_1\Psi(d_1 s + d_3) - d_2\Psi(d_2 s + d_4). \tag{3.20}$$

Applying Lemma 2.1 yields

$$\begin{aligned} g_2(s) &< g_3(s) \\ &= \log(s+\mu) - \frac{1}{2(s+\mu)} - d_1\log(d_1 s + d_3) + \frac{d_1}{d_1 s + d_3} - d_2\log(d_2 s + d_4) + \frac{d_2}{d_2 s + d_4}. \end{aligned} \tag{3.21}$$

Differentiating,

$$g_3'(s) = \frac{1}{s+\mu} + \frac{1}{2(s+\mu)^2} - \frac{d_1^2}{d_1 s + d_3} - \frac{d_1^2}{(d_1 s + d_3)^2} - \frac{d_2^2(1 + d_2 s + d_4)}{(d_2 s + d_4)^2}.$$

After simplification, we get

$$\begin{aligned} g_3'(s) = &\frac{d_1(1-d_1)s + (d_3 - d_1^2\mu)}{(\mu + s)(d_1 s + d_3)} \\ &+ \frac{(d_1(1-\sqrt{2})s + (d_3 - \sqrt{2}\mu d_1))(d_1(1+\sqrt{2})s + (d_3 + \sqrt{2}\mu d_1))}{2(s+\mu)^2(d_1 s + d_3)^2} \\ &- \frac{d_2^2(d_2 s + d_4 + 1)}{(d_2 s + d_4)^2} < 0 \end{aligned} \tag{3.22}$$

under the conditions

$$d_1 \geq 1, \quad d_3 \leq \min(d_1^2\mu,\ \sqrt{2}\mu d_1).$$

Hence, $g_3(s)$ is decreasing on $[1, \infty)$ and $g_3(1) < 0$, because $g_3(1)$ is exactly the negative of the left-hand side of (3.16), so

$$g_3(s) < 0 \quad (s \geq 1).$$

Thus, $g_2(s) < 0$, and by (3.19), $g_1(s)$ is decreasing. Therefore, (c_k) is decreasing.

Using (3.17), we have

$$\left| \frac{F^{\mu}_{d_1,d_2;d_3,d_4}(z)}{z} - 1 \right| < c_1(e-1) < 1.$$

Finally, Lemma 2.3 implies that $F^{\mu}_{d_1,d_2;d_3,d_4}$ is univalent and starlike in $\mathbb{D}_{1/2}$. □

Corollary 3.5. *Let*

$$d_1 = d_2 = 2, \qquad d_4 = 2, \qquad \mu = 1.$$

If

$$\frac{-1 + \sqrt{1 + \frac{2(e-1)}{3}}}{2} < d_3 \leq 2\sqrt{2},$$

then $F^1_{2,2;d_3,2}(z)$ *is starlike in* $\mathbb{D}_{1/2}$.

Proof. We apply the preceding Theorem 3.2 with the choices

$$d_1 = d_2 = 2, \qquad d_4 = 2, \qquad \mu = 1.$$

From the hypothesis

$$d_3 \leq \min(d_1^2\mu,\ \sqrt{2}\mu d_1),$$

we obtain

$$d_3 \leq \min(4, 2\sqrt{2}) = 2\sqrt{2}.$$

The theorem requires

$$\mu(e-1)\Gamma(d_3)\Gamma(d_4) < \Gamma(d_1+d_3)\Gamma(d_2+d_4).$$

With the chosen parameters,

$$(e-1)\Gamma(d_3)\Gamma(2) < \Gamma(d_3+2)\Gamma(4).$$

Using $\Gamma(2) = 1$ and $\Gamma(4) = 6$, this becomes

$$(e-1)\Gamma(d_3) < 6\,\Gamma(d_3+2).$$

Since

$$\Gamma(d_3+2) = (d_3+1)d_3\Gamma(d_3),$$

we get

$$e - 1 < 6d_3(d_3+1).$$

Hence,

$$d_3^2 + d_3 - \frac{e-1}{6} > 0.$$

Solving this quadratic inequality yields

$$d_3 > \frac{-1 + \sqrt{1 + \frac{2(e-1)}{3}}}{2}.$$

Substituting the parameters into the logarithmic condition gives

$$\frac{1}{4} - \log 2 + 2\log(2+d_3) - \frac{2}{2+d_3} + 2\log 4 - \frac{1}{2} > 0.$$

Define

$$\varphi(d_3) = 2\log(2+d_3) + 2\log 4 - \log 2 - \frac{1}{4} - \frac{2}{2+d_3}.$$

Then the required condition is $\varphi(d_3) > 0$.

Differentiating,

$$\varphi'(d_3) = \frac{2}{2+d_3} + \frac{2}{(2+d_3)^2} > 0,$$

so φ is increasing on $(0, \infty)$. Evaluating at the lower bound

$$d_3 = \frac{-1 + \sqrt{1 + \frac{2(e-1)}{3}}}{2},$$

one verifies that $\varphi(d_3) \approx 2.54 > 0$. Hence, the logarithmic condition holds for all admissible d_3.

All hypotheses of the theorem are satisfied. Therefore,

$$F^1_{2,2;d_3,2}(z)$$

is starlike in $\mathbb{D}_{1/2}$. □

Example 3.3. Let $d_3 = \frac{1}{2}$. Since

$$\frac{-1 + \sqrt{1 + \frac{2(e-1)}{3}}}{2} < \tfrac{1}{2} \leq 2\sqrt{2},$$

the conditions of the above corollary are satisfied. Hence, the function

$$F^1_{2,2;\frac{1}{2},2}(z)$$

is starlike in $\mathbb{D}_{1/2}$.

Putting $d_2 = 0$ and $\mu = 1$, we get the following known Theorem 3.2:

Corollary 3.6. *[22] Let $d_1 \geq 1$ and $d_3, d_4 > 0$ such that*

$$d_3 \leq \min(d_1^2,\ \sqrt{2}\, d_1)$$

and

$$(e-1)\Gamma(d_3) < \Gamma(d_1 + d_3).$$

If

$$\frac{1}{4} - \log 2 + d_1 \log(d_1 + d_3) - \frac{d_1}{d_1 + d_3} > 0,$$

then the function

$$F^1_{d_1,0;d_3,d_4}(z)$$

is starlike in $\mathbb{D}_{1/2}$.

3.2. Convexity of the normalized five-parameter Mittag-Leffler function

Theorem 3.3. *Let $d_1, d_3, d_4, \mu > 0$ and $d_2 \geq 0$ such that*

$$d_1 \geq 1, \quad 2d_1\mu \geq 2\mu + 1, \quad d_3 \leq \min\left(d_1^2\mu,\ \frac{2d_1^2\mu^2}{2\mu + 1}\right).$$

If

$$\frac{1}{2(1+\mu)} + d_1 \log(d_1 + d_3) + d_2 \log(d_2 + d_4) - \log(1+\mu) - \frac{3}{2} - \frac{d_1}{d_1 + d_3} - \frac{d_2}{d_2 + d_4} > 0 \tag{3.23}$$

and

$$4\mu\Gamma(d_3)\Gamma(d_4)(e-1) < \Gamma(d_1 + d_3)\Gamma(d_2 + d_4), \tag{3.24}$$

then the function $F^{\mu}_{d_1,d_2;d_3,d_4}$ is convex in $\mathbb{D}$.

Proof. Let f is convex in $\mathbb{D}$ if and only if $zf'(z)$ is starlike. Thus, it enough to show that

$$G^{\mu}_{d_1,d_2;d_3,d_4}(z) = z(F^{\mu}_{d_1,d_2;d_3,d_4}(z))'$$

is starlike.

We obtain

$$(G^{\mu}_{d_1,d_2;d_3,d_4}(z))' - \frac{G^{\mu}_{d_1,d_2;d_3,d_4}(z)}{z} = \sum_{k=1}^{\infty} \frac{e_k z^k}{k!}, \tag{3.25}$$

where

$$e_k = \frac{k(k+1)\Gamma(\mu+k)\Gamma(d_3)\Gamma(d_4)}{\Gamma(\mu)\Gamma(d_1k+d_3)\Gamma(d_2k+d_4)}. \tag{3.26}$$

Define

$$h_1(s) = \frac{s(s+1)\Gamma(\mu+s)\Gamma(d_3)\Gamma(d_4)}{\Gamma(\mu)\Gamma(d_1s+d_3)\Gamma(d_2s+d_4)}.$$

Then

$$h_1'(s) = h_1(s)h_2(s), \tag{3.27}$$

where

$$h_2(s) = \frac{2s+1}{s(s+1)} + \Psi(s+\mu) - d_1\Psi(d_1s+d_3) - d_2\Psi(d_2s+d_4).$$

Using Lemma 2.1, we get

$$\begin{aligned} h_2(s) < h_3(s) = {} & \frac{2s+1}{s(s+1)} + \log(s+\mu) - \frac{1}{2(s+\mu)} \\ & - d_1\log(d_1s+d_3) + \frac{d_1}{d_1s+d_3} \\ & - d_2\log(d_2s+d_4) + \frac{d_2}{d_2s+d_4}. \end{aligned} \tag{3.28}$$

Differentiating with respect to s, we have

$$\begin{aligned} h_3'(s) = {} & -\frac{2s^2+2s+1}{s^2(s+1)^2} \\ & + \frac{(2d_1(1-d_1)s + d_1((2\mu+1) - 2\mu d_1))s}{2(s+\mu)^2(d_1s+d_3)} \\ & + \frac{2(d_3 - d_1^2\mu)s + ((2\mu+1)d_3 - 2d_1^2\mu^2)}{2(s+\mu)^2(d_1s+d_3)} \\ & - \frac{d_1^2}{(d_1s+d_3)^2} - \frac{d_2^2(1+d_2s+d_4)}{(d_2s+d_4)^2}. \end{aligned} \tag{3.29}$$

Shows that $h_3'(s) < 0$ for $s \geq 1$ under the hypotheses

$$d_1 \geq 1, \quad 2d_1\mu \geq 2\mu+1, \quad d_3 \leq \min\left(d_1^2\mu, \frac{2d_1^2\mu^2}{2\mu+1}\right)$$

(each coefficient of s^2, s and s^0 in the two middle numerators is non-positive, and the other terms are negative). Moreover, $h_3(1)$ is exactly the negative of the left-hand side of (3.23), so $h_3(1) < 0$ and so $h_3(s) < 0$ for $s \geq 1$. Hence, $h_2(s) < 0$ and $h_1(s)$ is decreasing. Thus, (e_k) is decreasing.

Therefore,

$$\left|(G^{\mu}_{d_1,d_2;d_3,d_4})'(z) - \frac{G^{\mu}_{d_1,d_2;d_3,d_4}}{z}\right| \leq e_1(e-1). \tag{3.30}$$

Additionally,

$$\left|\frac{G^{\mu}_{d_1,d_2;d_3,d_4}(z)}{z}\right| = |(F^{\mu}_{d_1,d_2;d_3,d_4})'(z)| \geq 1 - \sum_{k=1}^{\infty} \frac{(k+1)c_k}{k!} \geq 1 - e_1(e-1), \tag{3.31}$$

where we used $(k+1)c_k \leq k(k+1)c_k = e_k$ for $k \geq 1$.

Combining,

$$\left|\frac{z(G^{\mu}_{d_1,d_2;d_3,d_4})'(z)}{G^{\mu}_{d_1,d_2;d_3,d_4}(z)} - 1\right| \leq \frac{e_1(e-1)}{1 - e_1(e-1)} < 1. \tag{3.32}$$

Hence, $G^{\mu}_{d_1,d_2;d_3,d_4}$ is starlike, and therefore $F^{\mu}_{d_1,d_2;d_3,d_4}$ is convex in $\mathbb{D}$. □

Corollary 3.7. *Let*

$$d_1 = d_2 = 2, \qquad d_4 = 2, \qquad \mu = 1.$$

If

$$\frac{-1 + \sqrt{1 + \frac{8(e-1)}{3}}}{2} < d_3 \leq \frac{8}{3},$$

then $F^1_{2,2;d_3,2}(z)$ is convex in $\mathbb{D}$.

Proof. Proceeding with similar steps of Corollary 3.5, we can prove Corollary 3.7. □

Example 3.4. Let $d_3 = \frac{3}{4}$. Since

$$\frac{-1 + \sqrt{1 + \frac{8(e-1)}{3}}}{2} < \frac{3}{4} < \frac{8}{3},$$

the hypotheses of the above corollary are satisfied. Hence, the function $F^1_{2,2;\frac{3}{4},2}(z)$ is convex in $\mathbb{D}$.

Remark 3.2. When $\mu = 1$, the five-parameter ML function reduces to the four-parameter ML function

$$E_{d_1,d_2;d_3,d_4}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(d_1k + d_3)\Gamma(d_2k + d_4)}.$$

Theorem 3.4. *Let $d_1, d_3, d_4, \mu > 0$ and $d_2 \geq 0$ such that*

$$d_1 \geq 1, \quad 2\mu d_1 \geq 2\mu + 1, \quad d_3 \leq \min\left(d_1^2\mu, \frac{2d_1^2\mu^2}{2\mu + 1}\right),$$

and the following condition holds:

$$\frac{1}{2(1+\mu)} - \frac{1}{2} - \log(1+\mu) + d_1 \log(d_1 + d_3) - \frac{d_1}{d_1 + d_3} + d_2 \log(d_2 + d_4) - \frac{d_2}{d_2 + d_4} > 0. \tag{3.33}$$

(i) If

$$2\mu\,\Gamma(d_3)\Gamma(d_4)(e-1) < \Gamma(d_1+d_3)\Gamma(d_2+d_4),$$

then $F^{\mu}_{d_1,d_2;d_3,d_4}(z)$ *is convex in* $\mathbb{D}_{1/2}$.

(ii) If

$$\sqrt{5}\,\mu\,\Gamma(d_3)\Gamma(d_4)(e-1) < \Gamma(d_1+d_3)\Gamma(d_2+d_4),$$

then $F^{\mu}_{d_1,d_2;d_3,d_4}(z)$ *is starlike in* $\mathbb{D}$.

Proof. A direct computation gives

$$(F^{\mu}_{d_1,d_2;d_3,d_4}(z))' - 1 = \sum_{k=1}^{\infty} \frac{t_k(d_1,d_2,d_3,d_4,\mu)}{k!}\, z^k, \tag{3.34}$$

where

$$t_k = (k+1)c_k = \frac{(k+1)(\mu)_k\Gamma(d_3)\Gamma(d_4)}{\Gamma(d_1k+d_3)\Gamma(d_2k+d_4)}, \qquad k \geq 1,$$

with c_k as in (3.12). Note that $t_k = e_k/k$, where e_k is defined in (3.26), and in particular $t_1 = e_1 = \dfrac{2\mu\Gamma(d_3)\Gamma(d_4)}{\Gamma(d_1+d_3)\Gamma(d_2+d_4)}$.

Let

$$I_1(s) = \frac{(s+1)(\mu)_s\Gamma(d_3)\Gamma(d_4)}{\Gamma(d_1s+d_3)\Gamma(d_2s+d_4)}, \qquad s \geq 1,$$

or equivalently,

$$I_1(s) = \frac{(s+1)\Gamma(\mu+s)\Gamma(d_3)\Gamma(d_4)}{\Gamma(\mu)\Gamma(d_1s+d_3)\Gamma(d_2s+d_4)}.$$

Differentiating logarithmically yields

$$\begin{aligned} I_1'(s) &= I_1(s)\left[\frac{\Gamma'(\mu+s)}{\Gamma(\mu+s)} + \frac{1}{s+1} - d_1\frac{\Gamma'(d_1s+d_3)}{\Gamma(d_1s+d_3)} - d_2\frac{\Gamma'(d_2s+d_4)}{\Gamma(d_2s+d_4)}\right] \\ &= I_1(s)\left[\frac{1}{s+1} + \Psi(\mu+s) - d_1\Psi(d_1s+d_3) - d_2\Psi(d_2s+d_4)\right]. \end{aligned}$$

Hence,

$$I_1'(s) = I_1(s)I_2(s),$$

where

$$I_2(s) = \frac{1}{s+1} + \Psi(\mu+s) - d_1\Psi(d_1s+d_3) - d_2\Psi(d_2s+d_4).$$

By Lemma 2.1, we obtain

$$\begin{aligned} I_2(s) < I_3(s) = {} & \frac{1}{s+1} + \log(s+\mu) - \frac{1}{2(s+\mu)} \\ & - d_1\log(d_1s+d_3) + \frac{d_1}{d_1s+d_3} - d_2\log(d_2s+d_4) + \frac{d_2}{d_2s+d_4}. \end{aligned}$$

Differentiating and simplifying gives

$$I_3'(s) = \frac{(2d_1(1-d_1)s + d_1((2\mu+1) - 2\mu d_1))s}{2(s+\mu)^2(d_1 s + d_3)} + \frac{2s(d_3 - d_1^2\mu) + ((2\mu+1)d_3 - 2d_1^2\mu^2)}{2(s+\mu)^2(d_1 s + d_3)} - \frac{d_2^2(1 + d_4 + d_2 s)}{(d_2 s + d_4)^2} - \frac{d_1^2}{(d_1 s + d_3)^2} - \frac{1}{(s+1)^2} < 0,$$

for all

$$s \geq 1, \quad d_1 \geq 1, \quad 2\mu d_1 \geq 2\mu + 1,$$
$$d_3 \leq d_1^2\mu, \quad 2d_1^2\mu^2 \geq (2\mu+1)d_3.$$

Thus, $I_3(s)$ is decreasing on $[1, \infty)$; since $I_3(1)$ is exactly the negative of the left-hand side of (3.33), we have $I_3(1) < 0$ and therefore $I_3(s) < 0$ for all $s \geq 1$. Hence, $I_2(s) < 0$, which implies that $I_1(s)$ is decreasing. Consequently, the sequence $(t_k)_{k\geq 1}$ is decreasing.

Therefore, for all $z \in \mathbb{D}$,

$$\left|(F^{\mu}_{d_1,d_2;d_3,d_4}(z))' - 1\right| < t_1(d_1, d_2, d_3, d_4, \mu)(e-1) = \frac{2\mu\Gamma(d_3)\Gamma(d_4)}{\Gamma(d_1+d_3)\Gamma(d_2+d_4)}(e-1). \tag{3.35}$$

Under condition (i),

$$\left|(F^{\mu}_{d_1,d_2;d_3,d_4}(z))' - 1\right| < 1,$$

and from Lemma 2.4, the function is convex in $\mathbb{D}_{1/2}$.

Under condition (ii),

$$\left|(F^{\mu}_{d_1,d_2;d_3,d_4}(z))' - 1\right| < \frac{2}{\sqrt{5}},$$

which implies starlikeness in $\mathbb{D}$. Finally, by Lemma 2.2, the required result follows. □

Corollary 3.8. *Let*

$$d_1 = d_2 = 2, \qquad d_4 = 2, \qquad \mu = 1.$$

If

$$\frac{-1 + \sqrt{1 + \frac{4(e-1)}{3}}}{2} < d_3 \leq \frac{8}{3},$$

then $F^1_{2,2;d_3,2}(z)$ *is convex in* $\mathbb{D}_{1/2}$.

Proof. Verification of parameter conditions:

Thus,

$$d_1 = 2 \geq 1$$

and

$$2\mu d_1 = 4 \geq 3 = 2\mu + 1.$$

Further,

$$d_3 \leq \min\left(d_1^2\mu, \frac{2d_1^2\mu^2}{2\mu+1}\right) = \min\left(4, \frac{8}{3}\right) = \frac{8}{3}.$$

From Gamma inequality, we have

$$d_3 > \frac{-1 + \sqrt{1 + \frac{4(e-1)}{3}}}{2}.$$

Now, substituting the parameters into the logarithmic condition gives

$$\varphi(d_3) = 2\log(2 + d_3) + 2\log 4 - \log 2 - \frac{3}{4} - \frac{2}{2 + d_3}.$$

Then

$$\varphi'(d_3) = \frac{2}{2 + d_3} + \frac{2}{(2 + d_3)^2} > 0,$$

so φ is increasing on $(0, \infty)$. Evaluating at the lower bound shows $\varphi(d_3) > 0$, and so the logarithmic condition holds for all admissible d_3. Therefore, all hypotheses of the theorem are satisfied, and the result follows. □

Example 3.5. Let $d_3 = \frac{1}{2}$. Since

$$\frac{-1 + \sqrt{1 + \frac{4(e-1)}{3}}}{2} < \frac{1}{2} < 1,$$

the hypotheses of the above corollary are satisfied. Hence, $F^1_{2,2;\frac{1}{2},2}(z)$ is convex in $\mathbb{D}_{1/2}$.

Corollary 3.9. *Let*

$$d_1 = d_2 = 2, \qquad d_4 = 2, \qquad \mu = 1.$$

If

$$\frac{-1 + \sqrt{1 + \frac{2\sqrt{5}(e-1)}{3}}}{2} < d_3 \leq \frac{8}{3},$$

then $F^1_{2,2;d_3,2}(z)$ is starlike in $\mathbb{D}$.

Proof. Proceeding similar steps of Corollary 3.5, we obtain the Corollary 3.9. □

Example 3.6. Let $d_3 = \frac{1}{2}$. Since

$$\frac{-1 + \sqrt{1 + \frac{2\sqrt{5}(e-1)}{3}}}{2} < \frac{1}{2} < \frac{8}{3},$$

the hypotheses of the above corollary are satisfied. Hence, the function

$$F^1_{2,2;\frac{1}{2},2}(z)$$

is starlike in $\mathbb{D}$.

3.3. Uniform convexity

Theorem 3.5. *Let $d_1, d_3, d_4, \mu > 0$ and $d_2 \geq 0$ such that*

$$d_1 \geq \max\left(1, \frac{2\mu+1}{2\mu}\right), \quad d_3 \leq \min\left(d_1^2\mu, \frac{2d_1^2\mu^2}{2\mu+1}\right)$$

and

$$-\frac{3}{2} - \log(1+\mu) + \frac{1}{2(1+\mu)} + d_1\log(d_1+d_3) - \frac{d_1}{d_1+d_3} + d_2\log(d_2+d_4) - \frac{d_2}{d_2+d_4} > 0. \tag{3.36}$$

If

$$6\mu\Gamma(d_3)\Gamma(d_4)(e-1) < \Gamma(d_1+d_3)\Gamma(d_2+d_4), \tag{3.37}$$

then the function

$$F^{\mu}_{d_1,d_2;d_3,d_4}(z)$$

is uniformly convex in $\mathbb{D}$.

Proof. After some simple calculation, we get

$$z(F^{\mu}_{d_1,d_2;d_3,d_4}(z))'' = \sum_{k=1}^{\infty} \frac{x_k(d_1,d_2,d_3,d_4,\mu)}{k!} z^k, \tag{3.38}$$

where

$$x_k = \frac{k(k+1)\Gamma(\mu+k)\Gamma(d_3)\Gamma(d_4)}{\Gamma(\mu)\Gamma(d_1k+d_3)\Gamma(d_2k+d_4)}, \quad k \geq 1.$$

Define

$$J_1(s) = \frac{s(s+1)\Gamma(\mu+s)\Gamma(d_3)\Gamma(d_4)}{\Gamma(\mu)\Gamma(d_1s+d_3)\Gamma(d_2s+d_4)}, \quad s \geq 1.$$

Then

$$J_1'(s) = J_1(s)J_2(s),$$

where

$$J_2(s) = \frac{2s+1}{s(s+1)} + \Psi(s+\mu) - d_1\Psi(d_1s+d_3) - d_2\Psi(d_2s+d_4).$$

Using Lemma 2.1,

$$\begin{aligned} J_2(s) < J_3(s) = {} & \frac{2s+1}{s(s+1)} + \log(s+\mu) - \frac{1}{2(s+\mu)} \\ & - d_1\log(d_1s+d_3) + \frac{d_1}{d_1s+d_3} - d_2\log(d_2s+d_4) + \frac{d_2}{d_2s+d_4}. \end{aligned}$$

Differentiating and simplifying gives

$$\begin{aligned} J_3'(s) = {} & \frac{(2d_1(1-d_1)s + d_1((2\mu+1)-2\mu d_1))s}{2(s+\mu)^2(d_1s+d_3)} \\ & + \frac{2s(d_3-d_1^2\mu) + ((2\mu+1)d_3 - 2d_1^2\mu^2)}{2(s+\mu)^2(d_1s+d_3)} \\ & - \frac{d_2^2(1+d_4+d_2s)}{(d_2s+d_4)^2} - \frac{d_1^2}{(d_1s+d_3)^2} - \frac{2s^2+2s+1}{s^2(s+1)^2} < 0, \end{aligned}$$

for all

$$s \geq 1, \quad d_1 \geq 1, \quad 2\mu d_1 \geq 2\mu + 1,$$
$$d_3 \leq d_1^2\mu, \quad 2d_1^2\mu^2 \geq (2\mu + 1)d_3.$$

Thus, $J_3(s)$ is decreasing on $[1, \infty)$. Moreover,

$$J_3(1) = \frac{3}{2} + \log(1+\mu) - \frac{1}{2(1+\mu)} - d_1 \log(d_1 + d_3) + \frac{d_1}{d_1 + d_3} - d_2 \log(d_2 + d_4) + \frac{d_2}{d_2 + d_4},$$

which is exactly the negative of the left-hand side of (3.36); thus, $J_3(1) < 0$ and therefore $J_3(s) \leq J_3(1) < 0$ for all $s \geq 1$. Thus, J_1 is decreasing. Hence, (x_k) is decreasing.

Therefore,

$$\left|z(F^{\mu}_{d_1,d_2;d_3,d_4}(z))''\right| < (e-1)x_1. \tag{3.39}$$

Additionally,

$$\left|(F^{\mu}_{d_1,d_2;d_3,d_4}(z))'\right| > 1 - \sum_{k=1}^{\infty} \frac{y_k}{k!}, \tag{3.40}$$

where $y_k = x_k/k$. Since (x_k) decreases, so does (y_k), yielding

$$\left|(F^{\mu}_{d_1,d_2;d_3,d_4}(z))'\right| > 1 - y_1(e-1). \tag{3.41}$$

Put

$$A = \frac{\mu\Gamma(d_3)\Gamma(d_4)}{\Gamma(d_1 + d_3)\Gamma(d_2 + d_4)}.$$

Then $x_1 = y_1 = 2A$, and (3.37) means $6A(e-1) < 1$; in particular, $2A(e-1) < 1/3 < 1$. Combining (3.39) and (3.41),

$$\left|\frac{zF''(z)}{F'(z)}\right| < \frac{(e-1)x_1}{1-(e-1)y_1} = \frac{2A(e-1)}{1-2A(e-1)} < \frac{1}{2},$$

where the last inequality is equivalent to $6A(e-1) < 1$, i.e., to (3.37). By Lemma 2.5(i), the estimate

$$\left|\frac{zF''(z)}{F'(z)}\right| < \frac{1}{2}, \qquad z \in \mathbb{D},$$

is a sufficient condition for uniform convexity. Hence, $F^{\mu}_{d_1,d_2;d_3,d_4} \in UCV$ in $\mathbb{D}$. □

Remark 3.3. Since $UCV \subset K \subset S^*$, Theorem 3.5 also implies convexity and starlikeness of the normalized five-parameter Mittag–Leffler function under the same hypotheses.

Corollary 3.10. *Let*

$$d_1 = d_2 = 2, \qquad d_4 = 2, \qquad \mu = 1.$$

If

$$\frac{-1 + \sqrt{1 + 4(e-1)}}{2} < d_3 \leq \frac{8}{3},$$

then $F^1_{2,2;d_3,2}(z)$ *is uniformly convex in* $\mathbb{D}$.

Proof. Here, $d_1 = 2 \geq \max(1, 3/2)$ and $d_3 \leq \min(4, 8/3) = 8/3$. Condition (3.37) reads $6(e-1)\Gamma(d_3) < 6\,d_3(d_3+1)\Gamma(d_3)$, i.e., $d_3^2 + d_3 - (e-1) > 0$, which gives the stated lower bound ≈ 0.903. Condition (3.36) becomes

$$\varphi(d_3) = 2\log(2+d_3) - \frac{2}{2+d_3} + 2\log 4 - \log 2 - \frac{7}{4} > 0;$$

φ is increasing and $\varphi(0.903) \approx 1.77 > 0$, so (3.36) holds on the range. The result follows from Theorem 3.5. □

Example 3.7. Let $d_3 = 1$. Since

$$\frac{-1 + \sqrt{1 + 4(e-1)}}{2} \approx 0.903 < 1 < \frac{8}{3},$$

the conditions of the above corollary hold. Hence, $F^1_{2,2;1,2}(z)$ is uniformly convex in $\mathbb{D}$.

Figure 1 illustrates the mappings of the normalized five-parameter Mittag–Leffler function for the parameter choices considered in the examples. The four panels correspond to $F^1_{4,3;1/2,2}(z)$ on $\mathbb{D}$, $F^1_{2,2;1/2,2}(z)$ on $\mathbb{D}$, $F^1_{2,2;1/2,2}(z)$ on $\mathbb{D}_{1/2}$, and $F^1_{2,2;1,2}(z)$ on $\mathbb{D}$, respectively. The circular and radial grids are mapped under the corresponding normalized function, providing a visual illustration of the geometric behavior of these functions in the prescribed domains.

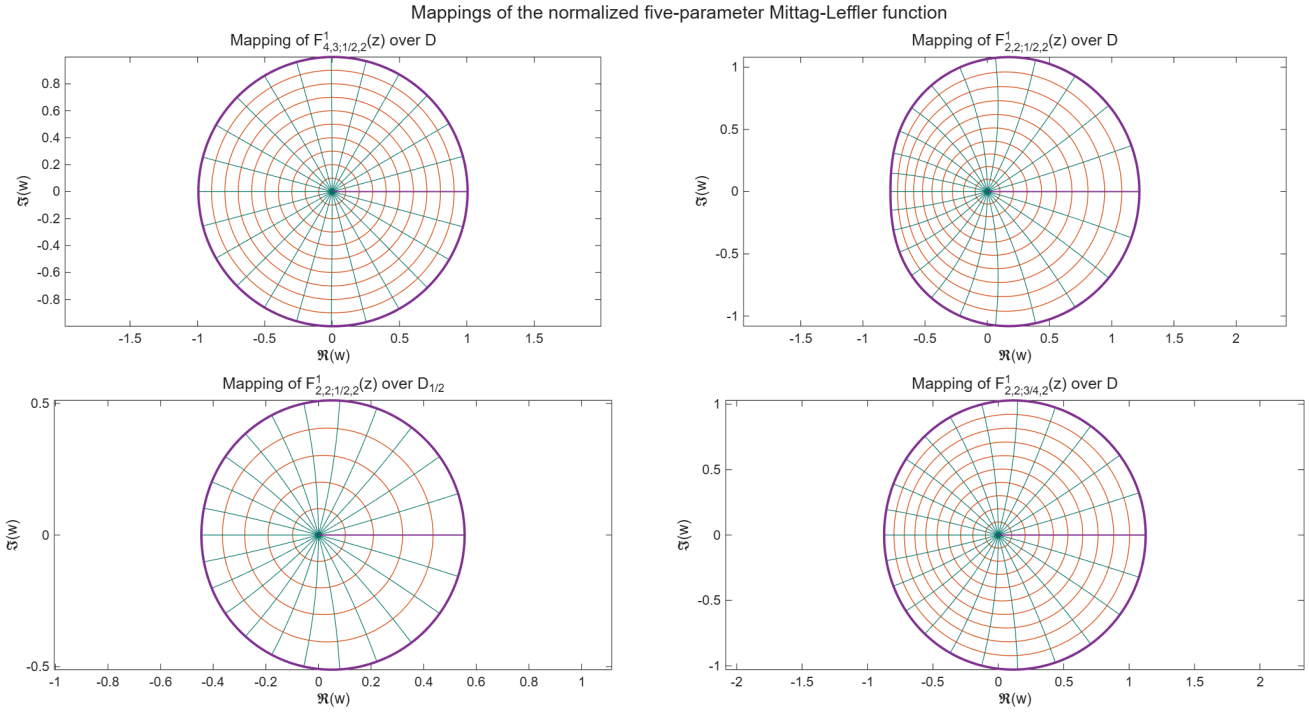


Figure 1. Mappings of the normalized five-parameter Mittag–Leffler function for selected parameter values.

3.4. Close-to-convexity of the normalized five-parameter Mittag–Leffler function with respect to certain convex functions

Theorem 3.6. *Let $d_1, d_3, d_4, \mu > 0$ and $d_2 \geq 0$. Let*

$$d_1 \geq \max\left(1, \frac{1+2\mu}{2\mu}\right), \quad d_3 \leq \min\left(d_1^2\mu, \frac{2d_1^2\mu^2}{2\mu+1}\right)$$

and

$$-\log(\mu+1) + \frac{1}{2(\mu+1)} + \log(2) - 1 + d_1\log(d_1+d_3) - \frac{d_1}{d_1+d_3} + d_2\log(d_2+d_4) - \frac{d_2}{d_2+d_4} > 0 \tag{3.42}$$

and

$$2\mu\,\Gamma(d_3)\Gamma(d_4) \leq \Gamma(d_1+d_3)\Gamma(d_2+d_4). \tag{3.43}$$

Then $F^{\mu}_{d_1,d_2;d_3,d_4}(z)$ is close-to-convex with respect to $-\log(1-z)$ in $\mathbb{D}$.

Proof. Writing $F^{\mu}_{d_1,d_2;d_3,d_4}(z) = z + \sum_{n=2}^{\infty} A_n z^n$, we have $A_{k+1} = a_k$, where (a_k) is defined in (1.8), so $(k+1)A_{k+1} = (k+1)a_k$ (see Remark 2.1). By Lemma 2.6, it suffices to show that

$$1 \geq 2a_1 \geq 3a_2 \geq \cdots \geq (k+1)a_k \geq \cdots \geq 0.$$

The initial inequality $2a_1 = \dfrac{2\mu\Gamma(d_3)\Gamma(d_4)}{\Gamma(d_1+d_3)\Gamma(d_2+d_4)} \leq 1$ is exactly (3.43), and positivity is clear. It remains to show that $\{(k+1)a_k\}_{k\geq 1}$ is decreasing. Define

$$K_1(s) = \frac{(s+1)(\mu)_s\Gamma(d_3)\Gamma(d_4)}{\Gamma(s+1)\,\Gamma(d_1s+d_3)\Gamma(d_2s+d_4)}, \qquad s \geq 1.$$

Equivalently,

$$K_1(s) = \frac{(s+1)\Gamma(\mu+s)\Gamma(d_3)\Gamma(d_4)}{\Gamma(s+1)\Gamma(\mu)\Gamma(d_1s+d_3)\Gamma(d_2s+d_4)}.$$

Differentiating logarithmically yields

$$K_1'(s) = K_1(s)\Bigg[\frac{1}{s+1} + \Psi(\mu+s) - \Psi(1+s) - d_1\Psi(d_1s+d_3) - d_2\Psi(d_2s+d_4)\Bigg].$$

Hence,

$$K_1'(s) = K_1(s)K_2(s),$$

where

$$K_2(s) = \frac{1}{s+1} + \Psi(\mu+s) - \Psi(1+s) - d_1\Psi(d_1s+d_3) - d_2\Psi(d_2s+d_4).$$

By Lemma 2.1, we have

$$K_2(s) < K_3(s),$$

where

$$K_3(s) = \frac{2}{s+1} + \log(\mu+s) - \frac{1}{2(\mu+s)} - \log(1+s) \\ - d_1 \log(d_1 s + d_3) + \frac{d_1}{d_1 s + d_3} - d_2 \log(d_2 s + d_4) + \frac{d_2}{d_2 s + d_4}.$$

Differentiating gives

$$K_3'(s) = -\frac{2}{(s+1)^2} - \frac{1}{s+1} \\ + \frac{[2d_1(1-d_1)s + d_1(1+2\mu-2\mu d_1) + 2(d_3 - d_1^2\mu)]s + (2\mu+1)d_3 - 2d_1^2\mu^2}{2(\mu+s)^2(d_1 s + d_3)} \\ - \frac{d_1^2}{(d_1 s + d_3)^2} - \frac{d_2^2(1 + d_4 + d_2 s)}{(d_2 s + d_4)^2}.$$

Under the given hypotheses, $K_3'(s) \le 0$. Thus, K_3 is decreasing on $[1, \infty)$. Moreover,

$$K_3(1) = 1 + \log(\mu+1) - \frac{1}{2(\mu+1)} - \log 2 - d_1 \log(d_1 + d_3) + \frac{d_1}{d_1 + d_3} - d_2 \log(d_2 + d_4) + \frac{d_2}{d_2 + d_4},$$

which is exactly the negative of the left-hand side of (3.42). Hence, $K_3(1) < 0$, and we obtain $K_3(s) < 0$ for all $s \ge 1$. Thus, $K_2(s) < 0$, implying $K_1(s)$ is decreasing. Therefore, $\{(k+1)a_k\}_{k\ge1}$ is decreasing. Lemma 2.6 completes the proof. □

Corollary 3.11. *Let*

$$d_1 = d_2 = 2, \qquad d_4 = 2, \qquad \mu = 1.$$

If

$$\frac{-1 + \sqrt{7/3}}{2} \le d_3 \le \frac{8}{3},$$

then $F^1_{2,2;d_3,2}(z)$ *is close-to-convex with respect to* $-\log(1-z)$ *in* $\mathbb{D}$.

Proof. Here, $d_1 = 2 \ge \max(1, 3/2)$ and $d_3 \le \min(4, 8/3) = 8/3$. Condition (3.43) reads $2\Gamma(d_3) \le 6\, d_3(d_3+1)\Gamma(d_3)$, i.e., $d_3^2 + d_3 \ge \frac{1}{3}$, which gives the lower bound $\frac{-1+\sqrt{7/3}}{2} \approx 0.264$. Condition (3.42) becomes

$$\varphi(d_3) = 2\log(2 + d_3) - \frac{2}{2 + d_3} + 2\log 4 - \frac{5}{4} > 0. \tag{3.44}$$

Since φ is increasing and $\varphi(0) \approx 1.91 > 0$, (3.44) holds for every $d_3 > 0$. The result follows from Theorem 3.6. □

Example 3.8. Let $d_3 = \frac{1}{2}$. Thus,

$$\frac{-1 + \sqrt{7/3}}{2} \approx 0.264 < \tfrac{1}{2} < 1 < \frac{8}{3}.$$

Moreover,

$$2\log\left(2 + \tfrac{1}{2}\right) - \frac{2}{2 + \frac{1}{2}} + 2\log 4 - \frac{5}{4} > 0.$$

Hence, all conditions of the above corollary are satisfied. Therefore, the function

$$F^1_{2,2;\frac{1}{2},2}(z)$$

is close-to-convex with respect to $-\log(1-z)$ in $\mathbb{D}$.

Theorem 3.7. *Let $d_1, d_3, d_4, \mu > 0$ and $d_2 \geq 0$. Suppose that*

$$d_1 \geq 1, \quad d_3 \leq \min(d_1\mu, d_1^2\mu),$$

and that

$$3\mu\,\Gamma(d_3)\Gamma(d_4) \leq \Gamma(d_1+d_3)\Gamma(d_2+d_4), \tag{3.45}$$

$$d_1\log(d_1+d_3) - \frac{d_1}{d_1+d_3} + d_2\log(d_2+d_4) - \frac{d_2}{d_2+d_4} - \log(\mu+1) + \frac{1}{2(\mu+1)} + \log 2 - \frac{7}{6} > 0. \tag{3.46}$$

Then the odd function

$$H^{\mu}_{d_1,d_2;d_3,d_4}(z) = z\Gamma(d_3)\Gamma(d_4)MF^{\mu}_{d_1,d_2;d_3,d_4}(z^2)$$

is close-to-convex with respect to

$$\frac{1}{2}\log\frac{1+z}{1-z}$$

in $\mathbb{D}$.

Proof. From the definition of the five-parameter Mittag–Leffler function,

$$MF^{\mu}_{d_1,d_2;d_3,d_4}(z) = \sum_{k=0}^{\infty} \frac{(\mu)_k}{k!\Gamma(d_1k+d_3)\Gamma(d_2k+d_4)} z^k.$$

Replacing z by z^2 gives

$$MF^{\mu}_{d_1,d_2;d_3,d_4}(z^2) = \sum_{k=0}^{\infty} \frac{(\mu)_k}{k!\Gamma(d_1k+d_3)\Gamma(d_2k+d_4)} z^{2k}.$$

Since this function is not in $\mathcal{A}$, we consider the normalization

$$H^{\mu}_{d_1,d_2;d_3,d_4}(z) = z\Gamma(d_3)\Gamma(d_4)MF^{\mu}_{d_1,d_2;d_3,d_4}(z^2).$$

Hence,

$$H^{\mu}_{d_1,d_2;d_3,d_4}(z) = z + \sum_{k=1}^{\infty} A_{2k+1}z^{2k+1},$$

where

$$A_{2k+1} = \frac{(\mu)_k\Gamma(d_3)\Gamma(d_4)}{k!\Gamma(d_1k+d_3)\Gamma(d_2k+d_4)} = a_k.$$

To prove the required result, by Lemma 2.7, it is enough to prove that

$$1 \geq 3A_3 \geq 5A_5 \geq \cdots \geq (2k+1)A_{2k+1} \geq \cdots \geq 0.$$

The initial inequality $3A_3 = \dfrac{3\mu\Gamma(d_3)\Gamma(d_4)}{\Gamma(d_1+d_3)\Gamma(d_2+d_4)} \leq 1$ is exactly (3.45), and positivity is clear. It remains to show that $\{(2k+1)A_{2k+1}\}_{k\geq 1}$ is decreasing.

Define

$$L_1(s) = \frac{(2s+1)(\mu)_s\Gamma(d_3)\Gamma(d_4)}{\Gamma(s+1)\Gamma(d_1s+d_3)\Gamma(d_2s+d_4)}, \quad s \geq 1.$$

Using $(\mu)_s = \Gamma(\mu+s)/\Gamma(\mu)$,

$$L_1(s) = \frac{(2s+1)\Gamma(\mu+s)\Gamma(d_3)\Gamma(d_4)}{\Gamma(s+1)\Gamma(\mu)\Gamma(d_1s+d_3)\Gamma(d_2s+d_4)}.$$

Taking logarithmic differentiation,

$$\frac{L_1'(s)}{L_1(s)} = \frac{2}{2s+1} + \Psi(\mu+s) - \Psi(s+1) - d_1\Psi(d_1s+d_3) - d_2\Psi(d_2s+d_4) = L_2(s).$$

By Lemma 2.1, we obtain $L_2(s) < L_3(s)$ where

$$\begin{aligned} L_3(s) =& \frac{2}{2s+1} + \log(\mu+s) - \frac{1}{2(\mu+s)} \\ & - \log(s+1) + \frac{1}{s+1} \\ & - d_1\log(d_1s+d_3) + \frac{d_1}{d_1s+d_3} \\ & - d_2\log(d_2s+d_4) + \frac{d_2}{d_2s+d_4}. \end{aligned}$$

Differentiation gives

$$\begin{aligned} L_3'(s) =& -\frac{4}{(2s+1)^2} + \frac{1}{\mu+s} + \frac{1}{2(\mu+s)^2} \\ & - \frac{1}{s+1} - \frac{1}{(s+1)^2} \\ & - \frac{d_1^2}{d_1s+d_3} - \frac{d_1^2}{(d_1s+d_3)^2} \\ & - \frac{d_2^2}{d_2s+d_4} - \frac{d_2^2}{(d_2s+d_4)^2}, \end{aligned}$$

that is,

$$\begin{aligned} L_3'(s) =& \frac{d_1(1-d_1)s + (d_3 - d_1^2\mu)}{(s+\mu)(d_1s+d_3)} + \frac{(d_1(1-\sqrt{2})s + (d_3 - \sqrt{2}\,\mu d_1))(d_1(1+\sqrt{2})s + (d_3 + \sqrt{2}\,\mu d_1))}{2(s+\mu)^2(d_1s+d_3)^2} \\ & - \frac{d_2^2(d_2s+d_4+1)}{(d_2s+d_4)^2} - \frac{1}{s+1} - \frac{1}{(s+1)^2} - \frac{4}{(2s+1)^2} < 0, \end{aligned}$$

for all

$$s \geq 1, \quad d_1 \geq 1, \quad d_3 \leq \min(d_1\mu, d_1^2\mu),$$

since $d_3 \le d_1^2\mu$ makes the first numerator non-positive and $d_3 \le d_1\mu < \sqrt{2}\,\mu d_1$ makes the second numerator negative.

Hence,

$$L_3'(s) < 0.$$

Thus, L_3 is decreasing on $[1,\infty)$. Moreover,

$$L_3(1) = \frac{7}{6} + \log(\mu+1) - \frac{1}{2(\mu+1)} - \log 2 - d_1\log(d_1+d_3) + \frac{d_1}{d_1+d_3} - d_2\log(d_2+d_4) + \frac{d_2}{d_2+d_4},$$

which is exactly the negative of the left-hand side of (3.46). Hence, $L_3(1) < 0$, and so $L_3(s) < 0$ for all $s \ge 1$. Therefore, $L_1'(s) < 0$, so $\{(2k+1)A_{2k+1}\}$ is decreasing. Lemma 2.7 completes the proof. □

Remark 3.4. The odd function $H^{\mu}_{d_1,d_2;d_3,d_4}$ is different from $F^{\mu}_{d_1,d_2;d_3,d_4}$ in (1.7); indeed $H^{\mu}_{d_1,d_2;d_3,d_4}(z) = F^{\mu}_{d_1,d_2;d_3,d_4}(z^2)/z$. The restrictions $d_1 \ge 1$ and $d_3 \le \min(d_1\mu, d_1^2\mu)$ alone are not sufficient for the conclusion of Theorem 3.7. For example, $d_1 = 1$, $d_2 = 0$, $d_3 = d_4 = \mu = 1$ satisfy these restrictions, but then $MF^1_{1,0;1,1}(z) = e^z$ and $H(z) = ze^{z^2}$, whose derivative $H'(z) = e^{z^2}(1+2z^2)$ vanishes at $z = \pm i/\sqrt{2} \in \mathbb{D}$; hence, H is not univalent in $\mathbb{D}$. In this case, (3.45) and (3.46) fail, since $3A_3 = 3 > 1$ and the left-hand side of (3.46) equals approximately $-0.72 < 0$.

Corollary 3.12. *Let $d_1, d_3, d_4 > 0$, $d_2 \ge 0$, with*

$$d_1 \ge 1, \quad d_3 \le \min(d_1, d_1^2),$$

$3\Gamma(d_3)\Gamma(d_4) \le \Gamma(d_1+d_3)\Gamma(d_2+d_4)$ *and*

$$d_1\log(d_1+d_3) - \frac{d_1}{d_1+d_3} + d_2\log(d_2+d_4) - \frac{d_2}{d_2+d_4} - \frac{11}{12} > 0.$$

Then the function

$$H^1_{d_1,d_2;d_3,d_4}(z) = z\Gamma(d_3)\Gamma(d_4)MF^1_{d_1,d_2;d_3,d_4}(z^2)$$

is close-to-convex with respect to

$$\frac{1}{2}\log\frac{1+z}{1-z}$$

in $\mathbb{D}$.

Proof. Setting $\mu = 1$ gives $(1)_k = k!$. Hence,

$$A_{2k+1} = \frac{\Gamma(d_3)\Gamma(d_4)}{\Gamma(d_1k+d_3)\Gamma(d_2k+d_4)}.$$

With $\mu = 1$, conditions (3.45) and (3.46) reduce to the two stated conditions. The result therefore follows from Theorem 3.7. □

Example 3.9. Let $d_1 = d_2 = 2$, $d_3 = 1$, $d_4 = 2$ and $\mu = 1$. Then $d_3 = 1 \le \min(2,4)$, $3\Gamma(1)\Gamma(2) = 3 \le \Gamma(3)\Gamma(4) = 12$, and $2\log 3 - \frac{2}{3} + 2\log 4 - \frac{1}{2} - \frac{11}{12} \approx 2.89 > 0$. Hence, $H^1_{2,2;1,2}(z)$ is close-to-convex with respect to $\frac{1}{2}\log\frac{1+z}{1-z}$ in $\mathbb{D}$.

4. Conclusions

In this paper, we have investigated a unified set of sufficient conditions for the geometric properties of the normalized five-parameter Mittag–Leffler function. More specifically, the results obtained in Theorems 3.1–3.7 provide sufficient conditions for starlikeness in the unit disk, starlikeness in a half unit disk, convexity, convexity in a half unit disk, starlikeness as a consequence of the convexity estimates, uniform convexity, and close-to-convexity, respectively. These results are established by combining coefficient estimates with gamma-function inequalities, digamma-function estimates, and monotonicity arguments adapted to the additional parameter structure of the function.

The sufficient conditions obtained in this paper are not claimed to be sharp. Their consistency with previously established criteria is confirmed through suitable parameter reductions to the classical two-parameter Mittag–Leffler function, while the determination of sharp parameter ranges remains an interesting problem for future investigation. Furthermore, the obtained results recover the classical two-parameter Mittag–Leffler function as a special case under suitable parameter reductions. Thus, this work complements earlier studies on lower-parameter Mittag–Leffler functions by extending the geometric-function-theoretic analysis to a broader parameter framework. The additional parameters provide greater flexibility in the resulting sufficient conditions and enable several geometric properties to be analyzed within a common analytical framework.

Author contributions

All authors contributed equally to this article. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

Bilal Khan is a guest editor of special issue ”Special Functions and q-Calculus in Geometric Function Theory” for AIMS Mathematics. Bilal Khan was not involved in the editorial review and the decision to publish this article. All authors declare that there are no conflicts of interest regarding the publication of this paper.

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