AIMS Mathematics, 11(11): 37815–37836.
DOI: 10.3934/math.20261462
Received: 02 September 2026
Revised: 19 October 2026
Accepted: 29 October 2026
Published: 13 November 2026

https://www.aimspress.com/journal/Math

*Research article*

# Pointwise-weighted chord Sobolev inequality for two distinct functions

**Wentao Guo***

School of Mathematics and Statistics, Chongqing Technology and Business University, Chongqing 400067, China

* **Correspondence:** Email: guowentao@ctbu.edu.cn.

**Abstract:** In a recent work, Baêta and Cai established the chord Sobolev inequality for a single function, revealing a deep link between Sobolev inequalities and integral geometry. We extend their single-function result to two distinct functions, obtaining pointwise-weighted chord Sobolev inequalities that exhibit the same control structure as the original ones. In contrast to the original integral-geometric approach, our proofs rely only on classical tools: symmetric decreasing rearrangement, convexity theory, and the Hardy–Littlewood–Sobolev (HLS) inequality. We also analyze the sharpness of the obtained constants, show that no non-trivial equality cases exist, and compare the two-function inequalities with the original single-function ones by means of explicit examples.

**Keywords:** chord Sobolev inequality; symmetric decreasing rearrangement; Hardy–Littlewood–Sobolev inequality; pointwise weights; Stein–Weiss inequality; layer cake representation; sharp constant
**Mathematics Subject Classification:** 26D10, 26D20, 52A40

## 1. Introduction

The Sobolev inequality lies at the interface of analysis and geometry, bridging isoperimetric phenomena and functional inequalities, and has been the subject of extensive and influential research. It plays a central role in potential theory, harmonic analysis, and partial differential equations. Various properties, extensions, and applications of the Sobolev inequality have been thoroughly investigated in the literature; we refer the interested reader to [1–3] for a more systematic account. We also refer to [4, 5] for further related results.

In a recent work, Baêta and Cai established the chord Sobolev inequality for a single function, which further reveals the profound connection between Sobolev inequalities and integral geometry; see [6, 7].

**Chord Sobolev inequalities for a single function.** For $\lambda \in (0, n)$ and nonnegative $f \in L^{\frac{n}{2n-\lambda}}(\mathbb{R}^n)$,

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{f(x), f(y)\}}{|x-y|^\lambda}\, dx\, dy \le \sigma_{n,\lambda} \|f\|_{\frac{n}{2n-\lambda}}. \tag{1.1}$$

For $\lambda < 0$ and non-zero nonnegative $f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$,

$$\sigma_{n,\lambda} \|f\|_1^{\frac{2n-\lambda}{n}} \|f\|_\infty^{\frac{\lambda-n}{n}} \le \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{f(x), f(y)\}}{|x-y|^\lambda}\, dx\, dy. \tag{1.2}$$

Here the sharp constant $\sigma_{n,\lambda}$ is given by

$$\sigma_{n,\lambda} = \frac{n\, 2^{n-\lambda}\, \pi^{\frac{\lambda-1}{2}} \Gamma\left(\frac{n-\lambda+1}{2}\right) \Gamma\left(\frac{n}{2}+1\right)^{\frac{n-\lambda}{n}}}{|n-\lambda|\, \Gamma\left(\frac{2n-\lambda}{2}+1\right)}.$$

In both cases, equality holds if and only if $f$ is a constant multiple of the characteristic function of a ball.

The proof of (1.1) and (1.2) relies on chord power integrals. For $\alpha \ge -1$ and a convex body $K \subset \mathbb{R}^n$, the chord power integral of $K$ is defined by

$$I_{\alpha+1}(K) = \int_{\mathrm{Aff}(n,1)} |K \cap l|^{\alpha+1}\, dl.$$

When $\alpha > 0$, it admits the equivalent representation

$$I_{\alpha+1}(K) = \frac{\alpha(\alpha+1)}{n\omega_n} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\mathbf{1}_K(x)\mathbf{1}_K(y)}{|x-y|^{n-\alpha}}\, dx\, dy.$$

The equivalence of the two definitions follows from a straightforward computation.

Motivated by (1.1) and (1.2), we establish two chord Sobolev inequalities for two distinct functions in this paper. Unlike the original approach, our proof does not rely on integral-geometric tools; instead, it employs the technique of symmetric decreasing rearrangement, convex geometry, and the classical Hardy–Littlewood–Sobolev (HLS) inequality. In particular, the two-function analogue of (1.1) involves both pointwise weights and pairwise interaction weights. As a consequence, its integral kernel is precisely that of the Stein–Weiss inequality (see [8,9]). Our results extend the single-function inequalities of Baêta and Cai and are obtained by more elementary methods.

**Theorem 1.1.** *Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ be two nonnegative functions, where $p = \frac{n}{2n-\lambda-2\alpha}$ and $q = \frac{n}{2n-\lambda-2\beta}$. The parameters satisfy*

$$\begin{cases} 0 \le 2\alpha < n;\ 0 \le 2\beta < n; \\ 0 < 2\lambda < n;\ \lambda + 2\max\{\alpha,\beta\} < n. \end{cases} \tag{1.3}$$

*Then*

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{f(x), g(y)\}}{|x|^\alpha |x-y|^\lambda |y|^\beta}\, dx\, dy \le C_{n,\lambda,\alpha,\beta} \|f\|_p^{\frac{1}{2}} \|g\|_q^{\frac{1}{2}}. \tag{1.4}$$

*Here, the constant $C_{n,\lambda,\alpha,\beta}$ has an explicit expression*

$$C_{n,\lambda,\alpha,\beta} = \left(\pi^{\lambda}\frac{\Gamma\left(\frac{n}{2}-\lambda\right)}{\Gamma(n-\lambda)}\left[\frac{\Gamma(n/2)}{\Gamma(n)}\right]^{\frac{2\lambda-n}{n}}\right)^{\frac{1}{2}} \cdot \frac{n\,|B_n|^{\frac{\alpha+\beta}{n}}}{\sqrt{(n-2\alpha)(n-2\beta)}},$$

*and $|B_n|$ denotes the volume of the n-dimensional Euclidean unit ball. Equality holds in* (1.4) *if and only if $f \equiv 0$ or $g \equiv 0$; for non-trivial $f$ and $g$, equality never occurs. Moreover, in the unweighted case $\alpha = \beta = 0$, the constant $C_{n,\lambda,0,0}$ is strictly larger than the optimal constant $\sigma_{n,\lambda}$ of* (1.1).

In particular, setting $\alpha = \beta = 0$ and $f = g$, inequality (1.4) reduces to (1.1) with the constant $C_{n,\lambda,0,0} = \sqrt{C(n,2\lambda)}$ in place of $\sigma_{n,\lambda}$, where $C(n,\cdot)$ denotes the sharp HLS constant of Theorem 2.1.

We will prove this inequality in Section 3 and provide the explicit representation of the constant $C_{n,\lambda,\alpha,\beta}$. Then, the equality cases and the non-optimality of constants will be discussed in Section 5. The following theorem generalizes inequality (1.2) for two distinct functions.

**Theorem 1.2.** *Let $f, g \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be two non-zero, nonnegative functions, and let $\lambda < 0$. Then*

$$C_{n,\lambda}\frac{\|f\|_1^{\frac{2n-\lambda}{2n}}\,\|g\|_1^{\frac{2n-\lambda}{2n}}}{\max\left\{\|f\|_\infty, \|g\|_\infty\right\}^{1-\frac{\lambda}{n}}} \le \iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\min\{f(x),g(y)\}}{|x-y|^{\lambda}}dx\,dy. \tag{1.5}$$

*Here, the constant $C_{n,\lambda}$ has an explicit expression*

$$C_{n,\lambda} = \frac{2^{n-\lambda}\pi^{\frac{2n-1}{2}}\Gamma\left(\frac{n-\lambda-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{2n-2-\lambda}{2}\right)} \cdot \frac{4\,|B_n|^{\frac{\lambda}{n}-2}}{(2n-\lambda)^2},$$

*and $|B_n|$ denotes the volume of the n-dimensional Euclidean unit ball. For every $\lambda < 0$, we have $C_{n,\lambda} < \sigma_{n,\lambda}$. Moreover, the constant $C_{n,\lambda}$ is not sharp.*

In particular, when $f = g$, inequality (1.5) reduces to (1.2) with exactly the same norm exponents and with $C_{n,\lambda}$ in place of $\sigma_{n,\lambda}$; we also study this reduction in detail in Section 5. Since $\sigma_{n,\lambda}$ is the optimal constant of (1.2), one has $C_{n,\lambda} \le \sigma_{n,\lambda}$, and in fact $C_{n,\lambda} < \sigma_{n,\lambda}$ (see Subsection 5.2).

This paper is organized in the following manner. We lay out essential background on function spaces, Schwarz symmetrization, and inequality theory in Section 2 that is needed in the sequel. Theorem 1.1 is proved in Section 3, while Section 4 contains the derivation of inequality (1.5). In Section 5, we study the sharpness of the constants and discuss the equality cases and the extremal functions of our main theorems. We also compare the two-function inequalities with the original single-function chord Sobolev inequalities of Baêta and Cai, and some explicit examples will be given in Section 5. Finally, we conclude the entire article with a brief summary in Section 6.

## 2. Preliminaries

We first collect some necessary results on function spaces, convexity theory, Schwarz symmetrization, and inequality theory, which are needed in our works. The books of Brezis [10], Lieb, and Loss [11] are nice general supportive references.

### 2.1. $L^p$ spaces

This subsection collects fundamental background material concerning function spaces. The definitions and auxiliary results presented below can be found in the references [10, 11]; we recap these elementary facts for the convenience of readers.

Let $p \geq 1$, and let $f : \mathbb{R}^n \to \mathbb{R}$ denote a measurable function. We define the standard $L^p$-norm of $f$ via

$$\|f\|_p = \left( \int_{\mathbb{R}^n} |f(x)|^p \, \mathrm{d}x \right)^{1/p}.$$

For any real number $t \in \mathbb{R}$, the superlevel set of $f$ at height $t$ is written as $\{f \geq t\} = \{x \in \mathbb{R}^n : f(x) \geq t\}$, and analogous notation will be adopted for all other superlevel set variants. A measurable function $f$ is called non-trivial whenever the set $\{f \neq 0\}$ carries positive Lebesgue measure. Throughout this paper, two measurable functions are treated as identical provided they coincide almost everywhere, i.e., off a null subset of $\mathbb{R}^n$.

For each exponent $p \geq 1$, the Lebesgue space $L^p(\mathbb{R}^n)$ is defined by

$$L^p(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{R} \ : \ f \text{ is measurable and } \|f\|_p < \infty \right\}.$$

### 2.2. Schwarz symmetrization

This subsection presents the classic Schwarz symmetrization for nonnegative measurable functions, and further relevant backgrounds are available in [12].

Given any subset $K \subseteq \mathbb{R}^n$, its *characteristic function* $\mathbf{1}_K$ is defined pointwise by $\mathbf{1}_K(x) = 1$ for $x \in K$, and $\mathbf{1}_K(x) = 0$ otherwise. For any Borel set $K \subseteq \mathbb{R}^n$ with finite Lebesgue measure, its Schwarz symmetrization $K^\star$ is defined as the closed origin-centered Euclidean ball that shares the same volume with $K$.

Let $f : \mathbb{R}^n \to [0, \infty)$ be a nonnegative measurable function satisfying that every superlevel set $\{f \geq t\}$ has finite Lebesgue measure for all $t > 0$. The *layer cake formula* asserts that

$$f(x) = \int_0^\infty \mathbf{1}_{\{f \geq t\}}(x) \, dt,$$

which holds for almost every point $x \in \mathbb{R}^n$. The *Schwarz symmetrization* $f^\star$ of $f$ is defined as

$$f^\star(x) = \int_0^\infty 1_{\{f \geq t\}^\star}(x) dt$$

for $x \in \mathbb{R}^n$. Consequently, $f^\star$ is characterized almost everywhere by being radially symmetric and possessing superlevel sets whose measures coincide with those of $f$. Note that $f^\star$ is also called the *symmetric decreasing rearrangement* of $f$. If for all $x$, $y$ with $|x| < |y|$, the inequality $f^\star(x) > f^\star(y)$ holds, we say that $f^\star$ is *strictly symmetric decreasing*. For more theory on rearrangement techniques, please refer to [11, 12].

### 2.3. Convexity theory

A subset $K \subseteq \mathbb{R}^n$ is called *convex* provided that $(1 - \lambda) x + \lambda y \in K$ for arbitrary $x, y \in K$ and all scalars $\lambda \in [0, 1]$. A *convex body* refers to a compact convex subset of $\mathbb{R}^n$ that possesses a nonempty interior.

For an arbitrary convex body $K$, its associated *gauge function* $\|\cdot\|_K : \mathbb{R}^n \to [0, \infty]$ is defined as

$$\|x\|_K = \inf\{\lambda > 0 : x \in \lambda K\}.$$

Each gauge function induced by a convex body $K$ satisfies positive homogeneity of degree one on the whole space $\mathbb{R}^n$. We also have $\|x\|_{B_n} = |x|$, where $B_n$ denotes the $n$-dimensional Euclidean unit ball. Given a constant $\alpha > 0$ and a vector $x \in \mathbb{R}^n$, the gauge function of the scaled set $\alpha K$ obeys the identity

$$\|x\|_{\alpha K} = \frac{1}{\alpha}\|x\|_K.$$

Let $C \subset \mathbb{R}^n$ be a convex set, and let $f : C \to \mathbb{R}$ be a real-valued function. The function $f$ is *convex* provided that

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2), \ \forall x_1, x_2 \in C, \ \lambda \in [0,1].$$

Meanwhile, $f$ is defined to be *strictly convex* on the convex set $C$ if

$$f(\lambda x_1 + (1-\lambda)x_2) < \lambda f(x_1) + (1-\lambda) f(x_2), \ \forall x_1, x_2 \in C, \ x_1 \neq x_2, \ \lambda \in (0,1).$$

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *concave* if $-f$ is convex on the whole space $\mathbb{R}^n$.

Readers can refer to the references [13, 14] for systematic introductions to fundamental notions and theories within convex geometric analysis.

### *2.4. Classical inequalities*

In this subsection, we introduce some essential inequalities. For detailed information, please refer to references [11, 12, 14].

#### 2.4.1. Hardy–Littlewood–Sobolev inequality

The classical HLS inequality plays a fundamental role in analysis and partial differential equations, describing the boundedness of fractional integral operators on $L^p$ spaces.

**Theorem 2.1** (The classical HLS inequality)**.** *Let $p, r > 1$ and $0 < \lambda < n$ with $1/p + \lambda/n + 1/r = 2$. Let $f \in L^p(\mathbb{R}^n)$ and $h \in L^r(\mathbb{R}^n)$. Then there exists a sharp constant $C(n, \lambda, p)$, independent of $f, h$, such that*

$$\left| \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{f(x)h(y)}{|x-y|^\lambda} dxdy \right| \le C(n,\lambda,p) \|f\|_p \|h\|_r. \tag{2.1}$$

*If $p = r = 2n/(2n-\lambda)$, then*

$$C(n,\lambda,p) = C(n,\lambda) = \pi^{\lambda/2} \frac{\Gamma(n/2 - \lambda/2)}{\Gamma(n - \lambda/2)} \left[\frac{\Gamma(n/2)}{\Gamma(n)}\right]^{-1+\lambda/n}.$$

*In this case, there is equality in* (2.1) *if and only if $f = Cg$, where $C$ is a constant, and*

$$f(x) = A\left(\gamma^2 + |x-a|^2\right)^{-(2n-\lambda)/2}$$

*for some $A \in \mathbb{C}, 0 \neq \gamma \in \mathbb{R}$, and $a \in \mathbb{R}^n$.*

2.4.2. Rearrangement inequalities

The following theorem states a rearrangement inequality for two distinct nonnegative functions.

**Theorem 2.2.** *Let $f$ and $g$ be nonnegative measurable functions on $\mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} f(x)g(x)dx \leq \int_{\mathbb{R}^n} f^\star(x)g^\star(x)dx,$$

*with the understanding that if the left-hand side is infinite, then so is the right-hand side.*

The proofs of our results rely on the Riesz rearrangement inequality as well as its general version.

**Theorem 2.3** (Riesz rearrangement inequality)**.** *Let $f, g, k : \mathbb{R}^n \to \mathbb{R}$ be nonnegative measurable functions with superlevel sets of finite measure. Then*

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} f(x)k(x-y)g(y)dxdy \leq \iint_{\mathbb{R}^n\times\mathbb{R}^n} f^\star(x)k^\star(x-y)g^\star(y)dxdy.$$

2.4.3. A corollary of Minkowski inequality

The following inequality, as a simple consequence of the Minkowski inequality, will be used in this paper. We have not provided a proof for Theorem 2.4 here; it can be found in reference [7], Lemma 3.1.

**Theorem 2.4.** *Let $\alpha \in (0, n)$ and let $f \in L^{\frac{n}{n+\alpha}}(\mathbb{R}^n)$ be nonnegative. Then*

$$\|f\|_{\frac{n}{n+\alpha}} \geq \int_0^\infty |\{f \geq t\}|^{\frac{n+\alpha}{n}} dt.$$

*Equality holds if and only if there exist some measurable set $E$ of finite measure and a constant $c > 0$ such that $f = c\mathbf{1}_E$.*

2.4.4. Jensen inequality

We next state the well-known Jensen inequality for convex functions, whose proof can be found in [15].

**Theorem 2.5** (Jensen inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a measurable set with finite Lebesgue measure, and let $G : \mathbb{R} \to (-\infty, +\infty]$ be a convex function satisfying $G \not\equiv +\infty$. Suppose $f \in L^1(\Omega)$ satisfies $f(x) \in \mathrm{dom}(G)$ for almost every $x \in \Omega$ and $G(f) \in L^1(\Omega)$. Then*

$$G\left(\frac{1}{|\Omega|}\int_\Omega f\right) \leq \frac{1}{|\Omega|}\int_\Omega G(f).$$

*In particular, if $G$ is strictly convex, equality holds if and only if $f$ is constant almost everywhere on $\Omega$. Furthermore, since $-G : \mathbb{R} \to [-\infty, +\infty)$ is concave, we obtain the reversed inequality*

$$-G\left(\frac{1}{|\Omega|}\int_\Omega f\right) \geq \frac{1}{|\Omega|}\int_\Omega -G(f).$$

2.4.5. Chebyshev integral inequality

We recall the classical Chebyshev integral inequality for monotone functions, which serves as a key ingredient in the proof of Theorem 1.2.

**Theorem 2.6** (Chebyshev integral inequality)**.** *Let $a, b \in \mathbb{R}$ with $a < b$, and let $f, g : [a, b] \to \mathbb{R}$ be integrable functions that are monotonic in the same direction (both non-increasing or both non-decreasing). Then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,dx \geq \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right).$$

*If $f$ and $g$ are monotonic in opposite directions, the inequality is reversed.*

## 3. The proof of Theorem 1.1

In this section, we aim to prove Theorem 1.1.

*Proof of Theorem 1.1.* Let $f$ and $g$ be two nonnegative functions. By the layer cake formula, we have

$$\begin{aligned} f(x) &= \int_0^\infty \mathbf{1}_{\{f\geq t\}}(x)dt, \\ g(x) &= \int_0^\infty \mathbf{1}_{\{g\geq t\}}(x)dt, \end{aligned}$$

and, since $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, Markov's inequality gives $|\{f \geq t\}| \leq t^{-p}\|f\|_p^p < \infty$ and $|\{g \geq t\}| \leq t^{-q}\|g\|_q^q < \infty$ for every $t > 0$; hence all these superlevel sets have finite measure. It is easy to verify that the following equation also holds:

$$\min\{f(x), g(y)\} = \int_0^\infty \mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\{g\geq t\}}(y)dt.$$

Substituting this into the left-hand side of (1.4) and keeping the parameters $\alpha, \beta, \lambda$ satisfying the condition (1.3), we arrive at

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{f(x), g(y)\}}{|x|^\alpha|x-y|^\lambda|y|^\beta}dxdy = \iint_{\mathbb{R}^n\times\mathbb{R}^n}\left(\int_0^\infty \frac{\mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\{g\geq t\}}(y)}{|x|^\alpha|x-y|^\lambda|y|^\beta}dt\right)dxdy.$$

Using the Tonelli theorem, we have

$$\begin{aligned} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{f(x), g(y)\}}{|x|^\alpha|x-y|^\lambda|y|^\beta}dxdy &= \iint_{\mathbb{R}^n\times\mathbb{R}^n}\left(\int_0^\infty \frac{\mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\{g\geq t\}}(y)}{|x|^\alpha|x-y|^\lambda|y|^\beta}dt\right)dxdy \\ &= \int_0^\infty\left(\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\{g\geq t\}}(y)}{|x|^\alpha|x-y|^\lambda|y|^\beta}dxdy\right)dt. \end{aligned} \tag{3.1}$$

We now investigate the inner integral of Eq (3.1). By the Cauchy–Schwarz inequality, we obtain

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\{g\geq t\}}(y)}{|x|^\alpha|x-y|^\lambda|y|^\beta}dxdy$$

$$
\begin{aligned}
&= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(\mathbf{1}^{1/2}_{\{f\geq t\}}(x)\mathbf{1}^{1/2}_{\{g\geq t\}}(y)\right)^2}{|x|^{\alpha}|x-y|^{\lambda}|y|^{\beta}}dxdy \\
&\leq \left(\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\{g\geq t\}}(y)}{|x|^{2\alpha}|y|^{2\beta}}dxdy\right)^{1/2}\left(\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\{g\geq t\}}(y)}{|x-y|^{2\lambda}}dxdy\right)^{1/2}. \qquad (3.2)
\end{aligned}
$$

Then, we denote

$$
\begin{aligned}
I_1 &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\{g\geq t\}}(y)}{|x|^{2\alpha}|y|^{2\beta}}dxdy, \\
I_2 &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\{g\geq t\}}(y)}{|x-y|^{2\lambda}}dxdy.
\end{aligned}
$$

We next estimate the upper bound of integrals $I_1$ and $I_2$ in two steps. First, for $I_1$, we have

$$
\begin{aligned}
I_1 &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\{g\geq t\}}(y)}{|x|^{2\alpha}|y|^{2\beta}}dxdy \\
&= \left(\int_{\mathbb{R}^n} \frac{\mathbf{1}_{\{f\geq t\}}(x)}{|x|^{2\alpha}}dx\right)\left(\int_{\mathbb{R}^n} \frac{\mathbf{1}_{\{g\geq t\}}(y)}{|y|^{2\beta}}dy\right). \qquad (3.3)
\end{aligned}
$$

If $2\alpha = 0$ (respectively $2\beta = 0$), then the bound (3.6) (respectively (3.7)) below is trivial, so we assume $2\alpha > 0$ and $2\beta > 0$ in the computation. We claim the following layer cake formula: For every $x \in \mathbb{R}^n$ and $\alpha > 0$,

$$
\frac{1}{|x|^{2\alpha}} = \int_0^{+\infty} \mathbf{1}_{\psi(s)B_n}(x)ds,
$$

where $\psi(s) := s^{-\frac{1}{2\alpha}}$ and $B_n$ denotes the $n$-dimensional Euclidean unit ball.

Indeed, for $x \neq 0$, since $|\cdot|$ is the gauge function of the convex body $B_n$, we have the equivalence

$$
x \in \psi(s)B_n \quad \Longleftrightarrow \quad |x| \leq \psi(s) = s^{-\frac{1}{2\alpha}}.
$$

The function $s \mapsto s^{-\frac{1}{2\alpha}}$ is strictly decreasing on $(0,\infty)$. Hence the above condition is equivalent to

$$
s \leq |x|^{-2\alpha} = \frac{1}{|x|^{2\alpha}}.
$$

Consequently,

$$
\int_0^{+\infty} \mathbf{1}_{\psi(s)B_n}(x)ds = \int_0^{|x|^{-2\alpha}} 1ds = \frac{1}{|x|^{2\alpha}},
$$

which proves the desired identity. The case $x = 0$ is trivial since both sides are infinite (or follow by a standard limiting argument). On this basis, using the Tonelli theorem, we establish the following result:

$$
\begin{aligned}
\int_{\mathbb{R}^n} \frac{\mathbf{1}_{\{f\geq t\}}(x)}{|x|^{2\alpha}}dx &= \int_{\mathbb{R}^n} \mathbf{1}_{\{f\geq t\}}(x)\left(\int_0^{+\infty} \mathbf{1}_{\psi(s)B_n}(x)ds\right)dx \\
&= \int_0^{+\infty}\left(\int_{\mathbb{R}^n} \mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\psi(s)B_n}(x)dx\right)ds. \qquad (3.4)
\end{aligned}
$$

Using Theorem 2.2, we obtain (note that $(\psi(s)B_n)^\star = \psi(s)B_n$, since $\psi(s)B_n$ is a ball centered at the origin)

$$\int_{\mathbb{R}^n} \mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\psi(s)B_n}(x)dx \leq \int_{\mathbb{R}^n} \mathbf{1}_{\{f\geq t\}^\star}(x)\mathbf{1}_{\psi(s)B_n}(x)dx. \tag{3.5}$$

By virtue of the theory of Schwarz symmetrization, the set $\{f \geq t\}^\star$ is a Euclidean ball of radius $r_0 = \left(\frac{|\{f\geq t\}|}{|B_n|}\right)^{1/n}$. Substituting (3.5) into (3.4), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{\mathbf{1}_{\{f\geq t\}}(x)}{|x|^{2\alpha}}dx &= \int_{\mathbb{R}^n} \mathbf{1}_{\{f\geq t\}}(x)\left(\int_0^{+\infty} \mathbf{1}_{\psi(s)B_n}(x)ds\right)dx \\
&= \int_0^{+\infty}\left(\int_{\mathbb{R}^n} \mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\psi(s)B_n}(x)dx\right)ds \\
&\leq \int_0^{+\infty}\left(\int_{\mathbb{R}^n} \mathbf{1}_{\{f\geq t\}^\star}(x)\mathbf{1}_{\psi(s)B_n}(x)dx\right)ds \\
&= \int_{\mathbb{R}^n} \frac{\mathbf{1}_{\{f\geq t\}^\star}(x)}{|x|^{2\alpha}}dx.
\end{aligned}$$

Using the spherical coordinate change of variables, we arrive at

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{\mathbf{1}_{\{f\geq t\}^\star}(x)}{|x|^{2\alpha}}dx &= \int_{\{f\geq t\}^\star} \frac{1}{|x|^{2\alpha}}dx \\
&= \int_{S^{n-1}}\int_0^{r_0} \frac{1}{r^{2\alpha}} r^{n-1}drdu \\
&= n\,|B_n| \int_0^{r_0} r^{n-2\alpha-1}dr.
\end{aligned}$$

Since $0 \leq 2\alpha < n$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{\mathbf{1}_{\{f\geq t\}^\star}(x)}{|x|^{2\alpha}}dx &= n\,|B_n| \int_0^{r_0} r^{n-2\alpha-1}dr \\
&= \frac{n\,|B_n|}{n-2\alpha}\left[r^{n-2\alpha}\right]_0^{r_0} \\
&= \frac{n\,|B_n|}{n-2\alpha}\left(\frac{|\{f\geq t\}|}{|B_n|}\right)^{\frac{n-2\alpha}{n}} \\
&= \frac{n}{n-2\alpha}\,|B_n|^{\frac{2\alpha}{n}}\,|\{f\geq t\}|^{\frac{n-2\alpha}{n}}.
\end{aligned}$$

Then, we obtain

$$\int_{\mathbb{R}^n} \frac{\mathbf{1}_{\{f\geq t\}}(x)}{|x|^{2\alpha}}dx \leq \frac{n}{n-2\alpha}\,|B_n|^{\frac{2\alpha}{n}}\,|\{f\geq t\}|^{\frac{n-2\alpha}{n}}. \tag{3.6}$$

By the same reasoning, for $0 \leq 2\beta < n$, we have

$$\int_{\mathbb{R}^n} \frac{\mathbf{1}_{\{g\geq t\}}(y)}{|y|^{2\beta}}dy \leq \frac{n}{n-2\beta}\,|B_n|^{\frac{2\beta}{n}}\,|\{g\geq t\}|^{\frac{n-2\beta}{n}}. \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.3), we arrive at

$$\begin{aligned} I_1 &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\{g\geq t\}}(y)}{|x|^{2\alpha}|y|^{2\beta}}dxdy \\ &\leq \frac{n^2}{(n-2\alpha)(n-2\beta)}\,|B_n|^{\frac{2\alpha+2\beta}{n}}\,|\{f\geq t\}|^{\frac{n-2\alpha}{n}}\,|\{g\geq t\}|^{\frac{n-2\beta}{n}}. \end{aligned} \tag{3.8}$$

Second, for $I_2$, we employ Theorem 2.1 (HLS inequality). Since $0 < 2\lambda < n$, we may apply Theorem 2.1 with the kernel exponent $2\lambda$ and with $p_0 = q_0 = \frac{n}{n-\lambda}$ (indeed $\frac{1}{p_0}+\frac{2\lambda}{n}+\frac{1}{q_0} = 2$), which gives

$$\begin{aligned} I_2 = \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\{g\geq t\}}(y)}{|x-y|^{2\lambda}}dxdy &\leq C(n,\lambda)\left\|\mathbf{1}_{\{f\geq t\}}\right\|_{\frac{n}{n-\lambda}}\left\|\mathbf{1}_{\{g\geq t\}}\right\|_{\frac{n}{n-\lambda}} \\ &\leq C(n,\lambda)\,|\{f\geq t\}|^{\frac{n-\lambda}{n}}\,|\{g\geq t\}|^{\frac{n-\lambda}{n}}. \end{aligned} \tag{3.9}$$

At this point, the optimal constant $C(n,\lambda)$ is given by

$$C(n,\lambda) = \pi^{\lambda}\frac{\Gamma(n/2-\lambda)}{\Gamma(n-\lambda)}\left[\frac{\Gamma(n/2)}{\Gamma(n)}\right]^{\frac{2\lambda-n}{n}}.$$

Finally, substituting (3.8) and (3.9) into (3.2), we obtain

$$\begin{aligned} &\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\{g\geq t\}}(y)}{|x|^{\alpha}|x-y|^{\lambda}|y|^{\beta}}dxdy \\ \leq\ &\left(\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\{g\geq t\}}(y)}{|x|^{2\alpha}|y|^{2\beta}}dxdy\right)^{1/2}\left(\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\{g\geq t\}}(y)}{|x-y|^{2\lambda}}dxdy\right)^{1/2} \\ \leq\ &C_{n,\lambda,\alpha,\beta}\,|\{f\geq t\}|^{\frac{2n-\lambda-2\alpha}{2n}}\,|\{g\geq t\}|^{\frac{2n-\lambda-2\beta}{2n}}. \end{aligned} \tag{3.10}$$

Here, the constant $C_{n,\lambda,\alpha,\beta}$ is given by

$$C_{n,\lambda,\alpha,\beta} = \left(\pi^{\lambda}\frac{\Gamma\left(\frac{n}{2}-\lambda\right)}{\Gamma\left(n-\lambda\right)}\left[\frac{\Gamma(n/2)}{\Gamma(n)}\right]^{\frac{2\lambda-n}{n}}\right)^{\frac{1}{2}}\cdot\frac{n\,|B_n|^{\frac{\alpha+\beta}{n}}}{\sqrt{(n-2\alpha)(n-2\beta)}}.$$

Inserting (3.10) into (3.1) and applying the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} &\int_0^{\infty}\left(\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\{g\geq t\}}(y)}{|x|^{\alpha}|x-y|^{\lambda}|y|^{\beta}}dxdy\right)dt \\ \leq\ &C_{n,\lambda,\alpha,\beta}\int_0^{\infty}|\{f\geq t\}|^{\frac{2n-\lambda-2\alpha}{2n}}\,|\{g\geq t\}|^{\frac{2n-\lambda-2\beta}{2n}}\,dt \\ \leq\ &C_{n,\lambda,\alpha,\beta}\left(\int_0^{\infty}|\{f\geq t\}|^{\frac{2n-\lambda-2\alpha}{n}}\,dt\right)^{1/2}\left(\int_0^{\infty}|\{g\geq t\}|^{\frac{2n-\lambda-2\beta}{n}}\,dt\right)^{1/2}. \end{aligned}$$

Since $\lambda+2\alpha < n$ and $\lambda+2\beta < n$, the numbers $n-\lambda-2\alpha$ and $n-\lambda-2\beta$ belong to $(0,n)$, and we may apply Theorem 2.4 with the parameter $\alpha$ of that theorem replaced by $n-\lambda-2\alpha$ and $n-\lambda-2\beta$, respectively. This yields

$$\int_0^{\infty}\left(\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\{g\geq t\}}(y)}{|x|^{\alpha}|x-y|^{\lambda}|y|^{\beta}}dxdy\right)dt$$

$$\begin{aligned} &\leq C_{n,\lambda,\alpha,\beta}\left(\int_0^\infty |\{f \geq t\}|^{\frac{2n-\lambda-2\alpha}{n}}\, dt\right)^{1/2}\left(\int_0^\infty |\{g \geq t\}|^{\frac{2n-\lambda-2\beta}{n}}\, dt\right)^{1/2} \\ &\leq C_{n,\lambda,\alpha,\beta}\, \|f\|_p^{\frac{1}{2}}\, \|g\|_q^{\frac{1}{2}}. \end{aligned}$$

That is,

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{f(x), g(y)\}}{|x|^\alpha |x-y|^\lambda |y|^\beta} dxdy \leq C_{n,\lambda,\alpha,\beta}\, \|f\|_p^{\frac{1}{2}}\, \|g\|_q^{\frac{1}{2}}.$$

This completes the proof of the inequality in Theorem 1.1. □

## 4. The proof of Theorem 1.2

In this section, we aim to prove Theorem 1.2. To this end, the following key lemma should be established first. An initial version of the proof for this lemma appears in a preprint by the author (see Lemma 4.3 in [16]), yet we include its proof here for completeness.

**Lemma 4.1.** *Let $f, g : \mathbb{R}^n \to [0,\infty)$ be integrable functions and let $q > 0$. Then*

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \min\{f(x), g(y)\}\, |x-y|^q\, dxdy \geq \iint_{\mathbb{R}^n\times\mathbb{R}^n} \min\{f^\star(x), g^\star(y)\}\, |x-y|^q\, dxdy.$$

*Proof.* For $z \in \mathbb{R}^n \setminus \{0\}$ and $q > 0$, the layer cake representation gives

$$|z|^q = \int_0^\infty \mathbf{1}_{\mathbb{R}^n \setminus r^{1/q} B_n}(z) dr,$$

and we shall also use the identity $\min\{f(x), g(y)\} = \int_0^\infty \mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\{g\geq t\}}(y)\, dt$. By the Tonelli theorem, we obtain

$$\begin{aligned} &\iint_{\mathbb{R}^n\times\mathbb{R}^n} \min\{f(x), g(y)\}\, |x-y|^q\, dxdy \\ = &\int_0^\infty \int_0^\infty \iint_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{1}_{\{f\geq t\}}(x)\left(\mathbf{1} - \mathbf{1}_{r^{1/q}B_n}(x-y)\right)\mathbf{1}_{\{g\geq t\}}(y) dxdydrdt. \end{aligned}$$

For every fixed $r > 0$ and $t > 0$, the Riesz rearrangement inequality (Theorem 2.3) applied to the kernel $\mathbf{1}_{r^{1/q}B_n}$ gives

$$\begin{aligned} &\iint_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{r^{1/q}B_n}(x-y)\mathbf{1}_{\{g\geq t\}}(y) dxdy \\ \leq &\iint_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{1}_{\{f\geq t\}^\star}(x)\mathbf{1}_{r^{1/q}B_n}(x-y)\mathbf{1}_{\{g\geq t\}^\star}(y) dxdy. \end{aligned}$$

On the other hand, the base integral is unchanged by symmetrization, because it depends only on the volumes of the superlevel sets. Then, we have

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\{g\geq t\}}(y) dxdy = |\{f \geq t\}|\, |\{g \geq t\}|$$

$$= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{1}_{\{f\geq t\}^\star}(x)\mathbf{1}_{\{g\geq t\}^\star}(y)dxdy.$$

Subtracting the rearranged upper bound from this invariant quantity reverses the inequality for the complementary kernels:

$$\begin{aligned} & \iint_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{1}_{\{f\geq t\}}(x)\left(\mathbf{1}-\mathbf{1}_{r^{1/q}B_n}(x-y)\right)\mathbf{1}_{\{g\geq t\}}(y)dxdy \\ \geq & \iint_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{1}_{\{f\geq t\}^\star}(x)\left(\mathbf{1}-\mathbf{1}_{r^{1/q}B_n}(x-y)\right)\mathbf{1}_{\{g\geq t\}^\star}(y)dxdy. \end{aligned}$$

Note that no finiteness assumption on the two double integrals is needed: Since $f, g \in L^1(\mathbb{R}^n)$, we have $|\{f \geq t\}|\,|\{g \geq t\}| \leq t^{-2}\|f\|_1\|g\|_1 < \infty$ for every $t > 0$, so that the subtraction above is legitimate; consequently the conclusion holds in $[0, +\infty]$, both sides being possibly infinite.

Integrating the last inequality in $r$ and $t$ and applying the Tonelli theorem once more, we obtain

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \min\{f(x), g(y)\}\,|x-y|^q\,dxdy \geq \iint_{\mathbb{R}^n\times\mathbb{R}^n} \min\{f^\star(x), g^\star(y)\}\,|x-y|^q\,dxdy,$$

which proves the lemma. □

*Proof of Theorem 1.2.* Let $f$ and $g$ be two non-zero, nonnegative functions. By the layer cake formula, we have

$$\begin{aligned} f(x) &= \int_0^\infty \mathbf{1}_{\{f\geq t\}}(x)dt, \\ g(x) &= \int_0^\infty \mathbf{1}_{\{g\geq t\}}(x)dt, \end{aligned}$$

and, since $f, g \in L^1(\mathbb{R}^n)$, Markov's inequality for $L^p (p > 0)$ gives $|\{f \geq t\}| \leq t^{-1}\|f\|_1 < \infty$ and $|\{g \geq t\}| \leq t^{-1}\|g\|_1 < \infty$ for every $t > 0$; hence all these superlevel sets have finite measure. It is easy to verify that the following equation also holds:

$$\min\{f(x), g(y)\} = \int_0^\infty \mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\{g\geq t\}}(y)dt.$$

Substituting this into the right-hand side of (1.5) and keeping the parameter $\lambda < 0$, we obtain

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{f(x), g(y)\}}{|x-y|^\lambda}dxdy = \iint_{\mathbb{R}^n\times\mathbb{R}^n} \left(\int_0^\infty \frac{\mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\{g\geq t\}}(y)}{|x-y|^\lambda}dt\right)dxdy.$$

Using the Tonelli theorem, we have

$$\begin{aligned} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{f(x), g(y)\}}{|x-y|^\lambda}dxdy &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \left(\int_0^\infty \frac{\mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\{g\geq t\}}(y)}{|x-y|^\lambda}dt\right)dxdy \\ &= \int_0^\infty \left(\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\{g\geq t\}}(y)}{|x-y|^\lambda}dxdy\right)dt. \end{aligned} \tag{4.1}$$

We now investigate the inner integral of (4.1). First, Lemma 4.1 gives

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\{g\geq t\}}(y)|x-y|^{-\lambda}dxdy \quad \geq \quad \iint_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{1}_{\{f\geq t\}^\star}(x)\mathbf{1}_{\{g\geq t\}^\star}(y)|x-y|^{-\lambda}dxdy.$$

Second, let $x, y \in \mathbb{R}^n$ and $x, y \neq o$. By the law of cosines and the arithmetic–geometric mean inequality, we have

$$\begin{aligned} |x-y|^2 &= |x|^2+|y|^2-2\cos\langle x,y\rangle\,|x||y| \\ &\geq 2\,(1-\cos\langle x,y\rangle)\,|x||y|. \end{aligned}$$

Taking $\xi = x/|x|$ and $\eta = y/|y|$, we have $\xi, \eta \in \mathbb{S}^{n-1}$. Moreover,

$$\begin{aligned} |\xi-\eta|^2 &= |\xi|^2+|\eta|^2-2\cos\langle \xi,\eta\rangle\,|\xi||\eta| \\ &= 2-2\cos\langle \xi,\eta\rangle\,. \end{aligned}$$

Since $\cos\langle \xi,\eta\rangle = \cos\langle x,y\rangle$, we obtain

$$\begin{aligned} |x-y|^2 &= |x|^2+|y|^2-2\cos\langle x,y\rangle\,|x||y| \\ &\geq |\xi-\eta|^2|x||y|. \end{aligned}$$

Then

$$\begin{aligned} &\iint_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{1}_{\{f\geq t\}^\star}(x)\mathbf{1}_{\{g\geq t\}^\star}(y)|x-y|^{-\lambda}dxdy \\ \geq\ &\iint_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{1}_{\{f\geq t\}^\star}(x)\mathbf{1}_{\{g\geq t\}^\star}(y)\left(|\xi-\eta|^2|x||y|\right)^{-\frac{\lambda}{2}}dxdy. \end{aligned}$$

Applying a change of variables to spherical coordinates in the preceding integral and noting that both $\{f \geq t\}^\star$ and $\{g \geq t\}^\star$ are balls centered at the origin, we set

$$R_f = \left(\frac{|\{f\geq t\}|}{|B_n|}\right)^{1/n},\ R_g = \left(\frac{|\{g\geq t\}|}{|B_n|}\right)^{1/n},$$

and obtain

$$\begin{aligned} &\iint_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{1}_{\{f\geq t\}^\star}(x)\mathbf{1}_{\{g\geq t\}^\star}(y)|x-y|^{-\lambda}dxdy \\ \geq\ &\iint_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{1}_{\{f\geq t\}^\star}(x)\mathbf{1}_{\{g\geq t\}^\star}(y)\left(|\xi-\eta|^2|x||y|\right)^{-\frac{\lambda}{2}}dxdy \\ =\ &\iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}}\int_0^{R_f}\int_0^{R_g}\left(|\xi-\eta|^2\,|r_1|\,|r_2|\right)^{-\frac{\lambda}{2}} r_1^{n-1}r_2^{n-1}dr_1dr_2d\xi d\eta \\ =\ &\left(\iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}}|\xi-\eta|^{-\lambda}d\xi d\eta\right)\left(\int_0^{R_f}\int_0^{R_g} r_1^{n-\frac{\lambda}{2}-1}r_2^{n-\frac{\lambda}{2}-1}dr_1dr_2\right). \end{aligned} \tag{4.2}$$

Since $\lambda < 0$, a direct computation (see, e.g., [14], or the change of variables $\xi \mapsto \xi - \eta$ together with the beta integral on the sphere) gives

$$\iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}}|\xi-\eta|^{-\lambda}d\xi d\eta = \frac{2^{n-\lambda}\pi^{\frac{2n-1}{2}}\Gamma\left(\frac{n-\lambda-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{2n-2-\lambda}{2}\right)}.$$

Furthermore, we have

$$\begin{aligned}\int_0^{R_f} r_1^{n-\frac{\lambda}{2}-1} dr_1 &= \frac{2}{2n-\lambda}\left(\frac{|\{f \geq t\}|}{|B_n|}\right)^{1-\frac{\lambda}{2n}},\\ \int_0^{R_g} r_2^{n-\frac{\lambda}{2}-1} dr_2 &= \frac{2}{2n-\lambda}\left(\frac{|\{g \geq t\}|}{|B_n|}\right)^{1-\frac{\lambda}{2n}}.\end{aligned}$$

Substituting them into (4.2), we arrive at

$$\begin{aligned}&\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\mathbf{1}_{\{f\geq t\}^\star}(x)\mathbf{1}_{\{g\geq t\}^\star}(y)}{|x-y|^\lambda} dxdy\\ \geq\ & \frac{2^{n-\lambda}\pi^{\frac{2n-1}{2}}\Gamma\left(\frac{n-\lambda-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{2n-2-\lambda}{2}\right)} \cdot \frac{4\,|B_n|^{\frac{\lambda}{n}-2}}{(2n-\lambda)^2} \,|\{f \geq t\}|^{1-\frac{\lambda}{2n}} \,|\{g \geq t\}|^{1-\frac{\lambda}{2n}}. \end{aligned} \tag{4.3}$$

Denoting the preceding coefficients by $C_{n,\lambda}$ and substituting (4.3) and (4.2) into (4.1), we obtain

$$\begin{aligned}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{f(x), g(y)\}}{|x-y|^\lambda} dxdy &= \int_0^\infty \left(\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\{g\geq t\}}(y)}{|x-y|^\lambda} dxdy\right) dt\\ &\geq \int_0^\infty \left(\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\mathbf{1}_{\{f\geq t\}^\star}(x)\mathbf{1}_{\{g\geq t\}^\star}(y)}{|x-y|^\lambda} dxdy\right) dt\\ &\geq C_{n,\lambda}\int_0^\infty |\{f \geq t\}|^{1-\frac{\lambda}{2n}} \,|\{g \geq t\}|^{1-\frac{\lambda}{2n}}\, dt.\end{aligned}$$

We also write $m = \max\{\|f\|_\infty, \|g\|_\infty\}$, and denote $r = 1 - \frac{\lambda}{2n}$. Note that $r > 1$ since $\lambda < 0$.

We first apply the Chebyshev integral inequality (Theorem 2.6). The superlevel set measures $|\{f \geq t\}|$ and $|\{g \geq t\}|$ are both non-increasing functions of $t$, and hence monotonic in the same direction on $[0, m]$. It follows that

$$\frac{1}{m}\int_0^m |\{f \geq t\}| \cdot |\{g \geq t\}|\, dt \geq \left(\frac{1}{m}\int_0^m |\{f \geq t\}|\, dt\right)\left(\frac{1}{m}\int_0^m |\{g \geq t\}|\, dt\right).$$

Since $r > 1$, the univariate function $\varphi(u) = u^r$ is convex on $[0, \infty)$. Applying Jensen's inequality (Theorem 2.5) to $\varphi$ yields

$$\frac{1}{m}\int_0^m (|\{f \geq t\}| \cdot |\{g \geq t\}|)^r\, dt \geq \left(\frac{1}{m}\int_0^m |\{f \geq t\}| \cdot |\{g \geq t\}|\, dt\right)^r.$$

Combining the two inequalities above, we conclude that

$$\frac{1}{m}\int_0^m |\{f \geq t\}|^r \,|\{g \geq t\}|^r\, dt \geq \left(\frac{1}{m}\int_0^m |\{f \geq t\}|\, dt\right)^r \left(\frac{1}{m}\int_0^m |\{g \geq t\}|\, dt\right)^r.$$

Multiplying both sides by $m$ and substituting $\int_0^m |\{f \geq t\}|\, dt = \|f\|_1$, $\int_0^m |\{g \geq t\}|\, dt = \|g\|_1$, we obtain

$$\int_0^\infty |\{f \geq t\}|^{1-\frac{\lambda}{2n}} \,|\{g \geq t\}|^{1-\frac{\lambda}{2n}}\, dt \geq \max\{\|f\|_\infty, \|g\|_\infty\}^{\frac{\lambda}{n}-1} \|f\|_1^{1-\frac{\lambda}{2n}} \|g\|_1^{1-\frac{\lambda}{2n}}.$$

That is,

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{f(x), g(y)\}}{|x-y|^\lambda} dxdy \geq C_{n,\lambda} \frac{\|f\|_1^{1-\frac{\lambda}{2n}} \|g\|_1^{1-\frac{\lambda}{2n}}}{\max\{\|f\|_\infty, \|g\|_\infty\}^{1-\frac{\lambda}{n}}}.$$

Now, the inequality (1.5) has been established. □

## 5. The study of equality cases and relations with the single-function inequalities

In this section, we study the sharpness of the constants in Theorems 1.1 and 1.2, and discuss the equality cases and the corresponding extremal functions. We also pinpoint the precise relations between the two-function inequalities and the original single-function chord Sobolev inequalities (1.1)–(1.2) of Baêta and Cai [6, 7].

### *5.1. The study of sharpness and equality cases for Theorem 1.1*

Our proof of Theorem 1.1 combines the layer-cake representation, two applications of the Cauchy–Schwarz inequality, the weighted rearrangement bound (3.6)–(3.7), the sharp HLS inequality, and Theorem 2.4. We note that each of these ingredients is essentially sharp in its own right. However, their equality conditions are mutually incompatible. This is why the constant $C_{n,\lambda,\alpha,\beta}$ is not attained.

**Proposition 5.1.** *Let the parameters satisfy* (1.3)*, and let $f$ and $g$ be admissible for Theorem 1.1. Equality holds in* (1.4) *if and only if $f \equiv 0$ or $g \equiv 0$; for non-trivial $f$ and $g$, equality never occurs. Moreover, in the unweighted case $\alpha = \beta = 0$, the constant $C_{n,\lambda,0,0} = \sqrt{C(n, 2\lambda)}$ is strictly larger than the optimal constant $\sigma_{n,\lambda}$ of* (1.1)*.*

*Proof.* We take $f \equiv 0$ or $g \equiv 0$. Then $\min\{f(x), g(y)\} = 0$ for almost every $(x, y)$. Hence both sides of (1.4) vanish, and equality holds trivially. Conversely, we suppose that $f \not\equiv 0$, $g \not\equiv 0$, and equality holds in (1.4). Then equality must hold in every step of our proof of Theorem 1.1. In particular, all the following equality conditions must hold simultaneously.

- By the equality case of Theorem 2.4 (see [7], Lemma 3.1), the functions must be characteristic functions: $f = c_f \mathbf{1}_{E_f}$ and $g = c_g \mathbf{1}_{E_g}$ for measurable sets $E_f, E_g$ of finite positive measure and constants $c_f, c_g > 0$.
- By the equality case of the rearrangement bound (3.6)–(3.7) (a weighted version of the two-function rearrangement inequality, with the strictly decreasing radial weight $|x|^{-2\alpha}$), the sets $E_f$ and $E_g$ must be, up to null sets, balls centered at the origin, whenever $\alpha > 0$ and $\beta > 0$.
- By the equality case of the HLS inequality (Theorem 2.1), equality in (3.9) forces $\mathbf{1}_{\{f \geq t\}}$ and $\mathbf{1}_{\{g \geq t\}}$ to have the extremal form $A\left(\gamma^2 + |x - a|^2\right)^{-(n-\lambda)}$, up to multiplicative constants, for almost every $t$.

The third condition is incompatible with the first. Indeed, a characteristic function of a set of finite measure is never of the extremal form $A\left(\gamma^2 + |x - a|^2\right)^{-(n-\lambda)}$, whose support is the whole space. Hence equality cannot occur when $f \not\equiv 0$ and $g \not\equiv 0$.

For the second assertion, we take $\alpha = \beta = 0$ and $f = g$. Then (1.4) becomes

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\min\{f(x), f(y)\}}{|x - y|^\lambda}\, dx\, dy \leq \sqrt{C(n, 2\lambda)}\, \|f\|_{\frac{n}{2n-\lambda}},$$

while (1.1) asserts the same upper bound with the optimal constant $\sigma_{n,\lambda}$. For $f = \mathbf{1}_B$, the characteristic function of a ball, equality holds in (1.1) by the results of Baêta and Cai [6, 7]:

$$\iint_{B \times B} |x - y|^{-\lambda}\, dx\, dy = \sigma_{n,\lambda}\, |B|^{\frac{2n-\lambda}{n}}.$$

If $\sqrt{C(n,2\lambda)} = \sigma_{n,\lambda}$ were true, then $\mathbf{1}_B$ would also saturate the reduced form of (1.4), and consequently equality would have to hold in every step of the proof, in particular in the HLS step (3.9)—impossible by the first part above. Hence, we obtain $\sqrt{C(n,2\lambda)} > \sigma_{n,\lambda}$. □

*Remark* 5.1. The extremal functions $A\left(\gamma^2 + |x-a|^2\right)^{-(n-\lambda)}$ of the HLS step are exactly the functions that saturate that step. Proposition 5.1 shows why no extremal function for (1.4) can exist: The rearrangement step and the HLS step pull in different directions. Therefore, we choose the constant multiples of characteristic functions of balls as the natural candidates. Then we find that those characteristic functions also do not attain the constant of Theorem 1.1.

**Example 5.1.** *Let* $n = 1$, $\lambda = \alpha = \beta = \frac{1}{4}$, *and* $f = g = \mathbf{1}_{[-1,1]}$. *The conditions* (1.3) *hold, and* $p = q = \frac{1}{2-\frac{1}{4}-\frac{1}{2}} = \frac{4}{5}$. *Theorem 1.1 gives*

$$\iint_{\mathbb{R}\times\mathbb{R}} \frac{\min\{f(x), g(y)\}}{|x|^{\frac{1}{4}}|x-y|^{\frac{1}{4}}|y|^{\frac{1}{4}}}\,dx\,dy = \int_{-1}^{1}\int_{-1}^{1} |x|^{-\frac{1}{4}}|x-y|^{-\frac{1}{4}}|y|^{-\frac{1}{4}}\,dx\,dy \le C_{1,\frac{1}{4},\frac{1}{4},\frac{1}{4}}\ \|f\|_{\frac{4}{5}},$$

*where*

$$C_{1,\frac{1}{4},\frac{1}{4},\frac{1}{4}} = \left(\frac{\Gamma(1/4)}{\Gamma(3/4)}\right)^{\frac{1}{2}} \times \frac{2^{\frac{1}{2}}}{1/2} \approx 4.8651, \qquad \|f\|_{\frac{4}{5}} = 2^{\frac{5}{4}} \approx 2.3784,$$

*so that the right-hand side of the inequality is approximately* $4.8651 \times 2.3784 \approx 11.5713$. *For the left-hand side, we split the square* $[-1,1]^2$ *into the regions* $\{xy > 0\}$ *and* $\{xy < 0\}$. *The integrand* $|x|^{-\frac{1}{4}}|x-y|^{-\frac{1}{4}}|y|^{-\frac{1}{4}}$ *is invariant under* $(x,y) \mapsto (-x,-y)$ *and symmetric in* $x$ *and* $y$; *hence the four congruent triangles of* $\{xy > 0\}$ *and the two congruent quadrants of* $\{xy < 0\}$ *give*

$$LHS \;=\; 4\int_0^1\int_0^x x^{-\frac{1}{4}}(x-y)^{-\frac{1}{4}}y^{-\frac{1}{4}}\,dy\,dx + 2\int_0^1\int_0^1 x^{-\frac{1}{4}}u^{-\frac{1}{4}}(x+u)^{-\frac{1}{4}}\,dx\,du,$$

*where* $u = -y$ *in the second integral. In the first integral, we substitute* $y = xs$ *with* $0 < s < 1$; *this yields*

$$\int_0^1\int_0^x x^{-\frac{1}{4}}(x-y)^{-\frac{1}{4}}y^{-\frac{1}{4}}\,dy\,dx \;=\; \int_0^1 x^{\frac{1}{4}}\,dx\int_0^1 s^{-\frac{1}{4}}(1-s)^{-\frac{1}{4}}\,ds = \frac{4}{5}B\Big(\tfrac{3}{4},\tfrac{3}{4}\Big),$$

*where* $B(\cdot,\cdot)$ *denotes the beta function. Then the first summand equals* $4 \times \frac{4}{5}B\Big(\frac{3}{4},\frac{3}{4}\Big) = \frac{16}{5}B\Big(\frac{3}{4},\frac{3}{4}\Big) \approx 5.42216$, *while the second one, evaluated by numerical quadrature, is approximately* 3.92946. *Consequently,*

$$LHS \approx 5.42216 + 3.92946 \approx 9.35162.$$

*Hence the inequality is strict, with ratio* $9.3516/11.5713 \approx 0.8082 < 1$, *in accordance with Proposition 5.1. For comparison, in the unweighted case* $\alpha = \beta = 0$, *the same interval* $[-1,1]$ *saturates the single-function inequality* (1.1) *with the sharp constant* $\sigma_{1,\frac{1}{4}} = \frac{32}{21}$; *see Subsection 5.3.*

### *5.2. The study of sharpness and equality cases for Theorem 1.2*

For Theorem 1.2, the situation is somewhat different, and it can be decided completely: The constant $C_{n,\lambda}$ is strictly smaller than the sharp single-function constant $\sigma_{n,\lambda}$.

**Proposition 5.2.** *For every $\lambda < 0$, we have $C_{n,\lambda} < \sigma_{n,\lambda}$. In particular, the constant $C_{n,\lambda}$ is not sharp. Moreover, equality in* (1.5) *can occur only in the trivial case $f \equiv 0$ or $g \equiv 0$, which the hypotheses of Theorem 1.2 exclude. Hence, within the admissible class of non-zero functions, equality never occurs.*

*Proof.* We take $f = g = \mathbf{1}_B$, where $B$ is a Euclidean ball. Since the single-function inequality (1.2) is sharp, with equality for constant multiples of characteristic functions of balls (Baêta–Cai, [6, 7]), we have

$$\iint_{B\times B} |x-y|^{-\lambda}\,dx\,dy = \sigma_{n,\lambda}\,|B|^{\frac{2n-\lambda}{n}}.$$

Next, we follow the proof of Theorem 1.2 with this particular pair. The layer-cake identity and the rearrangement step (Lemma 4.1) are equalities, because $\mathbf{1}_B$ is already symmetrized. However, we use the following elementary estimate

$$|x-y|^2 \geq |\xi-\eta|^2|x||y|, \qquad \xi = \frac{x}{|x|},\ \eta = \frac{y}{|y|}.$$

Here, equality holds if and only if $|x| = |y|$, and the set $\{(x,y) \in B\times B : |x| = |y|\}$ has measure zero in $\mathbb{R}^n \times \mathbb{R}^n$. Since $-\lambda/2 > 0$, the pointwise estimate $|x-y|^{-\lambda} \geq \left(|\xi-\eta|^2|x||y|\right)^{-\lambda/2}$ is strict on the set of positive measure where $|x| \neq |y|$, and therefore the integrated bound (4.3) is strict for every $t \in (0,1]$. Integrating in $t$ (the Chebyshev and Jensen steps are equalities for this pair, since $|\{f \geq t\}| = |\{g \geq t\}| = |B|$ for every $t \in (0,1]$, and both inequalities hold with equality for constant functions), we obtain the strict inequality

$$C_{n,\lambda}\,|B|^{\frac{2n-\lambda}{n}} < \iint_{B\times B} |x-y|^{-\lambda}\,dx\,dy = \sigma_{n,\lambda}\,|B|^{\frac{2n-\lambda}{n}},$$

that is, $C_{n,\lambda} < \sigma_{n,\lambda}$.

For the last assertion, we first suppose that equality holds in (1.5) for some non-zero $f, g$. Then equality must hold in every step of the proof. In particular, for almost every $t$ in the interval $(0, \min\{\|f\|_\infty, \|g\|_\infty\})$ (where both superlevel sets have positive measure), equality must hold in the estimate of (4.2). This forces $|x| = |y|$ for almost every $(x,y) \in \{f \geq t\}^\star \times \{g \geq t\}^\star$. But $\{f \geq t\}^\star$ and $\{g \geq t\}^\star$ are balls of positive radius, and the set $\{(x,y) : |x| = |y|\}$ has measure zero in the product of two such balls—a contradiction. Hence, the equality above cannot occur. □

*Remark* 5.2. We still state Theorem 1.2 with the sign $\leq$. Proposition 5.2 shows that equality never occurs for admissible (non-zero) functions: At every fixed $\lambda < 0$, inequality (1.5) is strict. A short discussion below will explain why we keep the sign $\leq$.

For $f = g = \mathbf{1}_B$, where $B$ is the Euclidean unit ball, we let $\lambda \to -\infty$. (We do not study the limiting cases of $\lambda$ in this paper, since the limiting cases of $\lambda$, especially near 0, give rise to further new research. We use $\lambda \to -\infty$ here only to explain why the sign $\leq$ in (1.5) is stated from the consideration of self-consistency.) We substitute the explicit expression of $C_{n,\lambda}$ from Theorem 1.2. Since $B$ is the Euclidean unit ball, we have $|B| = |B_n|$, and the powers of $|B_n|$ cancel. This gives

$$C_{n,\lambda}\,|B|^{\frac{2n-\lambda}{n}} = \frac{4\cdot 2^{n-\lambda}\,\pi^{\frac{2n-1}{2}}\,\Gamma\!\left(\frac{n-\lambda-1}{2}\right)}{\Gamma\!\left(\frac{n}{2}\right)\Gamma\!\left(\frac{2n-2-\lambda}{2}\right)(2n-\lambda)^2}.$$

By Stirling's formula, the ratio $\Gamma\!\left(\frac{n-\lambda-1}{2}\right)/\Gamma\!\left(\frac{2n-2-\lambda}{2}\right)$ is asymptotic to $\left(\frac{n-\lambda}{2}\right)^{-\frac{n-1}{2}}$ as $\lambda \to -\infty$. Since $2n-\lambda \sim n-\lambda$ as $\lambda \to -\infty$, the expression for $C_{n,\lambda}|B|^{\frac{2n-\lambda}{n}}$ above is of the order $2^{-\lambda}(n-\lambda)^{-\frac{n+3}{2}}$, and the exponential

factor $2^{-\lambda}$ dominates every power of $n-\lambda$; hence the left-hand side of (1.5) tends to $+\infty$. On the other hand, $|x-y| \le 2$ on $B \times B$ and $|x-y| \ge \frac{3}{2}$ on the set $S = \left\{(x,y) \in B \times B : \ x_1 \le -\frac{3}{4},\ y_1 \ge \frac{3}{4}\right\}$ (here $x_1$ and $y_1$ denote the first coordinates of $x$ and $y$), which has positive measure. Then, we have

$$\left(\tfrac{3}{2}\right)^{-\lambda} |S| \ \le \ \iint_{B\times B} |x-y|^{-\lambda}\, dx\, dy \ \le \ 2^{-\lambda}\, |B|^2,$$

and the right-hand side of (1.5) also tends to $+\infty$.

Thus both sides of (1.5) tend to $+\infty$ as $\lambda \to -\infty$. In fact the two sides are asymptotically equivalent, and not merely both divergent. Since $\mathbf{1}_B$ saturates (1.2), the right-hand side of (1.5) equals $\sigma_{n,\lambda}|B|^{\frac{2n-\lambda}{n}}$, so the ratio of the two sides of (1.5) is exactly

$$\frac{C_{n,\lambda}}{\sigma_{n,\lambda}} = 1 + \frac{\lambda}{(n-\lambda-1)(2n-\lambda)} \longrightarrow 1 \qquad (\lambda \to -\infty),$$

which also recovers $C_{n,\lambda} < \sigma_{n,\lambda}$ for $\lambda < 0$, in accordance with Proposition 5.2. Hence the two sides coincide in the limit, in the precise sense that they are asymptotically equivalent and both tend to $+\infty$. A strict inequality is not preserved under this limiting procedure: The limiting form of (1.5) reads $+\infty \le +\infty$, whereas $+\infty < +\infty$ would be false. Writing the theorem with the sign $\le$ therefore keeps the statement valid and self-consistent in the limiting regime, whereas the sign $<$ would degenerate. This is why we still state Theorem 1.2 with the sign $\le$.

**Example 5.2.** *Let $n = 2$, $\lambda = -\frac{1}{2}$, and $f = g = \mathbf{1}_B$, where $B$ is the unit disk. Theorem 1.2 gives the lower bound*

$$C_{2,-\frac{1}{2}}\, \pi^{\frac{9}{4}} \approx 8.4120 \ \le \ \iint_{B\times B} |x-y|^{\frac{1}{2}}\, dx\, dy,$$

*while the sharp single-function inequality* (1.2)*, saturated by $\mathbf{1}_B$, gives the value*

$$\iint_{B\times B} |x-y|^{\frac{1}{2}}\, dx\, dy = \sigma_{2,-\frac{1}{2}}\, \pi^{\frac{9}{4}} \approx 9.0849,$$

*which is also confirmed by an independent numerical quadrature of the double integral. Hence $C_{2,-\frac{1}{2}}/\sigma_{2,-\frac{1}{2}} \approx 0.9259 < 1$, in accordance with Proposition 5.2.*

*Remark* 5.3. We now summarize the sharpness status of the two main theorems: *We were not able to determine the sharp constants or the extremal functions of Theorems 1.1 and 1.2.* For Theorem 1.1, equality occurs exactly in the trivial cases $f \equiv 0$ or $g \equiv 0$. In the unweighted reduction $\alpha = \beta = 0$, $f = g$, the constant is strictly larger than the optimal single-function constant $\sigma_{n,\lambda}$. Whether $C_{n,\lambda,\alpha,\beta}$ is optimal for general $\alpha, \beta$ remains an open problem. For Theorem 1.2, equality never occurs for admissible (non-zero) functions, and the constant is strictly smaller than $\sigma_{n,\lambda}$, hence provably not optimal.

We also verified that the constant multiples of characteristic functions of balls do not attain our constants. Readers can see the computations of Examples 5.1 and 5.2. Determining the optimal constants of (1.4) and (1.5), and identifying the corresponding maximizing sequences (if any), remains interesting open problems.

*5.3. Differences and connections with the single-function chord Sobolev inequalities*

In this subsection, we display the reduced forms of the two-function inequalities. The explicit examples will be given here, which are compared with the single-function inequalities (1.1)–(1.2) and with the related literature.

**The reduced forms.** Setting $\alpha = \beta = 0$ in (1.4), the two-function inequality becomes

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{f(x), g(y)\}}{|x-y|^\lambda}\,dx\,dy \le C_{n,\lambda,0,0}\,\|f\|_{\frac{n}{2n-\lambda}}^{\frac{1}{2}}\,\|g\|_{\frac{n}{2n-\lambda}}^{\frac{1}{2}}, \qquad C_{n,\lambda,0,0} = \sqrt{C(n,2\lambda)}. \tag{5.1}$$

For $f = g$ in (5.1), the two square roots combine into one norm. Then, we obtain

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{f(x), f(y)\}}{|x-y|^\lambda}\,dx\,dy \le \sqrt{C(n,2\lambda)}\,\|f\|_{\frac{n}{2n-\lambda}}, \tag{5.2}$$

which is exactly (1.1) with $\sqrt{C(n,2\lambda)}$ in place of the sharp constant $\sigma_{n,\lambda}$.

Similarly, setting $f = g$ in (1.5), the two $L^1$-factors combine into $\|f\|_1^{(2n-\lambda)/n}$, and the denominator contributes $\|f\|_\infty^{1-\lambda/n} = \|f\|_\infty^{(n-\lambda)/n}$, so that

$$C_{n,\lambda}\,\|f\|_1^{\frac{2n-\lambda}{n}}\,\|f\|_\infty^{\frac{\lambda-n}{n}} \le \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{f(x), f(y)\}}{|x-y|^\lambda}\,dx\,dy, \tag{5.3}$$

which is exactly (1.2) with $C_{n,\lambda}$ in place of $\sigma_{n,\lambda}$ (note that $1 - \lambda/n = -(\lambda - n)/n$, which is why the exponents of the $\|f\|_\infty$-factors coincide). Thus the two-function inequalities reproduce the single-function ones *with the same norm exponents but with different constants*: $\sqrt{C(n,2\lambda)} > \sigma_{n,\lambda}$ (Proposition 5.1) for the upper bound, and $C_{n,\lambda} < \sigma_{n,\lambda}$ (Proposition 5.2) for the lower bound.

**An explicit one-dimensional example.** Let $n = 1$ and $f = \mathbf{1}_{[-1,1]}$. We first note the elementary one-dimensional identity

$$\begin{aligned}\iint_{[-1,1]^2} |x-y|^{-\lambda}\,dx\,dy &= \int_{-2}^{2}(2-|t|)\,|t|^{-\lambda}\,dt\\ &= 2\int_0^2 (2-t)\,t^{-\lambda}\,dt\\ &= \frac{2^{3-\lambda}}{(1-\lambda)(2-\lambda)}, \qquad \lambda < 1,\end{aligned}$$

obtained by the change of variables $t = x - y$ and a direct integration. For $\lambda = \frac{1}{4}$, this identity gives

$$\iint_{[-1,1]^2} |x-y|^{-\frac{1}{4}}\,dx\,dy = \frac{2^{3-\lambda}}{(1-\lambda)(2-\lambda)} = \frac{2^{\frac{11}{4}}}{\frac{3}{4}\times\frac{7}{4}} \approx 5.1255.$$

The single-function inequality (1.1) reads $5.1255 \le \sigma_{1,\frac{1}{4}}\,\|f\|_{\frac{4}{7}}$, and since $\sigma_{1,\lambda} = 2/((1-\lambda)(2-\lambda)) = \frac{32}{21}$ and $\|f\|_{\frac{4}{7}} = 2^{\frac{7}{4}} \approx 3.3636$, equality holds: $\frac{32}{21}\times 2^{\frac{7}{4}} \approx 5.1255$. By contrast, the reduced two-function bound (5.2) gives

$$\sqrt{C(1,\tfrac{1}{2})}\,\|f\|_{\frac{4}{7}} = \left(\frac{\Gamma(1/4)}{\Gamma(3/4)}\right)^{\frac{1}{2}}\times 2^{\frac{7}{4}} \approx 1.7201\times 3.3636 \approx 5.7856 > 5.1255.$$

Here the first equality follows from the explicit formula of $C(n,\lambda)$ in Theorem 2.1: For $n = 1$ and $\lambda = \frac{1}{2}$, one has $C(1,\frac{1}{2}) = \pi^{\frac{1}{4}}\frac{\Gamma(1/4)}{\Gamma(3/4)}\left[\frac{\Gamma(1/2)}{\Gamma(1)}\right]^{-\frac{1}{2}} = \frac{\Gamma(1/4)}{\Gamma(3/4)}$, since $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(1) = 1$; we also used $\|f\|_{\frac{4}{7}} = 2^{\frac{7}{4}}$ for $f = \mathbf{1}_{[-1,1]}$. Hence, we show that the two-function constant is strictly larger than the sharp one, as stated in Proposition 5.1. For the lower-bound direction, $\lambda = -\frac{1}{4}$, the same identity gives

$$\iint_{[-1,1]^2} |x-y|^{\frac{1}{4}}\,dx\,dy = \frac{2^{3-\lambda}}{(1-\lambda)(2-\lambda)} = \frac{2^{\frac{13}{4}}}{\frac{5}{4}\times\frac{9}{4}} \approx 3.3826.$$

Here (1.2) gives equality: $\sigma_{1,-\frac{1}{4}}\,\|f\|_1^{\frac{9}{4}} = \frac{32}{45}\times 2^{\frac{9}{4}} \approx 0.7111\times 4.7568 \approx 3.3826$. By the reduced two-function bound (5.3), we have

$$C_{1,-\frac{1}{4}}\,\|f\|_1^{\frac{9}{4}} = \frac{2}{(2-\lambda)^2}\times 2^{\frac{9}{4}} = \frac{32}{81}\times 4.7568 \approx 1.8792 < 3.3826.$$

Thus, the two-function constant is strictly smaller than the sharp one, as stated in Proposition 5.2. These examples also show explicitly how the equality cases of the single-function inequalities are lost in the two-function setting.

**Structural differences and comparison with the literature.**

- *The two-function functional.* In (1.4) and (1.5), the functional $\min\{f(x), g(y)\}$ couples the value of $f$ at $x$ with the value of $g$ at $y$: It is not symmetric in $(x, y)$ unless $f = g$, it never "sees" the diagonal $x = y$, and its layer-cake form is the product $\mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\{g\geq t\}}(y)$ of two level sets that are in general different. In the single-function theory, the pair $\mathbf{1}_{\{f\geq t\}}(x)\mathbf{1}_{\{f\geq t\}}(y)$ consists of two copies of the same set, and it is precisely this symmetry that the chord power integrals of Baêta–Cai exploit.
- *Pointwise weights.* The kernel $|x|^{-\alpha}|x-y|^{-\lambda}|y|^{-\beta}$ that we set in Theorem 1.1 is the Stein–Weiss kernel (see [8, 9]). The weights shift the integrability indices from $\frac{n}{2n-\lambda}$ to $\frac{n}{2n-\lambda-2\alpha}$ and $\frac{n}{2n-\lambda-2\beta}$. Hence $f$ and $g$ may lie in different $L^p$ spaces. This models two profiles with different behavior at the origin. This is one of our motivations for setting this kernel.
- *Methods of proof.* Baêta and Cai [6, 7] derive (1.1)–(1.2) from chord power integrals via integral geometry (Crofton-type and Blaschke–Petkantschin formulas). Our proofs use no integral geometry at all: only the layer-cake representation, rearrangement, the HLS inequality, and the elementary inequalities of Minkowski, Chebyshev, and Jensen.

Based on the above, we think the two-function framework may be useful for some physical problems involving two interacting profiles (for instance, estimates for min-type interactions or double-layer potentials), where the pointwise weights model different decay rates of the two profiles.

## 6. Conclusions

In this paper, we established two chord Sobolev inequalities for two distinct functions, extending the single-function chord Sobolev inequalities of Baêta and Cai [6, 7]. Theorem 1.1 is a pointwise-weighted upper bound whose kernel $|x|^{-\alpha}|x-y|^{-\lambda}|y|^{-\beta}$ is of Stein–Weiss type and whose right-hand side involves the product of two (possibly different) $L^p$ norms; Theorem 1.2 is a lower bound, valid for every $\lambda < 0$, involving the $L^1$ and $L^\infty$ norms. Both constants are explicit.

We also analyzed the sharpness of the constants. For Theorem 1.1, equality holds exactly in the trivial cases $f \equiv 0$ or $g \equiv 0$. In the unweighted reduction, the constant is strictly larger than the optimal single-function constant. For Theorem 1.2, equality never occurs for admissible functions, and the constant is strictly smaller than the sharp single-function constant $\sigma_{n,\lambda}$. The sharp constants and the extremal functions of the two-function inequalities could not be determined, and we stated this explicitly in Subsection 5.2, together with the corresponding open problem. Finally, we displayed the reductions of the two-function inequalities to the single-function ones.

We believe that the two-function pointwise-weighted framework opens several directions for future research, in particular the determination of the sharp constants and the study of affine-invariant two-function versions.

## Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant Nos. 12371137, 11971080), the Natural Science Foundation of Chongqing, China (Grant No. CSTB2025NSCQ-GPX0123), and the Doctoral Scientific Startup Fund of Hebei Normal University (Grant No. L2025B50). The author thanks the editor and the anonymous referees for their valuable comments and suggestions.

## Use of Generative-AI tools declaration

The author declares that no generative AI tools were used in the creation of this article.

## Conflict of interest

The author declares no conflict of interest in this paper.

## References

1. R. A. Adams, J. J. F. Fournier, *Sobolev spaces*, 2 Eds., Amsterdam: Academic Press, 2003. https://doi.org/10.1016/S0079-8169(03)X8001-0
2. J. Haddad, M. Ludwig, Affine fractional $L^p$ Sobolev inequalities, *Math. Ann.*, **388** (2024), 1091–1115. https://doi.org/10.1007/s00208-022-02540-3
3. J. Haddad, M. Ludwig, Affine fractional Sobolev and isoperimetric inequalities, *J. Differential Geom.*, **129** (2025), 695–724. https://doi.org/10.4310/jdg/1740137450
4. M. Ludwig, Anisotropic fractional perimeters, *J. Differential Geom.*, **96** (2014), 77–93. https://doi.org/10.4310/jdg/1391192693
5. F. Tröltzsch, *Optimal control of partial differential equations: theory, methods and applications*, Providence, RI: American Mathematical Society, 2010. https://doi.org/10.1090/gsm/112
6. F. M. Baêta, X. Cai, Affine chord Sobolev inequalities and radial mean bodies for functions, 2025, arXiv:2511.12866. https://doi.org/10.48550/arXiv.2511.12866

7. F. M. Baêta, X. Cai, Chord Sobolev inequalities, 2026, arXiv:2602.15160. https://doi.org/10.48550/arXiv.2602.15160
8. A. D. Gadjiev, V. S. Guliyev, A. Serbetci, E. V. Guliyev, The Stein–Weiss type inequalities for the B-Riesz potentials, *J. Math. Inequal.*, **5** (2011), 87–106. https://doi.org/10.7153/jmi-05-09
9. A. Kassymov, M. Ruzhansky, D. Suragan, Hardy–Littlewood–Sobolev and Stein–Weiss inequalities on homogeneous Lie groups, *Integral Transforms Spec. Funct.*, **30** (2019), 643–655. https://doi.org/10.1080/10652469.2019.1597080
10. H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, New York: Springer, 2011. https://doi.org/10.1007/978-0-387-70914-7
11. E. H. Lieb, M. Loss, *Analysis*, 2 Eds., Providence, RI: American Mathematical Society, 2001. https://doi.org/10.1090/gsm/014
12. A. Baernstein II, *Symmetrization in analysis*, Cambridge: Cambridge University Press, 2019. https://doi.org/10.1017/9781139020244
13. P. M. Gruber, *Convex and discrete geometry*, Berlin, Heidelberg: Springer, 2007. https://doi.org/10.1007/978-3-540-71133-9
14. R. Schneider, *Convex bodies: The Brunn–Minkowski theory*, 2 Eds., Cambridge: Cambridge University Press, 2013. https://doi.org/10.1017/CBO9781139003858
15. E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.*, **136** (2012), 521–573. https://doi.org/10.1016/j.bulsci.2011.12.004
16. W. Guo, Y. Li, Affine chord Sobolev inequalities for two functions, *ChinaXiv*, 2026. https://doi.org/10.12074/202608.00068




AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0)