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Research article

An adaptive Weibull positive operator with variable shape: Approximation and Voronovskaya-type asymptotics

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Abstract: In this paper, we introduced a new adaptive modification of the Weibull linear positive operator discussed in Montano et al., Positivity, 29 (2025), 4 by incorporating a bounded variable shape function into the Weibull kernel. More precisely, the proposed operator was constructed through the adaptive function $\phi(x) = \dfrac{x}{1+x}$ and a parameter $\lambda \geq 0$. The operator preserved positivity, linearity, and normalization, and reduced to the classical Weibull operator when $\lambda = 0$. We derived explicit expressions for its raw moments up to the fourth order and obtained the corresponding central moments. An asymptotic expansion of the associated Gamma-function ratio was established. A Korovkin-type theorem was proved, followed by a quantitative estimate in terms of the modulus of continuity. Numerical experiments illustrated the influence of the adaptive parameter on the second central moment and approximation errors, showing that the proposed adaptive modification can provide improved approximation compared with the classical Weibull operator. Furthermore, first and second-order Voronovskaya-type asymptotic formulas were established.

Keywords: Weibull operator; positive linear operator; adaptive operator; variable shape parameter; approximation; Korovkin theorem; modulus of continuity; Voronovskaya-type theorem
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1. Introduction

Approximation theory by positive linear operators constitutes an important area of mathematical analysis and has attracted considerable attention due to its wide range of applications in numerical analysis, functional analysis, probability theory, and computational mathematics. The basic objective is to construct sequences of operators which approximate a given function while preserving certain structural properties such as positivity, linearity, and, whenever possible, important test functions. Classical families of positive linear operators, including the Bernstein, Szász-Mirakjan, Baskakov, Gamma, and Post-Widder operators, have played a fundamental role in the development of modern approximation theory [2–6]. One of the main advantages of positive linear operators is that their convergence properties can often be investigated through a relatively small collection of test functions by means of Korovkin-type theorems. The theory of positive linear operators therefore provides a flexible framework for investigating both qualitative and quantitative properties of approximation processes.

Integral operators generated by probability densities and positive kernels form an important class of approximation processes. In particular, Gamma-type and Post–Widder-type operators have received considerable attention because of their natural probabilistic interpretation and the explicit representation of their moments in terms of the Gamma function. These properties are useful in investigating convergence, quantitative estimates, and asymptotic behavior. Several modified versions of classical operators have therefore been proposed with the objective of improving approximation properties or preserving additional functions. Motivated by this idea, we introduce in the present paper a new family of positive linear operators based on an adaptive modification of a Weibull-type kernel. The proposed construction incorporates a bounded spatial correction into the kernel through an adaptive function.

A significant line of research has focused on statistical approximation and generalized summability methods. Mursaleen and Mohiuddine [7] established a systematic framework for the statistical approximation of positive linear operators by proving Korovkin-type approximation theorems together with estimates for the rate of statistical convergence. Subsequently, Mohiuddine, Alotaibi, and Hazarika [8] investigated weighted A-statistical convergence and obtained weighted Korovkin-type approximation theorems along with corresponding convergence estimates. Further extensions of the classical Korovkin theory were presented by Mursaleen and Kiliçman [9], who established Korovkin's second theorem via $\mathcal{B}$-statistical A-summability. Later, Mohiuddine and Alotaibi [10] obtained new approximation results by employing statistical summability $(C, 1)$, thereby extending the applicability of Korovkin-type approximation through generalized summability methods.

Another active direction of research concerns the construction of modified positive linear operators possessing improved approximation behavior. Acu and Agrawal [11] introduced genuine Bernstein-Durrmeyer type operators and established quantitative approximation estimates together with Voronovskaya-type asymptotic formulas. Acu et al. [12] investigated positive linear operators preserving two test functions and obtained quantitative estimates as well as asymptotic approximation results. Agrawal et al. [13] proposed modified Bernstein-Durrmeyer operators and demonstrated an improved degree of approximation through refined convergence estimates. Cai [14] studied approximation properties of a new class of q-Bernstein-Kantorovich operators. More recently, Gupta and Muraru [15] introduced modified Szász-Durrmeyer operators by combining generalized Szász-

Mirakjan–Durrmeyer and Abel-Ivan type operators, and established several direct approximation results together with quantitative convergence estimates. Gupta and Anjali [16] introduced a new class of exponential operators connected with $2 + x^2$, generalizing the classical Post–Widder and Ismail-May operators. The study [17], presents a quadratic function-preserving modification of Baskakov operators to improve approximation properties. Acu et al. [18] investigated linear operators preserving polynomials and barycenters, along with corresponding difference estimates. Recently, Rao et al. [19] introduced adjoint Bernoulli's polynomials coupled with gamma functions and investigated their approximation properties.

Among the recently introduced integral-type approximation processes, the Weibull linear positive operator proposed by Cappelletti Montano and Leonessa [1] provides a new probabilistic framework for approximation on the unbounded interval $[0, \infty)$. The operator is generated from the Weibull kernel and preserves the fundamental properties of positivity, linearity, and normalization, making it suitable for the approximation of continuous functions. Furthermore, it satisfies the classical Korovkin test conditions and exhibits satisfactory approximation behavior through its associated moments and convergence properties. Owing to its simple analytical structure and effective approximation performance, the Weibull operator serves as an attractive alternative to several classical integral-type positive linear operators.

Motivated by the limitations of the classical Weibull operator, we introduce an adaptive modification by incorporating the bounded function $\phi(x) = \dfrac{x}{1+x}$ into the exponent of the Weibull kernel. The proposed operator preserves positivity, linearity, and normalization, and reduces to the classical Weibull operator when $\lambda = 0$. We investigate its approximation properties through Korovkin-type convergence, modulus of continuity estimates, first- and second-order Voronovskaya-type theorems, and numerical examples.

2. Construction of operator

We define space of continuous functions of at most polynomial growth as:

$$C_{\text{pol}}[0,\infty) = \left\{ f \in C[0,\infty) : |f(t)| \le M_f(1 + t^{r_f}) \text{ for all } t \ge 0, \text{ for some } M_f > 0,\ r_f \ge 0 \right\}.$$

Definition 2.1. *(Adaptive Weibull linear positive operator) Let $f \in C_{pol}[0,\infty)$, $x \ge 0$, $n \in \mathbb{N}$, and let $\lambda \ge 0$ be a fixed adaptive real parameter. Define the adaptive function*

$$\phi(x) = \frac{x}{1+x}, \qquad x \ge 0.$$

The adaptive Weibull linear positive operator is defined by

$$\mathcal{W}_n^{(\lambda)}(f;x) = \frac{n}{\Gamma\left(1 + \dfrac{\lambda\phi(x)}{n}\right)} \int_0^\infty u^{n+\lambda\phi(x)-1} e^{-u^n} f(xu)\, du. \tag{2.1}$$

This function $\phi(x)$ is chosen because it satisfies the following desirable properties:

1. $\phi(0) = 0$, so the proposed operator coincides with the classical Weibull operator near the origin, preserving its local behavior.

2. $0 \leq \phi(x) < 1$ for all $x \geq 0$, which guarantees that the exponent

$$n + \lambda\phi(x) - 1$$

remains nonnegative and uniformly bounded with respect to x. Consequently, the kernel remains well defined.

3. $\phi(x)$ is continuously differentiable and monotonically increasing since

$$\phi'(x) = \frac{1}{(1+x)^2} > 0,$$

allowing the adaptive correction to increase smoothly with x.

4. As $x \to \infty$,

$$\phi(x) \to 1,$$

so the adaptive effect is automatically bounded and does not excessively distort the original Weibull kernel.

These properties provide a simple, smooth, and bounded adaptive mechanism while preserving the positivity and normalization of the proposed operator.

Remark 2.1. *The normalization factor*

$$\frac{n}{\Gamma\left(1 + \dfrac{\lambda\phi(x)}{n}\right)}$$

is chosen so that

$$\mathcal{W}_n^{(\lambda)}(1; x) = 1,$$

for every $x \geq 0$. Furthermore, when $\lambda = 0$, the operator (2.1) *reduces to the classical Weibull operator*

$$\mathcal{W}_n(f; x) = n \int_0^\infty u^{n-1} e^{-u^n} f(xu)\, du.$$

Hence, the proposed operator is a genuine one-parameter generalization of the classical Weibull operator [1]. Here, the function

$$\phi(x) = \frac{x}{1+x}$$

determines the spatial variation of the adaptation, whereas the parameter $\lambda \geq 0$ controls the strength of the adaptive modification. In particular, $\lambda = 0$ reduces the proposed operator to the classical Weibull operator, while larger values of λ increase the adaptive effect without altering the functional form of $\phi(x)$.

3. Basic and main results of operator $\mathcal{W}_n^{(\lambda)}$

Lemma 3.1. *For $r \in \mathbb{N}_0$, the values of test functions t^r, $r = 0, 1, 2, 3, 4$ of the operator $\mathcal{W}_n^{(\lambda)}$ are given by*

$$\begin{aligned}
\mathcal{W}_n^{(\lambda)}(1; x) &= 1,\\
\mathcal{W}_n^{(\lambda)}(t; x) &= x\frac{\Gamma(1 + \frac{\lambda\phi(x)+1}{n})}{\Gamma(1 + \frac{\lambda\phi(x)}{n})},\\
\mathcal{W}_n^{(\lambda)}(t^2; x) &= x^2\frac{\Gamma(1 + \frac{\lambda\phi(x)+2}{n})}{\Gamma(\frac{\lambda\phi(x)}{n})},\\
\mathcal{W}_n^{(\lambda)}(t^3; x) &= x^3\frac{\Gamma(1 + \frac{\lambda\phi(x)+3}{n})}{\Gamma(\frac{\lambda\phi(x)}{n})},\\
\mathcal{W}_n^{(\lambda)}(t^4; x) &= x^4\frac{\Gamma(1 + \frac{\lambda\phi(x)+4}{n})}{\Gamma(\frac{\lambda\phi(x)}{n})}.
\end{aligned}$$

Proof. Using $t = xu$ and the definition of $\mathcal{W}_n^{(\lambda)}$, we obtain

$$\mathcal{W}_n^{(\lambda)}(t^r; x) = \frac{nx^r}{\Gamma(1 + \frac{\lambda\phi(x)}{n})}\int_0^\infty u^{n+\lambda\phi(x)+r-1}e^{-u^n}\,du.$$

With $v = u^n$, we have $du = \frac{1}{n}v^{1/n-1}dv$, and hence

$$\int_0^\infty u^{n+\lambda\phi(x)+r-1}e^{-u^n}du = \frac{1}{n}\Gamma\left(1 + \frac{\lambda\phi(x) + r}{n}\right).$$

Substitution gives the stated formula. The particular cases $r = 0, 1, 2, 3, 4$ follow immediately. □

Remark 3.1. *In general, we can easily conclude*

$$\mathcal{W}_n^{(\lambda)}(t^k; x) = x^k\frac{\Gamma\left(1 + \frac{\lambda\phi(x)+k}{n}\right)}{\Gamma\left(1 + \frac{\lambda\phi(x)}{n}\right)}. \tag{3.1}$$

Lemma 3.2. *(Central Moments calculation) Let*

$$\mu_{n,r}^{(\lambda)}(x) = \mathcal{W}_n^{(\lambda)}((t - x)^r; x), \qquad r \in \mathbb{N}.$$

Then,

$$\begin{aligned}
\mu_{n,1}^{(\lambda)}(x) &= x\left(\frac{\Gamma\left(1 + \frac{\lambda\phi(x)+1}{n}\right)}{\Gamma\left(1 + \frac{\lambda\phi(x)}{n}\right)} - 1\right),\\
\mu_{n,2}^{(\lambda)}(x) &= x^2\left(\frac{\Gamma\left(1 + \frac{\lambda\phi(x)+2}{n}\right)}{\Gamma\left(1 + \frac{\lambda\phi(x)}{n}\right)} - 2\frac{\Gamma\left(1 + \frac{\lambda\phi(x)+1}{n}\right)}{\Gamma\left(1 + \frac{\lambda\phi(x)}{n}\right)} + 1\right),
\end{aligned}$$

$$\mu_{n,3}^{(\lambda)}(x) = x^3\left(\frac{\Gamma\left(1+\frac{\lambda\phi(x)+3}{n}\right)}{\Gamma\left(1+\frac{\lambda\phi(x)}{n}\right)} - 3\frac{\Gamma\left(1+\frac{\lambda\phi(x)+2}{n}\right)}{\Gamma\left(1+\frac{\lambda\phi(x)}{n}\right)} + 3\frac{\Gamma\left(1+\frac{\lambda\phi(x)+1}{n}\right)}{\Gamma\left(1+\frac{\lambda\phi(x)}{n}\right)} - 1\right),$$

$$\mu_{n,4}^{(\lambda)}(x) = x^4\left(\frac{\Gamma\left(1+\frac{\lambda\phi(x)+4}{n}\right)}{\Gamma\left(1+\frac{\lambda\phi(x)}{n}\right)} - 4\frac{\Gamma\left(1+\frac{\lambda\phi(x)+3}{n}\right)}{\Gamma\left(1+\frac{\lambda\phi(x)}{n}\right)} + 6\frac{\Gamma\left(1+\frac{\lambda\phi(x)+2}{n}\right)}{\Gamma\left(1+\frac{\lambda\phi(x)}{n}\right)} - 4\frac{\Gamma\left(1+\frac{\lambda\phi(x)+1}{n}\right)}{\Gamma\left(1+\frac{\lambda\phi(x)}{n}\right)} + 1\right).$$

Proof. Using the binomial expansion

$$(t-x)^r = \sum_{k=0}^{r}\binom{r}{k}(-x)^{r-k}t^k$$

and Formula 3.1

$$\mathcal{W}_n^{(\lambda)}(t^k;x) = x^k\frac{\Gamma\left(1+\frac{\lambda\phi(x)+k}{n}\right)}{\Gamma\left(1+\frac{\lambda\phi(x)}{n}\right)},$$

the required identities follow immediately by linearity of $\mathcal{W}_n^{(\lambda)}$ and [1]. □

Lemma 3.3. *Let $m \in \mathbb{N}_0$ be fixed. Then, for every compact interval $[0,a] \subset [0,\infty)$,*

$$\lim_{n\to\infty}\sup_{x\in[0,a]}\left|\frac{\Gamma\left(1+\frac{\lambda\phi(x)+m}{n}\right)}{\Gamma\left(1+\frac{\lambda\phi(x)}{n}\right)} - 1\right| = 0,$$

where

$$\phi(x) = \frac{x}{1+x}.$$

Proof. Since

$$0 \le \phi(x) = \frac{x}{1+x} \le 1,$$

we have

$$\left|\frac{\lambda\phi(x)+m}{n}\right| \le \frac{|\lambda|+m}{n}, \qquad \left|\frac{\lambda\phi(x)}{n}\right| \le \frac{|\lambda|}{n},$$

for every $x \in [0,a]$. Hence,

$$\frac{\lambda\phi(x)+m}{n} \longrightarrow 0, \qquad \frac{\lambda\phi(x)}{n} \longrightarrow 0, \quad as\ n\to\infty$$

uniformly on $[0,a]$.

Since the Gamma function is continuous at 1 and

$$\Gamma(1) = 1,$$

it follows that

$$\Gamma\left(1+\frac{\lambda\phi(x)+m}{n}\right) \longrightarrow 1, \qquad \Gamma\left(1+\frac{\lambda\phi(x)}{n}\right) \longrightarrow 1, \quad as\ n\to\infty$$

uniformly for $x \in [0, a]$.
Moreover, since

$$\Gamma\left(1 + \frac{\lambda\phi(x)}{n}\right) \to 1 \quad as\ n \to \infty$$

uniformly on $[0, a]$, there exists $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$,

$$\inf_{x\in[0,a]} \Gamma\left(1 + \frac{\lambda\phi(x)}{n}\right) > \frac{1}{2}.$$

Hence, the denominator is uniformly bounded away from zero, and therefore the quotient converges uniformly to 1 on $[0, a]$. Therefore,

$$\frac{\Gamma\left(1 + \frac{\lambda\phi(x)+m}{n}\right)}{\Gamma\left(1 + \frac{\lambda\phi(x)}{n}\right)} \longrightarrow 1 \quad as\ n \to \infty$$

uniformly on $[0, a]$, which completes the proof. □

The following lemma gives a precise asymptotic expansion of the Gamma-function ratio required for the subsequent analysis.

Lemma 3.4. *Let $m \in \mathbb{N}_0$ be fixed and define*

$$R^{(\lambda)}_{n,m}(x) := \frac{\Gamma\left(1 + \dfrac{\lambda\phi(x) + m}{n}\right)}{\Gamma\left(1 + \dfrac{\lambda\phi(x)}{n}\right)}, \qquad \phi(x) = \frac{x}{1 + x}.$$

Then, uniformly for $x \in [0, a]$,

$$R^{(\lambda)}_{n,m}(x) = 1 - \frac{\gamma m}{n} + \frac{1}{n^2}\left[\frac{\pi^2}{6}\lambda\phi(x)m + \left(\frac{\pi^2}{12} + \frac{\gamma^2}{2}\right)m^2\right] + O(n^{-3}),$$

as $n \to \infty$, where γ denotes the Euler–Mascheroni constant (see [20]).

Proof. Put $A = \lambda\phi(x)$. Since $0 \leq \phi(x) \leq 1$, A is bounded uniformly for $x \in [0, a]$. Using the standard expansion [20, Eq (4.24)]

$$\log\Gamma(1 + z) = -\gamma z + \frac{\pi^2}{12}z^2 + O(z^3), \qquad z \to 0,$$

we obtain

$$\begin{aligned}
\log R^{(\lambda)}_{n,m}(x) &= \log\Gamma\left(1 + \frac{A + m}{n}\right) - \log\Gamma\left(1 + \frac{A}{n}\right) \\
&= -\frac{\gamma m}{n} + \frac{\pi^2}{12n^2}\left((A + m)^2 - A^2\right) + O(n^{-3}) \\
&= -\frac{\gamma m}{n} + \frac{\pi^2}{12n^2}(2Am + m^2) + O(n^{-3}).
\end{aligned}$$

Let

$$c_1 = -\gamma m, \qquad c_2 = \frac{\pi^2}{12}(2Am + m^2).$$

Then,

$$\log R_{n,m}^{(\lambda)}(x) = \frac{c_1}{n} + \frac{c_2}{n^2} + O(n^{-3}).$$

Using

$$e^y = 1 + y + \frac{y^2}{2} + O(y^3), \qquad y \to 0,$$

we get

$$R_{n,m}^{(\lambda)}(x) = 1 + \frac{c_1}{n} + \frac{1}{n^2}\left(c_2 + \frac{c_1^2}{2}\right) + O(n^{-3}).$$

Since $c_1^2 = \gamma^2 m^2$, it follows that

$$R_{n,m}^{(\lambda)}(x) = 1 - \frac{\gamma m}{n} + \frac{1}{n^2}\left[\frac{\pi^2}{6}Am + \left(\frac{\pi^2}{12} + \frac{\gamma^2}{2}\right)m^2\right] + O(n^{-3}).$$

Substituting $A = \lambda\phi(x)$ completes the proof. □

Lemma 3.5. *For $r = 3, 4$, the central moments of $\mathcal{W}_n^{(\lambda)}$ satisfy, uniformly for $x \in [0, a]$,*

$$\mu_{n,r}^{(\lambda)}(x) = O(n^{-r}), \qquad n \to \infty.$$

In particular,

$$\mu_{n,3}^{(\lambda)}(x) = O(n^{-3}), \qquad \mu_{n,4}^{(\lambda)}(x) = O(n^{-4}).$$

Proof. We consider two cases.

Case 1. $x > 0$**.**

Put

$$A = \lambda\phi(x), \qquad R_{n,k}^{(\lambda)}(x) = \frac{\Gamma\left(1 + \dfrac{A+k}{n}\right)}{\Gamma\left(1 + \dfrac{A}{n}\right)}.$$

Since $0 \le \phi(x) \le 1$, the quantity A is bounded uniformly for $x \in [0, a]$. From the Taylor expansion of $\log\Gamma(1+z)$, for every fixed k,

$$\log R_{n,k}^{(\lambda)}(x) = \sum_{j=1}^{r} \frac{L_j(A,k)}{n^j} + O(n^{-(r+1)}),$$

uniformly for $x \in [0, a]$, where $L_j(A, k)$ is a polynomial in k of degree at most j. Consequently,

$$R_{n,k}^{(\lambda)}(x) = 1 + \sum_{j=1}^{r} \frac{P_j(A,k)}{n^j} + O(n^{-(r+1)}),$$

where $P_j(A, k)$ is also a polynomial in k of degree at most j.

By the binomial expansion, for $x > 0$,

$$\frac{\mu_{n,r}^{(\lambda)}(x)}{x^r} = \sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}R_{n,k}^{(\lambda)}(x).$$

Substituting the above expansion gives

$$\frac{\mu_{n,r}^{(\lambda)}(x)}{x^r} = \sum_{j=1}^{r}\frac{1}{n^j}\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}P_j(A,k) + O(n^{-(r+1)}).$$

Since $P_j(A,k)$ is a polynomial in k of degree at most $j < r$ for $j = 1,\ldots,r-1$, its r-th finite difference vanishes. Hence,

$$\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}P_j(A,k) = 0, \qquad j < r.$$

Therefore,

$$\frac{\mu_{n,r}^{(\lambda)}(x)}{x^r} = O(n^{-r}),$$

uniformly for $x \in (0,a]$. Since $x \le a$, we obtain

$$\mu_{n,r}^{(\lambda)}(x) = O(n^{-r}).$$

Case 2. $x = 0$**.**

By the definition of the operator, $\mathcal{W}_n^{(\lambda)}(f;0) = f(0)$. Therefore, for $r = 3,4$,

$$\mu_{n,r}^{(\lambda)}(0) = \mathcal{W}_n^{(\lambda)}(t^r;0) = 0.$$

Combining both cases, we conclude that

$$\mu_{n,r}^{(\lambda)}(x) = O(n^{-r}),$$

uniformly for $x \in [0,a]$, for $r = 3,4$. This completes the proof. □

Let $\rho(x) = 1 + x^2$. We define the weighted space $C_\rho^0[0,\infty) \subset C_{\text{pol}}[0,\infty)$,

$$C_\rho^0[0,\infty) = \left\{ f \in C[0,\infty) : \|f\|_\rho < \infty, \quad \lim_{x\to\infty}\frac{f(x)}{\rho(x)} \text{ exists and is finite} \right\},$$

where

$$\|f\|_\rho = \sup_{x\ge 0}\frac{|f(x)|}{\rho(x)}.$$

Theorem 3.6. *(Weighted Korovkin type approximation theorem) Let $\rho(x) = 1 + x^2$ and let $C_\rho^0[0,\infty)$ be the weighted space defined above. Then the adaptive Weibull linear positive operators $\mathcal{W}_n^{(\lambda)}$ satisfy*

$$\lim_{n\to\infty}\left\|\mathcal{W}_n^{(\lambda)}(f;\cdot) - f\right\|_\rho = 0, \qquad f \in C_\rho^0[0,\infty),$$

where

$$\|g\|_\rho = \sup_{x\ge 0}\frac{|g(x)|}{1+x^2}.$$

Proof. The operators $\mathcal{W}_n^{(\lambda)}$ are linear and positive. Moreover, for each fixed n,

$$\mathcal{W}_n^{(\lambda)} : C_\rho^0[0,\infty) \to C_\rho^0[0,\infty).$$

By the weighted Korovkin theorem, it is sufficient to verify the convergence for the test functions 1, t, and t^2.

From the moment identities, we have

$$\mathcal{W}_n^{(\lambda)}(1;x) = 1$$

and

$$\left\|\mathcal{W}_n^{(\lambda)}(1) - 1\right\|_\rho = 0,$$

Next,

$$\mathcal{W}_n^{(\lambda)}(t;x) = x\frac{\Gamma\left(1+\dfrac{\lambda\phi(x)+1}{n}\right)}{\Gamma\left(1+\dfrac{\lambda\phi(x)}{n}\right)}.$$

Since $0 \le \phi(x) \le 1$ for all $x \ge 0$, choose $a = \lambda\phi(x)$.
Consequently,

$$\begin{aligned}\|\mathcal{W}_n^{(\lambda)}(t) - t\|_\rho &= \sup_{x\ge 0}\frac{x}{1+x^2}\left|\frac{\Gamma\left(1+\frac{\lambda\phi(x)+1}{n}\right)}{\Gamma\left(1+\frac{\lambda\phi(x)}{n}\right)} - 1\right| \\ &\le \frac{1}{2}\sup_{0\le a\le\lambda}\left|\frac{\Gamma\left(1+\frac{a+1}{n}\right)}{\Gamma\left(1+\frac{a}{n}\right)} - 1\right| \longrightarrow 0.\end{aligned}$$

In a similar way, we have

$$\left\|\mathcal{W}_n^{(\lambda)}(t^2) - t^2\right\|_\rho \longrightarrow 0.$$

Therefore, the weighted Korovkin theorem yields

$$\lim_{n\to\infty}\left\|\mathcal{W}_n^{(\lambda)}(f;\cdot) - f\right\|_\rho = 0.$$

This completes the proof. □

Definition 3.1. *Let $UC[0,\infty)$ denote the space of all uniformly continuous real-valued functions on $[0,\infty)$. For $f \in UC[0,\infty)$, the modulus of continuity is defined by*

$$\omega(f;\delta) = \sup_{\substack{x,t\ge 0\\ |t-x|\le\delta}}|f(t) - f(x)|, \qquad \delta > 0.$$

For every $f \in UC[0,\infty)$, we have

$$\lim_{\delta\to 0^+}\omega(f;\delta) = 0.$$

Theorem 3.7. *(Estimate via the modulus of continuity) Let $f \in UC[0,\infty)$. Then, for every $x \geq 0$,*

$$\left|\mathcal{W}_n^{(\lambda)}(f;x) - f(x)\right| \leq 2\,\omega\left(f;\ \sqrt{\mu_{n,2}^{(\lambda)}(x)}\right),$$

where

$$\mu_{n,2}^{(\lambda)}(x) = \mathcal{W}_n^{(\lambda)}((t-x)^2;x)$$

is the second-order central moment of $\mathcal{W}_n^{(\lambda)}$.

Proof. Let $x \geq 0$ and $\delta > 0$. By the definition of the modulus of continuity, for all $x, t \geq 0$ we have

$$|f(t) - f(x)| \leq \left(1 + \frac{|t-x|}{\delta}\right)\omega(f;\delta).$$

Since $\mathcal{W}_n^{(\lambda)}$ is a positive linear operator and $\mathcal{W}_n^{(\lambda)}(1;x) = 1$, it follows that

$$\begin{aligned}\left|\mathcal{W}_n^{(\lambda)}(f;x) - f(x)\right| &= \left|\mathcal{W}_n^{(\lambda)}(f(t) - f(x);x)\right| \\ &\leq \mathcal{W}_n^{(\lambda)}(|f(t) - f(x)|;x) \\ &\leq \left[1 + \frac{\mathcal{W}_n^{(\lambda)}(|t-x|;x)}{\delta}\right]\omega(f;\delta).\end{aligned}$$

By the Cauchy–Schwarz inequality,

$$\mathcal{W}_n^{(\lambda)}(|t-x|;x) \leq \left[\mathcal{W}_n^{(\lambda)}((t-x)^2;x)\right]^{1/2} = \sqrt{\mu_{n,2}^{(\lambda)}(x)}.$$

Consequently,

$$\left|\mathcal{W}_n^{(\lambda)}(f;x) - f(x)\right| \leq \left[1 + \frac{\sqrt{\mu_{n,2}^{(\lambda)}(x)}}{\delta}\right]\omega(f;\delta).$$

Choosing

$$\delta = \sqrt{\mu_{n,2}^{(\lambda)}(x)},$$

we obtain

$$\left|\mathcal{W}_n^{(\lambda)}(f;x) - f(x)\right| \leq 2\,\omega\left(f;\ \sqrt{\mu_{n,2}^{(\lambda)}(x)}\right).$$

This completes the proof. □

4. Numerical and graphical illustrations

The numerical values presented in Table 1 demonstrate the influence of the adaptive parameter λ on the second central moment of the proposed operator ($\mathcal{W}_n^{(\lambda)}$). It is observed that, for each fixed value of n and x, the second central moment decreases consistently as λ increases. Since the second central moment measures the spread of the kernel around the point of approximation, this numerical behavior indicates that the operator ($\mathcal{W}_n^{(\lambda)}$) becomes increasingly concentrated with larger values of λ. Furthermore, for fixed λ, the second central moment also decreases as n increases, confirming the

improved approximation behavior of the proposed operator $(\mathcal{W}_n^{(\lambda)})$. All numerical computations were performed using Python with the NumPy and SciPy libraries. The integrals defining the adaptive Weibull operators were evaluated numerically using the quad routine from scipy.integrate.

Table 1. Second central moment $\mu_{n,2}^{(\lambda)}(x)$ of the operator $(\mathcal{W}_n^{(\lambda)})$ for different values of n, x, and λ.

n	x	$\lambda = 0$	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$	$\lambda = 5$
20	0.5	0.0011	0.0011	0.0010	0.0010	0.0009
	1.0	0.0043	0.0041	0.0039	0.0038	0.0035
	2.0	0.0174	0.0163	0.0153	0.0145	0.0130
	5.0	0.1086	0.1002	0.0930	0.0866	0.0763
30	0.5	0.0005	0.0005	0.0005	0.0005	0.0005
	1.0	0.0020	0.0019	0.0019	0.0018	0.0017
	2.0	0.0080	0.0077	0.0074	0.0071	0.0065
	5.0	0.0503	0.0476	0.0451	0.0428	0.0389
50	0.5	0.0002	0.0002	0.0002	0.0002	0.0002
	1.0	0.0008	0.0007	0.0007	0.0007	0.0007
	2.0	0.0030	0.0029	0.0028	0.0028	0.0026
	5.0	0.0187	0.0181	0.0175	0.0169	0.0159

Figures 1 and 2 illustrate the numerical values of the second-order moment of the operators defined in (2.1), providing a clear visualization of their behavior.

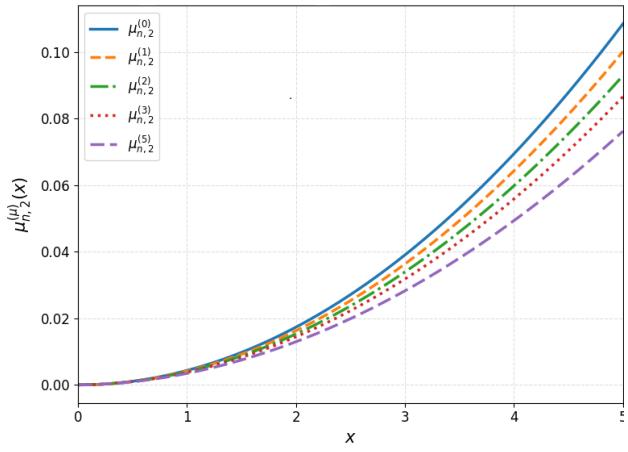


Figure 1. $\mu_{20,2}^{\lambda}$ for $\lambda = 0, 1, 2, 3, 5$.

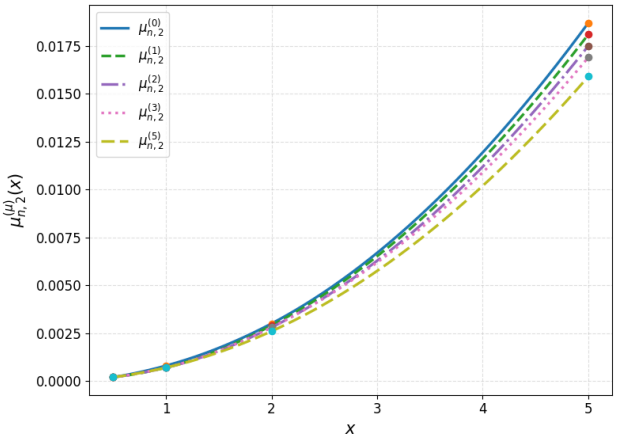


Figure 2. $\mu_{50,2}^{\lambda}$ for $\lambda = 0, 1, 2, 3, 5$.

The effect of the adaptive parameter λ on the approximation behavior of the proposed operator is investigated by fixing $n = 30$. The maximum approximation error is computed as

$$E_{30}(\lambda) = \max_{0 \le x \le 5} \left| W_{30}^{(\lambda)}(f; x) - f(x) \right|$$

for different values of λ using the test functions $f(x) = e^{-x} \sin x$, $f(x) = \log(1 + x)$, and $f(x) = \dfrac{x}{1 + x}$.

The numerical results presented in Table 2 clearly show that the approximation error decreases monotonically as the adaptive parameter λ increases. For the function $f(x) = e^{-x}\sin x$, the error decreases from 6.9384×10^{-3} when $\lambda = 0$ to 4.9146×10^{-3} when $\lambda = 5$. Similarly, for $f(x) = \log(1+x)$, the error is reduced from 1.5878×10^{-2} to 1.0143×10^{-2}, while for $f(x) = \dfrac{x}{1+x}$, it decreases from 4.8098×10^{-3} to 3.7701×10^{-3}.

Table 2. Influence of the adaptive parameter λ on the approximation error $E_{30}(\lambda) = \max_{0\le x\le 5}\left|W_{30}^{(\lambda)}(f;x) - f(x)\right|$.

λ	$f(x) = e^{-x}\sin x$	$f(x) = \log(1+x)$	$f(x) = \dfrac{x}{1+x}$
0	6.9384×10^{-3}	1.5878×10^{-2}	4.8098×10^{-3}
1	6.5082×10^{-3}	1.4643×10^{-2}	4.5805×10^{-3}
2	6.0913×10^{-3}	1.3455×10^{-2}	4.3621×10^{-3}
3	5.6871×10^{-3}	1.2311×10^{-2}	4.1544×10^{-3}
5	4.9146×10^{-3}	1.0143×10^{-2}	3.7701×10^{-3}

These numerical observations indicate that the adaptive parameter λ enhances the flexibility of the Weibull kernel and significantly improves the approximation performance of the operator $(\mathcal{W}_n^{(\lambda)})$. In particular, the adaptive operator consistently yields smaller approximation errors than the classical Weibull operator corresponding to $\lambda = 0$, thereby demonstrating the effectiveness of the proposed adaptive modification.

Next, Figure 3 illustrates the approximation of the function $f(x) = e^{-x^2}\sin(4x)$ by the adaptive Weibull operator $\mathcal{W}_n^{(2)}(f;x)$ for different values of n. It is observed that the approximation improves as n increases, and the operator curves gradually converge to the exact function, confirming the convergence behavior of the proposed operator.

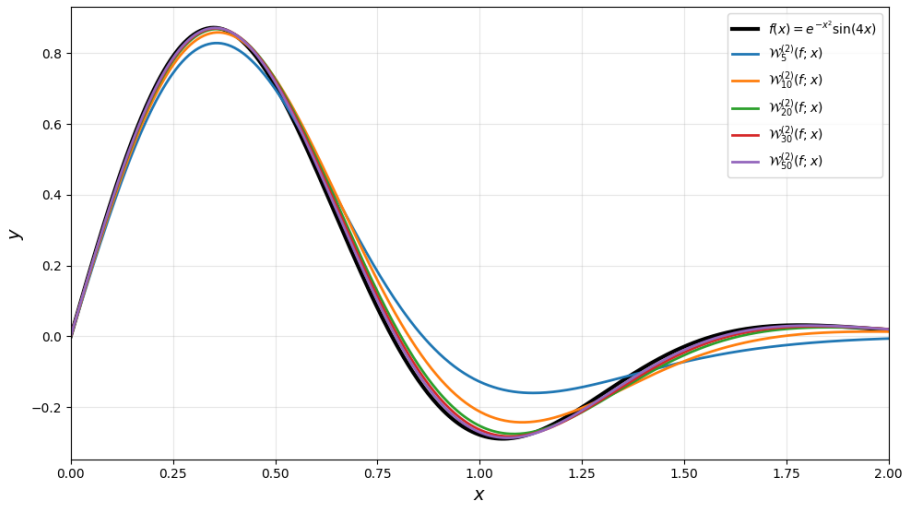


Figure 3. Approximation of $e^{-x^2}\sin(4x)$ by the operator $\mathcal{W}_n^{(2)}(f;x)$ for different values of n.

Figure 4 illustrates the approximation of $f(x) = \dfrac{x}{1+x}$ by the operator $(\mathcal{W}_n^{(\lambda)})$ for different values

of n. As n increases, the operator curves approach the exact function, demonstrating the convergence of the proposed operator.

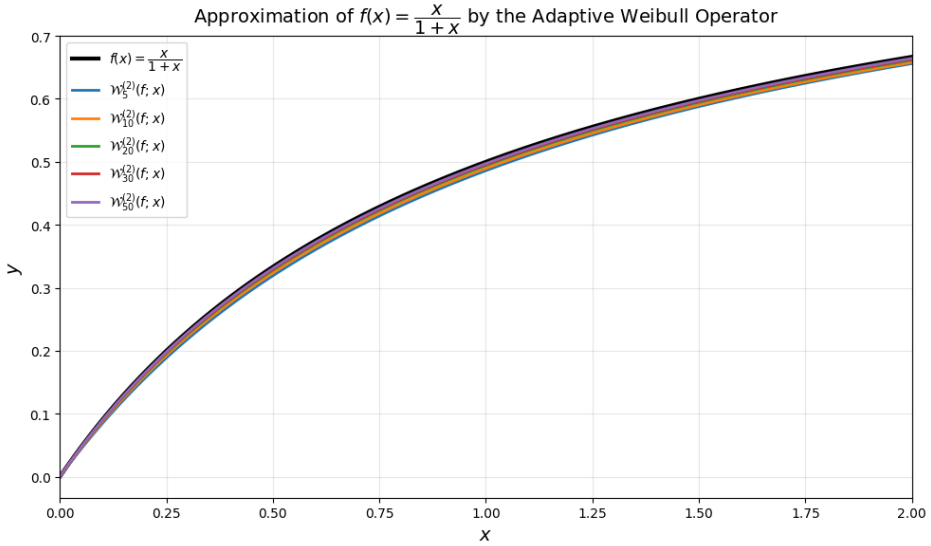


Figure 4. Approximation of $\dfrac{x}{1+x}$ by the operator $\mathcal{W}_n^{(2)}(f;x)$ for different values of n.

Figure 5 shows that the maximum absolute error $E_n^{(\lambda)} = \max_{x\in[0,5]} \left|\mathcal{W}_n^{(\lambda)}(f;x) - f(x)\right|$, where the maximum is evaluated over a uniform grid of points in the interval $[0,5]$, decreases steadily with increasing n, for the function $e^{-x}\sin x$, confirming the convergence of both the classical and proposed Weibull operators ($\mathcal{W}_n^{(\lambda)}$). Furthermore, larger values of the adaptive parameter λ yield smaller approximation errors, indicating that the proposed modification significantly enhances the approximation performance.

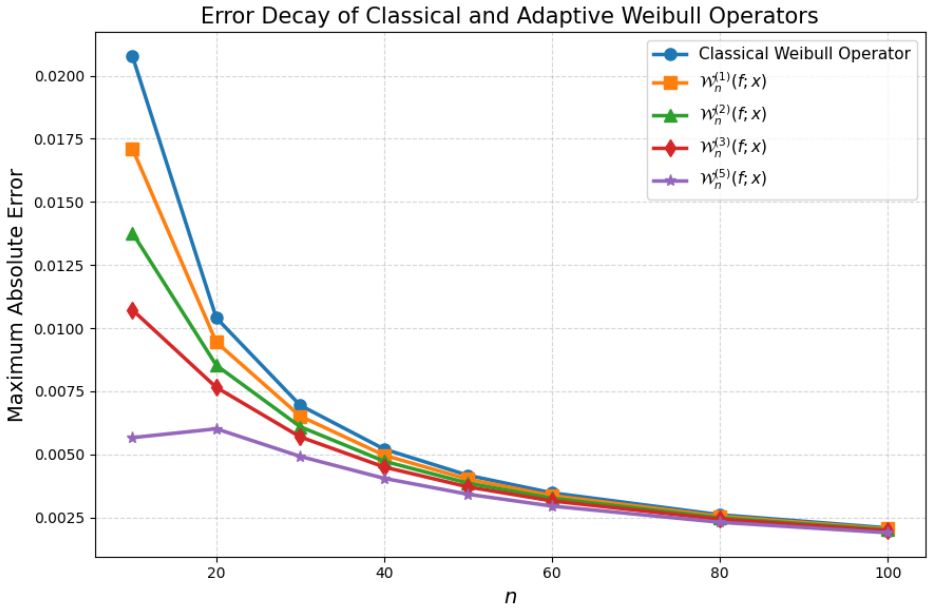


Figure 5. Error decay of the operator.

5. Voronovskaya formula of first and second kind

Asymptotic behavior of positive linear operators and the first-order approximation of a function in terms of its second derivative are described by the classical Voronovskaya-type result. Quantitative Voronovskaya as well as Grüss-Voronovskaya-type theorems using weighted moduli of continuity were investigated by Acar in [21]. The uniform version of the Voronovskaya theorem together with a second-order asymptotic formula for the Aldaz-Kounchev-Render operators were proved by Abel et al. [22]. In [23] Herzog established Voronovskaya-type and Grüss-Voronovskaya-type asymptotic results for squared basis operators related to Bessel functions, along with their convergence and approximation properties. The study by Mohiuddine et al. [24] presents approximation properties of Szász–Durrmeyer type operators involving Charlier polynomials, with particular emphasis on Grüss-Voronovskaya-type theorems that yield refined asymptotic estimates for the operators. Very recently, Cantarini and Costarelli [25] studied multivariate neural network operators and established simultaneous approximation results along with a Voronovskaya-type theorem. Some researchers have extended these results to various generalized operators, including Gamma, Kantorovich, Szász-type, and Jain-Durrmeyer operators [26–28] and references therein.

Theorem 5.1. *(Voronovskaya-type formula) Let $f \in C^2[0,\infty)$ and suppose that f'' is bounded. Then, for every compact interval $[0,a] \subset [0,\infty)$,*

$$\lim_{n\to\infty} \sup_{x\in[0,a]} \left| n\left(\mathcal{W}_n^{(\lambda)}(f;x) - f(x)\right) + \gamma x f'(x) \right| = 0.$$

Proof. By Taylor's formula,

$$f(t) = f(x) + (t-x)f'(x) + R(t,x),$$

where, since f'' is bounded,

$$|R(t,x)| \le \frac{M}{2}(t-x)^2, \qquad M = \|f''\|_\infty.$$

Applying $\mathcal{W}_n^{(\lambda)}$ and using $\mathcal{W}_n^{(\lambda)}(1;x) = 1$ gives

$$\mathcal{W}_n^{(\lambda)}(f;x) - f(x) = \mu_{n,1}^{(\lambda)}(x) f'(x) + \mathcal{W}_n^{(\lambda)}(R(\cdot,x);x).$$

By the asymptotic expansion of the first central moment,

$$n\mu_{n,1}^{(\lambda)}(x) = -\gamma x + O(n^{-1}),$$

uniformly on $[0,a]$. Moreover, positivity of $\mathcal{W}_n^{(\lambda)}$ yields

$$\left|\mathcal{W}_n^{(\lambda)}(R(\cdot,x);x)\right| \le \frac{M}{2}\mu_{n,2}^{(\lambda)}(x).$$

Since

$$\mu_{n,2}^{(\lambda)}(x) = O(n^{-2})$$

uniformly on $[0,a]$, we obtain

$$n\left|\mathcal{W}_n^{(\lambda)}(R(\cdot,x);x)\right| = O(n^{-1}) \to 0.$$

Consequently,

$$\lim_{n\to\infty}\sup_{x\in[0,a]}\left|n\left(\mathcal{W}_n^{(\lambda)}(f;x)-f(x)\right)+\gamma x f'(x)\right|=0.$$

□

Lemma 5.2. *Let $f\in C^3[0,\infty)$ with $\|f'''\|_\infty<\infty$. Then, uniformly for $x\in[0,a]$,*

$$\mathcal{W}_n^{(\lambda)}\left(|t-x|^3;x\right)=O(n^{-3}).$$

Proof. By the Cauchy–Schwarz inequality,

$$\mathcal{W}_n^{(\lambda)}\left(|t-x|^3;x\right)\le\left[\mathcal{W}_n^{(\lambda)}\left((t-x)^2;x\right)\right]^{1/2}\times\left[\mathcal{W}_n^{(\lambda)}\left((t-x)^4;x\right)\right]^{1/2}.$$

By the preceding moment estimates,

$$\mu_{n,2}^{(\lambda)}(x)=O(n^{-2}),\qquad \mu_{n,4}^{(\lambda)}(x)=O(n^{-4}),$$

uniformly for $x\in[0,a]$. Hence,

$$\mathcal{W}_n^{(\lambda)}\left(|t-x|^3;x\right)=O(n^{-3}),$$

uniformly on $[0,a]$. □

Theorem 5.3. *Let $f\in C^3[0,\infty)$ $\|f'''\|_\infty<\infty$, and let $a>0$. Then, uniformly for $x\in[0,a]$,*

$$\mathcal{W}_n^{(\lambda)}(f;x)=f(x)-\frac{\gamma x}{n}f'(x)+\frac{1}{n^2}\left[x\left(\frac{\pi^2}{6}\lambda\phi(x)+\frac{\pi^2}{12}+\frac{\gamma^2}{2}\right)f'(x)+\frac{x^2}{2}\left(\frac{\pi^2}{6}+\gamma^2\right)f''(x)\right]+o(n^{-2}),$$

as $n\to\infty$, where

$$\phi(x)=\frac{x}{1+x}$$

and γ denotes the Euler–Mascheroni constant.

Proof. By Taylor's formula,

$$f(t)=f(x)+(t-x)f'(x)+\frac{(t-x)^2}{2}f''(x)+R_3(t,x),$$

where

$$|R_3(t,x)|\le\frac{\|f'''\|_\infty}{6}|t-x|^3.$$

Applying $\mathcal{W}_n^{(\lambda)}(\cdot;x)$, we obtain

$$\mathcal{W}_n^{(\lambda)}(f;x)-f(x)=\mu_{n,1}^{(\lambda)}(x)f'(x)+\frac{1}{2}\mu_{n,2}^{(\lambda)}(x)f''(x)+\mathcal{R}_n(x),$$

where

$$|\mathcal{R}_n(x)|\le\frac{\|f'''\|_\infty}{6}\mathcal{W}_n^{(\lambda)}\left(|t-x|^3;x\right).$$

By Lemma 5.2,

$$\mathcal{R}_n(x)=O(n^{-3})=o(n^{-2}),$$

uniformly for $x \in [0, a]$.

From Lemma 3.4, with $m = 1$,

$$\mu_{n,1}^{(\lambda)}(x) = -\frac{\gamma x}{n} + \frac{x}{n^2}\left[\frac{\pi^2}{6}\lambda\phi(x) + \frac{\pi^2}{12} + \frac{\gamma^2}{2}\right] + O(n^{-3}).$$

Moreover, using Lemma 3.4 for $m = 1, 2$ in the formula for $\mu_{n,2}^{(\lambda)}$, we obtain

$$\mu_{n,2}^{(\lambda)}(x) = \frac{x^2}{n^2}\left(\frac{\pi^2}{6} + \gamma^2\right) + O(n^{-3}).$$

Consequently,

$$\mathcal{W}_n^{(\lambda)}(f; x) = f(x) - \frac{\gamma x}{n}f'(x) + \frac{1}{n^2}\left[x\left(\frac{\pi^2}{6}\lambda\phi(x) + \frac{\pi^2}{12} + \frac{\gamma^2}{2}\right)f'(x) + \frac{x^2}{2}\left(\frac{\pi^2}{6} + \gamma^2\right)f''(x)\right] + O(n^{-3}),$$

uniformly for $x \in [0, a]$. This completes the proof. □

In particular, the first-order asymptotic term is independent of the adaptive parameter, whereas the parameter λ appears explicitly in the second-order term.

6. Conclusions

In this paper, we introduced an adaptive modification of the classical Weibull positive linear operator using the bounded variable shape function $\phi(x) = x/(1 + x)$. The proposed operator preserves positivity, linearity, and normalization, and reduces exactly to the classical Weibull operator for $\lambda = 0$. Its moments and central moments were derived, leading to a Korovkin-type convergence theorem and a quantitative estimate via the modulus of continuity. The first- and second-order Voronovskaya-type asymptotic formulas were established. A notable feature is that λ does not affect the leading $O(n^{-1})$ term but appears explicitly in the $O(n^{-2})$ term, providing higher-order flexibility to the approximation process. Numerical experiments further illustrate the effect of the adaptive parameter on the approximation errors for the considered test functions. These results provide a basis for further studies involving weighted approximation, statistical convergence, higher-order asymptotics, and multivariate extensions of adaptive Weibull operators.

Author contributions

Sheetal Deshwal: Conceptualization, methodology, formal analysis, investigation, software, validation, visualization, writing–original draft; Harun Çiçek: Supervision, methodology, formal analysis, validation, writing–review and editing; Nadeem Rao: Methodology, formal analysis, and validation; Anita Kumari: Writing–review and editing. All authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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