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Research article

Solving nonlinear fractional Fredholm–Volterra integral equation by fixed point result in complex-valued *b*-suprametric spaces

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Abstract: The main objective of this study is to establish new common fixed point theorems in complex-valued *b*-suprametric spaces under generalized contractive conditions involving two-variable control functions. The proposed framework extends the scope of existing common fixed point results by replacing particular constant coefficients and specific contractive expressions with function-dependent conditions. Some established results are recovered as special cases and further consequences are derived through suitable choices of the control functions. A nontrivial example is presented to verify the applicability of the proposed hypotheses. Furthermore, the theoretical results are applied to a class of nonlinear fractional Fredholm–Volterra integral equations involving a Volterra term that accounts for dependence on the past and a Fredholm term that captures nonlocal interactions over the prescribed domain. Under appropriate assumptions, the existence and uniqueness of a continuous solution are established. These findings strengthen the theoretical framework of common fixed point theory in complex-valued *b*-suprametric spaces and demonstrate its applicability to nonlinear fractional integral equations.

Keywords: nonlinear fractional Fredholm–Volterra integral equations; fixed point; complex-valued *b*-suprametric spaces, generalized contractions
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1. Introduction

Fixed point (FP) theory is an important branch of nonlinear analysis, providing fundamental tools for establishing the existence and uniqueness of solutions to operator equations. Its development has

been closely connected with the generalization of metric spaces (MSs), in which modifications of the distance function or triangle inequality allow researchers to study operators under broader mathematical assumptions. The concept of a MS was introduced by Fréchet [1] in 1906. Subsequently, some extensions were developed, including b-metric spaces (b-MSs) introduced by Czerwik [2], cone metric spaces (CMSs) introduced by Huang and Zhang [3], and suprametric spaces (SMSs) introduced by Berzig [4]. These frameworks have expanded the applicability of classical FP results by allowing different forms of distance control. Subsequent developments by Berzig [5, 6] broadened the SMS framework by modifying its distance-related structure. In particular, relaxing the conventional triangle inequality led to the formulation of two generalized settings, namely, generalized SMSs and b-SMSs, thereby extending the scope of metric-type spaces. Nevertheless, the applicability of any generalized metric framework depends on the compatibility between its distance structure and the contractive assumptions imposed on the operator.

Complex-valued metric spaces (CVMSs), introduced by Azam et al. [7], represent another important development in metric FP theory. In these spaces, the distance between two points takes complex values, with an appropriate partial order used to formulate the distance axioms and associated inequalities. This framework has provided an alternative setting for studying contractive mappings and solving nonlinear equations. Subsequent investigations have extended the original results through different contractive methodologies. Rouzkard et al. [8] introduced rational expressions into contractive conditions, thereby broadening the range of mappings that can be treated. Sintunavarat et al. [9] developed related results using control functions of one variable, while Sitthikul et al. [10] employed control functions of two variables to formulate more flexible contraction conditions. These developments demonstrate the importance of replacing restrictive constant-based inequalities with function-dependent conditions. However, the effectiveness of such generalizations depends on the precise assumptions imposed on the control functions and their compatibility with the relevant FP theorem.

A further extension was provided by Mukheimer et al. [11], who introduced complex-valued b-metric spaces (CV b-MSs) and established common FP results in this setting. Kumar et al. [12] subsequently extended the results of Rouzkard et al. [8] from CVMSs to CV b-MSs. These contributions illustrate how relaxing the triangle inequality can enlarge the class of spaces in which common FP theorems can be established. Nevertheless, the use of a b-metric structure alone does not incorporate the additional flexibility associated with suprametric-type inequalities. This motivated the development of complex-valued suprametric spaces (CV SMSs) by Panda et al. [13], who combined the complex-valued and suprametric approaches, and established common FP results under rational contractive conditions. Their work also demonstrated the applicability of the resulting theory to complex nonlinear integral equations. More recently, Noorwali et al. [14] introduced complex-valued b-suprametric spaces (CV b-SMSs) and established common FP theorems under generalized contractive conditions. This framework combines complex-valued distances with a relaxed b-suprametric inequality, providing a broader setting for studying FP problems. These developments establish the theoretical foundation for further investigations into common FP results in CV b-SMSs. However, there remains scope to formulate more flexible contractive conditions using two-variable control functions and to examine the extent to which such conditions recover and extend existing results. The relevant developments and their relationships are discussed further in [15–19].

Motivated by these considerations, the present study develops common FP theorems in

complete CV b-SMSs by employing generalized contractive conditions formulated through two-variable control functions. Unlike approaches based on particular constant coefficients or specific rational expressions, the proposed framework allows the contractive behavior to be described through function-dependent estimates, subject to the stated hypotheses. This approach is intended to provide a unified treatment of some existing results while retaining the generalized distance structure of CV b-SMSs. In particular, the results recover established theorems as special cases and yield further consequences under suitable choices of the control functions. The relationships with relevant results in the literature are demonstrated through corollaries, including the recovery of the principal result of Noorwali et al. [14]. Thus, the contribution is not limited to introducing another contractive condition; it also establishes explicit connections between the proposed framework and existing common FP results.

To demonstrate the applicability of the theoretical findings, we provide a nontrivial example verifying the relevant hypotheses and illustrate the results through an existence and uniqueness problem for a nonlinear fractional Fredholm–Volterra integral equation. The application incorporates a Volterra term representing dependence on the past and a Fredholm term representing nonlocal interactions over a prescribed domain. The analysis establishes the existence and uniqueness of a continuous solution under the specified assumptions. The resulting contribution is therefore twofold: it advances the common FP theory of CV b-SMSs through generalized two-variable control functions and demonstrates how the resulting theorems can be applied to a class of fractional integral equations.

2. Preliminaries

The formal development of MSs can be traced back to Fréchet [1], who presented this framework in 1906. The definition adopted in this work is stated below:

Definition 1. [4] Let $\mathcal{X} \neq \emptyset$, and define a function $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ as follows:

(i) $0 \leq d(u, \mathfrak{l})$ and $d(u, \mathfrak{l}) = 0 \Longleftrightarrow u = \mathfrak{l}$,
(ii) $d(u, \mathfrak{l}) = d(\mathfrak{l}, u)$,
(iii) $d(u, \mathfrak{l}) \leq d(u, \varpi) + d(\varpi, \mathfrak{l})$,

$\forall\ u, \mathfrak{l}, \varpi \in \mathcal{X}$. Then $(\mathcal{X}, d)$ is referred to as a MS.

In [4], Berzig proposed the SMS framework, which may be formulated as follows.

Definition 2. [4] Let $\mathcal{X} \neq \emptyset$ and $\aleph \geq 0$. Define a mapping $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ that adheres to these axioms:

(i) $0 \leq d(u, \mathfrak{l})$ and $d(u, \mathfrak{l}) = 0 \Longleftrightarrow u = \mathfrak{l}$,
(ii) $d(u, \mathfrak{l}) = d(\mathfrak{l}, u)$,
(iii) $d(u, \mathfrak{l}) \leq d(u, \varpi) + d(\varpi, \mathfrak{l}) + \aleph d(u, \varpi) d(\varpi, \mathfrak{l})$,

$\forall\ u, \mathfrak{l}, \varpi \in \mathcal{X}$. Then $(\mathcal{X}, d)$ is called a SMS.

Let $z_1, z_2 \in \mathbb{C}$. It is well-known that

$$z_1 \precsim z_2 \ \Leftrightarrow\ \mathrm{Re}\,(z_1) \leqslant \mathrm{Re}\,(z_2)\,,\ \mathrm{Im}\,(z_1) \leqslant \mathrm{Im}\,(z_2)\,,$$

where $\precsim$ is a partial order on $\mathbb{C}$. Therefore,

$$z_1 \precsim z_2$$

provided that at least one of the conditions listed below is satisfied:

$$\begin{aligned}
&\text{(a) } \mathrm{Re}\,(z_1) = \mathrm{Re}\,(z_2)\,,\ \mathrm{Im}\,(z_1) < \mathrm{Im}\,(z_2)\,,\\
&\text{(b) } \mathrm{Re}\,(z_1) < \mathrm{Re}\,(z_2)\,,\ \mathrm{Im}\,(z_1) = \mathrm{Im}\,(z_2)\,,\\
&\text{(c) } \mathrm{Re}\,(z_1) < \mathrm{Re}\,(z_2)\,,\ \mathrm{Im}\,(z_1) < \mathrm{Im}\,(z_2)\,,\\
&\text{(d) } \mathrm{Re}\,(z_1) = \mathrm{Re}\,(z_2)\,,\ \mathrm{Im}\,(z_1) = \mathrm{Im}\,(z_2)\,.
\end{aligned}$$

Azam et al. [7] defined the notion of a CVMS in the following way.

Definition 3. [7] Let $\mathcal{X} \neq \emptyset$. Define a mapping $d : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ such that

(i) $0 \precsim d(u, \mathfrak{l})$ and $d(u, \mathfrak{l}) = 0 \Longleftrightarrow u = \mathfrak{l}$,
(ii) $d(u, \mathfrak{l}) = d(\mathfrak{l}, u)$,
(iii) $d(u, \mathfrak{l}) \precsim d(u, v) + d(v, \mathfrak{l})$,

$\forall\ u, \mathfrak{l}, v \in \mathcal{X}$, then $(\mathcal{X}, d)$ is said to be a CVMS.

Example 1. [7] Let $\mathcal{X} = [0, 1]$ and $u, \mathfrak{l} \in \mathcal{X}$. Define $d : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ by

$$d(u, \mathfrak{l}) = \begin{cases} 0, & \text{if } u = \mathfrak{l}, \\ \frac{i}{2}, & \text{if } u \neq \mathfrak{l}. \end{cases}$$

Then $(\mathcal{X}, d)$ is a CVMS.

Panda et al. [13] subsequently introduced the CVSMS framework, with its formal definition presented below.

Definition 4. [13] Let $\mathcal{X} \neq \emptyset$ and $\aleph \geq 0$. Define a complex function $d : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ such that

(i) $0 \precsim d(u, \mathfrak{l})$ and $d(u, \mathfrak{l}) = 0 \Longleftrightarrow u = \mathfrak{l}$,
(ii) $d(u, \mathfrak{l}) = d(\mathfrak{l}, u)$,
(iii) $d(u, \mathfrak{l}) \precsim d(u, \varpi) + d(\varpi, \mathfrak{l}) + \aleph d(u, \varpi) d(\varpi, \mathfrak{l})$,

$\forall\ u, \mathfrak{l}, \varpi \in \mathcal{X}$. Then $(\mathcal{X}, d)$ is designated as a CVSMS.

Example 2. Let $\mathcal{X} = \mathbb{R}$. Define a function $d : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ by

$$d(u, \mathfrak{l}) = (1 + i)\,|u - \mathfrak{l}|\,.$$

Then $(\mathcal{X}, d)$ is a CVSMS.

Recently, Noorwali et al. [14] defined the notion of a CV b-SMS in this way.

Definition 5. Let $\mathcal{X} \neq \emptyset$, $s \geq 1$, and $\aleph \geq 0$. Define a complex function $d : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ such that

(i) $0 \precsim d(u, \mathfrak{l})$ and $d(u, \mathfrak{l}) = 0 \Longleftrightarrow u = \mathfrak{l}$,
(ii) $d(u, \mathfrak{l}) = d(\mathfrak{l}, u)$,
(iii) $d(u, \mathfrak{l}) \precsim s\,(d(u, \varpi) + d(\varpi, \mathfrak{l})) + \aleph d(u, \varpi) d(\varpi, \mathfrak{l})$,

$\forall\ u, \mathfrak{l}, \varpi \in \mathcal{X}$. Then $(\mathcal{X}, d)$ is considered as a CV b-SMS.

Example 3. Let

$$\mathcal{X} = C([a,b], \mathbb{R})$$

and define

$$d(u, \mathfrak{l}) = \sup_{t\in[a,b]} |u(t) - \mathfrak{l}(t)|^p \, e^{i\frac{\pi}{4}},$$

where $p \geq 1$. We show that $(\mathcal{X},d)$ is a CV b-SMS with coefficient $s = 2^{p-1}$ and any $\aleph \geq 0$.

Since $|u(t) - \mathfrak{l}(t)|^p \geq 0$, for every $t \in [a,b]$, we have

$$\sup_{t\in[a,b]} |u(t) - \mathfrak{l}(t)|^p \geq 0.$$

Thus $0 \precsim d(u, \mathfrak{l})$. Moreover,

$$\begin{aligned} d(u, \mathfrak{l}) &= 0 \Longleftrightarrow \sup_{t\in[a,b]} |u(t) - \mathfrak{l}(t)|^p = 0 \\ &\Longleftrightarrow u(t) = \mathfrak{l}(t), \quad \forall t \in [a,b] \\ &\Longleftrightarrow u = \mathfrak{l}. \end{aligned}$$

Hence condition (i) holds. For all $u, \mathfrak{l} \in \mathcal{X}$, we have

$$\begin{aligned} d(u, \mathfrak{l}) &= \sup_{t\in[a,b]} |u(t) - \mathfrak{l}(t)|^p \, e^{i\frac{\pi}{4}} \\ &= \sup_{t\in[a,b]} |\mathfrak{l}(t) - u(t)|^p \, e^{i\frac{\pi}{4}} \\ &= d(\mathfrak{l},u). \end{aligned}$$

Therefore, condition (ii) holds. Let $u, \mathfrak{l}, \varpi \in \mathcal{X}$. Since

$$|u(t) - \mathfrak{l}(t)| \leq |u(t) - \varpi(t)| + |\varpi(t) - \mathfrak{l}(t)|,$$

and for $p \geq 1$, the inequality

$$(u + \mathfrak{l})^p \leq 2^{p-1} (u^p + \mathfrak{l}^p)$$

holds for all $u, \mathfrak{l} \geq 0$. Consequently,

$$|u(t) - \mathfrak{l}(t)|^p \leq 2^{p-1} \left(|u(t) - \varpi(t)|^p + |\varpi(t) - \mathfrak{l}(t)|^p\right).$$

Taking the supremum over $t \in [a,b]$, we obtain

$$\sup_{t\in[a,b]} |u(t) - \mathfrak{l}(t)|^p \leq 2^{p-1} \left(\sup_{t\in[a,b]} |u(t) - \varpi(t)|^p + \sup_{t\in[a,b]} |\varpi(t) - \mathfrak{l}(t)|^p \right).$$

Under the usual componentwise order on complex numbers, it follows that

$$\begin{aligned} d(u, \mathfrak{l}) &\precsim 2^{p-1} (d(u, \varpi) + d(\varpi, \mathfrak{l})) \\ &\precsim 2^{p-1} (d(u, \varpi) + d(\varpi, \mathfrak{l})) + \aleph d(u, \varpi) d(\varpi, \mathfrak{l}) \end{aligned}$$

because the product $d(u, \varpi)d(\varpi, \mathfrak{l})$ is a nonnegative multiple of $e^{i\frac{\pi}{2}} = i$ and hence is nonnegative in the componentwise order. Thus condition (iii) holds with $s = 2^{p-1}$ and any $\aleph \geq 0$.

Lemma 1. *[14] Let $(\mathcal{X}, d)$ be a CV b-SMS and let $\{u_n\} \subseteq \mathcal{X}$. Then $\{u_n\}$ converges to u if and only if $|d(u_n, u)| \to 0$ as $n \to \infty$.*

Lemma 2. *[14] Let $(\mathcal{X}, d)$ be a CV b-SMS and let $\{u_n\}$ be a sequence in $\mathcal{X}$. Then $\{u_n\}$ is a Cauchy sequence if and only if*

$$|d(u_n, u_m)| \to 0,$$

as $n, m \to \infty$.

3. Key results

This section is devoted to the development of common FP results in the setting of CV b-SMSs. The established theorems extend and unify a variety of existing FP principles available in the literature. Prior to presenting the principal results, we introduce a proposition together with an auxiliary lemma that play an essential role in the derivation of the subsequent theorems. These preliminary findings provide the foundation for the common FP analysis carried out thereafter.

Proposition 1. *Consider two self-mappings $\mathcal{O}, \mathcal{R} : \mathcal{X} \to \mathcal{X}$ and choose an arbitrary point $u_0 \in \mathcal{X}$. Construct a sequence $\{u_n\}$ by alternately applying the mappings $\mathcal{O}$ and $\mathcal{R}$ according to the relations*

$$u_{2n+1} = \mathcal{O}u_{2n} \text{ and } u_{2n+2} = \mathcal{R}u_{2n+1} \tag{3.1}$$

for all $n \in \mathbb{N} \cup \{0\}$.

Suppose that there exists a function $\mathcal{A} : \mathcal{X} \times \mathcal{X} \to [0, 1)$ satisfying

$$\mathcal{A}(\mathcal{R}\mathcal{O}u, \mathfrak{l}) \leq \mathcal{A}(u, \mathfrak{l}) \text{ and } \mathcal{A}(u, \mathcal{O}\mathcal{R}\mathfrak{l}) \leq \mathcal{A}(u, \mathfrak{l}),$$

$\forall\, u, \mathfrak{l} \in \mathcal{X}$. Then, for every nonnegative integer n and all $u, \mathfrak{l} \in \mathcal{X}$, the estimates

$$\mathcal{A}(u_{2n}, \mathfrak{l}) \leq \mathcal{A}(u_0, \mathfrak{l}) \text{ and } \mathcal{A}(u, u_{2n+1}) \leq \mathcal{A}(u, u_1)$$

are valid.

Proof. We establish the first inequality by induction on n. For $n = 0$, we have

$$\mathcal{A}(u_0, \mathfrak{l}) \leq \mathcal{A}(u_0, \mathfrak{l}),$$

which is immediate. Assume that

$$\mathcal{A}(u_{2n}, \mathfrak{l}) \leq \mathcal{A}(u_0, \mathfrak{l})$$

for some $n \geq 0$. Since the sequence is defined by

$$u_{2n+2} = \mathcal{R}\mathcal{O}u_{2n},$$

it follows from the assumption on $\mathcal{A}$ that

$$\mathcal{A}(u_{2n+2}, \mathfrak{l}) = \mathcal{A}(\mathcal{R}\mathcal{O}u_{2n}, \mathfrak{l}) \leq \mathcal{A}(u_{2n}, \mathfrak{l}) \leq \mathcal{A}(u_0, \mathfrak{l}).$$

Combining this with the induction hypothesis yields

$$\mathcal{A}(u_{2n+2}, \mathfrak{l}) \leq \mathcal{A}(u_0, \mathfrak{l}).$$

Thus the inequality holds for $n+1$. By induction, it holds for all $n \in \mathbb{N} \cup \{0\}$. Next, we prove the second inequality. For $n = 0$,

$$\mathcal{A}(u, u_1) \leq \mathcal{A}(u, u_1),$$

which is trivial. Suppose that

$$\mathcal{A}(u, u_{2n+1}) \leq \mathcal{A}(u, u_1),$$

for some $n \geq 0$. Since

$$u_{2n+3} = \mathcal{O}\mathcal{R}u_{2n+1},$$

the second condition imposed on $\mathcal{A}$ gives

$$\mathcal{A}(u, u_{2n+3}) = \mathcal{A}(u, \mathcal{O}\mathcal{R}u_{2n+1}) \leq \mathcal{A}(u, u_{2n+1}).$$

Using the induction hypothesis, we obtain

$$\mathcal{A}(u, u_{2n+3}) \leq \mathcal{A}(u, u_1).$$

Thus the inequality holds for $n+1$. By induction, it holds for all $n \in \mathbb{N} \cup \{0\}$. Hence, the proof is concluded. □

In the sequel, we state and prove our main result, which provides sufficient conditions guaranteeing a common FP for two self-mappings defined on a complete CV b-SMS, subject to an extended contraction criterion involving three two-variable control functions.

Theorem 1. *Let $(\mathcal{X}, d)$ be a complete CV b-SMS with the coefficient $s \geq 1$ and let $\mathcal{O}, \mathcal{R} : \mathcal{X} \to \mathcal{X}$ be two self-mappings. Suppose there exist functions*

$$\mathcal{A}, \mathcal{B}, \mathcal{C} : \mathcal{X} \times \mathcal{X} \to [0, 1)$$

such that the following hypotheses are fulfilled:

(H_1) For every $u, \mathfrak{l} \in \mathcal{X}$,

$$\begin{aligned}
\mathcal{A}(\mathcal{R}\mathcal{O}u, \mathfrak{l}) &\leq \mathcal{A}(u, \mathfrak{l}), \quad \mathcal{A}(u, \mathcal{O}\mathcal{R}\mathfrak{l}) \leq \mathcal{A}(u, \mathfrak{l}), \\
\mathcal{B}(\mathcal{R}\mathcal{O}u, \mathfrak{l}) &\leq \mathcal{B}(u, \mathfrak{l}), \quad \mathcal{B}(u, \mathcal{O}\mathcal{R}\mathfrak{l}) \leq \mathcal{B}(u, \mathfrak{l}), \\
\mathcal{C}(\mathcal{R}\mathcal{O}u, \mathfrak{l}) &\leq \mathcal{C}(u, \mathfrak{l}), \quad \mathcal{C}(u, \mathcal{O}\mathcal{R}\mathfrak{l}) \leq \mathcal{C}(u, \mathfrak{l}).
\end{aligned}$$

(H_2) The coefficients satisfy

$$s\mathcal{A}(u, \mathfrak{l}) + \mathcal{B}(u, \mathfrak{l}) + \mathcal{C}(u, \mathfrak{l}) < 1,$$

for all $u, \mathfrak{l} \in \mathcal{X}$.

(H_3) For any $u, \mathfrak{l} \in \mathcal{X}$,

$$d(\mathcal{O}u, \mathcal{R}\mathfrak{l}) \leq \mathcal{A}(u, \mathfrak{l})\, d(u, \mathfrak{l}) + \mathcal{B}(u, \mathfrak{l}) \frac{d(u, \mathcal{O}u)\, d(\mathfrak{l}, \mathcal{R}\mathfrak{l})}{1 + d(u, \mathfrak{l})} + \mathcal{C}(u, \mathfrak{l}) \frac{d(\mathfrak{l}, \mathcal{O}u)\, d(u, \mathcal{R}\mathfrak{l})}{1 + d(u, \mathfrak{l})}. \tag{3.2}$$

Then, the mappings $\mathcal{O}$ and $\mathcal{R}$ admit a unique common FP in $\mathcal{X}$.

Proof. Let $u_0 \in \mathcal{X}$ be arbitrary. Define a sequence $\{u_n\}$ inductively by

$$u_{2n+1} = \mathcal{O}u_{2n} \text{ and } u_{2n+2} = \mathcal{R}u_{2n+1}, \tag{3.3}$$

for all $n \in \mathbb{N} \cup \{0\}$. Using (3.2) and (3.3), we obtain

$$\begin{aligned} d(u_{2n+1}, u_{2n}) &= d(\mathcal{O}\mathcal{R}u_{2n-1}, \mathcal{R}u_{2n-1}) \\ &\leq \mathcal{A}(\mathcal{R}u_{2n-1}, u_{2n-1})\, d(\mathcal{R}u_{2n-1}, u_{2n-1}) \\ &\quad + \mathcal{B}(\mathcal{R}u_{2n-1}, u_{2n-1})\, \frac{d(\mathcal{R}u_{2n-1}, \mathcal{O}\mathcal{R}u_{2n-1})\, d(u_{2n-1}, \mathcal{R}u_{2n-1})}{1 + d(\mathcal{R}u_{2n-1}, u_{2n-1})} \\ &\quad + C(\mathcal{R}u_{2n-1}, u_{2n-1})\, \frac{d(u_{2n-1}, \mathcal{O}\mathcal{R}u_{2n-1})\, d(\mathcal{R}u_{2n-1}, \mathcal{R}u_{2n-1})}{1 + d(\mathcal{R}u_{2n-1}, u_{2n-1})} \\ &= \mathcal{A}(u_{2n}, u_{2n-1})\, d(u_{2n}, u_{2n-1}) \\ &\quad + \mathcal{B}(u_{2n}, u_{2n-1})\, \frac{d(u_{2n}, u_{2n+1})\, d(u_{2n-1}, u_{2n})}{1 + d(u_{2n}, u_{2n-1})}, \end{aligned}$$

which implies

$$\begin{aligned} |d(u_{2n+1}, u_{2n})| &\leq \mathcal{A}(u_{2n}, u_{2n-1})\, |d(u_{2n}, u_{2n-1})| + \mathcal{B}(u_{2n}, u_{2n-1})\, |d(u_{2n}, u_{2n+1})|\, \frac{|d(u_{2n-1}, u_{2n})|}{|1 + d(u_{2n}, u_{2n-1})|} \\ &\leq \mathcal{A}(u_{2n}, u_{2n-1})\, |d(u_{2n}, u_{2n-1})| + \mathcal{B}(u_{2n}, u_{2n-1})\, |d(u_{2n}, u_{2n+1})|. \end{aligned}$$

By Proposition (1), we get

$$\begin{aligned} |d(u_{2n+1}, u_{2n})| &\leq \mathcal{A}(u_0, u_{2n-1})\, |d(u_{2n}, u_{2n-1})| + \mathcal{B}(u_0, u_{2n-1})\, |d(u_{2n}, u_{2n+1})| \\ &\leq \mathcal{A}(u_0, u_1)\, |d(u_{2n}, u_{2n-1})| + \mathcal{B}(u_0, u_1)\, |d(u_{2n}, u_{2n+1})|. \end{aligned}$$

Hence,

$$|d(u_{2n+1}, u_{2n})| \leq \frac{\mathcal{A}(u_0, u_1)}{1 - \mathcal{B}(u_0, u_1)}\, |d(u_{2n}, u_{2n-1})|. \tag{3.4}$$

Likewise, by (3.2) and (3.3), we get

$$\begin{aligned} d(u_{2n+2}, u_{2n+1}) &= d(\mathcal{R}\mathcal{O}u_{2n}, \mathcal{O}u_{2n}) \\ &= d(\mathcal{O}u_{2n}, \mathcal{R}\mathcal{O}u_{2n}) \\ &\leq \mathcal{A}(u_{2n}, \mathcal{O}u_{2n})\, d(u_{2n}, \mathcal{O}u_{2n}) \\ &\quad + \mathcal{B}(u_{2n}, \mathcal{O}u_{2n})\, \frac{d(u_{2n}, \mathcal{O}u_{2n})\, d(\mathcal{O}u_{2n}, \mathcal{R}\mathcal{O}u_{2n})}{1 + d(u_{2n}, \mathcal{O}u_{2n})} \\ &\quad + C(u_{2n}, \mathcal{O}u_{2n})\, \frac{d(\mathcal{O}u_{2n}, \mathcal{O}u_{2n})\, d(u_{2n}, \mathcal{R}\mathcal{O}u_{2n})}{1 + d(u_{2n}, \mathcal{O}u_{2n})} \\ &= \mathcal{A}(u_{2n}, u_{2n+1})\, d(u_{2n}, u_{2n+1}) \\ &\quad + \mathcal{B}(u_{2n}, u_{2n+1})\, \frac{d(u_{2n}, u_{2n+1})\, d(u_{2n+1}, u_{2n+2})}{1 + d(u_{2n}, u_{2n+1})}, \end{aligned}$$

which implies

$$|d(u_{2n+2}, u_{2n+1})| \leq \mathcal{A}(u_{2n}, u_{2n+1})\, |d(u_{2n}, u_{2n+1})| + \mathcal{B}(u_{2n}, u_{2n+1})\, \frac{|d(u_{2n}, u_{2n+1})|}{|1 + d(u_{2n}, u_{2n+1})|}\, |d(u_{2n+1}, u_{2n+2})|$$

$$\leq \mathcal{A}(u_{2n}, u_{2n+1}) |d(u_{2n}, u_{2n+1})| + \mathcal{B}(u_{2n}, u_{2n+1}) |d(u_{2n+1}, u_{2n+2})|.$$

Applying Proposition 1, we obtain

$$\begin{aligned} |d(u_{2n+2}, u_{2n+1})| &\leq \mathcal{A}(u_0, u_{2n+1}) |d(u_{2n}, u_{2n+1})| + \mathcal{B}(u_0, u_{2n+1}) |d(u_{2n+1}, u_{2n+2})| \\ &\leq \mathcal{A}(u_0, u_1) |d(u_{2n}, u_{2n+1})| + \mathcal{B}(u_0, u_1) |d(u_{2n+1}, u_{2n+2})|, \end{aligned}$$

which leads to

$$|d(u_{2n+2}, u_{2n+1})| \leq \frac{\mathcal{A}(u_0, u_1)}{1 - \mathcal{B}(u_0, u_1)} |d(u_{2n}, u_{2n+1})|. \tag{3.5}$$

Rewriting (3.4) and (3.5), we obtain

$$|d(u_{2n}, u_{2n+1})| \leq \frac{\mathcal{A}(u_0,u_1)}{1 - \mathcal{B}(u_0,u_1)} |d(u_{2n-1}, u_{2n})| \tag{3.6}$$

and

$$|d(u_{2n+1}, u_{2n+2})| \leq \frac{\mathcal{A}(u_0, u_1)}{1 - \mathcal{B}(u_0, u_1)} |d(u_{2n}, u_{2n+1})|. \tag{3.7}$$

Let $\mu = \frac{\mathcal{A}(u_0,u_1)}{1-\mathcal{B}(u_0,u_1)}$. Note that from

$$s\mathcal{A}(u, \mathfrak{l}) + \mathcal{B}(u, \mathfrak{l}) + C(u, \mathfrak{l}) < 1$$

and $s \geq 1$, we have

$$s\mathcal{A}(u, \mathfrak{l}) + \mathcal{B}(u, \mathfrak{l}) \leq s\mathcal{A}(u, \mathfrak{l}) + \mathcal{B}(u, \mathfrak{l}) + C(u, \mathfrak{l}) < 1,$$

so

$$\frac{s\mathcal{A}(u, \mathfrak{l})}{1 - \mathcal{B}(u, \mathfrak{l})} < 1.$$

Thus, $s\mu < 1$. Then, from (3.6) and (3.7), we have

$$|d(u_{2n}, u_{2n+1})| \leq \mu |d(u_{2n-1}, u_{2n})| \tag{3.8}$$

and

$$|d(u_{2n+1}, u_{2n+2})| \leq \mu |d(u_{2n}, u_{2n+1})|. \tag{3.9}$$

Combining (3.8) and (3.9), we conclude that

$$|d(u_n, u_{n+1})| \leq \mu |d(u_{n-1}, u_n)|, \tag{3.10}$$

$\forall\, n \in \mathbb{N} \cup \{0\}$. Recursively, we can generate a sequence $\{u_n\}$ in $\mathcal{X}$ such that

$$|d(u_n, u_{n+1})| \leq \mu |d(u_{n-1}, u_n)| \leq \mu^2 |d(u_{n-2}, u_{n-1})| \leq \cdots \leq \mu^n |d(u_0, u_1)|,$$

that is,

$$|d(u_n, u_{n+1})| \leq \mu^n |d(u_0, u_1)|, \tag{3.11}$$

$\forall\, n \in \mathbb{N} \cup \{0\}$. For $m > n$, we obtain

$$|d(u_n, u_m)| \leq s|d(u_n, u_{n+1})| + s|d(u_{n+1}, u_m)| + \aleph |d(u_n, u_{n+1})| |d(u_{n+1}, u_m)|$$

$$= \quad s\left|d\left(u_n, u_{n+1}\right)\right| + \left(s + \aleph\left|d\left(u_n, u_{n+1}\right)\right|\right)\left|d\left(u_{n+1}, u_m\right)\right|. \tag{3.12}$$

Applying the triangle inequality once more to estimate $\left|d\left(u_{n+1}, u_m\right)\right|$, we get

$$\begin{aligned} \left|d\left(u_{n+1}, u_m\right)\right| &\leq s\left|d\left(u_{n+1}, u_{n+2}\right)\right| + s\left|d\left(u_{n+2}, u_m\right)\right| + \aleph\left|d\left(u_{n+1}, u_{n+2}\right)\right|\left|d\left(u_{n+2}, u_m\right)\right| \\ &= s\left|d\left(u_{n+1}, u_{n+2}\right)\right| + \left(s + \aleph\left|d\left(u_{n+1}, u_{n+2}\right)\right|\right)\left|d\left(u_{n+2}, u_m\right)\right|. \end{aligned}$$

Following the same argument for the remaining terms, we obtain

$$\begin{aligned} \left|d\left(u_{n+2}, u_m\right)\right| &\leq s\left|d\left(u_{n+2}, u_{n+3}\right)\right| + s\left|d\left(u_{n+3}, u_m\right)\right| + \aleph\left|d\left(u_{n+2}, u_{n+3}\right)\right|\left|d\left(u_{n+3}, u_m\right)\right| \\ &= s\left|d\left(u_{n+2}, u_{n+3}\right)\right| + \left(s + \aleph\left|d\left(u_{n+2}, u_{n+3}\right)\right|\right)\left|d\left(u_{n+3}, u_m\right)\right|. \end{aligned}$$

Continuing this recursive process, we obtain

$$\begin{aligned} \left|d\left(u_{m-2}, u_m\right)\right| &\leq s\left|d\left(u_{m-2}, u_{m-1}\right)\right| + s\left|d\left(u_{m-1}, u_m\right)\right| + \aleph\left|d\left(u_{m-2}, u_{m-1}\right)\right|\left|d\left(u_{m-1}, u_m\right)\right| \\ &= s\left|d\left(u_{m-2}, u_{m-1}\right)\right| + \left(s + \aleph\left|d\left(u_{m-2}, u_{m-1}\right)\right|\right)\left|d\left(u_{m-1}, u_m\right)\right|. \end{aligned}$$

Iterating the preceding estimates, substituting them into one another in (3.12), and then reducing the resulting expression, we obtain

$$\left|d\left(u_n, u_m\right)\right| \leq \sum_{k=n}^{m-1} s\left|d\left(u_n, u_{n+1}\right)\right| \prod_{j=n}^{k-1}\left(s + \aleph\left|d\left(u_j, u_{j+1}\right)\right|\right). \tag{3.13}$$

Using inequality (3.11) in inequality (3.13), we get

$$\left|d\left(u_n, u_m\right)\right| \leq s\left|d\left(u_0, u_1\right)\right| \sum_{k=n}^{m-1} \mu^k \left(\prod_{j=n}^{k-1}\left(s + \aleph\mu^j\left|d\left(u_0, u_1\right)\right|\right)\right). \tag{3.14}$$

Let

$$\Omega_k = \mu^k \prod_{j=n}^{k-1}\left(s + \aleph\mu^j\left|d\left(u_0, u_1\right)\right|\right),$$

and then

$$\Omega_{k+1} = \mu^{k+1} \prod_{j=n}^{k}\left(s + \aleph\mu^j\left|d\left(u_0, u_1\right)\right|\right).$$

Now

$$\lim_{k\to\infty} \frac{\Omega_{k+1}}{\Omega_k} = \lim_{k\to\infty} \mu\left(\left(s + \aleph\mu^k\left|d\left(u_0, u_1\right)\right|\right)\right) = \mu s < 1.$$

Hence, by the ratio test, the series $\sum_{k=0}^{\infty} \Omega_k$ converges. Thus, we conclude that

$$\left|d\left(u_n, u_m\right)\right| \to 0,$$

as $n, m \to \infty$. Hence, the sequence $\{u_n\}$ is a Cauchy sequence. Since $\mathcal{X}$ is complete, $\{u_n\}$ converges to some limit $u^* \in \mathcal{X}$, that is,

$$\lim_{n\to\infty} u_n = u^*. \tag{3.15}$$

Now, we show that u^* is a FP of $\mathcal{O}$. From (3.2), we have

$$\begin{aligned}
d(u^*, \mathcal{O}u^*) &\leq sd(u^*, u_{2n+2}) + sd(u_{2n+2}, \mathcal{O}u^*) + \aleph d(u^*, u_{2n+2})\, d(u_{2n+2}, \mathcal{O}u^*) \\
&= sd(u^*, u_{2n+2}) + sd(\mathcal{R}u_{2n+1}, \mathcal{O}u^*) + \aleph d(u^*, \mathcal{R}u_{2n+1})\, d(\mathcal{R}u_{2n+1}, \mathcal{O}u^*) \\
&= sd(u^*, \mathcal{R}u_{2n+1}) + sd(\mathcal{O}u^*, \mathcal{R}u_{2n+1}) + \aleph d(u^*, \mathcal{R}u_{2n+1})\, d(u_{2n+2}, \mathcal{O}u^*) \\
&\leq sd(u^*, u_{2n+2}) + s\begin{pmatrix} \mathcal{A}(u^*, u_{2n+1})\, d(u^*, u_{2n+1}) \\ + \mathcal{B}(u^*, u_{2n+1}) \frac{d(u^*, \mathcal{O}u^*) d(u_{2n+1}, \mathcal{R}u_{2n+1})}{1+d(u^*, u_{2n+1})} \\ + C(u^*, u_{2n+1}) \frac{d(u_{2n+1}, \mathcal{O}u^*) d(u^*, \mathcal{R}u_{2n+1})}{1+d(u^*, u_{2n+1})} \end{pmatrix} \\
&\quad + \aleph d(u^*, \mathcal{R}u_{2n+1})\, d(u_{2n+2}, \mathcal{O}u^*) \\
&= sd(u^*, u_{2n+2}) + s\begin{pmatrix} \mathcal{A}(u^*, u_{2n+1})\, d(u^*, u_{2n+1}) \\ + \mathcal{B}(u^*, u_{2n+1}) \frac{d(u^*, \mathcal{O}u^*) d(u_{2n+1}, u_{2n+2})}{1+d(u^*, u_{2n+1})} \\ + C(u^*, u_{2n+1}) \frac{d(u_{2n+1}, \mathcal{O}u^*) d(u^*, u_{2n+2})}{1+d(u^*, u_{2n+1})} \end{pmatrix} \\
&\quad + \aleph d(u^*, \mathcal{R}u_{2n+1})\, d(u_{2n+2}, \mathcal{O}u^*),
\end{aligned}$$

which implies by Proposition 1 that

$$\begin{aligned}
|d(u^*, \mathcal{O}u^*)| &\leq s|d(u^*, u_{2n+2})| + s\begin{pmatrix} \mathcal{A}(u^*, u_{2n+1})\, |d(u^*, u_{2n+1})| \\ + \mathcal{B}(u^*, u_{2n+1})\, |d(u^*, \mathcal{O}u^*)| \left|\frac{d(u_{2n+1}, u_{2n+2})}{1+d(u^*, u_{2n+1})}\right| \\ + C(u^*, u_{2n+1}) \left|\frac{d(u_{2n+1}, \mathcal{O}u^*)}{1+d(u^*, u_{2n+1})}\right| |d(u^*, u_{2n+2})| \end{pmatrix} \\
&\quad + \aleph |d(u^*, u_{2n+2})|\, |d(u_{2n+2}, \mathcal{O}u^*)|. \\
&\leq s|d(u^*, u_{2n+2})| + s\begin{pmatrix} \mathcal{A}(u^*, u_1)\, |d(u^*, u_{2n+1})| \\ + \mathcal{B}(u^*, u_1)\, |d(u^*, \mathcal{O}u^*)| \left|\frac{d(u_{2n+1}, u_{2n+2})}{1+d(u^*, u_{2n+1})}\right| \\ + C(u^*, u_1) \left|\frac{d(u_{2n+1}, \mathcal{O}u^*)}{1+d(u^*, u_{2n+1})}\right| |d(u^*, u_{2n+2})| \end{pmatrix} \\
&\quad + \aleph |d(u^*, u_{2n+2})|\, |d(u_{2n+2}, \mathcal{O}u^*)|.
\end{aligned}$$

Letting $n \to \infty$ in the above inequality and using the fact (3.15), we have

$$|d(u^*, \mathcal{O}u^*)| = 0,$$

which implies $u^* = \mathcal{O}u^*$. To verify that u^* constitutes a FP of $\mathcal{R}$, we use relation (3.2) to obtain

$$\begin{aligned}
d(u^*, \mathcal{R}u^*) &\leq sd(u^*, u_{2n+1}) + sd(u_{2n+1}, \mathcal{R}u^*) + \aleph d(u^*, u_{2n+1})\, d(u_{2n+1}, \mathcal{R}u^*) \\
&\leq sd(u^*, u_{2n+1}) + sd(\mathcal{O}u_{2n}, \mathcal{R}u^*) + \aleph d(u^*, u_{2n+1})\, d(u_{2n+1}, \mathcal{R}u^*) \\
&\leq sd(u^*, u_{2n+1}) + s\begin{pmatrix} \mathcal{A}(u_{2n}, u^*)\, d(u_{2n}, u^*) \\ + \mathcal{B}(u_{2n}, u^*) \frac{d(u_{2n}, \mathcal{O}u_{2n}) d(u^*, \mathcal{R}u^*)}{1+d(u_{2n}, u^*)} \\ + C(u_{2n}, u^*) \frac{d(u^*, \mathcal{O}u_{2n}) d(u_{2n}, \mathcal{R}u^*)}{1+d(u_{2n}, u^*)} \end{pmatrix} + \aleph d(u^*, u_{2n+1})\, d(u_{2n+1}, \mathcal{R}u^*) \\
&\leq sd(u^*, u_{2n+1}) + s\begin{pmatrix} \mathcal{A}(u_{2n}, u^*)\, d(u_{2n}, u^*) \\ + \mathcal{B}(u_{2n}, u^*) \frac{d(u_{2n}, u_{2n+1}) d(u^*, \mathcal{R}u^*)}{1+d(u_{2n}, u^*)} \\ + C(u_{2n}, u^*) \frac{d(u^*, u_{2n+1}) d(u_{2n}, \mathcal{R}u^*)}{1+d(u_{2n}, u^*)} \end{pmatrix} + \aleph d(u^*, u_{2n+1})\, d(u_{2n+1}, \mathcal{R}u^*),
\end{aligned}$$

which implies by Proposition 1 that

$$|d(u^*, \mathcal{R}u^*)| \leq s|d(u^*, u_{2n+1})| + s\begin{pmatrix} \mathcal{A}(u_{2n}, u^*)\, |d(u_{2n}, u^*)| \\ + \mathcal{B}(u_{2n}, u^*) \left|\frac{d(u_{2n}, u_{2n+1})}{1+d(u_{2n}, u^*)}\right| |d(u^*, \mathcal{R}u^*)| \\ + C(u_{2n}, u^*) \left|\frac{d(u^*, u_{2n+1})}{1+d(u_{2n}, u^*)}\right| |d(u_{2n}, \mathcal{R}u^*)| \end{pmatrix}$$

$$
\begin{aligned}
&+ \aleph \left| d\left(u^*, u_{2n+1}\right) \right| \left| d\left(u_{2n+1}, \mathcal{R}u^*\right) \right| \\
\leq\ & s \left| d\left(u^*, u_{2n+1}\right) \right| + s \left(\begin{array}{c} \mathcal{A}\left(u_0, u^*\right) \left| d\left(u_{2n}, u^*\right) \right| \\ + \mathcal{B}\left(u_0, u^*\right) \left| \frac{d(u_{2n}, u_{2n+1})}{1+d(u_{2n}, u^*)} \right| \left| d\left(u^*, \mathcal{R}u^*\right) \right| \\ + C\left(u_0, u^*\right) \left| \frac{d(u^*, u_{2n+1})}{1+d(u_{2n}, u^*)} \right| \left| d\left(u_{2n}, \mathcal{R}u^*\right) \right| \end{array} \right) \\
&+ \aleph \left| d\left(u^*, u_{2n+1}\right) \right| \left| d\left(u_{2n+1}, \mathcal{R}u^*\right) \right|.
\end{aligned}
$$

Letting $n \to \infty$ in the above inequality and using the fact (3.15), we have

$$
\left| d\left(u^*, \mathcal{R}u^*\right) \right| = 0,
$$

which implies $u^* = \mathcal{R}u^*$. Thus, u^* is a CFP of $\mathcal{O}$ and $\mathcal{R}$. To prove that u^* is the unique CFP, assume for contradiction that $u^{/}$ is another CFP of the operators $\mathcal{O}$ and $\mathcal{R}$. It follows that

$$
u^{/} = \mathcal{O}u^{/} = \mathcal{R}u^{/},
$$

but $u^* \neq u^{/}$. Now from (3.2), we have

$$
\begin{aligned}
d\left(u^*, u^{/}\right) &= d\left(\mathcal{O}u^*, \mathcal{R}u^{/}\right) \\
&\leq \mathcal{A}\left(u^*, u^{/}\right) d\left(u^*, u^{/}\right) \\
&\quad + \mathcal{B}\left(u^*, u^{/}\right) \frac{d\left(u^*, \mathcal{O}u^*\right) d\left(u^{/}, \mathcal{R}u^{/}\right)}{1 + d\left(u^*, u^{/}\right)} \\
&\quad + C\left(u^*, u^{/}\right) \frac{d\left(u^{/}, \mathcal{O}u^*\right) d\left(u^*, \mathcal{R}u^{/}\right)}{1 + d\left(u^*, u^{/}\right)} \\
&= \mathcal{A}\left(u^*, u^{/}\right) d\left(u^*, u^{/}\right) + C\left(u^*, u^{/}\right) \frac{d\left(u^{/}, u^*\right) d\left(u^*, u^{/}\right)}{1 + d\left(u^*, u^{/}\right)}.
\end{aligned}
$$

This implies that we have

$$
\begin{aligned}
\left| d\left(u^*, u^{/}\right) \right| &\leq \mathcal{A}\left(u^*, u^{/}\right) \left| d\left(u^*, u^{/}\right) \right| + C\left(u^*, u^{/}\right) \left| d\left(u^{/}, u^*\right) \right| \left| \frac{d\left(u^*, u^{/}\right)}{1 + d\left(u^*, u^{/}\right)} \right| \\
&\leq \mathcal{A}\left(u^*, u^{/}\right) \left| d\left(u^*, u^{/}\right) \right| + C\left(u^*, u^{/}\right) \left| d\left(u^{/}, u^*\right) \right|.
\end{aligned}
$$

Since

$$
s\mathcal{A}\left(u^*, u^{/}\right) + \mathcal{B}\left(u^*, u^{/}\right) + C\left(u^*, u^{/}\right) < 1,
$$

$s \geq 1$, and $\mathcal{A}\left(u^*, u^{/}\right) \geq 0$, we have

$$
\mathcal{A}\left(u^*, u^{/}\right) \leq s\mathcal{A}\left(u^*, u^{/}\right).
$$

Therefore,

$$
\mathcal{A}\left(u^*, u^{/}\right) + \mathcal{B}\left(u^*, u^{/}\right) + C\left(u^*, u^{/}\right) < 1.
$$

Because $0 \leq \mathcal{B}\left(u^*, u^{/}\right) < 1$, we get

$$\mathcal{A}\left(u^*, u^{/}\right) + C\left(u^*, u^{/}\right) \leq \mathcal{A}\left(u^*, u^{/}\right) + \mathcal{B}\left(u^*, u^{/}\right) + C\left(u^*, u^{/}\right) < 1.$$

Consequently,

$$1 - \mathcal{A}\left(u^*, u^{/}\right) - C\left(u^*, u^{/}\right) > 0,$$

and thus

$$\left|d\left(u^*, u^{/}\right)\right| = 0.$$

Hence $u^* = u^{/}$. □

By setting $O = \mathcal{R}$ in Theorem 1, we deduce the following corollary.

Corollary 1. *Let $(\mathcal{X}, d)$ be a complete CV b-SMS with the coefficient $s \geq 1$ and let $\mathcal{R} : \mathcal{X} \to \mathcal{X}$ be a self-mapping. Suppose there exist functions*

$$\mathcal{A}, \mathcal{B}, C : \mathcal{X} \times \mathcal{X} \to [0, 1)$$

such that the following hypotheses are fulfilled:

(H_1) For every $u, \mathfrak{l} \in \mathcal{X}$,

$$\begin{aligned}
\mathcal{A}(\mathcal{R}u, \mathfrak{l}) &\leq \mathcal{A}(u, \mathfrak{l}), \quad \mathcal{A}(u, \mathcal{R}\mathfrak{l}) \leq \mathcal{A}(u, \mathfrak{l}), \\
\mathcal{B}(\mathcal{R}u, \mathfrak{l}) &\leq \mathcal{B}(u, \mathfrak{l}), \quad \mathcal{B}(u, \mathcal{R}\mathfrak{l}) \leq \mathcal{B}(u, \mathfrak{l}), \\
C(\mathcal{R}u, \mathfrak{l}) &\leq C(u, \mathfrak{l}), \quad C(u, \mathcal{R}\mathfrak{l}) \leq C(u, \mathfrak{l}).
\end{aligned}$$

(H_2) The coefficients satisfy

$$s\mathcal{A}(u, \mathfrak{l}) + \mathcal{B}(u, \mathfrak{l}) + C(u, \mathfrak{l}) < 1,$$

for all $u, \mathfrak{l} \in \mathcal{X}$.

(H_3) For any $u, \mathfrak{l} \in \mathcal{X}$,

$$d(\mathcal{R}u, \mathcal{R}\mathfrak{l}) \leq \mathcal{A}(u, \mathfrak{l})\, d(u, \mathfrak{l}) + \mathcal{B}(u, \mathfrak{l}) \frac{d(u, \mathcal{R}u)\, d(\mathfrak{l}, \mathcal{R}\mathfrak{l})}{1 + d(u, \mathfrak{l})} + C(u, \mathfrak{l}) \frac{d(\mathfrak{l}, \mathcal{R}u)\, d(u, \mathcal{R}\mathfrak{l})}{1 + d(u, \mathfrak{l})}.$$

Then the mapping $\mathcal{R}$ admits a unique FP in $\mathcal{X}$.

Remark 1. By setting one or more of the control functions $\mathcal{A}$, $\mathcal{B}$, and C equal to zero, we can derive several special cases of the above theorem. In particular, taking

$$\mathcal{B}(u, \mathfrak{l}) = C(u, \mathfrak{l}) = 0,$$

$$\mathcal{A}(u, \mathfrak{l}) = C(u, \mathfrak{l}) = 0,$$

or

$$\mathcal{A}(u, \mathfrak{l}) = \mathcal{B}(u, \mathfrak{l}) = 0$$

yields different contractive conditions involving the remaining control function. Thus, Theorem 1 provides a unified framework for obtaining some common FP results in complete CV b-SMSs.

Example 4. Let $\mathcal{X} = \mathbb{R}$, and define $d : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ by

$$d(u, \mathfrak{l}) = (1 + i)\,|u - \mathfrak{l}|^2$$

for all $u, \mathfrak{l} \in \mathcal{X}$. Then, $(\mathcal{X}, d)$ is a complete CV b-SMS with coefficients $s = 2$ and $\aleph = 2$ (see [14]).

Define the self mappings $\mathcal{O}$,$\mathcal{R} : \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{O}u = \mathcal{R}u = \frac{1}{5}u.$$

Now, define the function $\varphi : \mathcal{X} \times \mathcal{X} \to [0, 1)$ by

$$\varphi(u, \mathfrak{l}) = \frac{u^2 + \mathfrak{l}^2}{1 + u^2 + \mathfrak{l}^2}.$$

Consider the control functions $\mathcal{A}, \mathcal{B}, C : \mathcal{X} \times \mathcal{X} \to [0, 1)$ in this way:

$$\mathcal{A}(u, \mathfrak{l}) = \frac{1}{10} + \frac{1}{100}\varphi(u, \mathfrak{l}),$$

$$\mathcal{B}(u, \mathfrak{l}) = \frac{1}{20} + \frac{1}{100}\varphi(u, \mathfrak{l}),$$

$$C(u, \mathfrak{l}) = \frac{1}{30} + \frac{1}{100}\varphi(u, \mathfrak{l}).$$

Now since

$$\mathcal{R}\mathcal{O}u = \frac{u}{25} \text{ and } \mathcal{O}\mathcal{R}\mathfrak{l} = \frac{\mathfrak{l}}{25},$$

we have

$$\varphi\left(\mathcal{R}\mathcal{O}u, \mathfrak{l}\right) = \frac{\frac{u^2}{625} + \mathfrak{l}^2}{1 + \frac{u^2}{625} + \mathfrak{l}^2} \leq \varphi\left(u, \mathfrak{l}\right).$$

Similarly,

$$\varphi\left(u, \mathcal{O}\mathcal{R}\mathfrak{l}\right) = \frac{u^2 + \frac{\mathfrak{l}^2}{625}}{1 + u^2 + \frac{\mathfrak{l}^2}{625}} \leq \varphi\left(u, \mathfrak{l}\right).$$

Therefore,

$$\begin{aligned}
\mathcal{A}\left(\mathcal{R}\mathcal{O}u, \mathfrak{l}\right) &\leq \mathcal{A}\left(u, \mathfrak{l}\right), \text{ and } \mathcal{A}\left(u, \mathcal{O}\mathcal{R}\mathfrak{l}\right) \leq \mathcal{A}\left(u, \mathfrak{l}\right), \\
\mathcal{B}\left(\mathcal{R}\mathcal{O}u, \mathfrak{l}\right) &\leq \mathcal{B}\left(u, \mathfrak{l}\right), \text{ and } \mathcal{B}\left(u, \mathcal{O}\mathcal{R}\mathfrak{l}\right) \leq \mathcal{B}\left(u, \mathfrak{l}\right), \\
C\left(\mathcal{R}\mathcal{O}u, \mathfrak{l}\right) &\leq C\left(u, \mathfrak{l}\right), \text{ and } C\left(u, \mathcal{O}\mathcal{R}\mathfrak{l}\right) \leq C\left(u, \mathfrak{l}\right).
\end{aligned}$$

Thus (H_1) holds immediately. Now since $0 \leq \varphi\left(u, \mathfrak{l}\right) < 1$, we obtain

$$\begin{aligned}
2\mathcal{A}(u, \mathfrak{l}) + \mathcal{B}(u, \mathfrak{l}) + C(u, \mathfrak{l}) &< \frac{1}{5} + \frac{1}{20} + \frac{1}{30} + \frac{4}{100} \\
&= \frac{97}{300} < 1.
\end{aligned}$$

So (H_2) is satisfied for all $u, \mathfrak{l} \in \mathcal{X}$. Moreover,

$$\begin{aligned} d(\mathcal{O}u, \mathcal{R}\mathfrak{l}) &= (1+i)\left|\frac{1}{5}u - \frac{1}{5}\mathfrak{l}\right|^2 \\ &= (1+i)\frac{1}{25}|u-\mathfrak{l}|^2 \\ &= \frac{1}{25}d(u,\mathfrak{l}). \end{aligned}$$

Moreover, since $\varphi(u,\mathfrak{l}) \geq 0$,

$$\mathcal{A}(u,\mathfrak{l}) \geq \frac{1}{10} > \frac{1}{25},$$

and we obtain

$$d(\mathcal{O}u, \mathcal{R}\mathfrak{l}) \leq \mathcal{A}(u,\mathfrak{l})\, d(u,\mathfrak{l}).$$

Moreover, since

$$d(u, \mathcal{O}u) = (1+i)\frac{16u^2}{25}$$

and

$$d(\mathfrak{l}, \mathcal{R}\mathfrak{l}) = (1+i)\frac{16\mathfrak{l}^2}{25},$$

then,

$$\begin{aligned} d(u,\mathcal{O}u)\, d(\mathfrak{l},\mathcal{R}\mathfrak{l}) &= (1+i)^2\frac{256u^2\mathfrak{l}^2}{625} \\ &= \frac{512u^2\mathfrak{l}^2}{625}i. \end{aligned}$$

Let $r = |u-\mathfrak{l}|^2 \geq 0$ and $Q = \frac{256u^2\mathfrak{l}^2}{625} \geq 0$. Then

$$\frac{d(u,\mathcal{O}u)\, d(\mathfrak{l},\mathcal{R}\mathfrak{l})}{1+d(u,\mathfrak{l})} = \frac{2iQ}{1+(1+i)\,r}.$$

Multiplying the numerator and denominator by $1 + r - ir$ gives

$$\frac{d(u,\mathcal{O}u)\, d(\mathfrak{l},\mathcal{R}\mathfrak{l})}{1+d(u,\mathfrak{l})} = \frac{2rQ + 2iQ(1+r)}{(1+r)^2 + r^2}.$$

Both the real and imaginary parts are nonnegative. $\mathcal{B}(u,\mathfrak{l}) \geq 0$, thus

$$\mathcal{B}(u,\mathfrak{l})\,\frac{d(u,\mathcal{O}u)\, d(\mathfrak{l},\mathcal{R}\mathfrak{l})}{1+d(u,\mathfrak{l})} \geq 0.$$

Similarly,

$$C(u,\mathfrak{l})\,\frac{d(\mathfrak{l},\mathcal{O}u)\, d(u,\mathcal{R}\mathfrak{l})}{1+d(u,\mathfrak{l})} \geq 0.$$

Hence we conclude that

$$d(\mathcal{O}u,\mathcal{R}\mathfrak{l}) \leq \mathcal{A}(u,\mathfrak{l})\, d(u,\mathfrak{l}) + \mathcal{B}(u,\mathfrak{l})\,\frac{d(u,\mathcal{O}u)\, d(\mathfrak{l},\mathcal{R}\mathfrak{l})}{1+d(u,\mathfrak{l})} + C(u,\mathfrak{l})\,\frac{d(\mathfrak{l},\mathcal{O}u)\, d(u,\mathcal{R}\mathfrak{l})}{1+d(u,\mathfrak{l})},$$

which is exactly condition (3.2). Hence, every assumption of Theorem 1 is satisfied. Therefore, the mappings $\mathcal{O}$ and $\mathcal{R}$ possess a unique common FP, namely, 0.

4. Fixed point theorems involving control functions depending on one variable

Corollary 2. *Let $(\mathcal{X}, d)$ be a complete CV b-SMS with the coefficient $s \geq 1$ and let $\mathcal{O}, \mathcal{R} : \mathcal{X} \to \mathcal{X}$ be two self-mappings. Suppose there exist functions*

$$\curlywedge_1, \curlywedge_2, \curlywedge_3 : \mathcal{X} \to [0, 1)$$

such that the following hypotheses are fulfilled:

(H_1) For every $u, \mathfrak{l} \in \mathcal{X}$,

$$\begin{aligned} \curlywedge_1(\mathcal{O}u) &\leq \curlywedge_1(u), \quad \curlywedge_1(\mathcal{R}u) \leq \curlywedge_1(u), \\ \curlywedge_2(\mathcal{O}u) &\leq \curlywedge_2(u), \quad \curlywedge_2(\mathcal{R}u) \leq \curlywedge_2(u), \\ \curlywedge_3(\mathcal{O}u) &\leq \curlywedge_3(u), \quad \curlywedge_3(\mathcal{R}u) \leq \curlywedge_3(u). \end{aligned}$$

(H_2) The coefficients satisfy

$$s\curlywedge_1(u) + \curlywedge_2(u) + \curlywedge_3(u) < 1$$

for all $u, \mathfrak{l} \in \mathcal{X}$.

(H_3) For any $u, \mathfrak{l} \in \mathcal{X}$,

$$d(\mathcal{O}u, \mathcal{R}\mathfrak{l}) \leq \curlywedge_1(u)\, d(u, \mathfrak{l}) + \curlywedge_2(u) \frac{d(u, \mathcal{O}u)\, d(\mathfrak{l}, \mathcal{R}\mathfrak{l})}{1 + d(u, \mathfrak{l})} + \curlywedge_3(u) \frac{d(\mathfrak{l}, \mathcal{O}u)\, d(u, \mathcal{R}\mathfrak{l})}{1 + d(u, \mathfrak{l})}.$$

Then the mappings $\mathcal{O}$ and $\mathcal{R}$ admit a unique common FP in $\mathcal{X}$.

Proof. Let $\mathcal{X}$ be a set. For every pair $(u, \mathfrak{l})$ of elements from $\mathcal{X}$, we define three real-valued functions $\mathcal{A}, \mathcal{B}, \mathcal{C} : \mathcal{X} \times \mathcal{X} \to [0, 1)$ as follows:

$$\mathcal{A}(u, \mathfrak{l}) = \curlywedge_1(u),$$

$$\mathcal{B}(u, \mathfrak{l}) = \curlywedge_2(u),$$

$$\mathcal{C}(u, \mathfrak{l}) = \curlywedge_3(u),$$

where $\curlywedge_1$, $\curlywedge_2$, and $\curlywedge_3$ are given functions from $\mathcal{X}$ to $[0, 1)$. For all $u, \mathfrak{l} \in \mathcal{X}$, the following inequalities hold:

$$\mathcal{A}(\mathcal{RO}u, \mathfrak{l}) = \curlywedge_1(\mathcal{RO}u) \leq \curlywedge_1(\mathcal{O}u) \leq \curlywedge_1(u) = \mathcal{A}(u, \mathfrak{l})$$

and

$$\mathcal{A}(u, \mathcal{OR}\mathfrak{l}) \leq \curlywedge_1(u) = \mathcal{A}(u, \mathfrak{l}).$$

In a completely analogous manner, we obtain: □

$$\mathcal{B}(\mathcal{RO}u, \mathfrak{l}) \leq \mathcal{B}(u, \mathfrak{l}) \text{ and } \mathcal{B}(u, \mathcal{OR}\mathfrak{l}) \leq \mathcal{B}(u, \mathfrak{l}),$$

$$\mathcal{C}(\mathcal{RO}u, \mathfrak{l}) \leq \mathcal{C}(u, \mathfrak{l}) \text{ and } \mathcal{C}(u, \mathcal{OR}\mathfrak{l}) \leq \mathcal{C}(u, \mathfrak{l}).$$

Thus condition (H_1) of Theorem 1 is satisfied.

For any $u, \mathfrak{l} \in \mathcal{X}$,

$$s\mathcal{A}(u,\mathfrak{l}) + \mathcal{B}(u,\mathfrak{l}) + C(u,\mathfrak{l}) = s\,\wedge_1(u) + \wedge_2(u) + \wedge_3(u) < 1.$$

Hence, hypothesis (H_2) of Theorem 1 holds. Now for every $u, \mathfrak{l} \in \mathcal{X}$, we obtain

$$\begin{aligned} d(\mathcal{O}u, \mathcal{R}\mathfrak{l}) &\leq \wedge_1(u)\, d(u,\mathfrak{l}) + \wedge_2(u)\, \frac{d(u,\mathcal{O}u)\, d(\mathfrak{l},\mathcal{R}\mathfrak{l})}{1 + d(u,\mathfrak{l})} + \wedge_3(u)\, \frac{d(\mathfrak{l},\mathcal{O}u)\, d(u,\mathcal{R}\mathfrak{l})}{1 + d(u,\mathfrak{l})} \\ &= \mathcal{A}(u,\mathfrak{l})\, d(u,\mathfrak{l}) + \mathcal{B}(u,\mathfrak{l})\, \frac{d(u,\mathcal{O}u)\, d(\mathfrak{l},\mathcal{R}\mathfrak{l})}{1 + d(u,\mathfrak{l})} + C(u,\mathfrak{l})\, \frac{d(\mathfrak{l},\mathcal{O}u)\, d(u,\mathcal{R}\mathfrak{l})}{1 + d(u,\mathfrak{l})}. \end{aligned}$$

Since all the required conditions from Theorem 1 are satisfied (including the contraction-type inequality above), it follows that the pair of mappings $\mathcal{O}$ and $\mathcal{R}$ possesses a unique common FP in $\mathcal{X}$.

Corollary 3. *Let $(\mathcal{X}, d)$ be a complete CV b-SMS with the coefficient $s \geq 1$ and let $\mathcal{O}, \mathcal{R} : \mathcal{X} \to \mathcal{X}$ be two self-mappings. Suppose there exist functions*

$$\wedge_1, \wedge_2 : \mathcal{X} \to [0, 1)$$

such that the following hypotheses are fulfilled:

(H_1) For every $u, \mathfrak{l} \in \mathcal{X}$,

$$\begin{aligned} \wedge_1(\mathcal{O}u) &\leq \wedge_1(u), \quad \wedge_1(\mathcal{R}u) \leq \wedge_1(u), \\ \wedge_2(\mathcal{O}u) &\leq \wedge_2(u), \quad \wedge_2(\mathcal{R}u) \leq \wedge_2(u). \end{aligned}$$

(H_2) The coefficients satisfy

$$s\,\wedge_1(u) + \wedge_2(u) < 1$$

for all $u, \mathfrak{l} \in \mathcal{X}$.

(H_3) For any $u, \mathfrak{l} \in \mathcal{X}$,

$$d(\mathcal{O}u, \mathcal{R}\mathfrak{l}) \leq \wedge_1(u)\, d(u,\mathfrak{l}) + \wedge_2(u)\, \frac{d(u,\mathcal{O}u)\, d(\mathfrak{l},\mathcal{R}\mathfrak{l})}{1 + d(u,\mathfrak{l})}.$$

Then the mappings $\mathcal{O}$ and $\mathcal{R}$ admit a unique common FP in $\mathcal{X}$.

Proof. Take $\wedge_3 : \mathcal{X} \to [0, 1)$ by $\wedge_3(u) = 0$ in Corollary 2. □

Corollary 4. *Let $(\mathcal{X}, d)$ be a complete CV b-SMS with the coefficient $s \geq 1$ and let $\mathcal{O}, \mathcal{R} : \mathcal{X} \to \mathcal{X}$ be two self-mappings. Suppose there exist functions*

$$\wedge_1, \wedge_3 : \mathcal{X} \to [0, 1)$$

such that the following hypotheses are fulfilled:

(H_1) For every $u, \mathfrak{l} \in \mathcal{X}$,

$$\begin{aligned} \wedge_1(\mathcal{O}u) &\leq \wedge_1(u), \quad \wedge_1(\mathcal{R}u) \leq \wedge_1(u), \\ \wedge_3(\mathcal{O}u) &\leq \wedge_3(u), \quad \wedge_3(\mathcal{R}u) \leq \wedge_3(u). \end{aligned}$$

(H_2) The coefficients satisfy

$$s \wedge_1 (u) + \wedge_3 (u) < 1$$

for all $u, \mathfrak{l} \in \mathcal{X}$.

(H_3) For any $u, \mathfrak{l} \in \mathcal{X}$,

$$d(\mathcal{O}u, \mathcal{R}\mathfrak{l}) \leq \wedge_1 (u)\, d(u, \mathfrak{l}) + \wedge_3 (u)\, \frac{d(\mathfrak{l}, \mathcal{O}u)\, d(u, \mathcal{R}\mathfrak{l})}{1 + d(u, \mathfrak{l})}.$$

Then the mappings $\mathcal{O}$ and $\mathcal{R}$ admit a unique common FP in $\mathcal{X}$.

Proof. Take $\wedge_2 : \mathcal{X} \to [0, 1)$ by $\wedge_2(u) = 0$ in Corollary 2. □

Remark 2. It is worth noting that the hypotheses labeled (H_1) and (H_2) in Corollary 2 are not strictly necessary in their original form; they can be replaced by the following weaker condition.

Corollary 5. *Let $(\mathcal{X}, d)$ be a complete CV b-SMS with the coefficient $s \geq 1$ and let $\mathcal{O}, \mathcal{R} : \mathcal{X} \to \mathcal{X}$ be two self-mappings. Suppose there exist functions*

$$\wedge_1, \wedge_2, \wedge_3 : \mathcal{X} \to [0, 1)$$

such that the following hypothesess are fulfilled:

(H_1) For every $u, \mathfrak{l} \in \mathcal{X}$,

$$\begin{aligned} \wedge_1 (\mathcal{R}\mathcal{O}u) &\leq \wedge_1 (u), \\ \wedge_2 (\mathcal{R}\mathcal{O}u) &\leq \wedge_2 (u), \\ \wedge_3 (\mathcal{R}\mathcal{O}u) &\leq \wedge_3 (u). \end{aligned}$$

(H_2) The coefficients satisfy

$$s \wedge_1 (u) + \wedge_2 (u) + \wedge_3 (u) < 1.$$

(H_3) For any $u, \mathfrak{l} \in \mathcal{X}$,

$$d(\mathcal{O}u, \mathcal{R}\mathfrak{l}) \leq \wedge_1 (u)\, d(u, \mathfrak{l}) + \wedge_2 (u)\, \frac{d(u, \mathcal{O}u)\, d(\mathfrak{l}, \mathcal{R}\mathfrak{l})}{1 + d(u, \mathfrak{l})} + \wedge_3 (u)\, \frac{d(\mathfrak{l}, \mathcal{O}u)\, d(u, \mathcal{R}\mathfrak{l})}{1 + d(u, \mathfrak{l})}.$$

Then the mappings $\mathcal{O}$ and $\mathcal{R}$ admit a unique common FP in $\mathcal{X}$.

Proof. Let three functions $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ be defined on the Cartesian product $\mathcal{X} \times \mathcal{X}$, each taking values in the interval $[0, 1)$. Specifically, for arbitrary elements $u, \mathfrak{l} \in \mathcal{X}$, set

$$\begin{aligned} \mathcal{A}(u, \mathfrak{l}) &= \wedge_1 (u), \\ \mathcal{B}(u, \mathfrak{l}) &= \wedge_2 (u), \\ \mathcal{C}(u, \mathfrak{l}) &= \wedge_3 (u), \end{aligned}$$

where $\wedge_1, \wedge_2, \wedge_3 : \mathcal{X} \to [0, 1)$ are given functions. For any $u, \mathfrak{l} \in \mathcal{X}$, the following chain of inequalities holds for $\mathcal{A}$:

$$\mathcal{A}(\mathcal{R}\mathcal{O}u, \mathfrak{l}) = \wedge_1 (\mathcal{R}\mathcal{O}u) \leq \wedge_1 (u) = \mathcal{A}(u, \mathfrak{l}).$$

Moreover

$$\mathcal{A}(u, \mathcal{O}\mathcal{R}\mathfrak{l}) \leq \wedge_1 (u) = \mathcal{A}(u, \mathfrak{l}).$$

Exactly the same reasoning yields analogous estimates for the other two functions: □

$$\mathcal{B}(\mathcal{R}\mathcal{O}u, \mathfrak{l}) \leq \mathcal{B}(u, \mathfrak{l}), \quad \mathcal{B}(u, \mathcal{O}\mathcal{R}\mathfrak{l}) \leq \mathcal{B}(u, \mathfrak{l}),$$

$$C(\mathcal{R}\mathcal{O}u, \mathfrak{l}) \leq C(u, \mathfrak{l}), \quad C(u, \mathcal{O}\mathcal{R}\mathfrak{l}) \leq C(u, \mathfrak{l}).$$

Now for every pair $u, \mathfrak{l} \in \mathcal{X}$, we have

$$s\mathcal{A}(u, \mathfrak{l}) + \mathcal{B}(u, \mathfrak{l}) + C(u, \mathfrak{l}) = s \wedge_1 (u) + \wedge_2 (u) + \wedge_3 (u) < 1.$$

Also

$$\begin{aligned} d(\mathcal{O}u, \mathcal{R}\mathfrak{l}) &\leq \wedge_1 (u)\, d(u, \mathfrak{l}) + \wedge_2 (u) \frac{d(u, \mathcal{O}u)\, d(\mathfrak{l}, \mathcal{R}\mathfrak{l})}{1 + d(u, \mathfrak{l})} + \wedge_3 (u) \frac{d(\mathfrak{l}, \mathcal{O}u)\, d(u, \mathcal{R}\mathfrak{l})}{1 + d(u, \mathfrak{l})} \\ &= \mathcal{A}(u, \mathfrak{l})\, d(u, \mathfrak{l}) + \mathcal{B}(u, \mathfrak{l}) \frac{d(u, \mathcal{O}u)\, d(\mathfrak{l}, \mathcal{R}\mathfrak{l})}{1 + d(u, \mathfrak{l})} + C(u, \mathfrak{l}) \frac{d(\mathfrak{l}, \mathcal{O}u)\, d(u, \mathcal{R}\mathfrak{l})}{1 + d(u, \mathfrak{l})}. \end{aligned}$$

Thus, all necessary conditions in Theorem 1 are fulfilled. Therefore, $\mathcal{O}$ and $\mathcal{R}$ have exactly one common FP.

Corollary 6. *Let $(\mathcal{X}, d)$ be a complete CV b-SMS with the coefficient $s \geq 1$ and let $\mathcal{O}, \mathcal{R} : \mathcal{X} \to \mathcal{X}$ be two self-mappings. Suppose there exist the functions*

$$\wedge_1, \wedge_2 : \mathcal{X} \to [0, 1)$$

such that the following hypotheses are fulfilled:

(H_1) For every $u, \mathfrak{l} \in \mathcal{X}$,

$$\begin{aligned} \wedge_1 (\mathcal{R}\mathcal{O}u) &\leq \wedge_1 (u), \\ \wedge_2 (\mathcal{R}\mathcal{O}u) &\leq \wedge_2 (u). \end{aligned}$$

(H_2) The coefficients satisfy

$$s \wedge_1 (u) + \wedge_2 (u) < 1.$$

(H_3) For any $u, \mathfrak{l} \in \mathcal{X}$,

$$d(\mathcal{O}u, \mathcal{R}\mathfrak{l}) \leq \wedge_1 (u)\, d(u, \mathfrak{l}) + \wedge_2 (u) \frac{d(u, \mathcal{O}u)\, d(\mathfrak{l}, \mathcal{R}\mathfrak{l})}{1 + d(u, \mathfrak{l})}.$$

Then the mappings $\mathcal{O}$ and $\mathcal{R}$ admit a unique common FP in $\mathcal{X}$.

Proof. Define $\wedge_3 (u) = 0$ in Corollary 5. □

Corollary 7. *Let $(\mathcal{X}, d)$ be a complete CV b-SMS with the coefficient $s \geq 1$ and let $\mathcal{O}, \mathcal{R} : \mathcal{X} \to \mathcal{X}$ be two self-mappings. Suppose there exist functions*

$$\wedge_1, \wedge_3 : \mathcal{X} \to [0, 1)$$

such that the following hypotheses are fulfilled:

(H_1) For every $u, \mathfrak{l} \in \mathcal{X}$,

$$\begin{aligned} \wedge_1(\mathcal{R}\mathcal{O}u) &\leq \wedge_1(u), \\ \wedge_3(\mathcal{R}\mathcal{O}u) &\leq \wedge_3(u). \end{aligned}$$

(H_2) The coefficients satisfy

$$s\wedge_1(u) + \wedge_3(u) < 1.$$

(H_3) For any $u, \mathfrak{l} \in \mathcal{X}$,

$$d(\mathcal{O}u, \mathcal{R}\mathfrak{l}) \leq \wedge_1(u)\, d(u, \mathfrak{l}) + \wedge_3(u)\, \frac{d(\mathfrak{l}, \mathcal{O}u)\, d(u, \mathcal{R}\mathfrak{l})}{1 + d(u, \mathfrak{l})}.$$

Then the mappings $\mathcal{O}$ and $\mathcal{R}$ admit a unique common FP in $\mathcal{X}$.

Proof. Take $\wedge_2(u) = 0$ in Corollary 5. □

Remark 3. By choosing the control functions in Corollary 5 to be constants, namely,

$$\wedge_1(\cdot) = \wedge_1, \wedge_2(\cdot) = \wedge_2, \wedge_3(\cdot) = \wedge_3,$$

we obtain the following consequence.

The following theorem of Noorwali et al. [14] can be recovered as a direct consequence of our result.

Corollary 8. *Let $(\mathcal{X}, d)$ be a complete CV b-SMS with the coefficient $s \geq 1$ and let $\mathcal{O}$ and $\mathcal{R}$ be self-mappings on $\mathcal{X}$. Assume that constants $\wedge_1, \wedge_2, \wedge_3 \in [0, 1)$ can be chosen so that*

$$s\wedge_1 + \wedge_2 + \wedge_3 < 1.$$

Further, assume that for every pair $u, \mathfrak{l} \in \mathcal{X}$, the following contractive condition is satisfied:

$$d(\mathcal{O}u, \mathcal{R}\mathfrak{l}) \leq \wedge_1 d(u, \mathfrak{l}) + \wedge_2 \frac{d(u, \mathcal{O}u)\, d(\mathfrak{l}, \mathcal{R}\mathfrak{l})}{1 + d(u, \mathfrak{l})} + \wedge_3 \frac{d(\mathfrak{l}, \mathcal{O}u)\, d(u, \mathcal{R}\mathfrak{l})}{1 + d(u, \mathfrak{l})}.$$

Under these assumptions, the mappings $\mathcal{O}$ and $\mathcal{R}$ admit exactly one common FP in $\mathcal{X}$.

Proof. The conclusion follows from Corollary 5 by choosing the functions $\wedge_1(\cdot)$, $\wedge_2(\cdot)$, and $\wedge_3(\cdot)$ to be the constants $\wedge_1$, $\wedge_2$, and $\wedge_3$, respectively. □

Corollary 9. *Let $(\mathcal{X}, d)$ be a complete CV b-SMS with the coefficient $s \geq 1$ and let $\mathcal{O}$ and $\mathcal{R}$ be self-mappings on $\mathcal{X}$. Assume that constants $\wedge_1, \wedge_2 \in [0, 1)$ can be chosen so that*

$$s\wedge_1 + \wedge_2 < 1.$$

Further, assume that for every pair $u, \mathfrak{l} \in \mathcal{X}$, the following contractive condition is satisfied:

$$d(\mathcal{O}u, \mathcal{R}\mathfrak{l}) \leq \wedge_1 d(u, \mathfrak{l}) + \wedge_2 \frac{d(u, \mathcal{O}u)\, d(\mathfrak{l}, \mathcal{R}\mathfrak{l})}{1 + d(u, \mathfrak{l})}.$$

Under these assumptions, the mappings $\mathcal{O}$ and $\mathcal{R}$ admit exactly one common FP in $\mathcal{X}$.

Proof. Take $\lambda_3 = 0$ in Corollary 8. □

Corollary 10. *Let $(\mathcal{X}, d)$ be a complete CV b-SMS with the coefficient $s \geq 1$ and let $\mathcal{O}$ and $\mathcal{R}$ be self-mappings on $\mathcal{X}$. Assume that constants $\lambda_1, \lambda_3 \in [0, 1)$ can be chosen so that*

$$s\,\lambda_1 + \lambda_3 < 1.$$

Further, assume that for every pair $u, \mathfrak{l} \in \mathcal{X}$, the following contractive condition is satisfied:

$$d(\mathcal{O}u, \mathcal{R}\mathfrak{l}) \leq \lambda_1 d(u, \mathfrak{l}) + \lambda_3 \frac{d(\mathfrak{l}, \mathcal{O}u)\, d(u, \mathcal{R}\mathfrak{l})}{1 + d(u, \mathfrak{l})}.$$

Under these assumptions, the mappings $\mathcal{O}$ and $\mathcal{R}$ admit exactly one common FP in $\mathcal{X}$.

Proof. Take $\lambda_2 = 0$ in Corollary 8. □

5. Relationship with existing results in the literature

The results established in this article encompass, extend, and unify some well-known FP theorems available in the existing literature. By selecting suitable values of the parameters involved in the proposed framework, a number of previously published results emerge as particular cases. The following observations illustrate the generality and applicability of our findings.

(a) Setting $s = 1$ in Definition 5, Corollary 8 reduces to the main theorem of Panda et al. [13].

(b) By choosing $s = 1$ in Theorem 1, one recovers the principal result established by Abdou et al. [16].

(c) Taking $s = 1$ in Corollary 2 yields one of the FP results obtained by Shammaky et al. [17].

(d) When $\aleph = 0$ in Definition 5 the structure of the CV b-SMS reduces to that of a CV b-MS. Consequently, Corollary 9 specializes to the main theorem of Mukheimer et al. [11].

(e) The fundamental theorem of Kumar et al. [12] is obtained from Corollary 8 by setting $\aleph = 0$.

(f) Corollary 6 coincides with the primary result of Dubey et al. [18] whenever $\aleph = 0$.

(g) Choosing $s = 1$ and $\aleph = 0$ in Definition 5 transforms the CV b-SMS framework into a CV MS. Under this setting, Theorem 1 reproduces the main result of Sitthikul et al. [10].

(h) By taking $s = 1$ and $\aleph = 0$ in Corollary 6, one obtains the principal theorem of Sintunavarat et al. [9].

(i) Corollary 8 with $s = 1$ and $\aleph = 0$ reduces to the corresponding result of Rouzkard et al. [8].

(j) The principal FP theorem of Azam et al. [7] follows from Corollary 9 when $s = 1$ and $\aleph = 0$.

(k) Restricting $\mathbb{C}$ to $\mathbb{R}$ in Definition 5, that is, considering a vanishing imaginary part, the notion of a CV b-SMS reduces to a b-SMS. In this case, Corollary 3.1 of Berzig [6] is recovered from Corollary 9 by taking $\lambda_2 = 0$.

(l) If $s = 1$ and $\mathbb{C}$ is restricted to $\mathbb{R}$ in Definition 5, then the underlying space becomes a SMS, and the corresponding result of Berzig [4] is obtained as a special case.

Therefore, the theoretical framework introduced in this work not only extends a variety of known FP results but also establishes a unified approach for their treatment. This highlights the versatility of the CV b-SMS and confirms the wider applicability of the obtained theorems.

6. Applications

In this section, we demonstrate the applicability of the common FP results established in the preceding sections by considering a class of nonlinear fractional Fredholm–Volterra integral equations. Such equations play an important role in the mathematical modeling of various phenomena involving memory effects and nonlocal interactions, particularly in engineering control systems, viscoelasticity, diffusion processes, and other applied sciences. By formulating the integral equation as an operator equation on a suitable complete CV b-SMS, we show that the associated operator satisfies the hypotheses of our FP theorem. Consequently, the existence and uniqueness of a continuous solution are established. This application illustrates the effectiveness of the proposed framework and highlights its potential for studying nonlinear fractional integral equations arising in both theoretical investigations and practical applications.

Consider a nonlinear fractional Fredholm–Volterra integral equation

$$u(t) = f(t) + \frac{\lambda_1}{\Gamma(\alpha)} \int_a^t K_1(t, s, u(s)) (t-s)^{\alpha-1} ds + \frac{\lambda_2}{\Gamma(\beta)} \int_a^b K_2(t, s, u(s)) |t-s|^{\beta-1} ds,$$

where t belongs to $[a, b]$, $0 < \alpha, \beta < 1$, and λ_1, λ_2 are real constants. The absolute value in the Fredholm integral ensures that the kernel remains real-valued throughout the integration interval, including when $s > t$.

Let

$$\mathcal{X} = C([a, b], \mathbb{R}).$$

For all $u, \mathfrak{l} \in \mathcal{X}$, define

$$d(u, \mathfrak{l}) = \|u - \mathfrak{l}\|_\infty^p e^{i\frac{\pi}{4}}$$

and

$$\|u - \mathfrak{l}\|_\infty = \sup_{t \in [a,b]} |u(t) - \mathfrak{l}(t)|,$$

where $p \geq 1$ and $\|\cdot\|_\infty$ denotes the usual supremum norm. As established in Section 2, $(\mathcal{X}, d)$ is a CV b-SMS with coefficient $s = 2^{p-1}$ and $\aleph \geq 0$. Moreover, $(\mathcal{X}, d)$ is complete because convergence with respect to d is equivalent to convergence in the supremum norm, and $C([a, b], \mathbb{R})$ is complete under that norm.

Theorem 2. *Let $(\mathcal{X}, d)$ be a complete CV b-SMS, and let $\mathcal{R} : \mathcal{X} \to \mathcal{X}$ be a mapping defined by*

$$\mathcal{R}u(t) = f(t) + \frac{\lambda_1}{\Gamma(\alpha)} \int_a^t K_1(t, s, u(s)) (t-s)^{\alpha-1} ds + \frac{\lambda_2}{\Gamma(\beta)} \int_a^b K_2(t, s, u(s)) |t-s|^{\beta-1} ds.$$

If the following conditions hold:

(i) Suppose there exist constants $L_1, L_2 > 0$ such that

$$|K_1(t, s, u(s)) - K_1(t, s, \mathfrak{l}(s))| \leq L_1 |u(s) - \mathfrak{l}(s)|$$

and

$$|K_2(t, s, u(s)) - K_2(t, s, \mathfrak{l}(s))| \leq L_2 |u(s) - \mathfrak{l}(s)|$$

for all $s \in [a, b]$.

(ii) The constant

$$q = \left(\frac{|\lambda_1| L_1 (b-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{|\lambda_2| L_2 2^{1-\beta} (b-a)^{\beta}}{\Gamma(\beta+1)} \right)$$

satisfies $sq^p < 1$. *Then* $\mathcal{R}$ *admits a unique FP* $u^*(t)$, *i.e.,*

$$\mathcal{R}(u^*)(t) = u^*(t),$$

which solves the nonlinear fractional Fredholm–Volterra integral equation

$$u(t) = f(t) + \frac{\lambda_1}{\Gamma(\alpha)} \int_a^t K_1(t, s, u(s)) (t-s)^{\alpha-1} ds + \frac{\lambda_2}{\Gamma(\beta)} \int_a^b K_2(t, s, u(s)) |t-s|^{\beta-1} ds.$$

Proof. Let $u, \mathfrak{l} \in \mathcal{X}$. Since f is continuous and K_1, K_2 are continuous, the functions $s \mapsto K_1(t, s, u(s))$ and $s \mapsto K_2(t, s, u(s))$ are continuous and bounded on $[a, b]$. The fractional kernels are integrable because $0 < \alpha, \beta < 1$. Standard continuity properties of weakly singular Volterra and Fredholm integral operators therefore imply that $\mathcal{R}u \in \mathcal{X}$. Hence $\mathcal{R}$ maps $\mathcal{X}$ into itself. Now for every $t \in [a, b]$, we have

$$\begin{aligned} |\mathcal{R}(u)(t) - \mathcal{R}(\mathfrak{l})(t)| &\leq \frac{|\lambda_1|}{\Gamma(\alpha)} \int_a^t |K_1(t, s, u(s)) - K_1(t, s, \mathfrak{l}(s))| (t-s)^{\alpha-1} ds \\ &\quad + \frac{|\lambda_2|}{\Gamma(\beta)} \int_a^b |K_2(t, s, u(s)) - K_2(t, s, \mathfrak{l}(s))| |t-s|^{\beta-1} ds. \end{aligned}$$

Using condition (i), we have

$$\begin{aligned} |\mathcal{R}(u)(t) - \mathcal{R}(\mathfrak{l})(t)| &\leq \frac{|\lambda_1| L_1}{\Gamma(\alpha)} \int_a^t |u(s) - \mathfrak{l}(s)| (t-s)^{\alpha-1} ds \\ &\quad + \frac{|\lambda_2| L_2}{\Gamma(\beta)} \int_a^b |u(s) - \mathfrak{l}(s)| |t-s|^{\beta-1} ds. \end{aligned} \tag{6.1}$$

Then

$$|u(s) - \mathfrak{l}(s)| \leq \sup_{s \in [a,b]} |u(s) - \mathfrak{l}(s)| = \|u - \mathfrak{l}\|_{\infty}.$$

Using the substitution $\omega = t - s$, we obtain

$$\begin{aligned} \int_a^t (t-s)^{\alpha-1} ds &= \int_0^{t-a} (\omega)^{\alpha-1} d\omega \\ &= \left[\frac{\omega^{\alpha}}{\alpha} \right]_0^{t-a} \\ &= \frac{(t-a)^{\alpha}}{\alpha}. \end{aligned}$$

Since $a \leq t \leq b$, we have

$$0 \leq t - a \leq b - a.$$

Consequently,

$$\int_a^t (t-s)^{\alpha-1} ds = \frac{(t-a)^{\alpha}}{\alpha} \leq \frac{(b-a)^{\alpha}}{\alpha}.$$

Furthermore, since $a \leq t \leq b$, we have

$$\int_a^b |t-s|^{\beta-1}\, ds = \int_a^t (t-s)^{\beta-1}\, ds + \int_t^b (s-t)^{\beta-1}\, ds.$$

Evaluating both integrals gives

$$\begin{aligned} \int_a^b |t-s|^{\beta-1}\, ds &= \frac{(t-a)^\beta}{\beta} + \frac{(b-t)^\beta}{\beta} \\ &= \frac{(t-a)^\beta + (b-t)^\beta}{\beta}. \end{aligned}$$

Set

$$\varsigma = t-a \text{ and } \tau = b-t.$$

Since $a \leq t \leq b$, both $\varsigma \geq 0$ and $\tau \geq 0$, and

$$\varsigma + \tau = b-a.$$

For $0 < \beta < 1$, the function z^β is concave on the nonnegative real numbers. By concavity,

$$\frac{\varsigma^\beta + \tau^\beta}{2} \leq \left(\frac{\varsigma+\tau}{2}\right)^\beta.$$

Multiplying both sides by 2, we obtain

$$\varsigma^\beta + \tau^\beta \leq 2^{1-\beta}(\varsigma+\tau)^\beta.$$

Substituting $\varsigma = t-a$, $\tau = b-t$, and $\varsigma + \tau = b-a$, we get

$$(t-a)^\beta + (b-t)^\beta \leq 2^{1-\beta}(b-a)^\beta.$$

Therefore,

$$\int_a^b |t-s|^{\beta-1}\, ds = \frac{(t-a)^\beta + (b-t)^\beta}{\beta} \leq \frac{2^{1-\beta}(b-a)^\beta}{\beta}.$$

Thus inequality (6.1) becomes

$$|\mathcal{R}(u)(t) - \mathcal{R}(\mathfrak{l})(t)| \leq q\, \|u-\mathfrak{l}\|_\infty,$$

where

$$q = \left(\frac{|\lambda_1| L_1 (b-a)^\alpha}{\Gamma(\alpha+1)} + \frac{|\lambda_2| L_2 2^{1-\beta}(b-a)^\beta}{\Gamma(\beta+1)}\right).$$

Taking the supremum over t, we have

$$\|\mathcal{R}u - \mathcal{R}\mathfrak{l}\|_\infty \leq q\, \|u-\mathfrak{l}\|_\infty.$$

Raising to the power p,

$$\|\mathcal{R}u - \mathcal{R}\mathfrak{l}\|_\infty^p \leq q^p\, \|u-\mathfrak{l}\|_\infty^p.$$

Multiplying by $e^{i\frac{\pi}{4}}$, we have

$$d(\mathcal{R}u, \mathcal{R}\mathfrak{l}) \leq q^p d(u, \mathfrak{l}).$$

Define the control functions

$$\mathcal{A}, \mathcal{B}, C : \mathcal{X} \times \mathcal{X} \to [0, 1)$$

by

$$\mathcal{A}(u, \mathfrak{l}) = q^p, \ \mathcal{B}(u, \mathfrak{l}) = 0, \ \ C(u, \mathfrak{l}) = 0.$$

Since $sq^p < 1$, by assumption (ii), it follows that

$$s\mathcal{A}(u, \mathfrak{l}) + \mathcal{B}(u, \mathfrak{l}) + C(u, \mathfrak{l}) = sq^p < 1,$$

for all $u, \mathfrak{l} \in \mathcal{X}$. Hence condition ($H_2$) of Corollary 1 is satisfied. Furthermore, since $\mathcal{A}$ is constant, we have

$$\mathcal{A}(\mathcal{R}u, \mathfrak{l}) \leq \mathcal{A}(u, \mathfrak{l})$$

and

$$\mathcal{A}(u, \mathcal{R}\mathfrak{l}) \leq \mathcal{A}(u, \mathfrak{l}).$$

Also

$$\begin{aligned} \mathcal{B}(\mathcal{R}u, \mathfrak{l}) &\leq \mathcal{B}(u, \mathfrak{l}), \quad \mathcal{B}(u, \mathcal{R}\mathfrak{l}) \leq \mathcal{B}(u, \mathfrak{l}), \\ C(\mathcal{R}u, \mathfrak{l}) &\leq C(u, \mathfrak{l}), \quad C(u, \mathcal{R}\mathfrak{l}) \leq C(u, \mathfrak{l}). \end{aligned}$$

Therefore condition (H_1) is fulfilled. Moreover,

$$d(\mathcal{R}u, \mathcal{R}\mathfrak{l}) \leq q^p d(u, \mathfrak{l}) = \mathcal{A}(u, \mathfrak{l}) d(u, \mathfrak{l}).$$

Since $\mathcal{B}(u, \mathfrak{l}) = C(u, \mathfrak{l}) = 0$,

$$d(\mathcal{R}u, \mathcal{R}\mathfrak{l}) \leq \mathcal{A}(u, \mathfrak{l}) d(u, \mathfrak{l}) + \mathcal{B}(u, \mathfrak{l}) \frac{d(u, \mathcal{R}u) d(\mathfrak{l}, \mathcal{R}\mathfrak{l})}{1 + d(u, \mathfrak{l})} + C(u, \mathfrak{l}) \frac{d(\mathfrak{l}, \mathcal{R}u) d(u, \mathcal{R}\mathfrak{l})}{1 + d(u, \mathfrak{l})}.$$

Hence condition (H_3) is satisfied. Consequently, all hypotheses of Corollary 1 hold. Therefore, the operator $\mathcal{R}$ possesses a unique FP $u^* \in \mathcal{X}$. Equivalently, the nonlinear fractional Fredholm–Volterra integral equation admits a unique continuous solution u^* on $[a, b]$. □

The following example illustrates the applicability of Theorem 2. By verifying each of its hypotheses, we show that the proposed FP approach provides an effective tool for establishing the existence and uniqueness of solutions to nonlinear fractional Fredholm–Volterra integral equations.

Example 5. Consider the nonlinear fractional Fredholm–Volterra integral equation on $[0, 1]$:

$$u(t) = 1 + \frac{1}{\Gamma(\frac{1}{2})} \int_0^t \frac{u(s)}{20(1 + |u(s)|)} (t - s)^{-\frac{1}{2}} ds + \frac{1}{\Gamma(\frac{1}{3})} \int_0^1 \frac{u(s)}{30(1 + |u(s)|)} |t - s|^{-\frac{2}{3}} ds.$$

Choose

$$a = 0, \quad b = 1,$$

$$\alpha = \frac{1}{2}, \quad \beta = \frac{1}{3},$$
$$\lambda_1 = \lambda_2 = 1,$$
$$f(t) = 1,$$

and define

$$K_1(t, s, u) = \frac{u}{20(1 + |u|)}, \quad K_2(t, s, u) = \frac{u}{30(1 + |u|)}.$$

Let $\mathcal{X} = C([0, 1], \mathbb{R})$ with

$$d(u, \mathfrak{l}) = \sup_{t \in [0,1]} |u(t) - \mathfrak{l}(t)|^2 e^{i\frac{\pi}{4}},$$

where $p = 2$. Then $(\mathcal{X}, d)$ is a complete CV b-SMS. For every $u, \mathfrak{l} \in \mathcal{X}$, we have

$$|K_1(t, s, u) - K_1(t, s, \mathfrak{l})| \leq \frac{1}{20} |u - \mathfrak{l}|.$$

Hence

$$L_1 = \frac{1}{20}.$$

Similarly,

$$|K_2(t, s, u) - K_2(t, s, \mathfrak{l})| \leq \frac{1}{30} |u - \mathfrak{l}|,$$

so,

$$L_2 = \frac{1}{30}.$$

Thus condition (i) holds. Compute

$$q = \left(\frac{|\lambda_1| L_1 (b - a)^\alpha}{\Gamma(\alpha + 1)} + \frac{|\lambda_2| L_2 2^{1-\beta} (b - a)^\beta}{\Gamma(\beta + 1)} \right).$$

Substituting the selected values gives

$$q = \left(\frac{\frac{1}{20}}{\Gamma(\frac{3}{2})} + \frac{\frac{1}{30} 2^{\frac{2}{3}}}{\Gamma(\frac{4}{3})} \right).$$

Using

$$\Gamma(\frac{3}{2}) \approx 0.88623,$$
$$\Gamma(\frac{4}{3}) \approx 0.89298,$$

we obtain

$$\frac{\frac{1}{20}}{\Gamma(\frac{3}{2})} \approx 0.05642$$

and

$$\frac{\frac{1}{30}2^{\frac{2}{3}}}{\Gamma(\frac{4}{3})} \approx 0.05927.$$

Therefore

$$\begin{aligned} q &= (0.05642 + 0.05927) \\ &= 0.11569. \end{aligned}$$

Moreover,

$$q^2 = (0.11569)^2 = 0.01338$$

and

$$sq^2 = 2(0.01338) = 0.0267 < 1.$$

Hence condition (ii) is satisfied. Therefore, all of the hypotheses of Theorem 2 are satisfied. Consequently, the associated operator $\mathcal{R}$ defined by

$$(\mathcal{R}u)(t) = 1 + \frac{1}{\Gamma(\frac{1}{2})}\int_0^t \frac{u(s)}{20\,(1+|u(s)|)}(t-s)^{-\frac{1}{2}}\,ds + \frac{1}{\Gamma(\frac{1}{3})}\int_0^1 \frac{u(s)}{30\,(1+|u(s)|)}|t-s|^{-\frac{2}{3}}\,ds,$$

possesses a unique FP $u^* \in \mathcal{X}$, which is the unique continuous solution of the given nonlinear fractional Fredholm–Volterra integral equation. This example confirms the applicability and effectiveness of the proposed common FP results in solving nonlinear fractional integral equations.

To further investigate the convergence of the proposed FP scheme, consider the Picard iteration

$$u_{n+1}(t) = u_n(t), \quad n = 0, 1, 2, ...$$

with the initial approximation $u_0(t) = 1$ for every $t \in [0, 1]$. Since $\mathcal{R}$ is a contraction with constant

$$q = 0.11569 < 1,$$

the Banach FP theorem guarantees that the sequence $\{u_n\}$ converges uniformly to the unique FP u^* of $\mathcal{R}$. The standard a priori error estimate is

$$\|u_n - u^*\|_\infty \leq \frac{q^n}{1-q}\|u_1 - u_0\|_\infty.$$

The corresponding a posteriori error estimate is

$$\|u_n - u^*\|_\infty \leq \frac{q}{1-q}\|u_n - u_{n-1}\|_\infty.$$

For $q = 0.11569$, we obtain

$$\frac{q}{1-q} = \frac{0.11569}{1-0.11569} \approx 0.1308,$$

$$\frac{1}{1-q} = \frac{1}{1-0.11569} \approx 1.1308.$$

We now calculate the first Picard iterate. Since $u_0(t) = 1$, the kernels satisfy

$$K_1(t, s, u_0(s)) = \frac{1}{20(1+1)} = \frac{1}{40},$$
$$K_2(t, s, u_0(s)) = \frac{1}{30(1+1)} = \frac{1}{60}.$$

Consequently,

$$u_1(t) = 1 + \frac{1}{\Gamma(\frac{1}{2})} \int_0^t \frac{1}{40} (t-s)^{-\frac{1}{2}} ds + \frac{1}{\Gamma(\frac{1}{3})} \int_0^1 \frac{1}{60} |t-s|^{-\frac{2}{3}} ds.$$

Using the integral identities

$$\int_0^t (t-s)^{-\frac{1}{2}} ds = 2\sqrt{t}$$

and

$$\int_0^1 |t-s|^{-\frac{2}{3}} ds = 3\left(t^{\frac{1}{3}} + (1-t)^{\frac{1}{3}}\right),$$

together with $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we obtain

$$u_1(t) = 1 + \frac{\sqrt{t}}{20\sqrt{\pi}} + \frac{t^{\frac{1}{3}} + (1-t)^{\frac{1}{3}}}{20\Gamma(\frac{1}{3})}.$$

It follows that

$$u_1(t) - u_0(t) = \frac{\sqrt{t}}{20\sqrt{\pi}} + \frac{t^{\frac{1}{3}} + (1-t)^{\frac{1}{3}}}{20\Gamma(\frac{1}{3})}.$$

Taking the supremum over $t \in [0, 1]$, we obtain

$$\|u_1 - u_0\|_\infty \leq \frac{1}{20\sqrt{\pi}} + \frac{2^{\frac{2}{3}}}{20\Gamma(\frac{1}{3})} \approx 0.0579.$$

Substituting this bound into the a priori error estimate gives

$$\begin{aligned} \|u_n - u^*\| &\leq (0.11569)^n (1.1308)(0.0579) \\ &\leq 0.06536 (0.11569)^n. \end{aligned}$$

Therefore, Table 1 presents numerical upper bounds for the error in the Picard iterates.

Thus the Picard iteration converges linearly with the contraction ratio $q \approx 0.11567$. In particular, the error is already below 10^{-6} after five iterations and below 10^{-7} after six iterations. This confirms that the proposed FP scheme converges rapidly to the unique solution u^*. To provide a visual illustration of the convergence behavior, Figure 1 presents the convergence analysis of the proposed method and shows how the iterative sequence approaches the FP.

Table 1. Convergence analysis.

n	Upper bound for $\|u_n - u^*\|$
0	6.536×10^{-2}
1	7.561×10^{-3}
2	8.747×10^{-4}
3	1.012×10^{-4}
4	1.171×10^{-5}
5	1.355×10^{-6}
6	1.568×10^{-7}

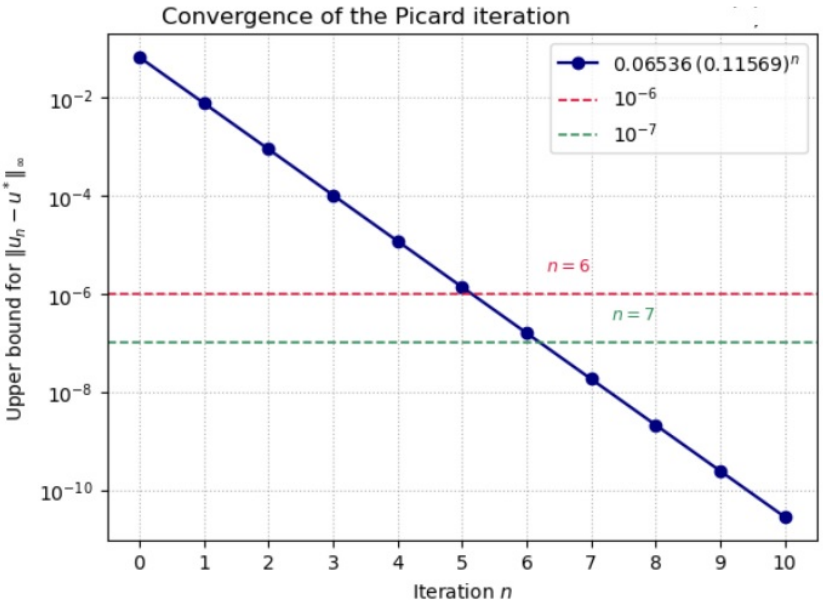


Figure 1. Convergence analysis.

6.1. Application to engineering control systems

Many engineering systems exhibit memory-dependent behavior, meaning that the present state depends not only on the current input but also on the entire history of the system. Examples include viscoelastic actuators, robotic manipulators with joint friction, electrical circuits with supercapacitors, thermal systems, and vibration control structures. Such systems are often described more accurately by fractional-order models than by classical differential equations.

Consider the nonlinear fractional Fredholm–Volterra integral equation

$$u(t) = f(t) + \frac{\lambda_1}{\Gamma(\alpha)} \int_a^t K_1(t, s, u(s)) (t-s)^{\alpha-1} ds + \frac{\lambda_2}{\Gamma(\beta)} \int_a^b K_2(t, s, u(s)) |t-s|^{\beta-1} ds,$$

where

- $u(t)$ denotes the system state or controller output,
- $f(t)$ is the external reference signal or forcing input,

- K_1 models nonlinear feedback depending on the past history of the system,
- K_2 represents distributed interactions among system components,
- $\alpha, \beta \in (0, 1)$ characterize the memory strength of the system.

The first (Volterra) integral

$$\frac{\lambda_1}{\Gamma(\alpha)} \int_a^t K_1 (t, s, u(s)) (t - s)^{\alpha-1} ds$$

describes history-dependent feedback, which naturally appears in adaptive control, fractional proportional integral derivative controllers, viscoelastic damping, and systems with actuator memory.

The second (Fredholm) integral

$$\frac{\lambda_2}{\Gamma(\beta)} \int_a^b K_2 (t, s, u(s)) |t - s|^{\beta-1} ds$$

models nonlocal interactions, such as distributed sensors, interconnected actuators, flexible mechanical structures, or networked control systems where the state at one location depends on the behavior over the entire operating domain.

7. Conclusions and future works

The present study focuses on the development of common FP theory in the setting of CV b-SMSs, employing generalized contractive conditions defined via control functions of two variables. The obtained results provide a more flexible and unified framework that not only generalizes some existing FP theorems in the literature but also encompasses them as particular cases.

To demonstrate the applicability and strength of the theoretical findings, a nontrivial illustrative example is constructed, verifying the validity of the proposed contractive conditions in a concrete setting. Furthermore, the theoretical outcomes are effectively utilized to investigate a class of nonlinear fractional Fredholm–Volterra integral equations. In particular, the existence and uniqueness of solutions are established by transforming the integral equation into an equivalent FP problem and applying the developed common FP results.

Overall, the obtained results highlight the significance of the proposed framework in extending classical FP theory and underline its practical relevance in analyzing nonlinear fractional integral equations arising in applied mathematics and engineering sciences.

The results of this study suggest some directions for future research. These include extending the proposed common FP theorems to other generalized MSs and investigating broader contractive conditions involving simulation functions, interpolation techniques, and admissibility concepts. Another direction is to apply the established results to more general fractional integral and integro-differential equations, including coupled systems, delay equations, and variable-order fractional models. Further research may also address the development of numerical methods for approximating solutions of fractional Fredholm–Volterra integral equations arising in engineering and other applied sciences.

Author contributions

Hadeel Z. Alzumi: Conceptualization, methodology, supervision; Wafa Shammakh: Investigation, validation, formal analysis; Jamshaid Ahmad: Writing–original draft and editing. All authors collectively contributed to every part of the work and have reviewed and approved the submitted manuscript. All authors have also read and approved the final version for publication.

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Conflicts of interest

The authors declare no conflicts of interest.

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