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Research article

Reduction for block-transitive 3-$(v, k, 3)$ designs

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Abstract: This paper contributed to the classification of block-transitive 3-designs. Using the O'Nan-Scott Theorem for the classification of primitive permutation groups, we proved that the block-transitive point-primitive automorphism group G of a nontrivial 3-$(v, k, 3)$ design must be of affine type, almost simple type, or product type with $G \leq H \wr S_2$ acting on $\mathcal{P} = \Gamma \times \Gamma$, where H was a primitive group on Γ.

Keywords: block-transitive; point-primitive; 3-designs; automorphism group; permutation group
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1. Introduction

A t-(v, k, λ) design $\mathcal{D}$ is a pair $(\mathcal{P}, \mathcal{B})$, where $\mathcal{P}$ is a set of v points and $\mathcal{B}$ is a collection of k-subsets of $\mathcal{P}$, known as blocks, such that each t-subset of $\mathcal{P}$ is contained in exactly λ blocks. We say $\mathcal{D}$ is nontrivial if $t < k < v$, and all designs considered in this paper are assumed to be nontrivial. An automorphism of $\mathcal{D}$ is a permutation of $\mathcal{P}$ that preserves the block set $\mathcal{B}$. The full set of such automorphisms forms a group under permutation composition, denoted by $\mathrm{Aut}(\mathcal{D})$, and any subgroup of $\mathrm{Aut}(\mathcal{D})$ is called an automorphism group of $\mathcal{D}$. A flag of $\mathcal{D}$ is a pair consisting of a point α and a block B such that $\alpha \in B$, where $\alpha \in \mathcal{P}$ and $B \in \mathcal{B}$. A group G is said to be point-primitive, point-transitive, block-transitive, or flag-transitive if it acts primitively on points, transitively on points, transitively on blocks, or transitively on flags, respectively. By Block's result [3], any block-transitive group G is necessarily point-transitive. Let $\mathcal{D}$ be a 3-$(v, k, 3)$ design admitting a point-primitive automorphism group G. According to the O'Nan-Scott Theorem, G is permutation isomorphic to a group of affine type, almost simple type, product type, simple diagonal type, or twisted wreath type. A detailed treatment of this classification can be found in [19].

Research on the classification of block-transitive designs began a long time ago. In [4, Theorem 1.1], a crucial conclusion states that if $\mathcal{D}$ is a block-transitive t-(v, k, λ) design, then $t \leq 7$. Moreover, for any t-design with $t \geq 4$ and $t \leq \min\{k, v-k\}$, it follows from a result of Ray-Chaudhuri and Wilson [22] and Cameron-Praeger [4, Theorem 2.1] that any block-transitive automorphism group of the design must be 2-homogeneous, and hence point-primitive of affine or almost simple type. Subsequently, the focus of research on block-transitive point-primitive t-designs has been on the cases with $t = 2, 3$. One of the most notable contributions is the reduction result for block-transitive point-primitive 2-$(v, k, 1)$ designs presented in [7]. In addition, Camina [5,6,8], Liu [17,18], Zhang [28], and other researchers have obtained several further results concerning block-transitive point-primitive 2-designs. Recently, Zhan and Pang [27] showed that if a block-transitive point-primitive automorphism group G acts on a nontrivial 3-(v, k, λ) design with $k \leq 6$, then G must be either of affine type or almost simple type. Building on this result, Pang [20] further completed the classification of non-trivial 3-$(v, 4, \lambda)$ designs admitting a block-transitive point-primitive automorphism group $G \leq \mathrm{Aut}(\mathcal{D})$ whose socle is sporadic or alternating. Furthermore, Liu and his collaborators have investigated block-transitive point-primitive 3-(v, k, λ) designs, with a focus on the cases $\lambda = 1, 2$. The results for $\lambda = 1$ can be found in [10, 15, 16], and those for $\lambda = 2$ are given in [11]. In this paper, we extend these classifications to 3-$(v, k, 3)$ designs. To overcome the limitations of standard counting arguments for $\lambda = 3$, we derive a new fixed-point inequality. Applying this tool, we obtain the following result.

Theorem 1.1. *Let G be an automorphism group of a nontrivial $3-(v, k, 3)$ design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$. If G is primitive on $\mathcal{P}$ and transitive on $\mathcal{B}$, then the following statements hold:*

(i) G is of either affine, almost simple, or product action type with $\mathrm{Soc}(G) = K \times K$, where K is a non-abelian group;

(ii) Assume that G is of product action type on $\mathcal{P} = \Gamma^2$ with $G \leq H \wr S_2$, where H has a primitive action of almost simple or simple diagonal type on Γ of size $\omega \geq 5$. Then $k(k-1)(k-2) = y(\omega^2-2)(\omega+1)$, where $y \in \{2, 3, 4, 5\}$ and H has rank s with $2 \leq s \leq 11$.

2. Preliminaries

In this section, notations and definitions from design theory and group theory are standard and can be found in [2,9,13]. Put $b = |\mathcal{B}|$ and let r be the number of blocks containing a point. We now present some useful results about a t-(v, k, λ) design.

Lemma 2.1. *([2, Theorem 3.2.2]) Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a t-(v, k, λ) design. Then, for any integer s satisfying $0 \leq s \leq t$, $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is also an s-(v, k, λ_s) design, where*

$$\lambda_s = \lambda \frac{(v-s)(v-s-1)\cdots(v-t+1)}{(k-s)(k-s-1)\cdots(k-t+1)}.$$

In particular, $\lambda_0 = b$ and $\lambda_1 = r$.

Lemma 2.2. *([14, Lemma 3]) Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a t-(v, k, λ) design with $t \geq 2$, then the following hold:*

(i) $vr = bk$;
(ii) $\lambda\binom{v}{t} = b\binom{k}{t}$;

(iii) $r(k-1) = \lambda_2(v-1)$, $\lambda_2 = \lambda\frac{(v-2)(v-3)\cdots(v-t+1)}{(k-2)(k-3)\cdots(k-t+1)}$;
(iv) $r(k-1)(k-2) = \lambda(v-1)(v-2)$ *if* $t = 3$.

Lemma 2.3. *([12, Lemma 2.2]) Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a t-(v, k, λ) design. Let $G \leq \mathrm{Aut}(\mathcal{D})$ be block-transitive, $\alpha \in \mathcal{P}$, and let d be a nontrivial subdegree of G. Then the following statements hold:*

(i) *r divides $k \cdot |G_\alpha|$;*
(ii) *r divides $k\lambda_s\binom{d}{s-1}$, and then $(v-1)(v-2)\cdots(v-s+1)$ divides $k(k-1)\cdots(k-s+1)\binom{d}{s-1}$ for every s with $2 \leq s \leq t$.*

In particular, if $\mathcal{D}$ is a 3-$(v, k, 3)$ design, we derive the following two divisibility relations:

(iii) $3(v-1)(v-2) \mid k(k-1)(k-2)|G_\alpha|$;
(iv) $(v-1)(v-2) \mid k(k-1)(k-2)d(d-1)$.

For a group L acting on the set $\mathcal{P}$, we define $\mathrm{Fix}_{\mathcal{P}}(L) = \{\alpha \in \mathcal{P} : \alpha^h = \alpha, \forall h \in L\}$ to be the fixed-point set of L on $\mathcal{P}$.

Lemma 2.4. *Let $\mathcal{D}$ be a nontrivial t-(v, k, λ) design with $t \geq 2$, and let $1 \neq L \leq \mathrm{Aut}(\mathcal{D})$. If L has a point orbit of length greater than λ, then*

$$|\mathrm{Fix}_{\mathcal{P}}(L)| \leq \frac{(t-1)(v-1)}{k-1} + \frac{\lambda(\lambda+1)(k-t-1)}{2}.$$

Proof. Suppose that L has a point orbit $\mathcal{O}$ of length greater than λ. Choose α in $\mathcal{O}$, and let $\mathcal{P}(\alpha)$ be the blocks through α. Then $|\alpha^L| > \lambda$ and

$$\lambda_2|\mathrm{Fix}_{\mathcal{P}}(L)| \leq \sum_{B\in\mathcal{P}(\alpha)} |B \cap \mathrm{Fix}_{\mathcal{P}}(L)| = \sum_{B\in\mathcal{P}(\alpha)} |\mathrm{Fix}_B(L)|,$$

for each fixed point is contained in exactly λ_2 blocks of $P(\alpha)$.

Let $\mathcal{Y} = \{B : B \in \mathcal{P}(\alpha), |B^L| \leq \lambda\}$. Since $|\alpha^L| > \lambda$, there exist $g_0 = 1, g_1, \cdots, g_\lambda \in L$ such that $\alpha^{g_i} \neq \alpha^{g_j}$ for $i \neq j$. Thus, $\alpha^{g_i g_j^{-1}} \neq \alpha$ for $0 \leq i < j \leq \lambda$. Let

$$\mathcal{X} = \{(\alpha, \alpha^{g_i g_j^{-1}}) : 0 \leq i < j \leq \lambda\},$$

then $|\mathcal{X}| \leq \lambda(\lambda+1)/2$. For each $B \in \mathcal{Y}$, there exist i, j satisfied $i < j$ such that $B^{g_i} = B_i^{g_j}$ for $|B^L| \leq \lambda$. Then $B^{g_i g_j^{-1}} = B$, which implies that $\{\alpha, \alpha^{g_i g_j^{-1}}\} \subseteq B$. Note that there are at most λ_2 blocks containing $\{\alpha, \alpha^{g_i g_j^{-1}}\}$ in $\mathcal{Y}$. Thus,

$$|\mathcal{Y}| \leq \lambda_2|\mathcal{X}| \leq \lambda_2\lambda(\lambda+1)/2.$$

Moreover, if $|B^L| > \lambda$, then $|\mathrm{Fix}_B(L)| \leq t-1$. In fact, if $|\mathrm{Fix}_B(L)| \geq t$, since each t-subset is contained in exactly λ block, we have $|B^L| \leq \lambda$, a contradiction. On the other hand, if $|B^L| \leq \lambda$, then $|\mathrm{Fix}_B(L)| \leq k-2$. This follows from the facts that $\alpha \in B$ is not fixed point, and $|\mathrm{Fix}_B(L)| = k-1$ implies that $|B^L| = |\alpha^L| > \lambda$.

Therefore, we can conclude

$$|\mathrm{Fix}_{\mathcal{P}}(L)| \leq \frac{1}{\lambda_2}\left(\sum_{B\in\mathcal{P}(\alpha)\setminus\mathcal{Y}} |\mathrm{Fix}_B(L)| + \sum_{B\in\mathcal{Y}} |\mathrm{Fix}_B(L)|\right)$$

$$\leq \frac{|\mathcal{Y}|(k-2)+(t-1)(r-|\mathcal{Y}|)}{\lambda_2}$$
$$\leq \frac{(t-1)(v-1)}{k-1}+\frac{\lambda(\lambda+1)(k-t-1)}{2}.$$

□

According to Lemma 2.4, we derive the following result immediately.

Corollary 2.5. *If* $1 \neq L \leq \mathrm{Aut}(\mathcal{D})$ *has an orbit of length greater than* 3 *on a nontrivial* 3-$(v,k,3)$ *design, then*

$$|\mathrm{Fix}_{\mathcal{P}}(L)| \leq \frac{2(v-1)}{k-1}+6(k-4).$$

Lemma 2.6. *Let* $\mathcal{D}=(\mathcal{P},\mathcal{B})$ *be a* 3-(v,k,λ) *design. Then the inequality*

$$\lambda(v-2) \geq (k-1)(k-2)$$

holds, and, in particular, we have $k < \sqrt{\lambda v}+3/2$.

Proof. Let $\alpha \in \mathcal{P}$, and let $\mathcal{P}(\alpha)$ denote the set of all blocks containing α. By removing α from each block in $\mathcal{P}(\alpha)$, we obtain a 2-$(v-1,k-1,\lambda)$ design with the point set $\mathcal{P}\setminus\{\alpha\}$ and block set $\{B\setminus\{\alpha\} : B \in \mathcal{P}(\alpha)\}$, which is known as the derived design with respect to α. Applying the Fisher's inequality, the number of blocks in this derived design satisfies $\lambda(v-2) \geq (k-1)(k-2)$. Thus, $k < \sqrt{\lambda v}+3/2$. The proof is therefore concluded. □

Finally, we present a result on finite simple groups to conclude this section.

Lemma 2.7. *([25, Lemma 2.3]) Let* T *be a non-abelian simple group. If* $|T| < 400|\mathrm{Out}(T)|^2$, *then*

$$T \cong PSL_2(q),\ q = 5,7,8,9,11,13,16,27 \text{ or } T \cong PSL_3(4).$$

3. Proof of main theorem

Let $\mathcal{D}=(\mathcal{P},\mathcal{B})$ be a nontrivial 3-$(v,k,3)$ design and G be a block-transitive point-primitive automorphism group. Then the product of all minimal normal subgroups of G, $\mathrm{Soc}(G)$ is isomorphic to the direct product of $T_1,T_2,\ldots,T_m$, where T_i $(i=1,2,\ldots,m)$ is isomorphic to T, a simple group. The O'Nan-Scott Theorem shows that $\mathrm{Soc}(G)$ necessarily falls into the following cases (see [19] for more details):

I. Affine groups: $\mathrm{Soc}(G)=Z_p^m$ for some prime p.

II. Almost simple groups: $\mathrm{Soc}(G)=T\leq G\leq \mathrm{Aut}(T)$ and T a non-abelian simple group.

III. In this case, $\mathrm{Soc}(G)=T^m$ with $m\geq 2$ and T is a non-abelian simple group. We distinguish three subtypes:

(a) Simple diagonal action;
(b) Product action;
(c) Twisted wreath action.

In this section, we will exclude the simple diagonal and twisted wreath subcases of Case III, and reduce the product action subcase to $n = 2$. It follows from [27, Theorem 1] that G is either almost simple or affine for $k \leq 6$. Consequently, we assume $k \geq 7$ for the remainder of this section.

Lemma 3.1. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a nontrivial 3-$(v, k, 3)$ design. If $G \leq \mathrm{Aut}(\mathcal{D})$ acts block-transitively and point-primitively on $\mathcal{D}$, then G is not of simple diagonal action type.*

Proof. Let G be a simple diagonal group, then there is a non-abelian simple group T such that $T^m \leq G \leq T^m : (\mathrm{Out}(T) \times S_m)$, where $m \geq 2$ and $v = |T|^{m-1}$. Let $\alpha \in \mathcal{P}$. From [10, 11], there is a nontrivial orbit Δ of G_α on $\mathcal{P} \setminus \{\alpha\}$ such that $|\Omega| \leq m|T|$. Write $d = |\Omega|$ and $X = m!|\mathrm{Out}(T)|$. Since $v \geq 60$ and Lemma 2.6, $k < 2\sqrt{v}$. Put $Q = k(k-1)(k-2)$, and we have $Q < 8v^{3/2}$. By Lemma 2.3(iv),

$$v^2 - 3v < (v-1)(v-2) \leq Qd(d-1) < 8m^2|T|^{(3m+1)/2}.$$

It follows that

$$60^{(m-5)/2} \leq |T|^{(m-5)/2} < 8m^2 + 1.$$

This forces $m \leq 8$. According to Lemma 2.3(iii), we get that $3(v-1)(v-2) \mid Q|G_\alpha|$, which in turn divides

$$Q|S_m| \cdot |\mathrm{Aut}(T)| = Q|S_m| \cdot |T| \cdot |\mathrm{Out}(T)| = QX|T|.$$

As $|T|$ is even and $v = |T|^{m-1}$, $\gcd((v-1)(v-2), |T|) = 2$. Therefore,

$$3(v-1)(v-2) | 2QX.$$

It follows that $3(v-1)(v-2) \leq 2QX < 16v^{3/2}X$. Since $(v-1)(v-2) > v^2 - 3v$ and $v \geq 60$, we obtain

$$\sqrt{v} < \frac{16X}{3} + \frac{3}{\sqrt{v}} < \frac{16X}{3} + 1,$$

and hence $v < 35X^2$. For $2 \leq m \leq 8$, the elementary constants satisfy

$$\left(35(m!)^2\right)^{1/(m-1)} \leq 140.$$

Consequently, $|T| < 140|\mathrm{Out}(T)|^{2/(m-1)} \leq 140\,\mathrm{Out}(T)^2 < 400\,\mathrm{Out}(T)^2$. By Lemma 2.7, we get

$$T \cong \mathrm{PSL}_2(q),\ q = 5, 7, 8, 9, 11, 13, 16, 27 \text{ or } T \cong \mathrm{PSL}_3(4).$$

Using GAP [24] commands, we compute the orders of these groups and their outer automorphism group orders; see Table 1. Apply the inequality $3(v-1)(v-2) \leq 2QX$ again, together with Table 1 and the bound $7 \leq k < \sqrt{3|T|^{m-1}} + 3/2$. Using a direct computation in MATLAB R2024b, this leaves only the following three cases for $(T, |T|, m, |\mathrm{Out}(T)|)$ and the possible values of k:

(1) $(T, |T|, m, |\mathrm{Out}(T)|) = (\mathrm{PSL}_2(5), 60, 2, 2)$ with $k \in \{12, 13, 14\}$,

(2) $(T, |T|, m, |\mathrm{Out}(T)|) = (\mathrm{PSL}_2(7), 168, 2, 2)$ with $k = 23$,

(3) $(T, |T|, m, |\mathrm{Out}(T)|) = (\mathrm{PSL}_2(9), 360, 2, 4)$ with $k \in \{30, 31, 32, 33, 34\}$.

For each of these three cases, the obtained values fail to satisfy

$$(k-1)(k-2) \mid 3(v-1)(v-2),$$

a contradiction. □

Table 1. The values of $|T|$ and $|\mathrm{Out}(T)|$ for the group T in in Lemma 2.7.

T	$\lvert T\rvert$	$\lvert\mathrm{Out}(T)\rvert$	T	$\lvert T\rvert$	$\lvert\mathrm{Out}(T)\rvert$	T	$\lvert T\rvert$	$\lvert\mathrm{Out}(T)\rvert$
$\mathrm{PSL}_2(5)$	60	2	$\mathrm{PSL}_2(7)$	168	2	$\mathrm{PSL}_2(8)$	504	3
$\mathrm{PSL}_2(9)$	360	4	$\mathrm{PSL}_2(11)$	660	2	$\mathrm{PSL}_2(13)$	1092	2
$\mathrm{PSL}_2(16)$	4080	4	$\mathrm{PSL}_2(27)$	9828	6	$\mathrm{PSL}_3(4)$	20160	12

Lemma 3.2. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a nontrivial 3-$(v, k, 3)$ design. If $G \leq \mathrm{Aut}(\mathcal{D})$ acts block-transitively and point-primitively on $\mathcal{D}$, then G is not of twisted wreath action type.*

Proof. Let G be a twisted wreath type automorphism group of $\mathcal{D}$. Let $N = T_1 \times T_2 \times \cdots \times T_m$, where each $T_i \cong T$ ($i \in \{1, 2, \ldots, m\}$) is a non-abelian simple group and $m \geq 6$ [10, 11]. Let $\alpha \in \mathcal{P}$, then $G = NG_\alpha$, the socle N is regular on $\mathcal{P}$ and $v = |T|^m$. Then there is a non-trivial orbit Ω of G_α on $\mathcal{P} \setminus \{\alpha\}$ such that $|\Omega| \leq m|T|$ by [10, 11]. As before, $k < 2\sqrt{v}$ and $Q < 8v^{3/2}$. According to Lemmas 2.3(iv), we have

$$60^{(m-4)/2} \leq |T|^{(m-4)/2} < 8m^2 + 1.$$

This implies that $m = 6$ or 7. On the other hand, Lemma 2.3(iii) gives

$$3(v-1)(v-2) \leq Qm! < 8v^{3/2}m!,$$

so $\sqrt{v} < 8m!/3 + 1$, and hence $v < 11(m!)^2$. Both $60^6 > 11(6!)^2$ and $60^7 > 11(7!)^2$, a contradiction. □

In the following Lemmas 3.3–3.5, suppose that G has a product action on $\mathcal{P}$. Then, there is a group H with a primitive action of almost simple or simple diagonal type on a set Γ of size $\omega \geq 5$, where $\mathcal{P} = \Gamma^n$, $G \leq H \wr S_n$, $n \geq 2$. We adopt the standard homogeneous wreath-product embedding for this product action, where H denotes the induced coordinate group. Let $K = \mathrm{Soc}(H) \cong T^\ell$ for a non-abelian simple group T, and write $\mathrm{Soc}(G) = K_1 \times \cdots \times K_n$ with $K_i \cong K$ for $i = 1, \ldots, n$.

Lemma 3.3. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a nontrivial 3-$(v, k, 3)$ design and $G \leq \mathrm{Aut}(\mathcal{D})$ acts block-transitively and point-primitively on $\mathcal{D}$. Suppose that G is of primitive product type action on $\mathcal{P} = \Gamma^n$, and $G \leq H \wr S_n, n \geq 2$, where H with a primitive action of almost simple or simple diagonal type on a set Γ of size $\omega \geq 5$. It follows that*

(i) if $n \geq 5$ and $\omega \geq 6$, then $k < 2\omega + 5/2$;
(ii) if $n = 4$ and $\omega \geq 12$, then $k < 9\omega/4 + 5/2$;
(iii) if $n = 3$ and $\omega \geq 134$, then $k < 5\omega/2 + 5/2$.

Proof. Let $\alpha = (\gamma, \ldots, \gamma) \in \mathcal{P}$ and $L = (K_1)_\alpha$. Since $K_1 \leq \mathrm{Soc}(G)$, we have $L \leq G$. It is clear that $|\mathrm{Fix}_{\mathcal{P}}(L)| \geq \omega^{n-1}$. The action of L on the set

$$\{(\xi, \gamma, \ldots, \gamma) : \xi \in \Gamma\}$$

is equivalent to that of K_γ on Γ.

Suppose that H is almost simple. Then K is a non-abelian simple group, transitive, and non-regular on Γ. Let $M = K_\gamma \neq 1$, and suppose, to the contrary, that all M-orbits have length at most 3. Every non-singleton orbit has prime length 2 or 3, and hence its induced permutation group is primitive. By [21, Corollary to Theorem 1], either there is a normal subgroup N of K containing M with $[N : M]$

equal to a subdegree, or M embeds in S_d for a subdegree $d > 1$. The first case is impossible, since simplicity yields $N = K$, contradicting $[K : M] = \omega \geq 5$. Therefore,

$$M \cong C_2,\ C_3 \text{ or } S_3.$$

Since $M = K \cap H_\gamma \trianglelefteq H_\gamma$, we have $H_\gamma \leq N_H(M) < H$. By the primitivity of H, H_γ is maximal in H, so $N_H(M) = H_\gamma$ and $N_K(M) = M$. From the fixed-point formula for transitive coset actions, we have

$$|\operatorname{Fix}_\Gamma(M)| = [N_K(M) : M] = 1.$$

If $M \cong C_2$ or C_3, then every nonidentity element of the point stabilizer fixes precisely one point. This forces K to be a Frobenius group, which possesses a nontrivial proper normal Frobenius kernel, contrary to the simplicity of K. Hence, $M \cong S_3$.

In addition, K is primitive on Γ. For any $a, b, c \in \Gamma$, the two-point stabilizers $K_{a,b}$ and $K_{a,c}$ have index at most 3 in $K_a \cong S_3$, and hence are nontrivial. If they are distinct, they generate K_a, since each proper subgroup of index at most 3 in S_3 is maximal. Thus, conditions (B2) and (C) of [21] are satisfied.The normal closure of M in K is K. According to [21, Theorem 3.2(i)], $\operatorname{Fix}_\Gamma(M)$ is the unique maximal proper block containing γ. Since $\operatorname{Fix}_\Gamma(M) = \{\gamma\}$, it follows that K is primitive.

Moreover, M has an orbit of length 3. Otherwise, its unique subgroup of order 3 would fix every orbit of length 1 or 2 pointwise, contradicting faithfulness. Thus, K is primitive with a suborbit of length 3. According to [26, Theorem], and also [1, Lemma 18], we may apply Table 2. Since K is non-abelian simple and $K_\gamma \cong S_3$, the only possibility is

$$(K, K_\gamma, |\Gamma|) = (A_5, S_3, 10).$$

From Table 2, this action has subdegrees $1, 3, 6$, which contradicts our assumption. Therefore, K_γ has an orbit of length greater than 3 in the almost simple case.

Suppose that H is of simple diagonal type. Write $K = T^\ell$, where $\ell \geq 2$ and T is non-abelian simple. We may identify Γ with the right cosets of $O = \{(t, \ldots, t) : t \in T\}$ in K, and take $\gamma = O$. For $1 \neq t \in T$, the O-orbit of $O(t, 1, \ldots, 1)$ has length $[T : C_T(t)] \geq 5$. Otherwise, T would embed in the soluble group S_4. Hence, K_γ has an orbit of length greater than 3 also in this case.

Table 2. Primitive groups K with non-abelian socle T and a suborbit of length 3.

K	K_γ	Degree	T	T_α	Subdegrees of T	Conditions
A_5	S_3	10	A_5	S_3	$1, 3, 6$	–
S_5	D_{12}	10	A_5	S_3	$1, 3, 6$	–
$\mathrm{PGL}_2(7)$	D_{12}	28	$\mathrm{PSL}_2(7)$	S_3	$1, 3^3, 6^3$	–
$\mathrm{PSL}_2(11)$	D_{12}	55	$\mathrm{PSL}_2(11)$	D_{12}	$1, 3^2, 6^4, 12^2$	–
$\mathrm{PSL}_2(13)$	D_{12}	91	$\mathrm{PSL}_2(13)$	D_{12}	$1, 3^2, 6^4, 12^5$	–
$\mathrm{PSL}_3(3)$	S_4	234	$\mathrm{PSL}_3(3)$	S_4	$1, 3, 4^2, 6, 12^8, 24^5$	–
$\mathrm{Aut}(\mathrm{PSL}_3(3))$	$S_4 \times Z_2$	234	$\mathrm{PSL}_3(3)$	S_4	$1, 3, 4^2, 6, 12^8, 24^5$	–
$\mathrm{PSL}_2(q)$	S_4	$\frac{q(q^2-1)}{48}$	$\mathrm{PSL}_2(q)$	S_4	–	$q \equiv \pm 1 \pmod{16}$

Combining the above argument with Corollary 2.5, applied to $L = (K_1)_\alpha \leq G$, we get

$$\omega^{n-1} \leq |\operatorname{Fix}_{\mathcal{P}}(L)| \leq \frac{2(\omega^n - 1)}{k - 1} + 6(k - 4).$$

Then we have following inequality

$$F_n(k) := 6k^2 - (\omega^{n-1} + 30)k + 2\omega^n + \omega^{n-1} + 22 \geq 0.$$

For the quadratic inequality in the variable k above, the discriminant of the corresponding equation is

$$\Delta = \omega^{2n-2} - 48\omega^n + 36\omega^{n-1} + 372.$$

If $n \geq 5$ and $\omega \geq 6$, then

$$\Delta = (\omega^{n-1} - 24\omega)^2 + 36\omega^2(\omega^{n-3} - 16) + 372 > (\omega^{n-1} - 24\omega)^2.$$

Thus, either

$$k > \omega^{n-1}/6 - 2\omega + 5/2 \quad \text{or} \quad k < 2\omega + 5/2.$$

Put $z = \omega^{n/2-1} \geq 6\sqrt{6} > 6+4\sqrt{3}$. Then $z^2/6-2 > 2z$. Combined with $k < 2\omega^{n/2}$, we have $k < 2\omega+5/2$.

If $n = 4$ and $\omega \geq 12$, we have

$$\Delta = (\omega^3 - 27\omega)^2 + 3\omega^2(2\omega^2 + 12\omega - 243) + 372 > (\omega^3 - 27\omega)^2.$$

It follows that either

$$k > \omega^3/6 - 9\omega/4 + 5/2 \quad \text{or} \quad k < 9\omega/4 + 5/2.$$

Since $\omega^3/6 - 9\omega/4 + 5/2 > \sqrt{3}\omega^2 + 3/2$, $k < 9\omega/4 + 5/2$ by Lemma 2.6.

If $n = 3$ and $\omega \geq 134$, then

$$\Delta = (\omega^2 - 30\omega)^2 + 12\omega^3 - 864\omega^2 + 372 > (\omega^2 - 30\omega)^2.$$

Then,

$$k > \omega^2/6 - 5\omega/2 + 5/2 \quad \text{or} \quad k < 5\omega/2 + 5/2.$$

Note that

$$F_3\left(\sqrt{3\omega^3} + 3/2\right) = 20\omega^3 - \frac{\omega^2 + 19}{2} - (\omega^2 + 12)\sqrt{3\omega^3} < 0.$$

Thus, $\omega^2/6 - 5\omega/2 + 5/2 > \sqrt{3\omega^3} + 3/2$. By Lemma 2.6 again, we get $k < \frac{5}{2}\omega + \frac{5}{2}$. □

Lemma 3.4. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a nontrivial 3-$(v, k, 3)$ design. If $G \leq \mathrm{Aut}(\mathcal{D})$ acts block-transitively and point-primitively on $\mathcal{D}$ and G is of primitive product action on $\mathcal{P} = \Gamma^n$ with $n \geq 2$, then $n = 2$ and $\mathrm{Soc}(G) = K \times K$, where $K = \mathrm{Soc}(H)$ is a non-abelian group.*

Proof. Let G be a product type group as described in Lemma 3.3 and $K = \mathrm{Soc}(H)$ non-abelian. Then $v = \omega^n$, $\alpha = (x_1, x_2, \ldots, x_n) \in \mathcal{P}$, and $h = (h_1, \ldots, h_n)\tau \in G$ where $\tau \in S_n$, and for all $i \in \{1, 2, \ldots, n\}$, $h_i \in H_i$, a primitive permutation group on a set Γ. Since $\mathcal{D}$ is a 3-$(v, k, 3)$ design, any 3-subset

$$\{(x_1, x_2, \ldots, x_{n-1}, y_1), (x_1, x_2, \ldots, x_{n-1}, y_2), (x_1, x_2, \ldots, x_{n-1}, y_3)\}$$

is contained in exactly three distinct blocks, where y_1, y_2, y_3 are distinct. This implies that any $B \in \mathcal{B}$ must contain at least three points with only one coordinate component different, for G is block-transitive. Moreover, each block B contains the same number of such triples denoted by ℓ_3. According

to the definition of a 3-$(v, k, 3)$ design, we perform a double count of the total number for such triples in the point set $\mathcal{P}$. Then,

$$\frac{\ell_3 b}{3} = \ell_3 \frac{\omega^n(\omega^n - 1)(\omega^n - 2)}{k(k-1)(k-2)} = \frac{\omega^n n(\omega - 1)(\omega - 2)}{3!},$$

where $\ell_3 \geq 1$. By Lemma 2.1, $\mathcal{D}$ also is a 2-(v, k, λ_2) design. Similarly, we define ℓ_2 as the number of pairs of points in a block that differ in exactly one coordinate position. Hence, we have

$$\frac{\ell_2 b}{\lambda_2} = \ell_2 \frac{\omega^n(\omega^n - 1)}{k(k-1)} = \frac{\omega^n n(\omega - 1)}{2},$$

where $\ell_2 \geq 3$. Thus,

$$2\ell_2 \left(\omega^{n-1} + \omega^{n-2} + \cdots + 1\right) = nk(k-1), \tag{3.1}$$

and

$$6\ell_3 \left(\omega^{n-1} + \omega^{n-2} + \cdots + 1\right) \frac{\omega^n - 2}{\omega - 2} = nk(k-1)(k-2). \tag{3.2}$$

If $n \geq 5$ and $\omega \geq 6$, by Eq (3.1) and Lemma 3.3(i), we get

$$2\ell_2 \left(\omega^{n-1} + \omega^{n-2} + \cdots + 1\right) = nk(k-1) < n\left(2\omega + \frac{5}{2}\right)\left(2\omega + \frac{3}{2}\right) < 5n(\omega + 1)^2.$$

Thus,

$$\omega^{n-4}(\omega - 1) < \frac{\omega^{n-2}}{\omega + 1} < \frac{\omega^{n-1} + \omega^{n-2} + \cdots + 1}{(\omega + 1)^2} < \frac{5n}{2\ell_2}.$$

It follows that

$$\frac{6^{n-4}}{n} \leq \frac{\omega^{n-4}}{n} < \frac{1}{2\ell_2} \leq \frac{1}{6},$$

which is a contradiction with $n \geq 5$.

If $n = 4$ and $\omega \geq 12$, by Eq (3.2) and Lemma 3.3(ii), we get

$$6\ell_3(\omega^3 + \omega^2 + \omega + 1)\frac{\omega^4 - 2}{\omega - 2} = 4k(k-1)(k-2) \leq \frac{1}{16}(9\omega + 10)(9\omega + 6)(9\omega + 2).$$

Thus,

$$\ell_3 \leq \frac{(9\omega + 10)(9\omega + 6)(9\omega + 2)(\omega - 2)}{96(\omega^3 + \omega^2 + \omega + 1)(\omega^4 - 2)} < \frac{243}{32\omega^3} < 1,$$

a contradiction.

If $n = 3$ and $\omega \geq 134$, again by Eq (3.2) and Lemma 3.3(iii), we get

$$6\ell_3(\omega^2 + \omega + 1)\frac{\omega^3 - 2}{\omega - 2} = 3k(k-1)(k-2) \leq \frac{3}{8}(5\omega + 5)(5\omega + 3)(5\omega + 1).$$

Thus,

$$\ell_3 \leq \frac{(5\omega + 5)(5\omega + 3)(5\omega + 1)(\omega - 2)}{16(\omega^2 + \omega + 1)(\omega^3 - 2)} < \frac{125}{16\omega} < 1,$$

which is impossible.

For $n \geq 5$ with $\omega = 5$, from Eq (3.2), $\ell_3 \geq 1$, and Lemma 2.6,

$$k \geq \frac{2(\omega^n - 1)}{n(\omega - 1)(\omega - 2)} > \frac{2\omega^{n-2}}{n}, \quad k < 2\omega^{n/2}.$$

Thus $5^{n/2-2} < n$, which forces $n = 5, 6$. However, MATLAB R2024b numerical checks reveal that neither case admits an integer k satisfying both Eqs (3.1) and (3.2) with $\ell_2 \geq 3$ and $\ell_3 \geq 1$. Similarly, the cases $n = 4$ ($5 \leq \omega \leq 11$) and $n = 3$ ($5 \leq \omega \leq 133$) can also be excluded. Thus, $n = 2$, and $\mathrm{Soc}(G) = K_1 \times K_2 \cong K \times K$. □

We will conduct a more detailed discussion of the case $n = 2$ in the following Lemma 3.5.

Lemma 3.5. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a nontrivial 3-$(v, k, 3)$ design. Suppose that $G \leq \mathrm{Aut}(\mathcal{D})$ acts block-transitively and point-primitively on $\mathcal{D}$, and G has a primitive product action on $\mathcal{P} = \Gamma^2$ with $G \leq H \wr S_2$, where H has a primitive action of almost simple or simple diagonal type on Γ of size $\omega \geq 5$. Then, the following hold:*

(i) $k(k-1)(k-2) = y(\omega^2 - 2)(\omega + 1)$ *for* $y \in \{2, 3, 4, 5\}$;
(ii) the rank s of H satisfies $2 \leq s \leq 11$.

Proof. Let G be a primitive group with product action on $\mathcal{P} = \Gamma^n$ where $n = 2$, as described in Lemmas 3.3 and 3.4. According to Eqs (3.1) and (3.2), we have

$$\ell_2(\omega + 1) = k(k - 1), \tag{3.3}$$

and

$$3\ell_3(\omega + 1)(\omega^2 - 2) = k(k - 1)(k - 2)(\omega - 2). \tag{3.4}$$

Put $Q = k(k - 1)(k - 2)$. Since $\gcd(\omega^2 - 2, \omega - 2) \mid 2$, every odd prime-power factor of $\omega^2 - 2$ is coprime to $\omega - 2$. Combined with Eq (3.4), we see that such prime-power factors divide Q. If ω is even, then $v_2(\omega^2 - 2) = 1$ and $2 \mid Q$; if ω is odd, then $\omega^2 - 2$ is odd. Thus $\omega^2 - 2 \mid Q$. On the other hand, $\gcd(\omega + 1, \omega^2 - 2) = 1$. By Eq (3.3), $\omega + 1 \mid k(k - 1)$, so

$$(\omega + 1)(\omega^2 - 2) \mid Q.$$

Consequently $x = \frac{2Q}{(\omega+1)(\omega^2-2)}$ is a positive even integer. By Lemmas 2.2(iv) and 2.6,

$$x \leq \frac{6k}{\omega + 1} < 6\sqrt{3}.$$

Hence, $x \in \{2, 4, 6, 8, 10\}$.

If $x = 2$, then $(\omega^2 - 2)(\omega + 1) = k(k - 1)(k - 2)$. Suppose that $\omega \leq k - 2$. Then

$$(\omega^2 - 2)(\omega + 1) \leq k^3 - 5k^2 + 6k - 2.$$

This implies

$$(\omega^2 - 2)(\omega + 1) - k(k - 1)(k - 2) = -2k^2 + 4k - 2 < 0,$$

which is a contradiction. Assume that $\omega \geq k-1$, and we have

$$(\omega^2-2)(\omega+1) \geq k^3-2k^2-k.$$

This yields

$$(\omega^2-2)(\omega+1)-k(k-1)(k-2) = k^2-3k > 0,$$

a contradiction. Therefore, $x \in \{4,6,8,10\}$, or equivalently,

$$Q = y(\omega+1)(\omega^2-2), \quad y \in \{2,3,4,5\}. \tag{3.5}$$

Suppose that H has rank s on Γ, $s \geq 2$. Then G has a suborbit of length $d \leq \frac{2(\omega-1)}{s-1}$ on $\mathcal{P}$ by [12, Proposition 3.2]. According to Lemma 2.3, we have $\omega^2-1 \mid k(k-1)d$, and thus

$$\omega-1 \mid x(\omega^2-2)d/2.$$

Since $2 \mid x$ and $\gcd(\omega-1, \omega^2-2) = 1$, we obtain $\omega-1 \mid \frac{xd}{2}$. It follows that

$$\omega-1 \leq \frac{xd}{2} \leq \frac{x(\omega-1)}{s-1}.$$

Therefore, $2 \leq s \leq x+1 \leq 11$. □

Proof of Theorem 1.1. The assertion holds via the O'Nan-Scott Theorem together with Lemmas 3.1, 3.2, 3.4, and 3.5. □

4. Conclusions and open questions

This paper investigates the classification of block-transitive point-primitive 3-designs. Using the fixed-point inequality and the O'Nan-Scott Theorem for the classification of primitive permutation groups, we prove that the block-transitive point-primitive automorphism group G of a nontrivial 3-$(v,k,3)$ design is of affine type, almost simple type, or product type. For the product type case, $G \leq H \wr S_2$ acts on the point set $\mathcal{P} = \Gamma \times \Gamma$, where H is a primitive group on Γ. However, the existence of block-transitive point-primitive nontrivial 3-$(v,k,3)$ designs whose parameters satisfy the cubic Diophantine Equation (3.5) remains an open problem. This problem is rather challenging, and we only obtain the necessary parameter relations. A complete determination of all integer solutions requires a completeness argument, and methods for solving cubic Diophantine equations can be found in [23].

Author contributions

Guoqiang Xiong: conceptualization, writing-original draft, software, formal analysis; Yajie Wang: conceptualization, methodology, data curation, formal analysis; Haiyan Guan: conceptualization, supervision, resources, writing, reviewing and editing; Yunfei Zhu: writing-original draft, methodology. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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Appendix

In this appendix, we present the `GAP` and `MATLAB` codes utilized to verify the finite cases in our proofs. The computations were carried out in `GAP 4.16.1` and `MATLAB R2024b`. The codes are provided below.

A.1. GAP code used in Lemma 3.1

Computation of $|T|$ and $|\mathrm{Out}(T)|$

```
PrintOutOrder := function(T, name)
local A, innSize, outSize;
A := AutomorphismGroup(T);
innSize := Size(T) / Size(Center(T));   # |Inn(T)| = |T|/|Z(T)|
outSize := Size(A) / innSize;           # |Out(T)| = |Aut(T)|/|Inn(T)|
Print(name, " -> |T| = ", Size(T), ", |Out(T)| = ", outSize, "\n");
end;
for q in [5, 7, 8, 9, 11, 13, 16, 27] do
PrintOutOrder(PSL(2, q), Concatenation("PSL(2,", String(q), ")"));
od;
PrintOutOrder(PSL(3, 4), "PSL(3,4)");
```

A.2. MATLAB code used in Lemma 3.1

```
% t=|T|, o=|Out(T)|
data = {
'PSL(2,5)',   60,    2;
'PSL(2,7)',   168,   2;
'PSL(2,8)',   504,   3;
'PSL(2,9)',   360,   4;
'PSL(2,11)',  660,   2;
'PSL(2,13)',  1092,  2;
'PSL(2,16)',  4080,  4;
'PSL(2,27)',  9828,  6;
'PSL(3,4)',   20160, 12
};
fprintf('%-12s %6s %3s %3s | %-6s %-22s %s\n', ...
'Group','t','o','m','t^(m-1)','k range','valid k for Cond. 2');
fprintf('%s\n', repmat('=',1,110));
for i = 1:size(data,1)
nm = data{i,1};
t  = data{i,2};
o  = data{i,3};
for m = 2:8
L1 = sym(t)^(m-1);
L2 = 3 * (L1 - 1) * (L1 - 2);
coeff = 2 * sym(factorial(m)) * o;
% Upper bound for k: k < sqrt(3*L1) + 1.5
kUpperReal = sqrt(3 * double(L1)) + 1.5;
kMax = floor(kUpperReal);
if kMax == kUpperReal
```

```
kMax = kMax - 1;
end
if kMax < 7
kRangeStr = 'empty';
goodKStr  = 'none';
else
kRangeStr = sprintf('[7..%d]', kMax);
fMax = coeff * sym(kMax) * (kMax-1) * (kMax-2);
if fMax <= L2
goodKStr = 'none';
else
% Binary search for the smallest k0 such that f(k0) > L2
lo = 7; hi = kMax;
while lo < hi
mid = floor((lo + hi) / 2);
fmid = coeff * sym(mid) * (mid-1) * (mid-2);
if fmid > L2
hi = mid;
else
lo = mid + 1;
end
end
goodKStr = sprintf('[%d..%d]', lo, kMax);
end
end
fprintf('%-12s %6d %3d %3d | %-6.4e %-22s %s\n', ...
nm, t, o, m, double(L1), kRangeStr, goodKStr);
end
fprintf('%s\n', repmat('-',1,110));
end
```

A.3. MATLAB code used in Lemma 3.4

(**1**) $\mathbf{n = 3, 5 \le \omega \le 133}$.

```
% Check for integer solutions satisfying the given conditions
% Parameters: n = 3, 5 <= omega <= 133
% Constraints: 7 <= k < 2 * omega^(n/2), l2 >= 3, l3 >= 1
clear; clc;
n = 3;
omega_values = 5:133;
valid_solutions = [];
for omega = omega_values
k_max = floor(2 * omega^(n/2) - 1e-9);
for k = 7 : k_max
```

```
% From Eq  (1): l2 = [n * k * (k - 1) * (omega - 1)] / [2 * (omegaˆn - 1)]
num_l2 = n * k * (k - 1) * (omega - 1);
den_l2 = 2 * (omegaˆn - 1);
if mod(num_l2, den_l2) == 0
l2 = num_l2 / den_l2;
if l2 >= 3
% Substituting Eq (1) into Eq  (2) simplifies to:
% l3 = [l2 * (k - 2) * (omega - 2)] / [3 * (omegaˆn - 2)]
num_l3 = l2 * (k - 2) * (omega - 2);
den_l3 = 3 * (omegaˆn - 2);
if mod(num_l3, den_l3) == 0
l3 = num_l3 / den_l3;
if l3 >= 1
valid_solutions = [valid_solutions; omega, k, l2, l3];
end
end
end
end
end
end
if isempty(valid_solutions)
fprintf('No integer solutions exist for the given conditions.\n');
else
fprintf('Valid solutions found:\n');
fprintf('omega\tk\tl2\tl3\n');
fprintf('%d\t%d\t%d\t%d\n', valid_solutions');
end
```

(**2**) $\mathbf{n = 4, 5 \leq \omega \leq 11}$.

```
% Check for integer solutions satisfying the given conditions
% Parameters: n = 4, 5 <= omega <= 11
% Constraints: 7 <= k < 2 * omegaˆ(n/2), l2 >= 3, l3 >= 1

clear; clc;
n = 4;
omega_values = 5:11;
valid_solutions = [];
for omega = omega_values
k_max = floor(2 * omegaˆ(n/2) - 1e-9);
for k = 7 : k_max
num_l2 = n * k * (k - 1) * (omega - 1);
den_l2 = 2 * (omegaˆn - 1);
if mod(num_l2, den_l2) == 0
l2 = num_l2 / den_l2;
```

```
if l2 >= 3
num_l3 = l2 * (k - 2) * (omega - 2);
den_l3 = 3 * (omegaˆn - 2);
if mod(num_l3, den_l3) == 0
l3 = num_l3 / den_l3;
if l3 >= 1
valid_solutions = [valid_solutions; omega, k, l2, l3];
end
end
end
end
end
end
if isempty(valid_solutions)
fprintf('No integer solutions exist for the given conditions.\n');
else
fprintf('Valid solutions found:\n');
fprintf('omega\tk\tl2\tl3\n');
fprintf('%d\t%d\t%d\t%d\n', valid_solutions');
end
```

(**3**) $\omega = \mathbf{5}, \mathbf{n} \in \{\mathbf{5}, \mathbf{6}\}$.

```
clear; clc;
omega = 5;
n_values = [5, 6];
valid_solutions = [];
for n = n_values
k_max = floor(2 * omegaˆ(n/2) - 1e-9);
for k = 7 : k_max
num_l2 = 2 * n * k * (k - 1);
den_l2 = omegaˆn - 1;
if mod(num_l2, den_l2) == 0
l2 = num_l2 / den_l2;
if l2 >= 3
num_l3 = l2 * (k - 2);
den_l3 = omegaˆn - 2;
if mod(num_l3, den_l3) == 0
l3 = num_l3 / den_l3;
if l3 >= 1
valid_solutions = [valid_solutions; n, k, l2, l3];
end
end
end
end
end
```

```
end
end
if isempty(valid_solutions)
fprintf('No integer solutions exist for the given conditions.\n');
else
fprintf('Valid solutions found:\n');
fprintf('n\tk\tl2\tl3\n');
fprintf('%d\t%d\t%d\t%d\n', valid_solutions');
end
```



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