AIMS Mathematics, 11(11): 37585–37608.
DOI: 10.3934/math.20261454
Received: 21 July 2026
Revised: 13 October 2026
Accepted: 28 October 2026
Published: 13 November 2026

https://www.aimspress.com/journal/Math

Research article

A new development on Mercer–Hadamard–like inclusions of interval–valued mappings via a fractional version for Æ–convex functions

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Abstract: The theory of interval-valued mappings has become an important framework for modeling uncertainty in engineering, economics, optimization, and applied sciences, where exact numerical values are often unavailable. Motivated by the growing interest in fractional calculus and generalized convexity, this paper establishes several new Mercer–Hadamard-type inclusions for interval-valued mappings whose interval endpoints satisfy the properties of Æ-convexity. By employing a generalized fractional integral approach, we derive novel interval inclusions that extend and refine a variety of existing Mercer–Hadamard inequalities from real-valued settings to interval-valued frameworks. The results obtained provide tighter estimates for fractional interval integrals and reveal the influence of the Æ-convex structure on the behavior of interval-valued functions. Furthermore, several important consequences are discussed by considering suitable choices of the fractional-order parameter, through which many previously known inequalities are recovered as special cases. The proposed findings not only broaden the scope of Mercer–Hadamard theory within interval analysis but also establish a unified fractional framework that can serve as a foundation for future investigations involving generalized convexity, interval optimization, and uncertainty modeling. Finally, an application with a matrix is developed to charm the beauty of the result.

Keywords: convexity; interval valued Æ–convexity; Hermite–Hadamard inequalities; Jensen inequality; Mercer's inequality; fractional integrals; Hölder and power mean inequalities
Mathematics Subject Classification: 26D15, 26E60, 26A33, 60E05, 33F05

1. Introduction

Integral inequalities constitute an important branch of mathematical analysis that studies relationships involving integrals and inequality signs. They provide estimates, bounds, and comparisons for functions when exact evaluations are difficult or impossible to obtain. The theory of integral inequalities has developed into a powerful mathematical tool with applications in pure mathematics, applied mathematics, physics, engineering, economics, statistics, and many other scientific disciplines. The central idea behind integral inequalities is to determine how the integral of a function behaves under certain conditions and to establish upper and lower bounds that help in understanding the qualitative and quantitative properties of mathematical models. The origin of integral inequalities can be traced to the broader development of calculus and mathematical analysis. As mathematicians began investigating the behavior of functions and their integrals, it became evident that exact formulas were not always available. In such situations, inequalities provided an effective alternative by offering useful approximations and estimates. Rather than finding the precise value of an integral, one could determine a range within which the value must lie. This approach proved particularly valuable in solving differential equations, studying convergence, and analyzing mathematical structures where exact solutions are difficult to obtain. Among the most productive results in the theory of integral inequalities are Hölder's inequality and Minkowski's inequality. The first of these inequalities establishes an upper bound for the integral of the product of two functions. This important property is the basis for the study of many questions in the theory of functions. Hölder's inequality is one of the most frequently used tools in real analysis, functional analysis, and the study of partial differential equations. Minkowski's inequality is the extension of the triangle inequality for real numbers to integrals. It is also used in the context of the Lebesgue integral and the theory of normed linear spaces, as a fundamental inequality. Using these examples of integral inequalities, it is possible to formulate and prove a wide range of other properties of integrals.

In addition to their analytical usefulness, integral inequalities have many applications in numerical analysis. For example, in the context of numerical integration, such inequalities are used to bound the error of approximation of an integral by some numerical method. If an integral cannot be taken exactly, it is often necessary to use approximate values. This can be done, for example, by using the trapezoidal rule or Simpson's rule. In such cases, integral inequalities help to estimate how large the error can be, and thus the accuracy of the solution. Therefore, integral inequalities are widely used in numerical analysis, which in turn is the basis of scientific computing. In modern mathematical research, the study of integral inequalities continues to expand through the introduction of new concepts such as generalized convexity, fractional calculus, and dynamic equations on time scales. Researchers develop novel inequalities to address emerging problems in science and technology. Fractional integral inequalities, for example, have attracted considerable attention due to their applications in modeling memory-dependent processes and anomalous diffusion phenomena. These developments demonstrate the adaptability and continuing relevance of integral inequalities in contemporary mathematics.

Integral inequalities have a key role to play in the formulation of different mathematical theories as they provide needed estimates for confirming existence, uniqueness, and stability results (see [1–3]).

Interval analysis refers to a mathematical theory that handles uncertainties that arise when quantitative descriptions of the characteristics of an object lack precision to some extent. In other words, instead of using a precise value, such data is represented by an interval that contains all

possible values that can represent a particular quantity. In general, it can be said that interval analysis allows a mathematician to work with intervals rather than points. This is especially important in the computational mathematics industry, which covers many branches of science, from physics to optimization or approximate numerical methods. Therefore, using interval analysis provides researchers with calculation results that are guaranteed to contain true solutions and provide accurate information about the accuracy of calculations and the nature of any inaccuracies. Interval analysis also plays an important role in solving equations and optimization problems. When searching for the roots of a function, interval techniques can identify regions that definitely contain a solution and eliminate regions that cannot contain one. Similarly, in optimization, interval methods can be used to determine reliable bounds for the minimum or maximum value of a function. These features make interval analysis a valuable tool in situations where reliability and verification are more important than obtaining a single approximate estimate (see [4–6]).

2. Calculus with interval analysis

This section introduces the basic notation and fundamental concepts of interval analysis that will be used throughout this work. Let $\mathcal{I}_c$ denote the collection of all closed intervals on the set of real numbers $\mathbb{R}$. An arbitrary interval in $\mathcal{I}_c$ is represented by

$$\mathcal{J} = [\underline{\rho}, \overline{\rho}] = \left\{\nu \in \mathbb{R} : \underline{\rho} \leq \nu \leq \overline{\rho}\right\}, \tag{2.1}$$

where $\underline{\rho}, \overline{\rho} \in \mathbb{R}$ satisfy $\underline{\rho} \leq \overline{\rho}$.

The length (or width) of the interval $\mathcal{J}$ is defined by

$$L(\mathcal{J}) = \overline{\rho} - \underline{\rho}.$$

The values $\underline{\rho}$ and $\overline{\rho}$ are referred to as the left and right endpoints of the interval, respectively.

An interval is called *degenerate* whenever its endpoints coincide, that is, $\underline{\rho} = \overline{\rho}$. In this case, the interval reduces to the single real number

$$\mathcal{J} = [\rho, \rho] = \rho.$$

Furthermore, an interval is said to be *positive* if $\underline{\rho} > 0$, while it is called *negative* whenever $\overline{\rho} < 0$.

Let $\mathcal{I}_c^+$ and $\mathcal{I}_c^-$ denote the families of all positive and negative closed intervals in $\mathbb{R}$, respectively. For two intervals

$$\mathcal{J} = [\underline{\rho}, \overline{\rho}] \quad \text{and} \quad \Lambda = [\underline{\nu}, \overline{\nu}],$$

their Pompeiu–Hausdorff distance is defined as

$$P_G(\mathcal{J}, \Lambda) = P_G\Big([\underline{\rho}, \overline{\rho}], [\underline{\nu}, \overline{\nu}]\Big) = \max\left\{\left|\underline{\rho} - \underline{\nu}\right|, |\overline{\rho} - \overline{\nu}|\right\}. \tag{2.2}$$

The set $(\mathcal{I}_c, \mathcal{P}_G)$ is a complete metric space. To our knowledge (see [7]), the absolute value of $\bigodot$ is $|\bigodot|$, which is the maximum value of the end points:

$$|\bigodot| = max\{|\underline{\varrho}, |\bar{\varrho}\}. \tag{2.3}$$

The primary interval arithmetic operations for two closed intervals are given as follows:

$$\bigodot +\Lambda = \underline{\varrho} + \underline{\upsilon}, \bar{\varrho} + \bar{\upsilon},$$
$$\bigodot -\Lambda = \underline{\varrho} - \underline{\upsilon}, \bar{\varrho} - \bar{\upsilon},$$
$$\bigodot /\Lambda = [min\mathcal{V}, max\mathcal{V}], \quad where \ \mathcal{V} = \{\underline{\varrho}/\underline{\upsilon}, \underline{\varrho}/\bar{\upsilon}, \bar{\varrho}/\underline{\upsilon}, \bar{\varrho}/\bar{\upsilon}\}; 0 \neq \Lambda,$$
$$\bigodot .\Lambda = [min\mathcal{U}, max\mathcal{U}], \quad where \ \mathcal{U} = \{\underline{\varrho}\underline{\upsilon}, \underline{\varrho}\bar{\upsilon}, \bar{\varrho}\underline{\upsilon}, \bar{\varrho}\bar{\upsilon}\}.$$

$\bigodot$ is defined in scalar multiplication as

$$\gamma \bigodot = \gamma[\underline{\varrho}, \bar{\varrho}] = \begin{cases} [\gamma\underline{\varrho}, \gamma\bar{\varrho}], & \gamma > 0; \\ \{0\}, & \gamma = 0; \\ [\gamma\underline{\varrho}, \gamma\bar{\varrho}], & \gamma < 0, \end{cases}$$

where $\gamma \in \mathfrak{R}$. The interval $\bigodot$ in an opposite form is

$$- \bigodot := (-1) \bigodot = [-\underline{\varrho}, -\bar{\varrho}],$$

where $\gamma = -1$. In general, $- \bigodot$ is not additive inverse for $\bigodot$, i.e., $\bigodot - \bigodot \neq 0$.

Definition 1. *For some kind of intervals* $\bigodot, \Lambda \in \mathcal{I}_c$, *we denote the G-difference of* $\bigodot, \Lambda$ *as the* $\Omega \in \mathcal{I}_c$, *thus we have*

$$\bigodot -\Lambda = \Omega \Leftrightarrow \begin{cases} (i) \quad \bigodot = \Lambda + \Omega, \\ \qquad or \\ (ii) \quad \Lambda = \Lambda + (-\Omega). \end{cases}$$

It appears beyond controversy that

$$\bigodot -\Lambda = \begin{cases} [\underline{\varrho} - \underline{\upsilon}, \bar{\varrho} - \bar{\upsilon}], if \quad L\left(\bigodot\right) \geq L(\Lambda), \\ [\bar{\varrho} - \bar{\upsilon}, \underline{\varrho} - \underline{\upsilon}], if \quad L\left(\bigodot\right) \geq L(\Lambda), \end{cases}$$

where $L(\bigodot) = \bar{\varrho} - \underline{\varrho}$ and $L(\Lambda) = \bar{\upsilon} - \underline{\upsilon}$. A large number of algebraic properties are produced by the operation definitions, allowing $\mathcal{I}_c$ to be a quasi−linear space (see [7]).

Theorem 1. *Let*

$$\hbar : [\check{h}_3, \check{h}_4] \to \mathcal{I}_c$$

be an interval-valued mapping defined by

$$\hbar(\chi) = \left[\underline{\hbar}(\chi), \overline{\hbar}(\chi)\right].$$

Assume that

$$\underline{\hbar}, \overline{\hbar} \in IR_{[(\check{h}_3, \check{h}_4)]}.$$

Then, for every $\chi \in [\check{h}_3, \check{h}_4]$, *the following relation holds:*

$$(\mathbb{IR}) \int_{\check{h}_3}^{\check{h}_4} \hbar(\chi)\, d\chi = \left[\int_{\check{h}_3}^{\check{h}_4} \underline{\hbar}(\chi)\, d\chi, \ \int_{\check{h}_3}^{\check{h}_4} \overline{\hbar}(\chi)\, d\chi\right].$$

Reference [8] introduced the following notion of an interval-valued hs-convex function.

Definition 2. *Let*

$$\nu, \upsilon \in [\check{h}_3, \check{h}_4], \qquad h : (0,1) \to \mathbb{R},$$

where h is nonnegative and not identically zero. An interval-valued mapping

$$\hbar : [\check{h}_3, \check{h}_4] \to \mathcal{I}_{A^+}(\mathbb{R})$$

is called an h s-convex interval-valued function *if, for every* $\lambda \in (0,1)$*,*

$$h(\lambda)\hbar(\nu) + h(1-\lambda)\hbar(\upsilon) \subseteq \hbar(\lambda\nu + (1-\lambda)\upsilon)\,. \tag{2.4}$$

The collection of all interval-valued functions satisfying (2.4) *is denoted by*

$$SX(h, [\check{h}_3, \check{h}_4], \mathcal{I}_{A^+}(\mathbb{R})).$$

Definition 2.4 extends the classical concept of convexity to interval-valued mappings. In particular, if the auxiliary function is chosen as

$$h(\lambda) = \lambda,$$

then (2.4) reduces to the usual definition of an interval-valued convex function (see [8]). Furthermore, selecting

$$h(\lambda) = \lambda^s, \qquad s \in (0,1],$$

yields the class of interval-valued $s-$convex functions discussed in [7]. Hence, the above definition provides a unified framework that includes several well-known convexity concepts as special cases.

Zhao et al. used the h-convexity of interval-valued functions in [10] and obtained the following H-H inclusion.

Theorem 2. *If* $SX(h, [\check{h}_3, \check{h}_4], \mathcal{I}_c^+)$ *and* $h(\frac{1}{2}) \neq 0$*, then we have*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\hbar\left(\frac{\check{h}_3 + \check{h}_4}{2}\right) \supseteq \frac{1}{\check{h}_4 - \check{h}_3}(IR)\int_{\check{h}_3}^{\check{h}_4} \hbar(\chi)\,d\chi \supseteq \left[\hbar\left(\check{h}_3\right) + \hbar\left(\check{h}_4\right)\right]\int_0^1 h(\chi)\,d\chi. \tag{2.5}$$

Remark 1. *The inclusion (2.5) becomes the following if* $h(\vartheta) = \vartheta$ *:*

$$\hbar\left(\frac{\check{h}_3 + \check{h}_4}{2}\right) \supseteq \frac{1}{\check{h}_4 - \check{h}_3}(IR)\int_{\check{h}_3}^{\check{h}_4} \hbar(x)\,dx \supseteq \frac{\hbar\left(\check{h}_3\right) + \hbar\left(\check{h}_4\right)}{2}.$$

Sadowska developed in the [8].

The inclusion (2.5) generates the following if $h(\vartheta) = \vartheta^s$:

$$2^{s-1}\hbar\left(\frac{\check{h}_3 + \check{h}_4}{2}\right) \supseteq \frac{1}{\check{h}_4 - \check{h}_3}(IR)\int_{\check{h}_3}^{\check{h}_4} \hbar(x)\,dx \supseteq \frac{\hbar\left(\check{h}_3\right) + \hbar\left(\check{h}_4\right)}{s+1},$$

which was discovered by Osuna-Gómez et al. in [8].

Definition 3. *A mapping* $\hbar : [\check{h}_3, \check{h}_4] \to \mathcal{I}_c$ *is said to be convex interval-valued if for all* $x, v \in [\check{h}_3, \check{h}_4], \nu \in (0,1)$, *we (see [8]).*

$$\upsilon\hbar(x) + (1-\nu)\hbar(\upsilon) \subseteq \hbar(\nu x + (1-\nu)\upsilon). \tag{2.6}$$

Theorem 3. *A mapping* $\hbar : [\check{h}_3, \check{h}_4] \to \mathcal{I}_c$ *is said to be convex interval-valued if and only if* $\underline{\hbar}$ *is a convex function on* $[\check{h}_3, \check{h}_4]$ *and* $\bar{\hbar}$ *is a concave function on* $[\check{h}_3, \check{h}_4]$.

Theorem 4. *(Interval−valued inclusion (IVI) in Jensen [7]- [10]) Let* $0 < \zeta_1 \leq \zeta_2 \leq ... \leq \zeta_n$ *and* $\hbar$ *be a convex function on the interval-valued mapping on an interval containing* ρ_k, *then the following inclusion holds:*

$$\hbar\left(\sum_{j=1}^{n} \rho_j \zeta_j\right) \supseteq \left(\sum_{j=1}^{n} \rho_j \hbar\left(\zeta_j\right)\right), \tag{2.7}$$

where $\sum_{j=1}^{n} \rho_j = 1, \rho_j \in [0,1]$.

The inequality (2.7) is extended to convex interval-valued functions by the authors in [8] as follows.

Theorem 5. *(IVI in Jensen Mercer [8]) Let* $\hbar$ *be a convex function on the interval-valued mapping on* $[\check{h}_3, \check{h}_4]$ *such that* $L(\check{h}_4) \geq L(\check{h}_0)$ *for all* $\check{h}_0 \in [\check{h}_3, \check{h}_4]$, *then the following inclusion*

$$\hbar\left(\check{h}_3 + \check{h}_4 - \sum_{j=1}^{n} \rho_j \zeta_j\right) \supseteq \quad \hbar\left(\check{h}_3\right) + \hbar\left(\check{h}_4\right) - \sum_{j=1}^{n} \rho_j \zeta_j, \tag{2.8}$$

is true.

In [9], the idea of the Jensen–Mercer inequality was used by Kian and Moslehian, and the following Hermite−Hadamard−Mercer inequalities were demonstrated:

$$\begin{aligned}\hbar\left(\check{h}_3 + \check{h}_4 - \frac{\varkappa + v}{2}\right) &\leq \hbar(\check{h}_3) + \hbar(\check{h}_4) - \frac{1}{v-\varkappa}\int_{\varkappa}^{v} \hbar(\vartheta)\, d\vartheta \\ &\leq \hbar(\check{h}_3) + \hbar(\check{h}_4) - \hbar\left(\frac{\varkappa + v}{2}\right)\end{aligned}$$

and

$$\begin{aligned}\hbar\left(\check{h}_3 + \check{h}_4 - \frac{\varkappa + v}{2}\right) &\leq \frac{1}{v-\varkappa}\int_{\varkappa}^{v} \hbar(\check{h}_3 + \check{h}_4 - \vartheta)\, d\vartheta \\ &\leq \frac{\hbar(\check{h}_3 + \check{h}_4 - \varkappa) + \hbar(\check{h}_3 + \check{h}_4 - v)}{2} \\ &\leq \hbar(\check{h}_3) + \hbar(\check{h}_4) - \frac{\hbar(\varkappa) + \hbar(v)}{2},\end{aligned}$$

where $\hbar$ is a convex function on $[\check{h}_3, \check{h}_4]$. For some recent studies linked to the Jensen–Mercer inequality, one can consult [11, 12].

Fractional integrals have been used to expand and improve Mercer−Hadamard−type inequalities, especially when discussing Æ−convex functions, which are a generalization of standard convex

functions. The class of interval−valued Æ−convex functions is a generalization of convex functions. These functions satisfy the inequality

$$\mathfrak{T}(\lambda x + (1-\lambda)y) \supseteq \mathcal{A}(\lambda)\mathfrak{T}(x) + \mathcal{E}(\lambda)\mathfrak{T}(y), \quad \forall x, y \in I, \lambda \in [0,1],$$

where $\mathcal{A}(\lambda)$ and $\mathcal{E}(\lambda)$ are weight functions that typically satisfy $\mathcal{A}(\lambda) + \mathcal{E}(\lambda) = 1$ and reduce to λ and $1-\lambda$ in the standard convex case. This generalization allows handling more flexible functional behaviors than strict convexity, which is useful in modeling, optimization, and inequality theory. Fractional integral approaches have been used to generalize the conventional Hadamard inequality, which gives upper bounds for the determinant of positive definite matrices, resulting in new inequalities that take fractional powers of integrals into account (see [8]). Sharper bounds for convex functions, particularly those that meet Æ−convexity, a more generic type of convexity defined in particular functional spaces, are established by these expansions by utilizing the characteristics of fractional integrals. These inequalities can be used more broadly thanks to the use of fractional integrals, such as in higher-dimensional spaces or in situations where conventional convexity assumptions are violated. These inequalities can handle more intricate convex structures by utilizing fractional calculus, providing more accurate estimates that are helpful in functional analysis, mathematical physics, and optimization. In addition to enhancing the theoretical landscape of integral inequalities, the combination of Mercer theory and fractional integrals for Æ-convex functions offers useful tools for handling non-linear and higher-order convex functions in a variety of scientific and engineering applications (see [15, 16]).

Let $0 < \nu_1 \leq \nu_2 \leq ... \leq \nu_n$ and let $\omega = (\omega_1, \omega_2, ..., \omega_n)$ be non-negative weights such that $\sum_{z=1}^{n} \omega_z = 1$. The most popular inequality due to Jensen (see [14]) states that $\mathfrak{T}$ is a convex function on the interval $[\theta_1, \theta_2]$, then

$$\mathfrak{T}\left(\sum_{z=1}^{n} \omega_z \, \nu_z\right) \supseteq \left(\sum_{z=1}^{n} \omega_z \, \mathfrak{T}(\nu_z)\right) \tag{2.9}$$

for all $nu_z \in [\theta_1, \theta_2]$ *and each* $\omega_z \in [0,1]$, $\left(z = \overline{1,n}\right)$.

A convex function is one whose graph bends upward, forming a bowl-shaped curve. Equivalently, the line segment joining any pair of points on the graph remains on or above the graph throughout its length.

Definition 4. *[22] The* $\mathfrak{T} : I \to \Re$ *is named as a convex function if*

$$\mathfrak{T}\left(\Omega\check{h}_3 + (1-\Omega)\check{h}_4\right) \supseteq \Omega\mathfrak{T}\left(\check{h}_3\right) + (1-\Omega)\mathfrak{T}\left(\check{h}_4\right), \tag{2.10}$$

for all $\check{h}_3, \check{h}_4 \in I$ *and* $\Omega \in [0,1]$.

For the reversed inequality, $\mathfrak{T}$ is called concave.

The Hermite−Hadamard inequality asserts that if $\mathfrak{T} : I \subseteq \Re \to \Re$ is a convex function in I, then

$$\mathfrak{T}\left(\frac{\check{h}_3 + \check{h}_4}{2}\right) \supseteq \frac{1}{\check{h}_4 - \check{h}_3} \int_{\check{h}_3}^{\check{h}_4} \mathfrak{T}(\Omega)\, d\Omega \supseteq \frac{\mathfrak{T}(\check{h}_3) + \mathfrak{T}(\check{h}_4)}{2}$$

for all $\check{h}_3, \check{h}_4 \in I$ with $\check{h}_3 < \check{h}_4$. The reversed inequality is true if $\mathfrak{T}$ is known to be concave.

Theorem 6. *(IVI in Jensen Mercer [13]) Let $\hbar$ be a convex function on the interval-valued mapping on $[\check{h}_3, \check{h}_4]$ such that $L(\check{h}_4) \geq L(\check{h}_o)$ for all $\check{h}_o \in [\check{h}_3, \check{h}_4]$, then the following inclusion holds:*

$$\mathfrak{T}\left(\check{h}_3 + \check{h}_4 - \sum_{z=1}^{n} \omega_z \nu_z\right) \supseteq \mathfrak{T}\left(\check{h}_3\right) + \mathfrak{T}\left(\check{h}_4\right) - \sum_{z=1}^{n} \omega_z \mathfrak{T}(\nu_z) \tag{2.11}$$

for all $\nu_z \in \left[\check{h}_3, \check{h}_4\right]$ and all $\omega_z \in [0,1]$, $\quad (z = 1, 2, ..., n)$.

For more related data connected with Mercer inequality, (see [14]).

Definition 5. *[19] Consider $Æ : [\check{h}_3, \check{h}_4] \subset \mathfrak{R} \to [\check{h}_3, \check{h}_4]$. A mapping $\mathfrak{T} : [\check{h}_3, \check{h}_4] \to \mathfrak{R}$ is known to be interval−valued Æ−convex on $[\check{h}_3, \check{h}_4]$ if for every $x, y \in [\check{h}_3, \check{h}_4]$ and $\Omega \in [0,1]$, then*

$$\mathfrak{T}(\Omega Æ(x) + (1-\Omega) Æ(y)) \supseteq \Omega\, \mathfrak{T}(Æ(x)) \ + \ (1-\Omega)\ \mathfrak{T}(Æ(y)). \tag{2.12}$$

Otherwise, $-\mathfrak{T}$ is called interval−valued Æ−concave on $[\check{h}_3, \check{h}_4]$.

It is clear that if $Æ(\theta) = \theta$, then we get the classical form of convexity .

Æ−convexity is useful in a variety of fields such as optimization theory, economics, and applied mathematics, especially in places where classical assumptions of convexity are overly stringent (see [16–21]).

Overall, the Riemann-Liouville fractional integral operator (RLFIO) is an important tool in fractional calculus, with various applications in different fields. Its properties and limitations make it a valuable tool for mathematical analysis and modeling.

Definition 6. *[24] Let $\mathfrak{T} \in L_1[\check{h}_3, \check{h}_4]$. The RLFIO $J^{\varrho}_{\check{h}_3^+}\mathfrak{T}$ and $J^{\ell}_{\check{h}_4^-}\mathfrak{T}$ of order $\ell > 0$ with $\check{h}_3 \geq 0$ are defined by*

$$J^{\ell}_{\check{h}_3^+}\mathfrak{T}\ (x) = \frac{1}{\Gamma(\ell)} \int_{\check{h}_3}^{x} (x-\Omega)^{\ell-1}\ \mathfrak{T}(\Omega)\ d\Omega, \quad x > \check{h}_3$$

and

$$J^{\ell}_{\check{h}_4^-}\mathfrak{T}\ (x) = \frac{1}{\Gamma(\ell)} \int_{x}^{\check{h}_4} (\Omega - x)^{\ell-1}\ \mathfrak{T}(\Omega)\ d\Omega, \quad x < \check{h}_4,$$

where $J^0_{\check{h}_3^+} h\ (x) = J^0_{\check{h}_4^-}\mathfrak{T}\ (x) = \mathfrak{T}\ (x)$.

In [14], there is a Hermite–Hadamard–Mercer−type inclusion for a convex interval−valued function via Riemann–Liouville fractional integrals.

Theorem 7. *Let $\mathfrak{T} : [\check{h}_3, \check{h}_4] \to I_c^+$ be an interval-valued convex function such that $\mathfrak{T}(\vartheta) = [\underline{\mathfrak{T}}(\vartheta), \overline{\mathfrak{T}}(\vartheta)]$ and $L(\check{h}_4) \geq L(\check{h}_0)$ for all $\check{h}_0 \in [\check{h}_3, \check{h}_4]$. Then, we have the following inclusions:*

$$\begin{aligned}
&\mathfrak{T}\left(\check{h}_3 + \check{h}_4 - \frac{\varkappa + \upsilon}{2}\right) \\
&\supseteq \frac{\Gamma(\ell+1)}{2(\upsilon-\varkappa)^{\ell}} \left[J^{\ell}_{(\check{h}_3+\check{h}_4)-\mathfrak{T}}(\check{h}_3 + \check{h}_4 - \upsilon) + J^{\ell}_{(\check{h}_3+\check{h}_4)+\mathfrak{T}}(\check{h}_3 + \check{h}_4 - \varkappa)\right] \\
&\supseteq \frac{\mathfrak{T}(\check{h}_3 + \check{h}_4 - \varkappa) + \mathfrak{T}(\check{h}_3 + \check{h}_4 - \upsilon)}{2} \supseteq \mathfrak{T}(\check{h}_3) + \mathfrak{T}(\check{h}_4) \odot_g \frac{\mathfrak{T}(\varkappa) + \mathfrak{T}(\upsilon)}{2}.
\end{aligned} \tag{2.13}$$

Throughout the paper, $Q1$ corresponds to the hypothesis: J is an interval with $\check{h}_3, \check{h}_4 \in J$ so that $\check{h}_4 > \check{h}_3 \geq 0$ and $Æ : J = \left[\check{h}_3, \check{h}_4\right] \to \mathfrak{R}$ is a continuous increasing function. Let $\mathfrak{T} : I = \left[Æ\left(\check{h}_3\right), Æ\left(\check{h}_4\right)\right] \subset \mathfrak{R} \to \mathfrak{R}$ be a differentiable function on the interior I^o of I.

While A_1 corresponds to the hypothesis: Suppose that if $\mathfrak{T} : [Æ\left(\check{h}_3\right), Æ\left(\check{h}_4\right)] \to \mathcal{I}_c^+$ is an interval−valued Æ−convex function such that $\mathfrak{T}(\Omega) = [\underline{\mathfrak{T}}(\Omega), \bar{\mathfrak{T}}(\Omega)]$ and $\mathcal{L}(\check{h}_4) \geq \mathcal{L}(\check{h}_0), \forall \check{h}_0 \in [\check{h}_3, \check{h}_4]$, for all $x, y \in \left[\check{h}_3, \check{h}_4\right], \ \ell > 0, \ \ \Omega \in [0, 1]$.

3. Main results

We first show the compulsory and basic results.

Theorem 8. *(Jensen−type inequality with IVI) Let $M \subset \mathfrak{R}$ and $\mathfrak{T} : M \to \mathfrak{R}$ be an interval−valued Æ−convex mapping. Suppose there exists a mapping $Æ : \mathfrak{R} \longrightarrow \mathfrak{R}$ so that M is an interval−valued Æ−convex set. Then the Jensen-type inequality*

$$\mathfrak{T}\left(\sum_{z=1}^{n} \omega_z Æ(x_z)\right) \supseteq \sum_{z=1}^{n} \omega_z \mathfrak{T}\left(Æ(x_z)\right),$$

holds for all $x_z \in M$ and $\omega_z \in [0, 1]$, $\left(z = \overline{1, n}\right)$ with $\sum_{z=1}^{n} \omega_z = 1$.

Proof. The above inequality is easy when we set $n = 2$. Now, let us assume that for $n = k \geq 3$, the inequality is also true. Then for any $x_1, x_2, ..., x_k \in M$ and $\gamma_z > 0$, $z = \overline{1, k}$ with $\sum_{z=1}^{k} \gamma_z = 1$, one has

$$\mathfrak{T}\left(\sum_{z=1}^{k} \gamma_z Æ(x_z)\right) \supseteq \sum_{z=1}^{k} \gamma_z \mathfrak{T}\left(Æ(x_z)\right).$$

If $x_1, x_2, ..., x_k, x_{(k+1)} \in M$ and $\omega_z > 0$ for $z = \overline{1, k+1}$ with $\sum_{z=1}^{k+1} \omega_z = 1$, then one sets up

$$\gamma_z = \frac{\omega_z}{1 - \omega_{k+1}},$$

for all $z = \overline{1, k}$. With this, one finds $\sum_{z=1}^{k} \gamma_z = 1$. Thus,

$$\begin{aligned}
\mathfrak{T}\left(\sum_{z=1}^{k+1} \omega_z Æ(x_z)\right) &= \mathfrak{T}(\omega_1 Æ(x_1) + \cdots + \omega_k Æ(x_k) + \omega_{k+1} Æ(x_{k+1})) \\
&= \mathfrak{T}\left((1 - \omega_{k+1}) \frac{\omega_1 Æ(x_1) + \cdots + \omega_k Æ(x_k)}{1 - \omega_{k+1}} + \omega_{k+1} Æ(x_{k+1})\right) \\
&\supseteq (1 - \omega_{k+1}) \mathfrak{T}(\gamma_1 Æ(x_1) + \cdots + \gamma_k Æ(x_k)) + \omega_{k+1} \mathfrak{T}(Æ(x_{k+1}))
\end{aligned}$$

$$\begin{aligned}
&\supseteq (1-\omega_{k+1})[\gamma_1 \mathfrak{T}(Æ(x_1)) + \cdots + \gamma_k \mathfrak{T}(Æ(x_k))] + \omega_{k+1}\mathfrak{T}(Æ(x_{k+1}))\\
&= (1-\omega_{k+1})\\
&\times \left[\frac{\omega_1}{1-\omega_{k+1}}\mathfrak{T}(Æ(x_1)) + \cdots + \frac{\omega_k}{1-\omega_{k+1}}\mathfrak{T}(Æ(x_k))\right] + \omega_{k+1}\mathfrak{T}(Æ(x_{k+1}))\\
&= \sum_{z=1}^{k+1}\omega_z \mathfrak{T}(Æ(x_z)).
\end{aligned}$$

So, the mathematical induction yields the proof. □

4. Jensen−Mercer inequality for Æ-convex functions with IVI

Lemma 1. *Let $M \subset \mathfrak{R}$ be an interval, $\{Æ(x_k)\}_{k=1}^n \in M$ be a finite positive increasing sequence, and $\mathfrak{T} : M \to \mathfrak{R}$ be an interval−valued Æ−convex function on M, then*

$$\mathfrak{T}\left(Æ(x_1) + Æ(x_n) - Æ(x_k)\right) \supseteq \mathfrak{T}\left(Æ(x_1)\right) + \mathfrak{T}\left(Æ(x_n)\right) - \mathfrak{T}\left(Æ(x_k)\right), \tag{4.1}$$

for all $1 \le k \le n$.

Proof. Let $Æ(y_k) = Æ(x_1) + Æ(x_n) - Æ(x_k)$. Then $Æ(x_1) + Æ(x_n) = Æ(x_k) + Æ(y_k)$ so that the pairs $(Æ(x_1), Æ(x_n))$ and $(Æ(x_k), Æ(y_k)$(possess the central point. Recall that there exists λ such that

$$Æ(x_k) = \lambda Æ(x_1) + (1-\lambda)Æ(x_n),$$

$$Æ(y_k) = (1-\lambda)Æ(x_1) + \lambda Æ(x_n),$$

where $0 < \lambda < 1$ and $1 \le k \le n$. Using the Æ-convexity of $\mathfrak{T}$, one has

$$\begin{aligned}
\mathfrak{T}(Æ(y_k)) &= \mathfrak{T}((1-\lambda)Æ(x_1) + \lambda Æ(x_n))\\
&\supseteq (1-\lambda)\mathfrak{T}(Æ(x_1)) + \lambda\mathfrak{T}(Æ(x_n))\\
&= \mathfrak{T}(Æ(x_1)) + \mathfrak{T}(Æ(x_n)) - (\lambda\mathfrak{T}(Æ(x_1)) + (1-\lambda)\mathfrak{T}(Æ(x_n)))\\
&\supseteq \mathfrak{T}(Æ(x_1)) + \mathfrak{T}(Æ(x_n)) - \mathfrak{T}(\lambda Æ(x_1) + (1-\lambda)Æ(x_n))\\
&= \mathfrak{T}(Æ(x_1)) + \mathfrak{T}(Æ(x_n)) - \mathfrak{T}(Æ(x_k)).
\end{aligned}$$

Therefore, the inequality (4.1) follows from $Æ(y_k) = Æ(x_1) + Æ(x_n) - Æ(x_k)$. □

Theorem 9. *(Jensen−Mercer inequality for interval−valued Æ−convexity) Let $M \subset \mathfrak{R}$ be an interval and $\mathfrak{T} : M \to \mathfrak{R}$ be an interval−valued Æ-convex function. Then*

$$\mathfrak{T}\left(Æ(x_1) + Æ(x_n) - \sum_{k=1}^{n}\omega_k\, Æ(x_k)\right) \supseteq \mathfrak{T}\left(Æ(x_1)\right) + \mathfrak{T}\left(Æ(x_n)\right) - \sum_{k=1}^{n}\omega_k \mathfrak{T}\left(Æ(x_k)\right) \tag{4.2}$$

for a positive finite sequence $\{Æ(x_k)\}_{k=1}^n \in M$ and $\omega_k \in [0,1]$, $\left(k = \overline{1,n}\right)$ with $\sum_{k=1}^{n}\omega_k = 1$.

Proof. It follows from Theorem 6 and Lemma 1 together with the interval−valued Æ−convexity of Ω on the interval M that

$$\begin{aligned}\mathfrak{T}\left(Æ(x_1)+Æ(x_n)-\sum_{k=1}^{n}\omega_k\, Æ(x_k)\right) &= \mathfrak{T}\left(\sum_{k=1}^{n}\omega_k(Æ(x_1)+Æ(x_n)-\ Æ(x_k))\right)\\ &\supseteq \sum_{k=1}^{n}\omega_k\mathfrak{T}(Æ(x_1)+Æ(x_n)-Æ(x_k))\\ &\supseteq \sum_{k=1}^{n}\omega_k[\mathfrak{T}(Æ(x_1))+\Omega(Æ(x_n))-\mathfrak{T}(Æ(x_k)]\\ &= \mathfrak{T}(Æ(x_1))+\mathfrak{T}(Æ(x_n))-\sum_{k=1}^{n}\omega_k\mathfrak{T}(Æ(x_k)),\end{aligned}$$

which completes the proof. □

5. Mercer−Hadamard inequalities for fractional integrals with IVI

By using Mercer inequalities, Hermite−Hadamard inequalities in fractional integrals form are given as follows:

Theorem 10. *Let $\mathfrak{T}: I=\left[\check{h}_3,\check{h}_4\right]\to\Re$ be an interval−valued Æ−convex positive function with $0\le\check{h}_3<\check{h}_4$ and $Æ: J=\left[\check{h}_3,\check{h}_4\right]\to\Re$ be a continuous increasing function under the condition as defined in A_1, then*

$$\begin{aligned}\mathfrak{T}\left(Æ\left(\check{h}_3\right)+Æ\left(\check{h}_4\right)-\frac{Æ(x)+Æ(y)}{2}\right) &\supseteq \mathfrak{T}\left(Æ\left(\check{h}_3\right)\right)+\mathfrak{T}\left(Æ\left(\check{h}_4\right)\right)\\ &-\frac{\Gamma(\ell+1)}{2\left(Æ(y)-Æ(x)\right)^{\ell}}\left\{J^{\ell}_{[Æ(x)]^{+}}\mathfrak{T}\left(Æ(y)\right)\ +\ J^{\ell}_{[Æ(y)]^{-}}\mathfrak{T}\left(Æ(x)\right)\right\}\\ &\supseteq \mathfrak{T}\left(Æ\left(\check{h}_3\right)\right)+\mathfrak{T}\left(Æ\left(\check{h}_4\right)\right)-\mathfrak{T}\left(\frac{Æ(x)+Æ(y)}{2}\right).\end{aligned}\tag{5.1}$$

Proof. The Mercer inequality yields that

$$\mathfrak{T}\left(Æ\left(\check{h}_3\right)+Æ\left(\check{h}_4\right)-\frac{Æ(u_1)+Æ(v_1)}{2}\right)\supseteq\mathfrak{T}\left(Æ\left(\check{h}_3\right)\right)+\mathfrak{T}\left(Æ\left(\check{h}_4\right)\right)-\frac{\mathfrak{T}\left(Æ(u_1)\right)+\mathfrak{T}\left(Æ(v_1)\right)}{2}\tag{5.2}$$

for all $Æ(u_1), Æ(v_1)\in\left[Æ\left(\check{h}_3\right), Æ\left(\check{h}_4\right)\right]$.

Now, by change of variables $u_1=\Omega Æ(x)+(1-\Omega)Æ(y)$ and $v_1=(1-\Omega)Æ(x)+\Omega Æ(y)$, $\forall\ Æ(x), Æ(y)\in\left[Æ\left(\check{h}_3\right), Æ\left(\check{h}_4\right)\right]$, and $\Omega\in[0,1]$ in (5.2), we get

$$\mathfrak{T}\left(Æ\left(\check{h}_3\right)+Æ\left(\check{h}_4\right)-\frac{Æ(x)+Æ(y)}{2}\right)$$

$$\geq \mathfrak{T}\left(\text{Æ}\left(\check{h}_3\right)\right)+\mathfrak{T}\left(\text{Æ}\left(\check{h}_4\right)\right)-\frac{\mathfrak{T}\left(\Omega\text{Æ}\left(x\right)+\left(1-\Omega\right)\text{Æ}\left(y\right)\right)+\mathfrak{T}\left(\left(1-\Omega\right)\text{Æ}\left(x\right)+\Omega\text{Æ}\left(y\right)\right)}{2}. \tag{5.3}$$

Product with the term $\Omega^{\ell-1}$ and then anti-derivative the obtained inequality with respect to (wrt) Ω over $[0,1]$ to get

$$\begin{aligned}\frac{1}{\ell}\mathfrak{T}\left(\text{Æ}\left(\check{h}_3\right)+\text{Æ}\left(\check{h}_4\right)-\frac{\text{Æ}\left(x\right)+\text{Æ}\left(y\right)}{2}\right) &\geq \frac{1}{\ell}\left\{\mathfrak{T}\left(\text{Æ}\left(\check{h}_3\right)\right)+\mathfrak{T}\left(\text{Æ}\left(\check{h}_4\right)\right)\right\}\\ &-\frac{1}{2}\left\{\int_0^1 \Omega^{\ell-1}\left(\mathfrak{T}\left(\Omega\text{Æ}\left(x\right)+\left(1-\Omega\right)\text{Æ}\left(y\right)\right)+\mathfrak{T}\left(\left(1-\Omega\right)\text{Æ}\left(x\right)+\Omega\text{Æ}\left(y\right)\right)\right)d\Omega\right\}\end{aligned}$$

$$\begin{aligned}\mathfrak{T}\left(\text{Æ}\left(\check{h}_3\right)+\text{Æ}\left(\check{h}_4\right)-\frac{\text{Æ}\left(x\right)+\text{Æ}\left(y\right)}{2}\right) &\geq \mathfrak{T}\left(\text{Æ}\left(\check{h}_3\right)\right)+\mathfrak{T}\left(\text{Æ}\left(v\right)\right)\\ &-\frac{\Gamma\left(\ell+1\right)}{2\left(\text{Æ}\left(y\right)-\text{Æ}\left(x\right)\right)^{\ell}}\left\{J^{\ell}_{[\text{Æ}(x)]^{+}}\mathfrak{T}\left(\text{Æ}\left(y\right)\right)\ +\ J^{\ell}_{[\text{Æ}(y)]^{-}}\mathfrak{T}\left(\text{Æ}\left(x\right)\right)\right\},\end{aligned} \tag{5.4}$$

and so the first part of (5.1) is proved. Now, we will go with (5.1)′*s* second part. We first note that if convexity on $\mathfrak{T}$ holds, then for $\Omega\in[0,1]$, it yields that

$$\begin{aligned}\mathfrak{T}\left(\frac{\text{Æ}\left(x\right)+\text{Æ}\left(y\right)}{2}\right)=\mathfrak{T}\Bigg(&\frac{\Omega\text{Æ}\left(x\right)+\left(1-\Omega\right)\text{Æ}\left(y\right)+\left(1-\Omega\right)\text{Æ}\left(x\right)+\Omega\text{Æ}\left(y\right)}{2}\Bigg)\\ &\frac{\mathfrak{T}\left(\Omega\text{Æ}\left(x\right)+\left(1-\Omega\right)\text{Æ}\left(y\right)\right)+\ \mathfrak{T}\left(\left(1-\Omega\right)\text{Æ}\left(x\right)+\Omega\text{Æ}\left(y\right)\right)}{2}.\end{aligned} \tag{5.5}$$

Product with the term $\Omega^{\ell-1}$ and then anti-derivative the obtained inequality wrt Ω over $[0,1]$ to obtain

$$\begin{aligned}&\frac{1}{\ell}\,\mathfrak{T}\left(\frac{\text{Æ}\left(x\right)+\text{Æ}\left(y\right)}{2}\right)\\ &\qquad\geq\frac{1}{2}\left\{\int_0^1\Omega^{\ell-1}\left(\mathfrak{T}\left(\Omega\text{Æ}\left(x\right)+\left(1-\Omega\right)\text{Æ}\left(y\right)\right)+\mathfrak{T}\left(\left(1-\Omega\right)\text{Æ}\left(x\right)+\Omega\text{Æ}\left(y\right)\right)\right)\right\}d\Omega\\ &\mathfrak{T}\left(\frac{\text{Æ}\left(x\right)+\text{Æ}\left(y\right)}{2}\right)\ \ \geq\frac{\Gamma\left(\ell+1\right)}{2\left(\text{Æ}\left(y\right)-\text{Æ}\left(x\right)\right)^{\ell}}\left\{J^{\ell}_{[\text{Æ}(x)]^{+}}\mathfrak{T}\left(\text{Æ}\left(y\right)\right)\ +\ J^{\ell}_{[\text{Æ}(y)]^{-}}\mathfrak{T}\left(\text{Æ}\left(x\right)\right)\right\}.\end{aligned} \tag{5.6}$$

Multiplying by (-1), one gets

$$-\ \frac{\Gamma\left(\ell+1\right)}{2\left(\text{Æ}\left(y\right)-\text{Æ}\left(x\right)\right)^{\ell}}\left\{J^{\ell}_{[\text{Æ}(x)]^{+}}\mathfrak{T}\left(\text{Æ}\left(y\right)\right)\ +\ J^{\ell}_{[\text{Æ}(y)]^{-}}\mathfrak{T}\left(\text{Æ}\left(x\right)\right)\right\}\ \ \geq\ \ -\ \mathfrak{T}\left(\frac{\text{Æ}\left(x\right)+\text{Æ}\left(y\right)}{2}\right). \tag{5.7}$$

Summing the $\mathfrak{T}(\text{Æ}\left(\check{h}_3\right))+\mathfrak{T}(\text{Æ}\left(\check{h}_4\right))$ on both sides of (5.7), we get the second inequality of (5.1). □

Remark 2. *If we choose $\ell=1, x=\check{h}_3$, and $y=\check{h}_4$ with $\underline{\hbar}(\Omega)=\bar{\hbar}(\Omega)$ in Theorem 10, we get [24, Theorem 3].*

Remark 3. *If we choose $\ell=1$ and $\text{Æ}(\theta)=\theta$ with $\underline{\hbar}(\Omega)=\bar{\hbar}(\Omega)$ in Theorem 10, we get [9, Theorem 2.1].*

Lemma 2. *Under considerations with Q1 and A_1, if $\mathfrak{T}' \in L\left[\text{Æ}\left(\check{h}_3\right), \text{Æ}\left(\check{h}_4\right)\right]$ for $\text{Æ}\left(\check{h}_3\right), \text{Æ}\left(\check{h}_4\right) \in I$, then*

$$\frac{\mathfrak{T}(\text{Æ}(\check{h}_3) + \text{Æ}(\check{h}_4) - \text{Æ}(x)) + \mathfrak{T}(\text{Æ}(\check{h}_3) + \text{Æ}(\check{h}_4) - \text{Æ}(y))}{2} - \frac{\Gamma(\ell+1)}{2(\text{Æ}(y) - \text{Æ}(x))^{\ell}}$$

$$\times\left\{ J^{\ell}_{\left(\text{Æ}\left(\check{h}_3\right)+\text{Æ}\left(\check{h}_4\right)-\text{Æ}(x)\right)^{-}} \mathfrak{T}(\text{Æ}(\check{h}_3) + \text{Æ}(\check{h}_4) - \text{Æ}(y)) + J^{\ell}_{\left(\text{Æ}\left(\check{h}_3\right)+\text{Æ}\left(\check{h}_4\right)-\text{Æ}(y)\right)^{+}} \mathfrak{T}(\text{Æ}(\check{h}_3) + \text{Æ}(\check{h}_4) - \text{Æ}(x)) \right\}$$

$$= \frac{\text{Æ}(y) - \text{Æ}(x)}{2} \int_0^1 \left[\Omega^{\ell} - (1-\Omega)^{\ell}\right] \mathfrak{T}'\left(\text{Æ}\left(\check{h}_3\right) + \text{Æ}\left(\check{h}_4\right) - (\Omega\text{Æ}(x)) + (1-\Omega)\,\text{Æ}(y)\right) d\Omega. \quad (5.8)$$

Proof. It will be noticed that

$$\frac{\text{Æ}(y) - \text{Æ}(x)}{2} \int_0^1 \left[\Omega^{\ell} - (1-\Omega)^{\ell}\right] \mathfrak{T}'\left(\text{Æ}\left(\check{h}_3\right) + \text{Æ}\left(\check{h}_4\right) - (\Omega\text{Æ}(x)) + (1-\Omega)\,\text{Æ}(y)\right) d\Omega, \quad (5.9)$$

where

$$\begin{aligned}
&\int_0^1 \Omega^{\ell} \mathfrak{T}'\left(\text{Æ}\left(\check{h}_3\right) + \text{Æ}\left(\check{h}_4\right) - (\Omega\text{Æ}(x) + (1-\Omega)\,\text{Æ}(y))\right) d\Omega \\
&= \frac{\Omega^{\ell} \mathfrak{T}\left(\text{Æ}\left(\check{h}_3\right) + \text{Æ}\left(\check{h}_4\right) - (\Omega\text{Æ}(x) + (1-\Omega)\,\text{Æ}(y))\right)}{\text{Æ}(y) - \text{Æ}(x)}\Bigg|_0^1 \\
&- \int_0^1 \frac{\mathfrak{T}\left(\text{Æ}\left(\check{h}_3\right) + \text{Æ}\left(\check{h}_4\right) - (\Omega\text{Æ}(x) + (1-\Omega)\,\text{Æ}(y))\right)}{\text{Æ}(y) - \text{Æ}(x)} . \ell\Omega^{\ell-1} d\Omega \\
&= \frac{\mathfrak{T}\left(\text{Æ}\left(\check{h}_3\right) + \text{Æ}\left(\check{h}_4\right) - \text{Æ}(x)\right)}{\text{Æ}(y) - \text{Æ}(x)} \\
&- \frac{\ell}{\text{Æ}(y) - \text{Æ}(x)} \int_0^1 \Omega^{\ell-1} \mathfrak{T}\left(\text{Æ}\left(\check{h}_3\right) + \text{Æ}\left(\check{h}_4\right) - (\Omega\text{Æ}(x) + (1-\Omega)\,\text{Æ}(y))\right) d\Omega
\end{aligned}$$

$$\begin{aligned}
&= \frac{\mathfrak{T}(\text{Æ}(\check{h}_3) + \text{Æ}(\check{h}_4) - \text{Æ}(x))}{\text{Æ}(y) - \text{Æ}(x)} \\
&- \frac{\Gamma(\ell+1)}{(\text{Æ}(y) - \text{Æ}(x))^{\ell+1}} J^{\ell}_{\left(\text{Æ}\left(\check{h}_3\right)+\text{Æ}\left(\check{h}_4\right)-\text{Æ}(x)\right)^{-}} \mathfrak{T}(\text{Æ}(\check{h}_3) + \text{Æ}(\check{h}_4) - \text{Æ}(y)).
\end{aligned} \quad (5.10)$$

Similarly,

$$\int_0^1 (1-\Omega)^{\ell} \mathfrak{T}'\left(\text{Æ}\left(\check{h}_3\right) + \text{Æ}\left(\check{h}_4\right) - (\Omega\text{Æ}(x)) + (1-\Omega)\,\text{Æ}(y)\right) d\Omega$$

$$= -\frac{\mathfrak{T}(\text{Æ}(\check{h}_3) + \text{Æ}(\check{h}_4) - \text{Æ}(y))}{\text{Æ}(y) - \text{Æ}(x)} + \frac{\ell}{\text{Æ}(y) - \text{Æ}(x)}$$

$$\times \int_0^1 (1-\Omega)^{\ell-1} \mathfrak{T}(Æ(\check{h}_3) + Æ(\check{h}_4) - (\Omega Æ(x) + (1-\Omega)Æ(y)))\, d\Omega$$

$$\begin{aligned} &= -\frac{\mathfrak{T}(Æ(\check{h}_3) + Æ(\check{h}_4) - Æ(y))}{Æ(y) - Æ(x)} \\ &+ \frac{\Gamma(\ell+1)}{(Æ(y) - Æ(x))^{\ell+1}} J^{\ell}_{(Æ(\check{h}_3)+Æ(\check{h}_4)-Æ(y))^+} \mathfrak{T}(Æ(\check{h}_3) + Æ(\check{h}_4) - Æ(x)). \end{aligned} \tag{5.11}$$

Combining (5.10) and (5.11) with (5.9), we get (5.8). □

Corollary 1. *By choosing $\ell = 1$ with $\underline{\hbar}(\Omega) = \bar{\hbar}(\Omega)$ in Lemma 2,*

$$\begin{aligned} &\frac{\mathfrak{T}\left(Æ(\check{h}_3) + Æ(\check{h}_4) - Æ(x)\right) + \mathfrak{T}\left(Æ(\check{h}_3) + Æ(\check{h}_4) - Æ(y)\right)}{2} \\ &- \frac{1}{Æ(y) - Æ(x)} \int_{Æ(\check{h}_3)+Æ(\check{h}_4)-Æ(y)}^{Æ(\check{h}_3)+Æ(\check{h}_4)-Æ(x)} \mathfrak{T}(\Omega)\, d\Omega \\ &= \frac{y-x}{2} \int_0^1 (1-2\Omega)\, \mathfrak{T}'\left(\check{h}_3 + \check{h}_4 - (\Omega x + (1-\Omega)\, y)\right) d\Omega. \end{aligned} \tag{5.12}$$

Remark 4. *By taking $Æ(\theta) = \theta$, $x = \check{h}_3$, and $y = \check{h}_4$ with $\underline{\hbar}(\Omega) = \bar{\hbar}(\Omega)$ in Corollary 1, [23, Lemma 2.1] is obtained.*

Theorem 11. *Let us consider with Q1 and A_1, and $\mathfrak{T}' \in L\left[Æ\left(\check{h}_3\right), Æ\left(\check{h}_4\right)\right]$ for $Æ\left(\check{h}_3\right), Æ\left(\check{h}_4\right) \in I$. If the interval−valued Æ−convexity of $|\mathfrak{T}'|$ on $[\check{h}_3, \check{h}_4]$ holds, then*

$$\begin{aligned} &\left|\frac{\mathfrak{T}(Æ(\check{h}_3) + Æ(\check{h}_4) - Æ(x)) + \mathfrak{T}(Æ(\check{h}_3) + Æ(\check{h}_4) - Æ(y))}{2} - \frac{\Gamma(\ell+1)}{2(Æ(y) - Æ(x))^{\ell}}\right. \\ &\times \left\{ J^{\ell}_{(Æ(\check{h}_3)+Æ(\check{h}_4)-Æ(x))^-}\ \mathfrak{T}(Æ(\check{h}_3) + Æ(\check{h}_4) - Æ(y)) \right. \\ &\left.\left. + J^{\ell}_{(Æ(\check{h}_3)+Æ(\check{h}_4)-Æ(y))^+}\ \mathfrak{T}(Æ(\check{h}_3) + Æ(\check{h}_4) - Æ(x))\right\}\right| \end{aligned}$$

$$\supseteq \frac{Æ(y) - Æ(x)}{\ell+1}\left(1 - \frac{1}{2^{\ell}}\right)\left[\left|\mathfrak{T}'\left(Æ\left(\check{h}_3\right)\right)\right| + \left|\mathfrak{T}'\left(Æ\left(\check{h}_4\right)\right)\right| - \frac{|\mathfrak{T}'(Æ(x))| + |\mathfrak{T}'(Æ(y))|}{2}\right]. \tag{5.13}$$

Proof. From Lemma 2 and the Mercer inequality with interval−valued Æ−convexity of $|\mathfrak{T}'|$, one has

$$\begin{aligned} &\left|\frac{\mathfrak{T}(Æ(\check{h}_3) + Æ(\check{h}_4) - Æ(x)) + \mathfrak{T}(Æ(\check{h}_3) + Æ(\check{h}_4) - Æ(y))}{2} - \frac{\Gamma(\ell+1)}{2(Æ(y) - Æ(x))^{\ell}}\right. \\ &\times \left\{ J^{\ell}_{(Æ(\check{h}_3)+Æ(\check{h}_4)-Æ(x))^-}\ \mathfrak{T}(Æ(\check{h}_3) + Æ(\check{h}_4) - Æ(y)) \right. \end{aligned}$$

$$
\begin{aligned}
&+ J^{\ell}_{\left(\text{Æ}\left(\check{h}_3\right)+\text{Æ}\left(\check{h}_4\right)-\text{Æ}(y)\right)^{+}} \ \mathfrak{T}(\text{Æ}(\check{h}_3)+\text{Æ}(\check{h}_4)-\text{Æ}(x))\Big\}\Big| \\
&\supseteq \frac{\text{Æ}(y)-\text{Æ}(x)}{2}\int_0^1 \left|\Omega^{\ell}-(1-\Omega)^{\ell}\right| \left|\mathfrak{T}'\left(\text{Æ}\left(\check{h}_3\right)+\text{Æ}\left(\check{h}_4\right)-(\Omega\text{Æ}(x)+(1-\Omega)\,\text{Æ}(y))\right)\right| d\Omega \\
&\supseteq \frac{\text{Æ}(y)-\text{Æ}(x)}{2}\Big[\int_0^{\frac{1}{2}}\left((1-\Omega)^{\ell}-\Omega^{\ell}\right) \\
&\times\left\{\left|\mathfrak{T}'\left(\text{Æ}\left(\check{h}_3\right)\right)\right|+\left|\mathfrak{T}'\left(\text{Æ}\left(\check{h}_4\right)\right)\right|-(\Omega\left|\mathfrak{T}'\left(\text{Æ}(x)\right)\right|+(1-\Omega)\left|\mathfrak{T}'\left(\text{Æ}(y)\right)\right|)\right\}d\Omega \\
&+\int_{\frac{1}{2}}^1\left(\Omega^{\ell}-(1-\Omega)^{\ell}\right) \\
&\times\left\{\left|\mathfrak{T}'\left(\text{Æ}\left(\check{h}_3\right)\right)\right|+\left|\mathfrak{T}'\left(\text{Æ}\left(\check{h}_4\right)\right)\right|-(\Omega\left|\mathfrak{T}'\left(\text{Æ}(x)\right)\right|+(1-\Omega)\left|\mathfrak{T}'\left(\text{Æ}(y)\right)\right|)\right\}d\Omega\Big] \\
&= \frac{\text{Æ}(y)-\text{Æ}(x)}{2}\left[v_1+v_2\right], \qquad (5.14)
\end{aligned}
$$

where

$$
\begin{aligned}
v_1 = &\left(\left|\mathfrak{T}'\left(\text{Æ}\left(\check{h}_3\right)\right)\right|+\left|\mathfrak{T}'\left(\text{Æ}\left(\check{h}_4\right)\right)\right|\right)\int_0^{\frac{1}{2}}\left((1-\Omega)^{\ell}-\Omega^{\ell}\right)d\Omega \\
&-\left\{\left|\mathfrak{T}'\left(\text{Æ}(x)\right)\right|\int_0^{\frac{1}{2}}\left(\Omega(1-\Omega)^{\ell}-\Omega^{\ell+1}\right)d\Omega+\left|\mathfrak{T}'\left(\text{Æ}(y)\right)\right|\int_0^{\frac{1}{2}}\left((1-\Omega)^{\ell+1}-\Omega^{\ell}(1-\Omega)\right)d\Omega\right\} \\
&=\left(\left|\mathfrak{T}'\left(\text{Æ}\left(\check{h}_3\right)\right)\right|+\left|\mathfrak{T}'\left(\text{Æ}\left(\check{h}_4\right)\right)\right|\right)\left(\frac{1}{(\ell+1)}-\frac{1}{(\ell+1)2^{\ell}}\right) \\
&-\left\{\left|\mathfrak{T}'\left(\text{Æ}(x)\right)\right|\left(\frac{1}{(\ell+1)(\ell+2)}-\frac{1}{(\ell+1)2^{\ell+1}}\right)+\left|\mathfrak{T}'\left(\text{Æ}(y)\right)\right|\left(\frac{1}{(\ell+2)}-\frac{1}{(\ell+1)2^{\ell+1}}\right)\right\} \qquad (5.15)
\end{aligned}
$$

and

$$
\begin{aligned}
v_2 = &\left(\left|\mathfrak{T}'\left(\text{Æ}\left(\check{h}_3\right)\right)\right|+\left|\mathfrak{T}'\left(\text{Æ}\left(\check{h}_4\right)\right)\right|\right)\int_{\frac{1}{2}}^1\left(\Omega^{\ell}-(1-\Omega)^{\ell}\right)d\Omega \\
&-\left\{\left|\mathfrak{T}'\left(\text{Æ}(x)\right)\right|\int_{\frac{1}{2}}^1\left(\Omega^{\ell+1}-\Omega(1-\Omega)^{\ell}\right)d\Omega+\left|\mathfrak{T}'\left(\text{Æ}(y)\right)\right|\int_{\frac{1}{2}}^1\left((1-\Omega)\Omega^{\ell}-(1-\Omega)^{\ell+1}\right)d\Omega\right\} \\
&=\left(\left|\mathfrak{T}'\left(\text{Æ}\left(\check{h}_3\right)\right)\right|+\left|\mathfrak{T}'\left(\text{Æ}\left(\check{h}_4\right)\right)\right|\right)\left(\frac{1}{(\ell+1)}-\frac{1}{(\ell+1)2^{\ell}}\right) \\
&-\left\{\left|\mathfrak{T}'\left(\text{Æ}(x)\right)\right|\left(\frac{1}{(\ell+2)}-\frac{1}{(\ell+1)2^{\ell+1}}\right)+\left|\mathfrak{T}'\left(\text{Æ}(y)\right)\right|\left(\frac{1}{(\ell+1)(\ell+2)}-\frac{1}{(\ell+1)2^{\ell+1}}\right)\right\}. \qquad (5.16)
\end{aligned}
$$

By combining (5.15) and (5.16) with (5.14), (5.13) follows. □

Remark 5. *If we choose* $\text{Æ}(\theta)=\theta$, $x=\check{h}_3$, *and* $y=\check{h}_4$ *with* $\underline{\hbar}(\Omega)=\bar{\hbar}(\Omega)$ *in Theorem 11, [22, Theorem 3] follows.*

Corollary 2. *If we choose* $\ell = 1$ *in Theorem* 11*, then*

$$\begin{aligned}
&\left|\frac{\mathfrak{T}\left(Æ(\check{h}_3)+Æ(\check{h}_4)-Æ(x)\right)+\mathfrak{T}\left(Æ(\check{h}_3)+Æ(\check{h}_4)-Æ(y)\right)}{2}\right.\\
&\left.-\frac{1}{Æ(y)-Æ(x)}\int_{Æ(\check{h}_3)+Æ(\check{h}_4)-Æ(y)}^{Æ(\check{h}_3)+Æ(\check{h}_4)-Æ(x)}\mathfrak{T}(\Omega)\,d\Omega\right|\\
&\geq \frac{Æ(y)-Æ(x)}{4}\left[\left(\left|\mathfrak{T}'\left(Æ(\theta_1)\right)\right|+\left|\mathfrak{T}'\left(Æ(\theta_2)\right)\right|\right)-\frac{1}{2}\left(\left|\mathfrak{T}'\left(Æ(x)\right)\right|+\left|\mathfrak{T}'\left(Æ(y)\right)\right|\right)\right].
\end{aligned}$$

Theorem 12. *Let us consider with Q1 and* A_1*, and* $\mathfrak{T}' \in L\left[Æ\left(\check{h}_3\right), Æ\left(\check{h}_4\right)\right]$ *for* $Æ\left(\check{h}_3\right), Æ\left(\check{h}_4\right) \in I$*. If an interval−valued Æ−convexity of* $|\mathfrak{T}'|^q$ *on* $[\check{h}_3, \check{h}_4]$ *(with* $q \geq 1$*) holds, then*

$$\begin{aligned}
&\left|\frac{\mathfrak{T}(Æ(\check{h}_3)+Æ(\check{h}_4)-Æ(x))+\mathfrak{T}(Æ(\check{h}_3)+Æ(\check{h}_4)-Æ(y))}{2}-\frac{\Gamma(\ell+1)}{2(Æ(y)-Æ(x))^{\ell}}\right.\\
&\times\left\{J^{\ell}_{\left(Æ\left(\check{h}_3\right)+Æ\left(\check{h}_4\right)-Æ\left(x\right)\right)^{-}}\ \mathfrak{T}(Æ(\check{h}_3)+Æ(\check{h}_4)-Æ(y))\right.\\
&\left.\left.+J^{\ell}_{\left(Æ\left(\check{h}_3\right)+Æ\left(\check{h}_4\right)-Æ\left(y\right)\right)^{+}}\ \mathfrak{T}(Æ(\check{h}_3)+Æ(\check{h}_4)-Æ(x))\right\}\right|\\
&\geq \frac{Æ(\check{h}_4)-Æ(\check{h}_3)}{2}\times 2^{1-\frac{1}{q}}\times\left[\left(\left|\mathfrak{T}'(Æ(\check{h}_3))\right|^q+\left|\mathfrak{T}'(Æ(\check{h}_4))\right|^q\right)\left(\frac{2}{\ell+1}\left(1-\frac{1}{2^{\ell}}\right)\right)\right.\\
&\left.-\left\{\left(\left|\mathfrak{T}'(Æ(x))\right|^q+\left|\mathfrak{T}'(Æ(y))\right|^q\right)\left(\frac{1}{\ell+1}\left(1-\frac{1}{2^{\ell}}\right)\right)\right\}\right].
\end{aligned}\tag{5.17}$$

Proof. Assume first that $q = 1$. Due to Lemma 2 and the Mercer inequality with an interval−valued Æ−convexity of $|\mathfrak{T}'|^q$, one has

$$\begin{aligned}
&\left|\frac{\mathfrak{T}(Æ(\check{h}_3)+Æ(\check{h}_4)-Æ(x))+\mathfrak{T}(Æ(\check{h}_3)+Æ(\check{h}_4)-Æ(y))}{2}-\frac{\Gamma(\ell+1)}{2(Æ(y)-Æ(x))^{\ell}}\right.\\
&\times\left\{J^{\ell}_{\left(Æ\left(\check{h}_3\right)+Æ\left(\check{h}_4\right)-Æ\left(x\right)\right)^{-}}\ \mathfrak{T}(Æ(\check{h}_3)+Æ(\check{h}_4)-Æ(y))\right.\\
&\left.\left.+J^{\ell}_{\left(Æ\left(\check{h}_3\right)+Æ\left(\check{h}_4\right)-Æ\left(y\right)\right)^{+}}\ \mathfrak{T}(Æ(\check{h}_3)+Æ(\check{h}_4)-Æ(x))\right\}\right|\\
&\geq \frac{Æ(y)-Æ(x)}{2}\int_0^1\left|\Omega^{\ell}-(1-\Omega)^{\ell}\right|\left|\mathfrak{T}'\left(Æ\left(\check{h}_3\right)+Æ\left(\check{h}_4\right)-(\Omega Æ(x)+(1-\Omega)Æ(y))\right)\right|d\Omega\\
&\geq \frac{Æ(y)-Æ(x)}{2}\int_0^1\left|\Omega^{\ell}-(1-\Omega)^{\ell}\right|\\
&\times\left\{\left|\mathfrak{T}'\left(Æ\left(\check{h}_3\right)\right)\right|+\left|\mathfrak{T}'\left(Æ\left(\check{h}_4\right)\right)\right|-(\Omega\left|\mathfrak{T}'\left(Æ(x)\right)\right|+(1-\Omega)\left|\mathfrak{T}'\left(Æ(y)\right)\right|)\right\}d\Omega\\
&\geq \frac{Æ(y)-Æ(x)}{2}\left[\int_0^{\frac{1}{2}}\left((1-\Omega)^{\ell}-\Omega^{\ell}\right)\right.\\
&\times\left\{\left|\mathfrak{T}'\left(Æ\left(\check{h}_3\right)\right)\right|+\left|\mathfrak{T}'\left(Æ\left(\check{h}_4\right)\right)\right|-(\Omega\left|\mathfrak{T}'\left(Æ(x)\right)\right|+(1-\Omega)\left|\mathfrak{T}'\left(Æ(y)\right)\right|)\right\}d\Omega
\end{aligned}$$

$$+\int_{\frac{1}{2}}^{1}\left(\Omega^{\ell}-(1-\Omega)^{\ell}\right)$$
$$\times\left\{\left|\mathfrak{T}'\left(Æ\left(\check{h}_3\right)\right)\right|+\left|\mathfrak{T}'\left(Æ\left(\check{h}_4\right)\right)\right|-\left(\Omega\left|\mathfrak{T}'\left(Æ\left(x\right)\right)\right|+\left(1-\Omega\right)\left|\mathfrak{T}'\left(Æ\left(y\right)\right)\right|\right)\right\}d\Omega\Bigg].$$

By using calculus tools, we obtain

$$\geq \frac{Æ\left(y\right)-Æ\left(x\right)}{2}\Bigg[\left(\left|\mathfrak{T}'\left(Æ\left(\check{h}_3\right)\right)\right|+\left|\mathfrak{T}'\left(Æ\left(\check{h}_4\right)\right)\right|\right)\left\{\frac{2}{\ell+1}\left(1-\frac{1}{2^{\ell}}\right)\right\}$$
$$-\left(\left|\mathfrak{T}'\left(Æ\left(x\right)\right)\right|+\left|\mathfrak{T}'\left(Æ\left(y\right)\right)\right|\right)\left\{\frac{1}{\ell+1}\left(1-\frac{1}{2^{\ell}}\right)\right\}\Bigg].$$

Second, suppose that $q>1$ and using Lemma 2 with the power mean inequality, we get

$$\int_0^1\left|\Omega^{\ell}-(1-\Omega)^{\ell}\right|\left|\mathfrak{T}'\left(Æ\left(\check{h}_3\right)+Æ\left(\check{h}_4\right)-\left(\Omega Æ\left(x\right)+\left(1-\Omega\right)Æ\left(y\right)\right)\right)\right|d\Omega$$
$$\geq\left(\int_0^1\left|(1-\Omega)^{\ell}-\Omega^{\ell}\right|d\Omega\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left|(1-\Omega)^{\ell}-\Omega^{\ell}\right|\left|\mathfrak{T}'\left(Æ\left(\check{h}_3\right)+Æ\left(\check{h}_4\right)-\left(\Omega Æ\left(x\right)+\left(1-\Omega\right)Æ\left(y\right)\right)\right)\right|^{q}d\Omega\right)^{\frac{1}{q}}.$$

By using an interval−valued Æ−convexity of $|\mathfrak{T}'|^q$, with the Mercer inequality, we obtain

$$\left|\frac{\mathfrak{T}(Æ(\check{h}_3)+Æ(\check{h}_4)-Æ(x))+\mathfrak{T}(Æ(\check{h}_3)+Æ(\check{h}_4)-Æ(y))}{2}-\frac{\Gamma(\ell+1)}{2(Æ(y)-Æ(x))^{\ell}}\right.$$
$$\times\Big\{J^{\ell}_{\left(Æ\left(\check{h}_3\right)+Æ\left(\check{h}_4\right)-Æ\left(x\right)\right)^{-}}\ \mathfrak{T}(Æ(\check{h}_3)+Æ(\check{h}_4)-Æ(y))$$
$$\left.+J^{\ell}_{\left(Æ\left(\check{h}_3\right)+Æ\left(\check{h}_4\right)-Æ\left(y\right)\right)^{+}}\ \mathfrak{T}(Æ(\check{h}_3)+Æ(\check{h}_4)-Æ(x))\Big\}\right|$$
$$\geq\ \frac{Æ\left(y\right)-Æ\left(x\right)}{2}\left(\int_0^1\left|(1-\Omega)^{\ell}-\Omega^{\ell}\right|d\Omega\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left|(1-\Omega)^{\ell}-\Omega^{\ell}\right|\left|\mathfrak{T}'\left(Æ\left(\check{h}_3\right)+Æ\left(\check{h}_4\right)-\left(\Omega Æ\left(x\right)+\left(1-\Omega\right)Æ\left(y\right)\right)\right)\right|^{q}d\Omega\right)^{\frac{1}{q}}$$
$$\geq\ \frac{Æ\left(y\right)-Æ\left(x\right)}{2}\left(\int_0^{\frac{1}{2}}\left((1-\Omega)^{\ell}-\Omega^{\ell}\right)d\Omega+\int_{\frac{1}{2}}^1\left(\Omega^{\ell}-(1-\Omega)^{\ell}\right)d\Omega\right)^{1-\frac{1}{q}}$$
$$\times\Bigg[\int_0^1\left|(1-\Omega)^{\ell}-\Omega^{\ell}\right|$$
$$\times\left\{\left|\mathfrak{T}'\left(Æ\left(\check{h}_3\right)\right)\right|^{q}+\left|\mathfrak{T}'\left(Æ\left(\check{h}_4\right)\right)\right|^{q}-\left(\Omega\left|\mathfrak{T}'\left(Æ\left(x\right)\right)\right|^{q}+\left(1-\Omega\right)\left|\mathfrak{T}'\left(Æ\left(y\right)\right)\right|^{q}\right)\right\}d\Omega\Bigg]^{\frac{1}{q}}$$
$$=\ \frac{Æ\left(y\right)-Æ\left(x\right)}{2}2^{1-\frac{1}{q}}$$

$$\times\left[\left(\left|\mathfrak{T}'\left(\text{Æ}\left(\check{h}_3\right)\right)\right|^q+\left|\mathfrak{T}'\left(\text{Æ}\left(\check{h}_4\right)\right)\right|^q\right)\left\{\frac{2}{\ell+1}\left(1-\frac{1}{2^\ell}\right)\right\}\right.$$
$$\left.-\left(\left|\mathfrak{T}'\left(\text{Æ}\left(x\right)\right)\right|^q+\left|\mathfrak{T}'\left(\text{Æ}\left(y\right)\right)\right|^q\right)\left\{\frac{1}{\ell+1}\left(1-\frac{1}{2^\ell}\right)\right\}\right]^{\frac{1}{q}}.$$

□

Remark 6. *If we choose $x=\check{h}_3$ and $y=\check{h}_4$ with $\underline{\hbar}(\Omega)=\bar{\hbar}(\Omega)$ in Theorem 12, then it reduces to [24, Theorem 5] follows.*

Theorem 13. *Let us consider with Q1 and A_1, and $\mathfrak{T}'\in L\left[\text{Æ}\left(\check{h}_3\right),\text{Æ}\left(\check{h}_4\right)\right]$ for $\text{Æ}\left(\check{h}_3\right),\text{Æ}\left(\check{h}_4\right)\in I$. If $|\mathfrak{T}'|^q$ is an interval−valued Æ−convexity on $[\check{h}_3,\check{h}_4]$, $q>1$, then*

$$\left|\frac{\mathfrak{T}(\text{Æ}(\check{h}_3)+\text{Æ}(\check{h}_4)-\text{Æ}(x))+\mathfrak{T}(\text{Æ}(\check{h}_3)+\text{Æ}(\check{h}_4)-\text{Æ}(y))}{2}-\frac{\Gamma(\ell+1)}{2(\text{Æ}(y)-\text{Æ}(x))^\ell}\right.$$
$$\times\left\{J^\ell_{\left(\text{Æ}\left(\check{h}_3\right)+\text{Æ}\left(\check{h}_4\right)-\text{Æ}\left(x\right)\right)^-}\ \mathfrak{T}(\text{Æ}(\check{h}_3)+\text{Æ}(\check{h}_4)-\text{Æ}(y))\right.$$
$$\left.+J^\ell_{\left(\text{Æ}\left(\check{h}_3\right)+\text{Æ}\left(\check{h}_4\right)-\text{Æ}\left(y\right)\right)^+}\ \mathfrak{T}(\text{Æ}(\check{h}_3)+\text{Æ}(\check{h}_4)-\text{Æ}(x))\right\}\Bigg|$$
$$\supseteq\ \frac{\text{Æ}\left(y\right)-\text{Æ}\left(x\right)}{2}\left(\frac{2}{\ell p+1}\left(1-\frac{1}{2^{\ell p}}\right)\right)^{\frac{1}{p}}$$
$$\times\left[\left(\left|\mathfrak{T}'\left(\text{Æ}\left(\check{h}_3\right)\right)\right|^q+\left|\mathfrak{T}'\left(\text{Æ}\left(\check{h}_4\right)\right)\right|^q\right)-\frac{|\mathfrak{T}'\left(\text{Æ}\left(x\right)\right)|^q+|\mathfrak{T}'\left(\text{Æ}\left(y\right)\right)|^q}{2}\right]^{\frac{1}{q}}. \tag{5.18}$$

Proof. From Lemma 2, and the Mercer inequality with an interval−valued Æ−convexity of $|\mathfrak{T}'|^q$, and the famous Hölder's integral inequality, we have

$$\left|\frac{\mathfrak{T}(\text{Æ}(\check{h}_3)+\text{Æ}(\check{h}_4)-\text{Æ}(x))+\mathfrak{T}(\text{Æ}(\check{h}_3)+\text{Æ}(\check{h}_4)-\text{Æ}(y))}{2}-\frac{\Gamma(\ell+1)}{2(\text{Æ}(y)-\text{Æ}(x))^\ell}\right.$$
$$\times\left\{J^\ell_{\left(\text{Æ}\left(\check{h}_3\right)+\text{Æ}\left(\check{h}_4\right)-\text{Æ}\left(x\right)\right)^-}\ \mathfrak{T}(\text{Æ}(\check{h}_3)+\text{Æ}(\check{h}_4)-\text{Æ}(y))\right.$$
$$\left.+J^\ell_{\left(\text{Æ}\left(\check{h}_3\right)+\text{Æ}\left(\check{h}_4\right)-\text{Æ}\left(y\right)\right)^+}\ \mathfrak{T}(\text{Æ}(\check{h}_3)+\text{Æ}(\check{h}_4)-\text{Æ}(x))\right\}\Bigg|$$
$$\supseteq\ \frac{\text{Æ}\left(y\right)-\text{Æ}\left(x\right)}{2}\int_0^1\left|\Omega^\ell-(1-\Omega)^\ell\right|\left|\mathfrak{T}'\left(\text{Æ}\left(\check{h}_3\right)+\text{Æ}\left(\check{h}_4\right)-\left(\Omega\text{Æ}\left(x\right)+(1-\Omega)\,\text{Æ}\left(y\right)\right)\right)\right|d\Omega$$
$$\supseteq\ \frac{\text{Æ}\left(y\right)-\text{Æ}\left(x\right)}{2}\left(\int_0^1\left|\Omega^\ell-(1-\Omega)^\ell\right|^p d\Omega\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\left|\mathfrak{T}'\left(\text{Æ}\left(\check{h}_3\right)+\text{Æ}\left(\check{h}_4\right)-\left(\Omega\text{Æ}\left(x\right)+(1-\Omega)\,\text{Æ}\left(y\right)\right)\right)\right|^q d\Omega\right)^{\frac{1}{q}}$$
$$\supseteq\ \frac{\text{Æ}\left(y\right)-\text{Æ}\left(x\right)}{2}\left(\int_0^{\frac{1}{2}}\left((1-\Omega)^\ell-\Omega^\ell\right)^p d\Omega+\int_{\frac{1}{2}}^1\left(\Omega^\ell-(1-\Omega)^\ell\right)^p d\Omega\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\left\{\left|\mathfrak{T}'\left(\text{Æ}\left(\check{h}_3\right)\right)\right|^q+\left|\mathfrak{T}'\left(\text{Æ}\left(\check{h}_4\right)\right)\right|^q-\left(\Omega\left|\mathfrak{T}'\left(\text{Æ}\left(x\right)\right)\right|^q+\left(1-\Omega\right)\left|\mathfrak{T}'\left(\text{Æ}\left(y\right)\right)\right|^q\right)\right\}d\Omega\right)^{\frac{1}{q}}.$$

Here, we use the fact that $(\mathfrak{U}-\mathfrak{V})^p\supseteq\mathfrak{U}^p-\mathfrak{V}^p$ for any $\mathfrak{U}>\mathfrak{V}\geq 0$ and $p\geq 1$. By using the calculus tools, we obtain

$$\begin{aligned}&\supseteq\ \frac{\text{Æ}\left(y\right)-\text{Æ}\left(x\right)}{2}\left(\frac{2}{\ell p+1}\left(1-\frac{1}{2^{\ell p}}\right)\right)^{\frac{1}{p}}\\&\times\left[\left(\left|\mathfrak{T}'\left(\text{Æ}\left(\check{h}_3\right)\right)\right|^q+\left|\mathfrak{T}'\left(\text{Æ}\left(\check{h}_4\right)\right)\right|^q\right)-\frac{\left|\mathfrak{T}'\left(\text{Æ}\left(x\right)\right)\right|^q+\left|\mathfrak{T}'\left(\text{Æ}\left(y\right)\right)\right|^q}{2}\right]^{\frac{1}{q}}.\end{aligned}$$

□

Remark 7. *If we choose $x=\check{h}_3$ and $y=\check{h}_4$ with $\underline{\hbar}(\Omega)=\bar{\hbar}(\Omega)$ in Theorem 13, then we get [24, Theorem 6].*

Corollary 3. *By choosing $\ell=1$ in Theorem* 13*, then*

$$\begin{aligned}&\left|\frac{\mathfrak{T}\left(\text{Æ}(\check{h}_3)+\text{Æ}(\check{h}_4)-\text{Æ}(x)\right)+\mathfrak{T}\left(\text{Æ}(\check{h}_3)+\text{Æ}(\check{h}_4)-\text{Æ}(y)\right)}{2}\right.\\&\left.-\frac{1}{\text{Æ}(y)-\text{Æ}(x)}\int_{\text{Æ}(\check{h}_3)+\text{Æ}(\check{h}_4)-\text{Æ}(y)}^{\text{Æ}(\check{h}_3)+\text{Æ}(\check{h}_4)-\text{Æ}(x)}\mathfrak{T}\left(\Omega\right)d\Omega\right|\\&\supseteq\ \frac{\text{Æ}\left(y\right)-\text{Æ}\left(x\right)}{2}\left(\frac{2}{p+1}\left(1-\frac{1}{2^p}\right)\right)^{\frac{1}{p}}\\&\times\left[\left(\left|\mathfrak{T}'\left(\text{Æ}\left(\theta_1\right)\right)\right|^q+\left|\mathfrak{T}'\left(\text{Æ}\left(\theta_2\right)\right)\right|^q\right)-\frac{1}{2}\left(\left|\mathfrak{T}'\left(\text{Æ}\left(x\right)\right)\right|^q+\left|\mathfrak{T}'\left(\text{Æ}\left(y\right)\right)\right|^q\right)\right]^{\frac{1}{q}}.\end{aligned}$$

Theorem 14. *Let us consider with Q1 and A_1, and $\mathfrak{T}'\in L\left[\text{Æ}\left(\check{h}_3\right),\text{Æ}\left(\check{h}_4\right)\right]$ for $\text{Æ}\left(\check{h}_3\right),\text{Æ}\left(\check{h}_4\right)\in I$. If $|\mathfrak{T}'|^q$ is an interval−valued Æ−convexity on $[\check{h}_3,\check{h}_4]$, $q\geq 1$, then*

$$\begin{aligned}&\left|\frac{\mathfrak{T}(\text{Æ}(\check{h}_3)+\text{Æ}(\check{h}_4)-\text{Æ}(x))+\mathfrak{T}(\text{Æ}(\check{h}_3)+\text{Æ}(\check{h}_4)-\text{Æ}(y))}{2}-\frac{\Gamma(\ell+1)}{2(\text{Æ}(y)-\text{Æ}(x))^{\ell}}\right.\\&\times\left\{J^{\ell}_{\left(\text{Æ}\left(\check{h}_3\right)+\text{Æ}\left(\check{h}_4\right)-\text{Æ}\left(x\right)\right)^{-}}\ \mathfrak{T}(\text{Æ}(\check{h}_3)+\text{Æ}(\check{h}_4)-\text{Æ}(y))\right.\\&\left.\left.+\ J^{\ell}_{\left(\text{Æ}\left(\check{h}_3\right)+\text{Æ}\left(\check{h}_4\right)-\text{Æ}\left(y\right)\right)^{+}}\ \mathfrak{T}(\text{Æ}(\check{h}_3)+\text{Æ}(\check{h}_4)-\text{Æ}(x))\right\}\right|\\&\supseteq\ \frac{\text{Æ}\left(y\right)-\text{Æ}\left(x\right)}{2}\left[\left(\left|\mathfrak{T}'\left(\text{Æ}\left(\check{h}_3\right)\right)\right|^q+\left|\mathfrak{T}'\left(\text{Æ}\left(\check{h}_4\right)\right)\right|^q\right)\left(\frac{2^{\ell q}-1}{(\ell q+1)\,2^{\ell q}}\right)\right.\\&-\left\{\left|\mathfrak{T}'\left(\text{Æ}\left(x\right)\right)\right|^q\left(\frac{2^{\ell q+2}-\ell q-1-(\ell q+3)\,2^{\ell q}e^{-\ln(2)\ell q}}{4\,(\ell q+1)\,(\ell q+2)\,2^{\ell q}}\right)\right.\\&\left.+\ \left|\mathfrak{T}'\left(\text{Æ}\left(y\right)\right)\right|^q\left(\frac{(\ell q+1)\,2^{\ell q+1}-(\ell q+2)}{(2\ell^2q^2+6\ell q+4)\,2^{\ell q}}\right)\right\}\\&\left.+\left(\left|\mathfrak{T}'\left(\text{Æ}\left(\check{h}_3\right)\right)\right|^q+\left|\mathfrak{T}'\left(\text{Æ}\left(\check{h}_4\right)\right)\right|^q\right)\left(\frac{2^{\ell q}-1}{(\ell q+1)\,2^{\ell q}}\right)\end{aligned}$$

$$- \left\{ \left| \mathfrak{T}' \left(\text{Æ} \left(x \right) \right) \right|^q \left(\frac{\left(\ell q + 1 \right) \left(2^{\ell q + 2} - 1 \right) - \left(\ell q + 3 \right) - \left(\ell q + 3 \right) e^{-\ln(2) \ell q}}{4 \left(\ell q + 1 \right) \left(\ell q + 2 \right)} \right) \right.$$

$$\left. + \left| \mathfrak{T}' \left(\text{Æ} \left(y \right) \right) \right|^q \left(\frac{2^{\ell q + 1} - \ell q - 2}{\left(2 \ell^2 q^2 + 6 \ell q + 4 \right) 2^{\ell q}} \right) \right\} \Bigg]^{\frac{1}{q}}. \tag{5.19}$$

Proof. From Lemma 2, the Mercer inequality with an interval−valued Æ−convexity of $|\mathfrak{T}'|^q$, and the famous Hölder integral inequality, one writes

$$\begin{aligned}
&\left| \frac{\mathfrak{T}(\text{Æ}(\check{h}_3) + \text{Æ}(\check{h}_4) - \text{Æ}(x)) + \mathfrak{T}(\text{Æ}(\check{h}_3) + \text{Æ}(\check{h}_4) - \text{Æ}(y))}{2} - \frac{\Gamma(\ell + 1)}{2(\text{Æ}(y) - \text{Æ}(x))^{\ell}} \right. \\
&\times \left\{ J^{\ell}_{\left(\text{Æ}(\check{h}_3) + \text{Æ}(\check{h}_4) - \text{Æ}(x) \right)^-} \ \mathfrak{T}(\text{Æ}(\check{h}_3) + \text{Æ}(\check{h}_4) - \text{Æ}(y)) \right. \\
&\left. \left. + J^{\ell}_{\left(\text{Æ}(\check{h}_3) + \text{Æ}(\check{h}_4) - \text{Æ}(y) \right)^+} \ \mathfrak{T}(\text{Æ}(\check{h}_3) + \text{Æ}(\check{h}_4) - \text{Æ}(x)) \right\} \right| \\
&\geq \frac{\text{Æ}(y) - \text{Æ}(x)}{2} \int_0^1 \left| \Omega^{\ell} - (1 - \Omega)^{\ell} \right| \left| \mathfrak{T}' \left(\text{Æ} \left(\check{h}_3 \right) + \text{Æ} \left(\check{h}_4 \right) - (\Omega \text{Æ}(x) + (1 - \Omega) \text{Æ}(y)) \right) \right| d\Omega \\
&\geq \frac{\text{Æ}(y) - \text{Æ}(x)}{2} \left(\int_0^1 1^p d\Omega \right)^{\frac{1}{p}} \\
&\times \left(\int_0^1 \left| \Omega^{\ell} - (1 - \Omega)^{\ell} \right|^q \left| \mathfrak{T}' \left(\text{Æ} \left(\check{h}_3 \right) + \text{Æ} \left(\check{h}_4 \right) - (\Omega \text{Æ}(x) + (1 - \Omega) \text{Æ}(y)) \right) \right|^q d\Omega \right)^{\frac{1}{q}}
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{E(y) - \text{Æ}(x)}{2} \left[\int_0^{\frac{1}{2}} \left((1 - \Omega)^{\ell} - \Omega^{\ell} \right)^q \left| \mathfrak{T}' \left(\text{Æ} \left(\check{h}_3 \right) + \text{Æ} \left(\check{h}_4 \right) - (\Omega \text{Æ}(x) + (1 - \Omega) \text{Æ}(y)) \right) \right|^q d\Omega \right. \\
&\left. + \int_{\frac{1}{2}}^1 \left(\Omega^{\ell} - (1 - \Omega)^{\ell} \right)^q \left| \mathfrak{T}' \left(\text{Æ} \left(\check{h}_3 \right) + \text{Æ} \left(\check{h}_4 \right) - (\Omega \text{Æ}(x) + (1 - \Omega) \text{Æ}(y)) \right) \right|^q d\Omega \right]^{\frac{1}{q}} \\
&\geq \frac{\text{Æ}(y) - \text{Æ}(x)}{2} \left[\int_0^{\frac{1}{2}} \left((1 - \Omega)^{\ell q} - \Omega^{\ell q} \right) \right. \\
&\times \left\{ \left| \mathfrak{T}' \left(\text{Æ} \left(\check{h}_3 \right) \right) \right|^q + \left| \mathfrak{T}' \left(\text{Æ} \left(\check{h}_4 \right) \right) \right|^q - \left(\Omega \left| \mathfrak{T}' \left(\text{Æ}(x) \right) \right|^q + (1 - \Omega) \left| \mathfrak{T}' \left(\text{Æ}(y) \right) \right|^q \right) \right\} d\Omega \\
&+ \left. \int_{\frac{1}{2}}^1 \left(\Omega^{\ell q} - (1 - \Omega)^{\ell q} \right) \right. \\
&\left. \times \left\{ \left| \mathfrak{T}' \left(\text{Æ} \left(\check{h}_3 \right) \right) \right|^q + \left| \mathfrak{T}' \left(\text{Æ} \left(\check{h}_4 \right) \right) \right|^q - \left(\Omega \left| \mathfrak{T}' \left(\text{Æ}(x) \right) \right|^q + (1 - \Omega) \left| \mathfrak{T}' \left(\text{Æ}(y) \right) \right|^q \right) \right\} d\Omega \right]^{\frac{1}{q}}.
\end{aligned}$$

Here, we use the fact that $(\mathfrak{U} - \mathfrak{V})^{\theta} \supseteq \mathfrak{U}^{\theta} - \mathfrak{V}^{\theta}$ for any $\mathfrak{U} > \mathfrak{V} \geq 0$ and $\theta \geq 1$.

$$= \frac{\text{Æ}(y) - \text{Æ}(x)}{2} \left[\left(\left| \mathfrak{T}' \left(\text{Æ} \left(\check{h}_3 \right) \right) \right|^q + \left| \mathfrak{T}' \left(\text{Æ} \left(\check{h}_4 \right) \right) \right|^q \right) \int_0^{\frac{1}{2}} \left((1 - \Omega)^{\ell q} - \Omega^{\ell q} \right) d\Omega \right.$$

$$
\begin{aligned}
&- \left\{ \left|\mathfrak{T}'\left(Æ\left(x\right)\right)\right|^q \int_0^{\frac{1}{2}} \left(\Omega\left(1-\Omega\right)^{\ell q} - \Omega^{\ell q+1}\right) d\Omega \right. \\
&+ \left. \left|\mathfrak{T}'\left(Æ\left(y\right)\right)\right|^q \int_0^{\frac{1}{2}} \left(\left(1-\Omega\right)^{\ell q+1} - \Omega^{\ell q}\left(1-\Omega\right)\right) d\Omega \right\} \\
&+ \left(\left|\mathfrak{T}'\left(Æ\left(\check{h}_3\right)\right)\right|^q + \left|\mathfrak{T}'\left(Æ\left(\check{h}_4\right)\right)\right|^q \right) \int_{\frac{1}{2}}^1 \left(\Omega^{\ell q} - \left(1-\Omega\right)^{\ell q}\right) d\Omega \\
&- \left\{ \left|\mathfrak{T}'\left(Æ\left(x\right)\right)\right|^q \int_{\frac{1}{2}}^1 \left(\Omega^{\ell q+1} - \Omega\left(1-\Omega\right)^{\ell q}\right) d\Omega \right. \\
&+ \left. \left|\mathfrak{T}'\left(Æ\left(y\right)\right)\right|^q \int_{\frac{1}{2}}^1 \left(\Omega^{\ell q}\left(1-\Omega\right) - \left(1-\Omega\right)^{\ell q+1}\right) d\Omega \right\} \Bigg]^{\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&\supseteq \frac{Æ\left(y\right) - Æ\left(x\right)}{2} \Bigg[\left(\left|\mathfrak{T}'\left(Æ\left(\check{h}_3\right)\right)\right|^q + \left|\mathfrak{T}'\left(Æ\left(\check{h}_4\right)\right)\right|^q \right) \left(\frac{2^{\ell q}-1}{\left(\ell q+1\right)2^{\ell q}} \right) \\
&- \left\{ \left|\mathfrak{T}'\left(Æ\left(x\right)\right)\right|^q \left(\frac{2^{\ell q+2} - \ell q - 1 - \left(\ell q+3\right)2^{\ell q} e^{-\ln(2)\ell q}}{4\left(\ell q+1\right)\left(\ell q+2\right)2^{\ell q}} \right) \right. \\
&+ \left. \left|\mathfrak{T}'\left(Æ\left(y\right)\right)\right|^q \left(\frac{\left(\ell q+1\right)2^{\ell q+1} - \left(\ell q+2\right)}{\left(2\ell^2 q^2 + 6\ell q + 4\right)2^{\ell q}} \right) \right\} \\
&+ \left(\left|\mathfrak{T}'\left(Æ\left(\check{h}_3\right)\right)\right|^q + \left|\mathfrak{T}'\left(Æ\left(\check{h}_4\right)\right)\right|^q \right) \left(\frac{2^{\ell q}-1}{\left(\ell q+1\right)2^{\ell q}} \right) \\
&- \left\{ \left|\mathfrak{T}'\left(Æ\left(x\right)\right)\right|^q \left(\frac{\left(\ell q+1\right)\left(2^{\ell q+2}-1\right) - \left(\ell q+3\right) - \left(\ell q+3\right)e^{-\ln(2)\ell q}}{4\left(\ell q+1\right)\left(\ell q+2\right)} \right) \right. \\
&+ \left. \left|\mathfrak{T}'\left(Æ\left(y\right)\right)\right|^q \left(\frac{2^{\ell q+1} - \ell q - 2}{\left(2\ell^2 q^2 + 6\ell q + 4\right)2^{\ell q}} \right) \right\} \Bigg]^{\frac{1}{q}}.
\end{aligned}
$$

Hence proved. □

6. Matrix application

We use $\mathbb{M}_n$ to represent the collection of all $n \times n$ complex matrices, while $\mathbb{H}_n$ denotes the algebra of complex matrices of the same order. Let $Æ : \mathbb{M}_n \to \mathbb{M}_n$ be a matrix-valued function. We define $\mathfrak{T} : I \to \mathbb{H}_n^+$ to be matrix Æ-convex if for all $x, y \in I$ and $\lambda \in [0,1]$, the following holds in the Loewner order:

$$\mathfrak{T}(\lambda x + (1-\lambda)y) \preceq_{Æ} \lambda\mathfrak{T}(x) + (1-\lambda)\mathfrak{T}(y).$$

Furthermore, let $\mathbb{H}_n^+$ denote the set of all strictly positive (positive definite) matrices in $\mathbb{H}_n$. In other words, a matrix $\mathcal{D}$ belongs to $\mathbb{H}_n^+$ whenever

$$\mathcal{D} \in \mathbb{H}_n^+ \quad \Longleftrightarrow \quad \langle \mathcal{D}x, x\rangle > 0, \qquad \forall\, x \neq 0, \tag{6.1}$$

for all nonzero $x \in \mathbb{M}_n$.

In [25], Sababheh proved that the function

$$\psi(\theta) = \left\|\hat{S}^{\theta}\mathcal{Y}\bar{U}^{1-\theta} + \hat{S}^{1-\theta}\mathcal{Y}\bar{U}^{\theta}\right\|, \qquad \hat{S}, \bar{U} \in \mathfrak{P}_n^+, \ \mathcal{Y} \in \mathfrak{P}_n,$$

is convex for all $\theta \in [0, 1]$.

Example 4.1. By using the conditions in Theorem 10, we have

$$\begin{aligned}
&\left\|\hat{S}^{Æ(h_3)+Æ(h_3)-\frac{Æ(x)+Æ(y)}{2}}Y\bar{U}^{1-\left(Æ(h_3)+Æ(h_3)-\frac{Æ(x)+Æ(y)}{2}\right)} + \hat{S}^{1-\left(Æ(h_3)+Æ(h_3)-\frac{Æ(x)+Æ(y)}{2}\right)}Y\bar{U}^{Æ(h_3)+Æ(h_3)-\frac{Æ(x)+Æ(y)}{2}}\right\| \\
&\supseteq \ \left\|\hat{S}^{Æ(h_3)}Y\bar{U}^{1-Æ(h_3)} + \hat{S}^{1-Æ(h_3)}Y\bar{U}^{Æ(h_3)}\right\| + \left\|\hat{S}^{Æ(h_4)}Y\bar{U}^{1-Æ(h_4)} + \hat{S}^{1-Æ(h_4)}Y\bar{U}^{Æ(h_4)}\right\| \\
&- \ \frac{\Gamma(l+1)}{2\left(Æ(y) - Æ(x)\right)^{l}} \\
&\times \left\{ J^{l}_{Æ(x)^+}\left\|\hat{S}^{Æ(y)}Y\bar{U}^{1-Æ(y)} + \hat{S}^{1-Æ(y)}Y\bar{U}^{E(y)}\right\| + J^{l}_{Æ(y)^-}\left\| \hat{S}^{Æ(x)}Y\bar{U}^{1-Æ(x)} + \hat{S}^{1-Æ(x)}Y\bar{U}^{Æ(x)}\right\| \right\} \\
&\supseteq \left\|\hat{S}^{E(h_3)}Y\bar{U}^{1-Æ(h_3)} + \hat{S}^{1-Æ(h_3)}Y\bar{U}^{Æ(h_3)}\right\| \\
&+ \left\|\hat{S}^{Æ(h_4)}Y\bar{U}^{1-Æ(h_4)} + \hat{S}^{1-Æ(h_4)}Y\bar{U}^{E(h_4)}\right\| - \left\|\hat{S}^{\frac{Æ(x)+Æ(y)}{2}}Y\bar{U}^{1-\frac{Æ(x)+Æ(y)}{2}} + \hat{S}^{1-\frac{Æ(x)+Æ(y)}{2}}Y\bar{U}^{\frac{Æ(x)+Æ(y)}{2}}\right\|.
\end{aligned}$$

7. Conclusions

In this paper, we have established new Mercer−Hadamard−type inclusions for interval−valued functions by employing a fractional integral framework under the concept of an interval−valued Æ-convexity. The proposed results significantly extend the classical Hermite–Hadamard and Mercer inequalities from real-valued settings to interval-valued analysis, thereby providing more comprehensive bounds in the presence of interval uncertainty. The incorporation of fractional integral operators allows the derived inclusions to capture nonlocal characteristics that cannot be described by classical integral inequalities. Future work could extend these inclusions to other fractional integral types (e.g., Caputo−Fabrizio, Atangana−Baleanu, or Hadamard operators) to examine how different nonlocal kernels affect the tightness of the bounds.

Author contributions

Samina Samina: Writing-reviewing and editing, software, validation; Jamshed Nasir: Visualization, methodology, writing-original draft preparation; Kashif Nazar: Visualization, methodology, software, writing-original draft preparation; Adil Jhangeer: Conceptualization, visualization, methodology, supervision, writing-reviewing and editing, investigation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This article has been produced with the financial support of the European Union under the REFRESH-Research Excellence For Region Sustainability and High-tech Industries project number CZ .10.03.01/00/22_003/0000048 via the Operational Programme Just Transition.

Funding

This research was supported by Natural Science Foundation of Fujian Province grant number 2026J0012057.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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