AIMS Mathematics, 11(11): 37532–37547.
DOI: 10.3934/math.20261452
Received: 25 September 2026
Revised: 04 November 2026
Accepted: 06 November 2026
Published: 12 November 2026

https://www.aimspress.com/journal/Math

Research article

The symmetrized sum of finite q-multiple harmonic sums with one exceptional index

Hideaki Ishikawa[1] **and Takao Komatsu**[2,*]

In memory of Florian Luca[*]

[1] School of Education, University of Toyama, Toyama 930-8555, Japan

[2] Department of Mathematics, Institute of Science Tokyo 152-8551, Japan

* **Correspondence:** Email: komatsu@zstu.edu.cn, komatsu.t.al@m.titech.ac.jp.

Abstract: In this paper, we give explicit expressions for q-harmonic sums on $1-\cdots-1, A, 1-\cdots-1$ indices. The case $A = 1$ has been extensively studied, and a variety of identities, explicit evaluations, and structural properties are known. Likewise, numerous explicit evaluations are available for q-multiple zeta values and q-harmonic sums with uniform indices $A-\cdots-A$. Although several methods are available for treating non-uniform indices, explicit evaluations for the pattern $1-\cdots-1, A, 1-\cdots-1$ remain comparatively scarce. The case $A = 2$ was settled only recently. In this paper, with the theory of the elementary symmetric functions, we extend this framework to the general case $A \geq 3$ for the sum over all positions of A, though we do not generally evaluate each summand at a prescribed position of A. We obtain explicit formulas in terms of binomial coefficients and degenerate Bernoulli numbers. Some expressions about $2-\cdots-2, A, 2-\cdots-2$ are also shown.

Keywords: multiple zeta functions; elementary symmetric functions; roots of unity; degenerate Bernoulli polynomials
Mathematics Subject Classification: 11M32, 05A15, 05A19, 05A30, 11B37, 11B73

1. Introduction

For positive integers $s_1, s_2, \ldots, s_m$, we consider the q-multiple harmonic sums of the form

$$\mathfrak{Z}_n(q;;s_1,s_2,\ldots,s_m) := \sum_{1\leq i_1<i_2<\cdots<i_m\leq n-1} \frac{1}{(1-q^{i_1})^{s_1}(1-q^{i_2})^{s_2}\cdots(1-q^{i_m})^{s_m}}. \tag{1.1}$$

[*] See also [16, 17]

In previous papers ([7,14,15]), when $s = s_1 = s_2 = \cdots = s_m$, we wrote $\beta_n(q;m;s) = \beta_n(q;;\underbrace{s,s,\ldots,s}_{m})$.

When $n \to \infty$, the infinite version was studied by Schlesinger [23]:

$$\beta(q;;s_1,s_2,\ldots,s_m) := \sum_{1\le i_1<i_2<\cdots<i_m} \frac{1}{(1-q^{i_1})^{s_1}(1-q^{i_2})^{s_2}\cdots(1-q^{i_m})^{s_m}} \quad (|q|>1)\,. \tag{1.2}$$

(For $0<|q|<1$ the terms do not tend to 0. Replacing q by q^{-1}, (1.2) becomes $(-1)^{s_1+\cdots+s_m}$ times the sum of $\prod_j q^{i_j s_j}/(1-q^{i_j})^{s_j}$ with $0<|q|<1$.) There are several different forms; see, e.g., [4,22,26,28]. For finite versions, see, e.g., [1,2,24,25].

Multiplying (1.1) by $(1-q)^{s_1+\cdots+s_m}$ and letting $q\to 1$ and then $n\to\infty$, one obtains, under the usual admissibility condition, the multiple zeta value

$$\zeta(s_1,s_2,\ldots,s_m) := \sum_{1\le i_1<i_2<\cdots<i_m} \frac{1}{i_1^{s_1} i_2^{s_2}\cdots i_m^{s_m}}\,, \tag{1.3}$$

which has been studied extensively, as well as its generalizations or modifications (see, e.g., [27]).

Although $s_1,s_2,\ldots,s_m$ may be arbitrary positive integers, considerable attention has been devoted to the case $s = s_1 = s_2 = \cdots = s_m$, that is, all the indices are the same. The same phenomenon occurs for (1.1) and its various generalizations. In particular, several explicit formulas are now known when all indices coincide ([14,15,17–19]). Some of the simplest and most fundamental results ([15]) are given as

$$\beta_n(\zeta_n;;\underbrace{1,\ldots,1}_{m}) = \frac{1}{m+1}\binom{n-1}{m} \tag{1.4}$$

and by applying symmetric functions ([20]), as a determinant

$$\beta_n(\zeta_n;;s) = \begin{vmatrix} \frac{n-1}{2} & 1 & 0 & \cdots & \\ \frac{2}{3}\binom{n-1}{2} & \frac{n-1}{2} & 1 & & \vdots \\ \vdots & & \ddots & & 0 \\ \frac{s-1}{s}\binom{n-1}{s-1} & \frac{1}{s-1}\binom{n-1}{s-2} & \cdots & \frac{n-1}{2} & 1 \\ \frac{s}{s+1}\binom{n-1}{s} & \frac{1}{s}\binom{n-1}{s-1} & \cdots & \frac{1}{3}\binom{n-1}{2} & \frac{n-1}{2} \end{vmatrix}. \tag{1.5}$$

Here, $\zeta_n = e^{2\pi\sqrt{-1}/n}$. In particular, by taking $s = 2,3,\ldots,9$ in (1.5), we have

$$\begin{aligned}
\beta_n(\zeta_n;;2) &= -\frac{(n-1)(n-5)}{12}\,,\\
\beta_n(\zeta_n;;3) &= -\frac{(n-1)(n-3)}{8}\,,\\
\beta_n(\zeta_n;;4) &= \frac{(n-1)(n^3+n^2-109n+251)}{6!}\,,\\
\beta_n(\zeta_n;;5) &= \frac{(n-1)(n-5)(n^2+6n-19)}{288}\,,\\
\beta_n(\zeta_n;;6) &= -\frac{(n-1)(2n^5+2n^4-355n^3-355n^2+11153n-19087)}{12\cdot 7!}\,,
\end{aligned}$$

$$\mathfrak{Z}_n(\zeta_n;;7) = -\frac{(n-1)(n-7)(2n^4+16n^3-33n^2-376n+751)}{24\cdot 6!},$$
$$\mathfrak{Z}_n(\zeta_n;;8) = \frac{(n-1)(3n^7+3n^6-917n^5-917n^4+39697n^3+39697n^2-744383n+1070017)}{10!},$$
$$\mathfrak{Z}_n(\zeta_n;;9) = \frac{27(n-1)(n-3)(n-9)(n^5+13n^4+10n^3-350n^2-851n+2857)}{2\cdot 10!}.$$

In the case of uniform exponents, one can directly utilize the properties of Stirling numbers ([12, 13]), and Stirling number identities, and the expansions of generating functions derived from symmetric functions. However, in the case of non-uniform exponents, these approaches are not applicable; consequently, there are relatively few explicit results. Even the apparently simplest non-uniform case $1-\cdots-1,2,1-\cdots-1$ proved to be delicate: the relevant real-part identity was conjectured in [2] and proved in [3].

Recently, in [3], the case $1-\cdots-1,2,1-\cdots-1$ was settled by evaluating the reversal pair, equivalently, the real part of each ordered sum. More precisely, for nonnegative integers a and b, the following was proved. For integers n and m with $n, m \geq 2$,

$$\mathfrak{Z}_n(\zeta_n;;\underbrace{\underbrace{1,\ldots,1}_{a},2,\underbrace{1,\ldots,1}_{b}}_{m}) + \mathfrak{Z}_n(\zeta_n;;\underbrace{\underbrace{1,\ldots,1}_{b},2,\underbrace{1,\ldots,1}_{a}}_{m}) = -\frac{m!(n-2m-3)}{(m+2)!}\binom{n-1}{m}.$$

In other words, by taking the real part,

$$\mathfrak{Re}\left(\mathfrak{Z}_n(\zeta_n;;\underbrace{\underbrace{1,\ldots,1}_{a},2,\underbrace{1,\ldots,1}_{b}}_{m})\right) = -\frac{m!(n-2m-3)}{2(m+2)!}\binom{n-1}{m}.$$

In the present paper, we develop the approach of [3] further, treat the general pattern $1-\cdots-1,A,1-\cdots-1$, and obtain explicit formulas for $A \geq 3$, though we do not generally evaluate each summand at a prescribed position of A.

2. Preliminaries

We first recall the following elementary facts from [3].

Complex conjugation reverses the order of the indices. Consequently, the corresponding sums have the same real part and opposite imaginary parts for the index strings $s_1 - s_2 - \cdots - s_m$ and those $s_m - \cdots - s_2 - s_1$.

Lemma 1.

$$\overline{\mathfrak{Z}_n(\zeta_n;;s_1,s_2,\ldots,s_m)} = \mathfrak{Z}_n(\zeta_n;;s_m,s_{m-1},\ldots,s_1).$$

For convenience, define

$$u_r := \frac{1}{1-\zeta_n^r} = \frac{1}{2} + \frac{\sqrt{-1}}{2}\cot\frac{r\pi}{n} \quad (1 \leq r \leq n-1). \tag{2.1}$$

For a positive integer A, put the sum

$$P_m^{(A,j)}(n) := \mathfrak{Z}_n(\zeta_n;;\underbrace{1,\dots,1}_{j-1},A,\underbrace{1,\dots,1}_{m-j}) = \sum_{1\le i_1<\cdots<i_m\le n-1} u_{i_1}\cdots u_{i_{j-1}}u_{i_j}^A u_{i_{j+1}}\cdots u_{i_m}.$$

We also consider the average of $P_m^{(A,j)}(n)$ over the position of the exceptional index:

$$Q_m^{(A)}(n) = \frac{1}{m}\sum_{j=1}^{m} P_m^{(A,j)}(n).$$

By using the elementary symmetric function, we write

$$e_k := e_k(u_1,\dots,u_{n-1}) = \sum_{1\le i_1<\cdots<i_k\le n-1} u_{i_1}\cdots u_{i_k}.$$

For a positive integer A, we also write

$$e_k^{(A)} := e_k(u_1^A,\dots,u_{n-1}^A),$$

so that $e_k = e_k^{(1)}$. Then, for $A \ge 2$ we have

$$\begin{aligned} e_1^{(A-1)}e_m &= \sum_{j=1}^{n-1} u_j^{A-1} \sum_{1\le i_1<\cdots<i_m\le n-1} u_{i_1}\cdots u_{i_m} \\ &= \sum_{j=1}^{m+1} P_{m+1}^{(A-1,j)}(n) + \sum_{|S|=m}\left(\sum_{j\in S} u_j^{A-1}\right)\prod_{h\in S} u_h \\ &= \sum_{j=1}^{m+1} P_{m+1}^{(A-1,j)}(n) + \sum_{j=1}^{m} P_m^{(A,j)}(n). \end{aligned} \tag{2.2}$$

Here, each term $u_j^{A-1}\prod_{h\in S} u_h$ with $j\notin S$ gives the monomial on $H = S\cup\{j\}$ in which j carries the exponent $A-1$; summing over the $m+1$ choices of $j\in H$ yields $\sum_{j=1}^{m+1} P_{m+1}^{(A-1,j)}(n)$. Each term with $j\in S$ gives the monomial on S in which j carries the exponent A, which yields $\sum_{j=1}^{m} P_m^{(A,j)}(n)$. In particular, for $A=2$ all $m+1$ terms coming from a given H are equal to $\prod_{h\in H} u_h$, so that $\sum_{j=1}^{m+1} P_{m+1}^{(1,j)}(n) = (m+1)e_{m+1}$.

3. Main results

We now obtain the following explicit polynomial evaluations.

Theorem 1. *For $n, m \ge 1$, we have*

$$\begin{aligned} &\sum_{j=1}^{m} \mathfrak{Z}_n(\zeta_n;;\underbrace{1,\dots,1}_{j-1},2,\underbrace{1,\dots,1}_{m-j}) \\ &= -\frac{m(n-2m-3)}{2(m+1)(m+2)}\binom{n-1}{m}, \end{aligned}$$

$$\sum_{j=1}^{m} \mathfrak{Z}_n(\zeta_n;;\underbrace{1,\dots,1}_{j-1},3,\underbrace{1,\dots,1}_{m-j})$$
$$= \frac{1}{m+1}\binom{n-1}{m}\left(-\frac{(n-1)(n-5)}{12}+\frac{(m+1)(n-2m-5)(n-m-1)}{2(m+2)(m+3)}\right),$$
$$\sum_{j=1}^{m} \mathfrak{Z}_n(\zeta_n;;\underbrace{1,\dots,1}_{j-1},4,\underbrace{1,\dots,1}_{m-j})$$
$$= -\frac{(n-1)(n-3)}{8(m+1)}\binom{n-1}{m}+\frac{(n-1)(n-5)}{12(m+2)}\binom{n-1}{m+1}-\frac{(m+2)(n-2m-7)}{2(m+3)(m+4)}\binom{n-1}{m+2},$$
$$\sum_{j=1}^{m} \mathfrak{Z}_n(\zeta_n;;\underbrace{1,\dots,1}_{j-1},5,\underbrace{1,\dots,1}_{m-j})$$
$$= \frac{(n-1)(n^3+n^2-109n+251)}{6!(m+1)}\binom{n-1}{m}+\frac{(n-1)(n-3)}{8(m+2)}\binom{n-1}{m+1}$$
$$-\frac{(n-1)(n-5)}{12(m+3)}\binom{n-1}{m+2}+\frac{(m+3)(n-2m-9)}{2(m+4)(m+5)}\binom{n-1}{m+3}.$$

Remark. For $A = 2$, the real part of $P_m^{(2,j)}(n)$ is the same for any j with $1 \le j \le m$ ([3]). For $A \ge 3$, reversal still gives $\Re\mathfrak{e}\, P_m^{(A,j)}(n) = \Re\mathfrak{e}\, P_m^{(A,m-j+1)}(n)$, but the real part depends on j in general. For example, for $n = 5$, $m = 3$ and $A = 3$, we have $\Re\mathfrak{e}\, P_3^{(3,1)}(5) = \Re\mathfrak{e}\, P_3^{(3,3)}(5) = -(3+\sqrt{5})/20$, whereas $P_3^{(3,2)}(5) = (\sqrt{5}-1)/10$. In particular, the real part at a fixed position need not be rational.

Proof. From [3], we know that for any j with $1 \le j \le m$, the real part of $P_m^{(2,j)}(n)$ is given as

$$-\frac{m!(n-2m-3)}{2(m+2)!}\binom{n-1}{m}$$

and the imaginary parts of $P_m^{(2,j)}(n)$ and $-P_m^{(2,m-j+1)}(n)$ are the same. Hence,

$$\sum_{j=1}^{m+1} P_{m+1}^{(2,j)}(n) = -\frac{(m+1)(m+1)!(n-2m-5)}{2(m+3)!}\binom{n-1}{m+1}.$$

By (1.4) and (1.5) with $s = 2$, we see that

$$e_m = \frac{1}{m+1}\binom{n-1}{m}$$

and

$$e_1^{(2)} = \mathfrak{Z}_n(\zeta_n;;2) = -\frac{(n-1)(n-5)}{12}.$$

Alternatively, since $P_{m+1}^{(1,j)}(n) = e_{m+1}$ for every j, (2.2) with $A = 2$ and (1.4) give directly

$$\sum_{j=1}^{m} P_m^{(2,j)}(n) = e_1 e_m - (m+1)e_{m+1} = -\frac{m(n-2m-3)}{2(m+1)(m+2)}\binom{n-1}{m};$$

the result of [3] is needed only for the stronger statement that each real part is independent of j. Hence, by (2.2), we have

$$\begin{aligned}
&\sum_{j=1}^{m} P_m^{(3,j)}(n) = e_1^{(2)} e_m - \sum_{j=1}^{m+1} P_{m+1}^{(2,j)}(n) \\
&= -\frac{(n-1)(n-5)}{12} \cdot \frac{1}{m+1}\binom{n-1}{m} + \frac{(m+1)(m+1)!(n-2m-5)}{2(m+3)!}\binom{n-1}{m+1} \\
&= -\frac{(n-1)(n-5)}{12(m+1)}\binom{n-1}{m} + \frac{(m+1)(n-2m-5)}{2(m+2)(m+3)}\binom{n-1}{m+1}. \qquad (3.1)
\end{aligned}$$

We next consider $A = 4$. By (1.5) with $s = 3$, we see that

$$e_1^{(3)} = \mathfrak{Z}_n(\zeta_n;;3) = -\frac{(n-1)(n-3)}{8}.$$

Hence, by (2.2) and (3.1), we have

$$\begin{aligned}
&\sum_{j=1}^{m} P_m^{(4,j)}(n) = e_1^{(3)} e_m - \sum_{j=1}^{m+1} P_{m+1}^{(3,j)}(n) \\
&= -\frac{(n-1)(n-3)}{8} \cdot \frac{1}{m+1}\binom{n-1}{m} \\
&\quad + \frac{(n-1)(n-5)}{12(m+2)}\binom{n-1}{m+1} - \frac{(m+2)(n-2m-7)}{2(m+3)(m+4)}\binom{n-1}{m+2} \\
&= -\frac{(n-1)(n-3)}{8(m+1)}\binom{n-1}{m} + \frac{(n-1)(n-5)}{12(m+2)}\binom{n-1}{m+1} - \frac{(m+2)(n-2m-7)}{2(m+3)(m+4)}\binom{n-1}{m+2}. \qquad (3.2)
\end{aligned}$$

Finally, let $A = 5$. By (1.5) with $s = 4$, we see that

$$e_1^{(4)} = \mathfrak{Z}_n(\zeta_n;;4) = \frac{(n-1)(n^3+n^2-109n+251)}{6!}.$$

Hence, by (2.2) and (3.2), we have

$$\begin{aligned}
&\sum_{j=1}^{m} P_m^{(5,j)}(n) = e_1^{(4)} e_m - \sum_{j=1}^{m+1} P_{m+1}^{(4,j)}(n) \\
&= \frac{(n-1)(n^3+n^2-109n+251)}{6!} \cdot \frac{1}{m+1}\binom{n-1}{m} \\
&\quad + \frac{(n-1)(n-3)}{8(m+2)}\binom{n-1}{m+1} - \frac{(n-1)(n-5)}{12(m+3)}\binom{n-1}{m+2} + \frac{(m+3)(n-2m-9)}{2(m+4)(m+5)}\binom{n-1}{m+3} \\
&= \frac{(n-1)(n^3+n^2-109n+251)}{6!(m+1)}\binom{n-1}{m} + \frac{(n-1)(n-3)}{8(m+2)}\binom{n-1}{m+1} \\
&\quad - \frac{(n-1)(n-5)}{12(m+3)}\binom{n-1}{m+2} + \frac{(m+3)(n-2m-9)}{2(m+4)(m+5)}\binom{n-1}{m+3}. \qquad (3.3)
\end{aligned}$$

□

Iterating (2.2) and using the depth-one evaluations above gives explicit polynomial expressions for these symmetrized sums for $A = 6, 7, \ldots$. The following subsection provides a uniform expression in terms of degenerate Bernoulli numbers. A uniform formula with explicit polynomial coefficients in n is less transparent, since for general s the depth-one values $\mathfrak{Z}_n(\zeta_n; ; s)$ themselves are available only through degenerate Bernoulli numbers, as in (3.6) below.

3.1. Degenerate Bernoulli numbers

Using degenerate Bernoulli numbers, we obtain a uniform expression for these symmetrized sums $\sum_{j=1}^{m} P_m^{(A,j)}(n)$.

The degenerate Bernoulli polynomials $\beta_n(x|\lambda)$ are defined by Carlitz [6] as

$$\frac{t(1+\lambda t)^{x/\lambda}}{(1+\lambda t)^{1/\lambda}-1} = \sum_{k=0}^{\infty} \beta_k(x|\lambda)\frac{t^k}{k!}. \tag{3.4}$$

When $x = 0$ in (3.4), the degenerate Bernoulli numbers $\beta_n(\lambda) = \beta_n(0|\lambda)$ are defined by Carlitz [5] as

$$\frac{t}{(1+\lambda t)^{1/\lambda}-1} = \sum_{k=0}^{\infty} \beta_k(\lambda)\frac{t^k}{k!}. \tag{3.5}$$

When $\lambda \to 0$, this generating function becomes $t/(e^t - 1)$, which is that of the classical Bernoulli numbers $B_n = \lim_{\lambda\to 0}\beta_n(\lambda)$. One of the expressions of degenerate Bernoulli numbers is given by Howard [10, Theorem 3.1] as

$$\beta_m(\lambda) = C_m\lambda^m + \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor} \frac{m}{2j} B_{2j} \begin{bmatrix} m-1 \\ 2j-1 \end{bmatrix} (-\lambda)^{m-2j} \quad (m \geq 2)$$

with

$$\beta_0(\lambda) = 1 \quad \text{and} \quad \beta_1(\lambda) = -\frac{1}{2} + \frac{\lambda}{2}.$$

Here, $\begin{bmatrix} n \\ k \end{bmatrix}$ denotes the (unsigned) Stirling numbers of the first kind, defined by

$$x(x-1)(x-2)\cdots(x-n+1) = \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} x^k.$$

C_m are Cauchy numbers ($b_m = C_m/m!$ are called Bernoulli numbers of the second kind), defined by the generating function

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} C_n \frac{t^n}{n!}$$

[8, 11, 21]. One of the expressions of Cauchy numbers is

$$C_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^{n-k}}{k+1} \quad (n \geq 1).$$

In [15, Theorem 4], the determinant expression of $\Im_n(\zeta_n;;s)$ in (1.5) is also given in terms of degenerate Bernoulli numbers as

$$\Im_n(\zeta_n;;s) = -\sum_{j=1}^{s}\binom{s-1}{j-1}\beta_j(n^{-1})\frac{n^j}{j!}\,. \tag{3.6}$$

For simplicity, for any positive integer A, put

$$\mathcal{F}_m^{(A)} = \sum_{j=1}^{m} P_m^{(A,j)}(n)\,.$$

We know that

$$\mathcal{F}_m^{(2)} = -\frac{m(n-2m-3)}{2(m+1)(m+2)}\binom{n-1}{m} \quad \text{and} \quad e_m = \frac{1}{m+1}\binom{n-1}{m}.$$

From (3.6), we see that

$$e_1^{(A)} = -\sum_{j=1}^{A}\binom{A-1}{j-1}\beta_j(n^{-1})\frac{n^j}{j!}\,.$$

As discussed above, we get

$$\begin{aligned}
\mathcal{F}_m^{(3)} &= e_1^{(2)}e_m - \mathcal{F}_{m+1}^{(2)}\,,\\
\mathcal{F}_m^{(4)} &= e_1^{(3)}e_m - \mathcal{F}_{m+1}^{(3)}\\
&= e_1^{(3)}e_m - e_1^{(2)}e_{m+1} + \mathcal{F}_{m+2}^{(2)}\,,\\
\mathcal{F}_m^{(5)} &= e_1^{(4)}e_m - e_1^{(3)}e_{m+1} + e_1^{(2)}e_{m+2} - \mathcal{F}_{m+3}^{(2)}\,.
\end{aligned}$$

Applying (2.2) repeatedly, we have

$$\begin{aligned}
\mathcal{F}_m^{(A)} &= \sum_{k=0}^{A-3}(-1)^k e_1^{(A-k-1)}e_{m+k} + (-1)^A\mathcal{F}_{m+A-2}^{(2)}\\
&= \sum_{k=0}^{A-3}(-1)^k(-1)\sum_{j=1}^{A-k-1}\binom{A-k-2}{j-1}\beta_j(n^{-1})\frac{n^j}{j!}\frac{1}{m+k+1}\binom{n-1}{m+k}\\
&\quad + (-1)^A(-1)\frac{(m+A-2)(n-2m-2A+1)}{2(m+A-1)(m+A)}\binom{n-1}{m+A-2}\\
&= \sum_{k=0}^{A-3}\sum_{j=1}^{A-k-1}\frac{(-1)^{k+1}n^j\beta_j(n^{-1})}{(m+k+1)j!}\binom{n-1}{m+k}\binom{A-k-2}{j-1}\\
&\quad + (-1)^{A+1}\frac{(m+A-2)(n-2m-2A+1)}{2(m+A-1)(m+A)}\binom{n-1}{m+A-2}.
\end{aligned}$$

We thus obtain the following explicit expression in terms of the degenerate Bernoulli numbers. The binomial coefficients outside their natural range are regarded as zero.

Theorem 2. *For an integer A with $A \geq 2$ and positive integers n and m, we have*

$$\sum_{j=1}^{m}\Im_n(\zeta_n;;\underbrace{1,\ldots,1}_{j-1},A,\underbrace{1,\ldots,1}_{m-j})$$

$$= \sum_{k=0}^{A-3} \sum_{i=1}^{A-k-1} \frac{(-1)^{k+1} n^i \beta_i(n^{-1})}{(m+k+1)i!} \binom{n-1}{m+k} \binom{A-k-2}{i-1} + (-1)^{A+1} \frac{(m+A-2)(n-2m-2A+1)}{2(m+A-1)(m+A)} \binom{n-1}{m+A-2}.$$

3.2. Another expression of $\mathfrak{Z}_n(\zeta_n;;s)$ in terms of degenerate Bernoulli polynomials

The degenerate Bernoulli polynomials provide a natural framework for a more compact formulation. The following result may also be useful in future extensions to more general index patterns. Theorem 3 expresses the depth-one sum through degenerate Bernoulli polynomials in (3.4) without the finite sum in (3.6).

Theorem 3. *For positive integers n and s, we have*

$$\mathfrak{Z}_n(\zeta_n;;s) = \frac{(-1)^s n^{s+1}}{(s+1)!} \left(\beta_{s+1}\left(\frac{n-1}{n} \middle| \frac{1}{n} \right) - \beta_{s+1}\left(0 \middle| \frac{1}{n} \right) \right).$$

To prove Theorem 3, we need the following identity. Here, define $\mathfrak{Z}_n(\zeta_n;;0) = n-1$. Both sides are regarded as power series in x convergent for $|x| < 2\sin(\pi/n)$.

Lemma 2. *For an integer n with $n \geq 2$, we have*

$$\sum_{s=0}^{\infty} \mathfrak{Z}_n(\zeta_n;;s) x^s = \frac{n(1-(1-x)^{n-1})}{1-(1-x)^n}.$$

Proof. We have

$$\sum_{s=0}^{\infty} \mathfrak{Z}_n(\zeta_n;;s) x^s = \sum_{s=0}^{\infty} \sum_{i=1}^{n-1} \frac{x^s}{(1-\zeta_n^i)^s} = \sum_{i=1}^{n-1} \frac{1-\zeta_n^i}{1-\zeta_n^i-x} = \sum_{i=0}^{n-1} \frac{1-\zeta_n^i}{1-\zeta_n^i-x}.$$

Now, by putting $y = 1-x$, we shall prove that

$$\sum_{i=0}^{n-1} \frac{1-\zeta_n^i}{1-\zeta_n^i-x} = \frac{n(1-y^{n-1})}{1-y^n}. \tag{3.7}$$

Since $y^n - 1 = \prod_{i=0}^{n-1}(y-\zeta_n^i)$, taking the logarithmic derivative with respect to y, we get

$$\frac{ny^{n-1}}{y^n-1} = \sum_{i=0}^{n-1} \frac{1}{y-\zeta_n^i}. \tag{3.8}$$

Next, put $f(t) = t^n - 1$ and $t_i = \zeta_n^i$ $(i = 0, 1, \ldots, n-1)$. Since $f'(t_i) = nt_i^{n-1} = n/t_i$, the partial fraction decomposition gives

$$\frac{1}{t^n - 1} = \sum_{i=0}^{n-1} \frac{1}{f'(t_i)(t - t_i)} = \frac{1}{n} \sum_{i=0}^{n-1} \frac{t_i}{t - t_i}.$$

Hence,

$$\frac{n}{y^n - 1} = \sum_{i=0}^{n-1} \frac{\zeta_n^i}{y - \zeta_n^i}. \tag{3.9}$$

By subtracting (3.9) from (3.8) side by side, we obtain (3.7). □

Proof of Theorem 3. By setting

$$(x, \lambda, t) = \left(\frac{n-1}{n}, \frac{1}{n}, -nx\right) \quad \text{and} \quad \left(0, \frac{1}{n}, -nx\right)$$

in degenerate Bernoulli polynomials in (3.4), we have

$$\frac{-nx(1-x)^{n-1}}{(1-x)^n - 1} = \sum_{s=0}^{\infty} \beta_s \left(\frac{n-1}{n} \middle| \frac{1}{n}\right) \frac{(-n)^s}{s!} x^s$$

and

$$\frac{-nx}{(1-x)^n - 1} = \sum_{s=0}^{\infty} \beta_s \left(0 \middle| \frac{1}{n}\right) \frac{(-n)^s}{s!} x^s,$$

respectively. Subtracting both sides of these two identities gives

$$\frac{-nx((1-x)^{n-1} - 1)}{(1-x)^n - 1} = \sum_{s=0}^{\infty} \frac{(-1)^s n^s}{s!} \left(\beta_s \left(\frac{n-1}{n} \middle| \frac{1}{n}\right) - \beta_s \left(0 \middle| \frac{1}{n}\right)\right) x^s.$$

Hence,

$$\frac{n((1-x)^{n-1} - 1)}{(1-x)^n - 1} = \sum_{s=0}^{\infty} \frac{(-1)^s n^{s+1}}{(s+1)!} \left(\beta_{s+1} \left(\frac{n-1}{n} \middle| \frac{1}{n}\right) - \beta_{s+1} \left(0 \middle| \frac{1}{n}\right)\right) x^s.$$

Together with Lemma 2, by comparing the coefficients on both sides, we get the desired result. □

4. Some expressions about $2 - \cdots - 2, A, 2 - \cdots - 2$

For convenience, define

$$R_m^{(A,j)}(n) = \sum_{1 \le i_1 < \cdots < i_m \le n-1} u_{i_1}^2 \cdots u_{i_{j-1}}^2 u_{i_j}^A u_{i_{j+1}}^2 \cdots u_{i_m}^2.$$

Then consider the sum

$$\mathcal{R}_m^{(A)} = \sum_{j=1}^{m} R_m^{(A,j)}(n).$$

In this setting, we have

$$e_1^{(A-1)}e_m^{(2)} = (u_1^{A-1} + \cdots + u_{n-1}^{A-1})\left(\sum_{1\le i_1<\cdots<i_m\le n-1} u_{i_1}^2\cdots u_{i_m}^2\right)$$
$$= \mathcal{R}_{m+1}^{(A-1)} + \mathcal{R}_m^{(A+1)}. \tag{4.1}$$

From [15, Theorem 5], we find that

$$e_m^{(2)} = \frac{1}{n(m+1)}\left(\binom{n-1}{m} + (-1)^m\binom{n-1}{2m+1}\right).$$

Also, for any j with $1 \le j \le m$, $R_m^{(2,j)}(n) = e_m^{(2)}$. Hence, when $A = 3$, together with the fact that $e_1^{(2)} = -(n-1)(n-5)/12$, by applying (4.1) we have

$$\begin{aligned}
\mathcal{R}_m^{(4)} &= e_1^{(2)}e_m^{(2)} - (m+1)e_{m+1}^{(2)}\\
&= -\frac{(n-1)(n-5)}{12}\frac{1}{n(m+1)}\left(\binom{n-1}{m} + (-1)^m\binom{n-1}{2m+1}\right)\\
&\quad - \frac{m+1}{n(m+2)}\left(\binom{n-1}{m+1} + (-1)^{m+1}\binom{n-1}{2m+3}\right)\\
&= -\frac{(n-1)(n-5)}{12n(m+1)}\left(\binom{n-1}{m} + (-1)^m\binom{n-1}{2m+1}\right)\\
&\quad - \frac{m+1}{n(m+2)}\left(\binom{n-1}{m+1} + (-1)^{m+1}\binom{n-1}{2m+3}\right).
\end{aligned}$$

Applying (4.1) as $A = 5$ and $A = 7$, we have

$$\begin{aligned}
\mathcal{R}_m^{(6)} &= e_1^{(4)}e_m^{(2)} - \mathcal{R}_{m+1}^{(4)}\\
&= \frac{(n-1)(n^3+n^2-109n+251)}{6!n(m+1)}\left(\binom{n-1}{m} + (-1)^m\binom{n-1}{2m+1}\right)\\
&\quad + \frac{(n-1)(n-5)}{12n(m+2)}\left(\binom{n-1}{m+1} + (-1)^{m+1}\binom{n-1}{2m+3}\right)\\
&\quad + \frac{m+2}{n(m+3)}\left(\binom{n-1}{m+2} + (-1)^m\binom{n-1}{2m+5}\right)\\
\mathcal{R}_m^{(8)} &= e_1^{(6)}e_m^{(2)} - \mathcal{R}_{m+1}^{(6)}\\
&= -\frac{(n-1)(2n^5+2n^4-355n^3-355n^2+11153n-19087)}{12\cdot 7!n(m+1)}\\
&\quad \times\left(\binom{n-1}{m} + (-1)^m\binom{n-1}{2m+1}\right)\\
&\quad - \frac{(n-1)(n^3+n^2-109n+251)}{6!n(m+2)}\left(\binom{n-1}{m+1} + (-1)^{m+1}\binom{n-1}{2m+3}\right)\\
&\quad - \frac{(n-1)(n-5)}{12n(m+3)}\left(\binom{n-1}{m+2} + (-1)^m\binom{n-1}{2m+5}\right)
\end{aligned}$$

$$-\frac{m+3}{n(m+4)}\left(\binom{n-1}{m+3}+(-1)^{m+1}\binom{n-1}{2m+7}\right).$$

The same recurrence gives explicit polynomial expressions for $\sum_{j=1}^{m} R_m^{(A,j)}(n)$ for every even A.

For example, when $m = 2$, we have

$$\begin{aligned}
&\mathfrak{Z}_n(\zeta_n;;2,4)+\mathfrak{Z}_n(\zeta_n;;4,2) = -\frac{(n-1)(n-2)(5n^4-27n^3-469n^2+5787n-13936)}{12\cdot 7!}\\
&\mathfrak{Z}_n(\zeta_n;;2,6)+\mathfrak{Z}_n(\zeta_n;;6,2)\\
&\quad = \frac{(n-1)(n-2)(7n^6-39n^5-946n^4+7950n^3+33743n^2-411111n+773596)}{10!}.
\end{aligned}$$

4.1. Odd case

For odd values of A, we instead use an explicit expression of $\mathcal{R}_m^{(1)}$, which appears in [9, Remark (14) of Theorem 6].

Lemma 3.

$$\mathcal{R}_m^{(1)} = \frac{1}{n}\left(\binom{n-1}{m}+(-1)^{m-1}\binom{n-1}{2m}\right). \tag{4.2}$$

By applying (4.1), together with Lemma 3 (4.2), we have

$$\begin{aligned}
\mathcal{R}_m^{(3)} &= e_1^{(1)}e_m^{(2)} - \mathcal{R}_{m+1}^{(1)}\\
&= \frac{n-1}{2}\frac{1}{n(m+1)}\left(\binom{n-1}{m}+(-1)^m\binom{n-1}{2m+1}\right)\\
&\quad -\frac{1}{n}\left(\binom{n-1}{m+1}+(-1)^m\binom{n-1}{2m+2}\right)\\
&= -\frac{n-2m-1}{2n^2}\binom{n}{m+1}+\frac{(2m+1)(-1)^m}{n^2}\binom{n}{2m+2}.
\end{aligned}$$

Applying (4.1) as $A = 4$ and $A = 6$, we have

$$\begin{aligned}
\mathcal{R}_m^{(5)} &= e_1^{(3)}e_m^{(2)} - \mathcal{R}_{m+1}^{(3)}\\
&= -\frac{(n-1)(n-3)}{8n(m+1)}\left(\binom{n-1}{m}+(-1)^m\binom{n-1}{2m+1}\right)\\
&\quad +\frac{n-2m-3}{2n^2}\binom{n}{m+2}-\frac{(2m+3)(-1)^{m+1}}{n^2}\binom{n}{2m+4},\\
\mathcal{R}_m^{(7)} &= e_1^{(5)}e_m^{(2)} - \mathcal{R}_{m+1}^{(5)}\\
&= \frac{(n-1)(n-5)(n^2+6n-19)}{288n(m+1)}\left(\binom{n-1}{m}+(-1)^m\binom{n-1}{2m+1}\right)\\
&\quad +\frac{(n-1)(n-3)}{8n(m+2)}\left(\binom{n-1}{m+1}+(-1)^{m+1}\binom{n-1}{2m+3}\right)\\
&\quad -\frac{n-2m-5}{2n^2}\binom{n}{m+3}+\frac{(2m+5)(-1)^m}{n^2}\binom{n}{2m+6}.
\end{aligned}$$

4.2. Expressions in terms of degenerate Bernoulli numbers

As in Theorem 2, a general expression for $\mathcal{R}_m^{(A)}$ can be obtained in terms of the degenerate Bernoulli numbers defined in (3.5). The binomial coefficients outside their natural range are regarded as zero.

Theorem 4. *For even A with $A \geq 2$ and positive integers n and m, we have*

$$\begin{aligned}
&\sum_{j=1}^{m} \mathfrak{Z}_n(\zeta_n;;\underbrace{2,\dots,2}_{j-1},A,\underbrace{2,\dots,2}_{m-j}) \\
&= \sum_{k=0}^{\frac{A}{2}-2} \sum_{i=1}^{A-2k-2} \frac{(-1)^{k+1} n^i \beta_i(n^{-1})}{n(m+k+1)i!} \binom{A-2k-3}{i-1} \left(\binom{n-1}{m+k} + (-1)^{m+k} \binom{n-1}{2m+2k+1} \right) \\
&\quad + (-1)^{\frac{A}{2}-1} \frac{m+\frac{A}{2}-1}{n(m+\frac{A}{2})} \left(\binom{n-1}{m+\frac{A}{2}-1} + (-1)^{m+\frac{A}{2}-1} \binom{n-1}{2m+A-1} \right).
\end{aligned}$$

For odd A with $A \geq 1$, we have

$$\begin{aligned}
&\sum_{j=1}^{m} \mathfrak{Z}_n(\zeta_n;;\underbrace{2,\dots,2}_{j-1},A,\underbrace{2,\dots,2}_{m-j}) \\
&= \sum_{k=0}^{\frac{A-3}{2}} \sum_{i=1}^{A-2k-2} \frac{(-1)^{k+1} n^i \beta_i(n^{-1})}{n(m+k+1)i!} \binom{A-2k-3}{i-1} \left(\binom{n-1}{m+k} + (-1)^{m+k} \binom{n-1}{2m+2k+1} \right) \\
&\quad + (-1)^{\frac{A-1}{2}} \frac{1}{n} \left(\binom{n-1}{m+\frac{A-1}{2}} + (-1)^{m+\frac{A-3}{2}} \binom{n-1}{2m+A-1} \right).
\end{aligned}$$

Proof. When $A = 2A'$ is even with $A' \geq 2$, we have

$$\begin{aligned}
&\mathcal{R}_m^{(2A')} \\
&= e_1^{(2A'-2)} e_m^{(2)} - \mathcal{R}_{m+1}^{(2A'-2)} \\
&= \sum_{k=0}^{A'-2} (-1)^k e_1^{(2A'-2k-2)} e_{m+k}^{(2)} + (-1)^{A'-1} \mathcal{R}_{m+A'-1}^{(2)} \\
&= \sum_{k=0}^{A'-2} (-1)^k (-1) \sum_{j=1}^{2A'-2k-2} \binom{2A'-2k-3}{j-1} \beta_j(n^{-1}) \frac{n^j}{j!} \\
&\qquad \times \frac{1}{n(m+k+1)} \left(\binom{n-1}{m+k} + (-1)^{m+k} \binom{n-1}{2m+2k+1} \right) \\
&\quad + (-1)^{A'-1} \frac{m+A'-1}{n(m+A')} \left(\binom{n-1}{m+A'-1} + (-1)^{m+A'-1} \binom{n-1}{2m+2A'-1} \right) \\
&= \sum_{k=0}^{A'-2} \sum_{j=1}^{2A'-2k-2} \frac{(-1)^{k+1} n^j \beta_j(n^{-1})}{n(m+k+1)j!} \binom{2A'-2k-3}{j-1} \left(\binom{n-1}{m+k} + (-1)^{m+k} \binom{n-1}{2m+2k+1} \right) \\
&\quad + (-1)^{A'-1} \frac{m+A'-1}{n(m+A')} \left(\binom{n-1}{m+A'-1} + (-1)^{m+A'-1} \binom{n-1}{2m+2A'-1} \right).
\end{aligned}$$

The case where $A = 2$ can be checked separately using $\mathcal{R}_m^{(2)} = me_m^{(2)}$.

When $A = 2A' + 1$ is odd with $A' \geq 1$, we have

$$\begin{aligned}
&\mathcal{R}_m^{(2A'+1)} \\
&= e_1^{(2A'-1)} e_m^{(2)} - \mathcal{R}_{m+1}^{(2A'-1)} \\
&= \sum_{k=0}^{A'-1} (-1)^k e_1^{(2A'-2k-1)} e_{m+k}^{(2)} + (-1)^{A'} \mathcal{R}_{m+A'}^{(1)} \\
&= \sum_{k=0}^{A'-1} (-1)^k (-1) \sum_{j=1}^{2A'-2k-1} \binom{2A'-2k-2}{j-1} \beta_j(n^{-1}) \frac{n^j}{j!} \\
&\quad \times \frac{1}{n(m+k+1)} \left(\binom{n-1}{m+k} + (-1)^{m+k} \binom{n-1}{2m+2k+1} \right) \\
&\quad + (-1)^{A'} \frac{1}{n} \left(\binom{n-1}{m+A'} + (-1)^{m+A'-1} \binom{n-1}{2m+2A'} \right) \\
&= \sum_{k=0}^{A'-1} \sum_{j=1}^{2A'-2k-1} \frac{(-1)^{k+1} n^j \beta_j(n^{-1})}{n(m+k+1)j!} \binom{2A'-2k-2}{j-1} \left(\binom{n-1}{m+k} + (-1)^{m+k} \binom{n-1}{2m+2k+1} \right) \\
&\quad + (-1)^{A'} \frac{1}{n} \left(\binom{n-1}{m+A'} + (-1)^{m+A'-1} \binom{n-1}{2m+2A'} \right).
\end{aligned}$$

This is also valid for $A' = 0$ because the first term is nullified and the second term is the right-hand side of the identity (4.2) in Lemma 3. □

5. Comments and future work

The recurrence strategy developed in this paper applies to the sum over the exceptional position. It seems possible to derive explicit formulas for the sum over j of

$$\sum_{1 \leq i_1 < \cdots < i_m \leq n-1} u_{i_1}^3 \cdots u_{i_{j-1}}^3 u_{i_j}^A u_{i_{j+1}}^3 \cdots u_{i_m}^3 . \tag{5.1}$$

Once explicit formulas for the basic cases $A = 1, 2, 3$ in (5.1) are established, the same strategy may lead to a systematic treatment of this broader class of sums.

A fixed-position evaluation $\mathfrak{Z}_n(\zeta_n; ; \underbrace{1, \ldots, 1}_{j-1}, A, \underbrace{1, \ldots, 1}_{m-j})$ for each j is a separate question.

6. Conclusions

In this paper we have established explicit expressions for q-harmonic sums on $1-\cdots-1, A, 1-\cdots-1$ indices when $A \geq 3$, though we do not generally evaluate each summand at a prescribed position of A. The cases when $A = 1, 2$ have already been treated in the previous articles. The explicit formulas are obtained in terms of binomial coefficients and degenerate Bernoulli numbers. Some expressions about $2-\cdots-2, A, 2-\cdots-2$ are also shown.

Author contributions

T. K.: Writing – original draft; H. I.: Writing – review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors have used Artificial Intelligence (AI) tools to check the English text and make sure the formulas are correct.

Acknowledgments

The authors thank the referees for carefully reading the manuscript and for giving constructive comments.

Conflict of interest

The authors declare that there is no conflict of interest.

References

1. H. Bachmann, Y. Takeyama, K. Tasaka, Cyclotomic analogues of finite multiple zeta values, *Compositio Math.*, **154** (2018), 2701–2721. https://doi.org/10.1112/S0010437X18007583
2. H. Bachmann, Y. Takeyama, K. Tasaka, *Special values of finite multiple harmonic q-series at roots of unity*, Algebraic combinatorics, resurgence, moulds and applications (CARMA), Vol. 2, 1–18. IRMA Lect. Math. Theor. Phys., 32; EMS Publishing House, Berlin, 2020.
3. Y. Bilu, H. Ishikawa, T. Komatsu, Some explicit values of a q-multiple zeta function whose denominator power is not uniform, *arXiv preprint*, 2025. https://doi.org/10.48550/arXiv.2512.06672
4. D. M. Bradley, Multiple q-zeta values, *J. Algebra*, **283** (2005), 752–798. https://doi.org/10.1016/j.jalgebra.2004.09.017
5. L. Carlitz, A degenerate Staudt-Clausen theorem, *Arch. Math.*, **7** (1956), 28–33. https://doi.org/10.1007/BF01900520
6. L. Carlitz, Degenerate Stirling, Bernoulli and Eulerian numbers, *Util. Math.*, **15** (1979), 51–88.
7. T. Chatterjee, T. Komatsu, Special values of a q-multiple t-function of general level at roots of unity, *Mediterr. J. Math.*, **22** (2025), 226. http://dx.doi.org/10.1007/s00009-025-02993-1
8. L. Comtet, *Advanced combinatorics*, Dordrecht, Holland, D. Reidel, 1974.
9. Z. Dong, T. Komatsu, Finite q-multiple harmonic sums at roots of unity: Symmetrized identities for 1-2 index multisets, *arXiv preprint*, 2026. https://doi.org/10.48550/arXiv.2601.21395
10. F. T. Howard, Explicit formulas for degenerate Bernoulli numbers, *Discrete Math.*, **162** (1996), 175–185. https://doi.org/10.1016/0012-365X(95)00284-4

11. T. Komatsu, Poly-Cauchy numbers, *Kyushu. J. Math.*, **67** (2013), 143–153. https://doi.org/10.2206/kyushujm.67.143
12. T. Komatsu, On s-Stirling transform and poly-Cauchy numbers of the second kind with level 2, *Aequationes Math.*, **97** (2023), 31–61. https://doi.org/10.1007/s00010-022-00931-0
13. T. Komatsu, On q-generalized (r, s)-Stirling numbers, *Aequationes Math.*, **98** (2024), 1281–1304. http://dx.doi.org/10.1007/s00010-023-00997-4
14. T. Komatsu, Some explicit values of a q-multiple zeta-star function at roots of unity, *RAIRO-Theor. Inf. Appl.*, **59** (2025). https://doi.org/10.1051/ita/2025005
15. T. Komatsu, Some explicit values of a q-multiple zeta function at roots of unity, *J. Math. Anal. Appl.*, **555** (2026), 130065. https://doi.org/10.1016/j.jmaa.2025.130065
16. T. Komatsu, Obituary – Florian Luca (1969–2026), *RAIRO-Theor. Inf. Appl.*, **60** (2026), 44. https://doi.org/10.1051/ita/2026044
17. T. Komatsu, F. Luca, Some explicit forms of special values of an alternating q-multiple t-function of general level at roots of unity, *Ramanujan J.*, **68** (2025). http://dx.doi.org/10.1007/s11139-025-01150-2
18. T. Komatsu, R. K. Pandey, Some explicit values of an alternative q-multiple zeta function at roots of unity, *Log. J. IGPL*, **33** (2025), jzaf075. http://dx.doi.org/10.1093/jigpal/jzaf075
19. T. Komatsu, T. Wang, Some explicit values of a q-multiple t-function at roots of unity, *Aequationes Math.*, **99** (2025), 2377–2400. http://dx.doi.org/10.1007/s00010-025-01235-9
20. I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2 Eds., Clarendon Press, Oxford, 1995.
21. D. Merlini, R. Sprugnoli, M. C. Verri, The Cauchy numbers, *Discrete Math.*, **306** (2006), 1906–1920. https://doi.org/10.1016/j.disc.2006.03.065
22. Y. Ohno, J. Okuda, W. Zudilin, Cyclic q-MZSV sum, *J. Number Theory*, **132** (2012), 144–155. https://doi.org/10.1016/j.jnt.2011.08.001
23. K. G. Schlesinger, Some remarks on q-deformed multiple polylogarithms, *arXiv preprint*, 2001. https://doi.org/10.48550/arXiv.math/0111022
24. Y. Takeyama, A q-analogue of non-strict multiple zeta values and basic hypergeometric series, *Proc. Amer. Math. Soc.*, **137** (2009), 2997–3002. https://doi.org/10.1090/S0002-9939-09-09931-6
25. K. Tasaka, Finite and symmetric colored multiple zeta values and multiple harmonic q-series at roots of unity, *Sel. Math.-New Ser.*, **27** (2021). https://doi.org/10.1007/s00029-021-00636-3
26. J. Zhao, Multiple q-zeta functions and multiple q-polylogarithms, *Ramanujan J.*, **14** (2007), 189–221. https://doi.org/10.1007/s11139-007-9025-9
27. J. Zhao, *Multiple zeta functions, multiple polylogarithms and their special values*, Series on Number Theory and Its Applications 12, Hackensack, NJ; World Scientific, 2016.
28. W. Zudilin, Algebraic relations for multiple zeta values (Russian), *Uspekhi Mat. Nauk*, **58** (2003), 3–32. https://doi.org/10.4213/rm592



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0)