AIMS Mathematics, 11(11): 37512–37531.
DOI: 10.3934/math.20261451
Received: 09 September 2026
Revised: 21 October 2026
Accepted: 28 October 20266
Published: 12 November 2026

https://www.aimspress.com/journal/Math

Research article

Global existence analysis for a two-species cross-diffusion system with volume-filling

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Abstract: The system of multispecies chemotaxis equations is a cross-diffusion system with volume-filling. In this work, we apply the entropy method to prove that the two-species chemotaxis system has a weak solution. Previous investigations indicate that traditional entropies in volume-filling cases may not be able to provide required gradient estimates. In this situation, we upgrade existing matrix computation methods to derive gradient estimates. We also apply the distance functional to show that when parameters of the chemotaxis system are identical, the weak solution is unique.

Keywords: two-species chemotaxis system; entropy method; cross-diffusion system with volume-filling
Mathematics Subject Classification: 35K40, 35K55, 35Q92

1. Introduction

1.1. Motivation

In this paper, we analyze a two-species cross-diffusion system with volume-filling. The system is called the two-species chemotaxis equations, and is used to describe the movement of two different cell types in crowded and uncrowded tissues. Refer to [1] for a detailed explanation of this model.

In [2], the two-species chemotaxis system was investigated as a cross-diffusion system with volume-filling, but it did not show that a weak solution of this system exists. This work is a continuation of these references, showing that the system has a weak solution.

The global existence analysis of this chemotaxis system in [2] indicated that the method to derive efficient gradient estimates is the key factor for the existence proof. The upgraded entropy estimate implies the existence of a weak solution, which is the main contributions of this work.

Let us denote $\mathcal{D} = \{u = (u_1, u_2) : u_1, u_2 > 0, u_1 + u_2 < 1\}$, where $\overline{\mathcal{D}}$ refers to the closure set of $\mathcal{D}$. For $u = (u_1, u_2) : \Omega \times (0, \infty) \to \mathcal{D}$, the two-species chemotaxis system refers to parabolic partial

differential equations in the form of

$$\partial_t u = \text{div}\,(\mathbf{A}(u)\nabla u) \text{ in } Q_T = \Omega \times (0,T),\ T > 0, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$, $d \geq 1$ is a bounded domain with Lipschitz boundary, and u_1, u_2 refer to cell densities. The diffusion matrix $\mathbf{A}(u) = (A_{ij}(u))$ is given by

$$\mathbf{A}(u) = \begin{pmatrix} \alpha_1(1-u_2)+u_2 & (\alpha_1-1)u_1 \\ (\alpha_2-1)u_2 & \alpha_2(1-u_1)+u_1 \end{pmatrix}, \quad \alpha_1, \alpha_2 > 0,\ \alpha_1, \alpha_2 \neq 1. \tag{1.2}$$

Equations subject to initial and no-flux boundary conditions are as follows:

$$\begin{aligned} &(A_{11}(u)\nabla u_1 + A_{12}(u)\nabla u_2)\cdot \nu = 0, \quad (A_{21}(u)\nabla u_1 + A_{22}(u)\nabla u_2)\cdot \nu = 0 \text{ on } \partial\Omega,\ t > 0, \\ &u_1(0) = u_1^0, u_2(0) = u_2^0 \text{ in } \Omega, \end{aligned} \tag{1.3}$$

where ν is an exterior unit normal vector to $\partial\Omega$, and $u^0 = (u_1^0, u_2^0)$ is the initial datum.

If the weak solution $u = (u_1, ..., u_n)$ of a cross-diffusion system satisfies that $u \in \overline{\mathcal{V}}$, $\mathcal{V} = \{u = (u_1, ..., u_n) : u_i > 0, \sum_{i=1}^n u_i < 1\}$, then we say that the system has the volume-filling effect. A chemotaxis system is a typical cross-diffusion system with volume-filling.

Cross-diffusion systems were divided into two groups in [2]. Degenerate systems [3, 4] and Maxwell-Stefan equations [5, 6] are typical cross-diffusion systems with volume-filling. The Shigesada-Kawasaki-Teramoto type population system [7, 8] and semiconductor model with electron-hole scattering [9, 10] are cross-diffusion systems without volume-filling.

After these preparations, we present the definition of weak solutions to (1.1)–(1.3) and Theorem 1.1.

Definition 1.1. *For every $T > 0$, $u^0 = (u_1^0, u_2^0) \in \mathcal{D}$, if $u = (u_1, u_2)$ satisfies $u \in \overline{\mathcal{D}}$, and for every test function $\phi = (\phi_1, \phi_2)$, $\phi_1, \phi_2 \in L^2(0,T;H^1(\Omega))$,*

$$\begin{aligned} &\int_0^T \langle \partial_t u_1, \phi_1\rangle dt + \int_{Q_T} (\alpha_1\nabla u_1 + (\alpha_1-1)(u_1\nabla u_2 - u_2\nabla u_1))\cdot\nabla\phi_1 dxdt = 0, \\ &\int_0^T \langle \partial_t u_2, \phi_2\rangle dt + \int_{Q_T} (\alpha_2\nabla u_2 - (\alpha_2-1)(u_1\nabla u_2 - u_2\nabla u_1))\cdot\nabla\phi_2 dxdt = 0, \end{aligned} \tag{1.4}$$

with the property that $u_i \in L^2(0,T;H^1(\Omega))$, $\partial_t u_i \in L^2(0,T;H^1(\Omega)')$, $i = 1, 2$, then we call u is a weak solution of (1.1)–(1.3).

Theorem 1.1. *Let $u^0 = (u_1^0, u_2^0)$ be such that $u^0 \in \mathcal{D}$, $u_1^0(x), u_2^0(x) \in [a,b]$ for $x \in \Omega$, where $0 < a < b < 1$. There exists a weak solution $u \in \overline{\mathcal{D}}$ of* (1.1)–(1.3).

How to show the weak solution of a cross-diffusion system is unique is very difficult due to the existence of the cross-diffusion phenomenon. So far, few uniqueness results have been derived.

Among these existing uniqueness results, we usually require that the diffusion coefficients, the parameters of a cross-diffusion system, satisfy certain assumptions. In this work, we will show Theorem 1.2.

Theorem 1.2. *Let $u^0 = (u_1^0, u_2^0)$ be such that $u^0 \in \mathcal{D}$. The weak solution $u \in \overline{\mathcal{D}}$ of* (1.1)–(1.3) *is unique, if $\alpha_1 = \alpha_2 = \alpha$.*

Once $\alpha_1 \neq \alpha_2$, we conjecture that the weak solution still exists uniquely, though it remains very difficult to prove this conjecture.

1.2. Major difficulties and novelties

The entropy method has been demonstrated as a powerful tool in global existence analysis of cross-diffusion systems. How to apply the entropy method to show the existence of weak solutions of cross-diffusion systems was discussed in [3–7]. Let us present these assumptions:

(H1) There exists a convex function $h \in C^2(\mathcal{V}, (0,\infty))$ such that its derivative $h' : \mathcal{V} \to \mathbb{R}^n$ is invertible on $\mathbb{R}^n$.

(H2) For every $z = (z_1, ..., z_n)^{\mathrm{T}} \in \mathbb{R}^n$ and $u = (u_1, ..., u_n) \in \mathcal{V}$, there exists a constant $c > 0$ independent of z and u, such that

$$z^{\mathrm{T}} h''(u)\mathbf{A}(u)z \geq c \sum_{i=1}^{n} z_i^2.$$

Here, $\mathbf{H}(u) = (H_{ij}(u)) = h''(u)$ is the Hessian matrix of $h(u)$, and $\mathbf{A}(u)$ is the diffusion matrix of the cross-diffusion system. For every $1 \leq i, j \leq n$, $H_{ij}(u) = \partial^2 h/\partial u_i \partial u_j$.

(H3) There exists a constant $c' > 0$ such that for $u \in \mathcal{V}$ and $i, j = 1, ..., n$, $|A_{ij}(u)| < c'$.

If $h(u)$ and $\mathbf{A}(u)$ satisfy Assumptions (H1)–(H3), then the entropy method can be applied to show that a weak solution of the cross-diffusion system exists. Assumptions (H1)–(H3) are intimately related to the concept of "entropy structure" in [11]. The entropy structure is an essential structural character of cross-diffusion systems.

In the global existence analysis of Maxwell-Stefan equations in [5], the entropy was chosen as

$$h(\rho') = k \sum_{i=1}^{N+1} x_i(\ln x_i - 1) + k, \quad \rho' = (\rho_1, ..., \rho_N). \tag{1.5}$$

In (1.5), x_i and ρ_i are related by $x_i = \rho_i/(kM_i)$, $\rho_i > 0$ with $k = \sum_{i=1}^{N+1}(\rho_i/M_i)$ and $\sum_{i=1}^{N+1} \rho_i = \sum_{i=1}^{N+1} x_i = 1$. $M_i > 0$, $i = 1, ..., N+1$ are constants.

For degenerate systems in [4], the entropy was

$$h(u) = \sum_{i=1}^{n}(u_i \log u_i - u_i + 1) + \int_a^{u_{n+1}} \log q(s)ds + \chi(u), \quad a \in (0,1], \ \sum_{i=1}^{n+1} u_i = 1, \ u_i > 0, \tag{1.6}$$

and $q(s), \chi(u)$ were functions defined in (7) and (8) of [4].

For the Shigesada-Kawasaki-Teramoto type population system in [7], the entropy was

$$h(u) = \sum_{i=1}^{n} \pi_i u_i(\log u_i - 1), \quad \pi_i > 0. \tag{1.7}$$

For the two-species chemotaxis system, let us denote $u_3 = 1 - u_1 - u_2$, and consider the entropy

$$h(u) = u_1(\log u_1 - 1) + u_2(\log u_2 - 1) + u_3(\log u_3 - 1). \tag{1.8}$$

We remark that $h(u)$ in (1.8) is a major functional in showing the existence of weak solutions of

volume-filling cross-diffusion systems. For $u \in \mathcal{D}$, we have

$$\begin{aligned} h''(u)\mathbf{A}(u) &= \left\{ \begin{pmatrix} \frac{1}{u_1} & 0 \\ 0 & \frac{1}{u_2} \end{pmatrix} + \frac{1}{u_3} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\} \cdot \begin{pmatrix} \alpha_1(1-u_2)+u_2 & (\alpha_1-1)u_1 \\ (\alpha_2-1)u_2 & \alpha_2(1-u_1)+u_1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{\alpha_1}{u_1} + (1-\alpha_1)\cdot\frac{u_2}{u_1} & \alpha_1 - 1 \\ \alpha_2 - 1 & \frac{\alpha_2}{u_2} + (1-\alpha_2)\cdot\frac{u_1}{u_2} \end{pmatrix} \\ &\quad + \frac{1}{u_3} \begin{pmatrix} \alpha_1 + (\alpha_2-\alpha_1)u_2 & \alpha_2 + (\alpha_1-\alpha_2)u_1 \\ \alpha_1 + (\alpha_2-\alpha_1)u_2 & \alpha_2 + (\alpha_1-\alpha_2)u_1 \end{pmatrix}. \end{aligned} \tag{1.9}$$

For the general $\alpha_1 \neq \alpha_2$ case, we cannot verify Assumption (H2) for $h(u)$ and $\mathbf{A}(u)$ in (1.9). The major novelty of this work is the matrix analysis in Section 2. For each α_1, α_2 case, we verify Assumptions (H1)–(H3). Then, we apply the entropy method to show that a weak solution of chemotaxis system exists.

In order to show that the weak solution of a cross-diffusion system is unique, we apply the distance functional to measure the "distance" between two weak solutions of the system. Once the "distance" between any two weak solutions of the system is zero, we deduce that the weak solution is unique.

The distance functional technique was applied in [4, 12–14]. In [12], the uniqueness result was shown for a weak solution that is not bounded in general. In [13], uniqueness results for several typical cross-diffusion systems were discussed.

In general, it is very difficult to construct an appropriate distance functional to show that the weak solution of a cross-diffusion system is unique. In [15,16], weaker uniqueness results for cross-diffusion systems were shown. We call such a kind of uniqueness weak-strong uniqueness.

In these references, so-called relative entropy functionals were applied to measure distances between weak and strong solutions. In this manuscript, we still focus on the uniqueness proof for weak solutions.

We assume that $v = (v_1, v_2)$ is also a weak solution of (1.1)–(1.3). For every test function $\phi = (\phi_1, \phi_2)$, $\phi_1, \phi_2 \in L^2(0,T;H^1(\Omega))$, we have

$$\begin{aligned} &\int_0^T \langle \partial_t v_1, \phi_1 \rangle dt + \int_{Q_T} (\alpha_1 \nabla v_1 + (\alpha_1 - 1)(v_1 \nabla v_2 - v_2 \nabla v_1)) \cdot \nabla \phi_1 dx dt = 0, \\ &\int_0^T \langle \partial_t v_2, \phi_2 \rangle dt + \int_{Q_T} (\alpha_2 \nabla v_2 - (\alpha_2 - 1)(v_1 \nabla v_2 - v_2 \nabla v_1)) \cdot \nabla \phi_2 dx dt = 0, \end{aligned} \tag{1.10}$$

with the property that $v_i \in L^2(0,T;H^1(\Omega))$, $\partial_t v_i \in L^2(0,T;H^1(\Omega)')$, $i = 1,2$. The initial datum is $v^0 = (v_1^0, v_2^0) = u^0$.

Let us take sums of the equations in (1.4) and (1.10). For every test function $\varphi \in L^2(0,T;H^1(\Omega))$, we have

$$\begin{aligned} \langle \partial_t((\alpha_2-1)u_1 + (\alpha_1-1)u_2), \varphi \rangle &= \langle \Delta(\alpha_1(\alpha_2-1)u_1 + \alpha_2(\alpha_1-1)u_2), \varphi \rangle, \\ \langle \partial_t((\alpha_2-1)v_1 + (\alpha_1-1)v_2), \varphi \rangle &= \langle \Delta(\alpha_1(\alpha_2-1)v_1 + \alpha_2(\alpha_1-1)v_2), \varphi \rangle. \end{aligned} \tag{1.11}$$

In (1.11), we notice that once $\alpha_1 = \alpha_2 = \alpha$, we have

$$\langle \partial_t(u_1+u_2), \varphi \rangle = \alpha \langle \Delta(u_1+u_2), \varphi \rangle, \quad \langle \partial_t(v_1+v_2), \varphi \rangle = \alpha \langle \Delta(v_1+v_2), \varphi \rangle, \quad u^0 = v^0 \in \mathcal{D}. \tag{1.12}$$

We can show that $u_1 + u_2 = v_1 + v_2$ for almost everywhere(a.e.) $(x,t) \in \Omega \times (0,T)$ merely by the relation (1.12). For the general $\alpha_1 \neq \alpha_2$ case, we cannot show that $(\alpha_2 - 1)u_1 + (\alpha_1 - 1)u_2 = (\alpha_2 - 1)v_1 + (\alpha_1 - 1)v_2$ for a.e. $(x,t) \in \Omega \times (0,T)$ merely by (1.11).

We conjecture that for every $\alpha_1, \alpha_2 > 0, \alpha_1, \alpha_2 \neq 1$, the relation $(\alpha_2 - 1)u_1 + (\alpha_1 - 1)u_2 = (\alpha_2 - 1)v_1 + (\alpha_1 - 1)v_2$ holds for a.e. $(x,t) \in \Omega \times (0,T)$. It remains one of the major challenges in the study of parabolic partial differential equations. Some further remarks will be provided in the final section.

In [4, 12], the distance functional was chosen as

$$d_\varepsilon(u,v) = \sum_{i=1,2} \int_\Omega (\xi_\varepsilon(u_i) + \xi_\varepsilon(v_i) - 2\xi_\varepsilon(\frac{u_i + v_i}{2}))dx, \quad 0 < \varepsilon < 1, \tag{1.13}$$

where

$$\xi_\varepsilon(s) = (s+\varepsilon)(\log(s+\varepsilon) - 1) + 1, \quad s \geq 0.$$

By the convexity of ξ_ε, we have $\xi_\varepsilon(u_i) + \xi_\varepsilon(v_i) - 2\xi_\varepsilon((u_i + v_i)/2) \geq 0$ in Ω; thus, $d_\varepsilon(u,v) \geq 0$. Because u_i, v_i, $i = 1, 2$ are only non-negative, we have to add the regularizer $0 < \varepsilon < 1$ in (1.13).

In Theorem 1 of [13], once we choose $\bar{u} = u_1 + u_2$, $p(s) = \alpha - (\alpha - 1)s$, $q(s) = \alpha - 1$ and $r = 0$ (notations are defined in [13]), we notice that this theorem directly yields the uniqueness result Theorem 1.2. For integrity purposes, we still apply the distance functional in (1.13) to show that the weak solution of (1.1)–(1.3) is unique in Section 4.

1.3. Main steps

We divide the existence proof into four steps.

Step 1. In Section 2, we define entropies $h(u)$ in (2.1)–(2.4). We verify that for each α_1, α_2 case, $h(u), \mathbf{A}(u)$ satisfy Assumptions (H1)–(H3).

The verification of assumption (H2) is reflected in Lemmas 2.1–2.4, which is the major step of this existence proof.

Step 2. In Section 3, we show that a weak solution of (1.1)–(1.3) exists. We rely on (H1)–(H2) to derive a sequence of approximated solutions indexed by $\tau, \varepsilon > 0$.

In this step, we mainly refer to Lemma 5 in [2]. This lemma showed the existence of approximated solutions by applying the Lax-Milgram lemma and fixed point theorem.

The proof of Lemma 5 in [2] takes several pages, and this classical result has been widely applied in the global existence analysis of cross-diffusion systems. For these reasons, we briefly state its proof in the Appendix section.

Step 3. We provide uniform estimates of approximated solutions indexed by $\tau, \varepsilon > 0$. By (H2) and (H3), we derive the estimation formulas (3.5)–(3.6).

Step 4. We let $(\tau, \varepsilon) \to 0$. We apply the Aubin-Lions lemma from [17,18] to show that the sequence of approximated solutions converges to a weak solution of (1.1)–(1.3).

We then divide the uniqueness proof into these three steps.

Step 1. In Section 4, we list Lemma 4.1 as an important preparation. Once $\alpha_1 = \alpha_2 = \alpha$, $u_1 + u_2 = v_1 + v_2$, a.e. $(x,t) \in \Omega \times (0,T)$. For convenience, we denote $u_3 = 1 - u_1 - u_2$, $v_3 = 1 - v_1 - v_2$.

Step 2. We apply the distance functional in (1.13) to measure the "distance" between $u = (u_1, u_2)$ and $v = (v_1, v_2)$. u and v are two weak solutions of (1.1)–(1.3).

Step 3. The distance functional in (1.13) is indexed by $0 < \varepsilon < 1$. We let $\varepsilon \to 0$ to show that $u = v$, a.e. $(x,t) \in \Omega \times (0,T)$.

2. Matrix analysis

2.1. The construction of $h(u)$

Let us consider the case when $0 < \alpha_1, \alpha_2 < 1$. We define the entropy $h(u)$ as

$$h(u) = \frac{\beta u_1(\log u_1 - 1)}{1-\alpha_1} + \frac{\beta u_2(\log u_2 - 1)}{1-\alpha_2} + u_3(\log u_3 - 1) + \frac{\beta}{1-\alpha_1} + \frac{\beta}{1-\alpha_2} + 1, \quad \beta > 0. \tag{2.1}$$

We verify that $h \in C^2(\mathcal{D}, (0,\infty))$ and that

$$\partial h/\partial u_i = \beta \log u_i/(1-\alpha_i) - \log u_3, \quad i = 1, 2.$$

For every $(w_1, w_2) \in \mathbb{R}^2$, we define

$$g(y) = e^{(1-\alpha_1)w_1/\beta}(1-y)^{(1-\alpha_1)/\beta} + e^{(1-\alpha_2)w_2/\beta}(1-y)^{(1-\alpha_2)/\beta}, \quad 0 < y < 1.$$

g is continuous and nonincreasing in $0 < y < 1$, and

$$g(0) = e^{(1-\alpha_1)w_1/\beta} + e^{(1-\alpha_2)w_2/\beta} > 0, \quad g(1) = 0.$$

We deduce that there exists a unique fixed point $0 < y_0 < 1$ such that $g(y_0) = y_0$.

Then, we define $u_i = e^{(1-\alpha_i)w_i/\beta}(1-y_0)^{(1-\alpha_i)/\beta} > 0 (i = 1, 2)$, which satisfies $u_1 + u_2 = g(y_0) = y_0 < 1$. Consequently, $u = (u_1, u_2) \in \mathcal{D}$, $u_3 = 1 - u_1 - u_2 = 1 - y_0$, and

$$w_i = \beta \log u_i/(1-\alpha_i) - \log u_3 = \partial h/\partial u_i, \quad i = 1, 2.$$

Then, we conclude that $h' : \mathcal{D} \to \mathbb{R}^2$ is invertible on $\mathbb{R}^2$, which satisfies Assumption (H1).

Entropies in (2.2)–(2.4) also satisfy Assumption (H1), with similar verification processes. In Lemmas 2.1–2.4, we verify Assumption (H2). We divide the verification into four cases.

2.2. Case 1: $0 < \alpha_1, \alpha_2 < 1$

Let us verify Assumption (H2) in Lemma 2.1.

Lemma 2.1. *For every $0 < \alpha_1, \alpha_2 < 1$, $z = (z_1, z_2) \in \mathbb{R}^2$, $u \in \mathcal{D}$, $h(u)$ given by (2.1), we have*

$$z^{\mathrm{T}} h''(u) \boldsymbol{A}(u) z \geq c(z_1^2 + z_2^2),$$

with $c > 0$ a constant independent of u and z.

Proof. By (2.1), we have

$$h''(u) = \begin{pmatrix} \frac{\beta}{(1-\alpha_1)u_1} & 0 \\ 0 & \frac{\beta}{(1-\alpha_2)u_2} \end{pmatrix} + \frac{1}{u_3}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

In the following computation, once $\alpha_1 < \alpha_2$, we decompose the matrix as

$$\begin{aligned} &\frac{1}{u_3}\begin{pmatrix} \alpha_1 + (\alpha_2 - \alpha_1)u_2 & \alpha_2 + (\alpha_1 - \alpha_2)u_1 \\ \alpha_1 + (\alpha_2 - \alpha_1)u_2 & \alpha_2 + (\alpha_1 - \alpha_2)u_1 \end{pmatrix} \\ &= \frac{\alpha_1 + (\alpha_2 - \alpha_1)u_2}{u_3}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + (\alpha_2 - \alpha_1)\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

If $\alpha_1 > \alpha_2$, the decomposition is given as

$$\begin{aligned}&\frac{1}{u_3}\begin{pmatrix} \alpha_1+(\alpha_2-\alpha_1)u_2 & \alpha_2+(\alpha_1-\alpha_2)u_1 \\ \alpha_1+(\alpha_2-\alpha_1)u_2 & \alpha_2+(\alpha_1-\alpha_2)u_1 \end{pmatrix}\\ =&\frac{\alpha_2+(\alpha_1-\alpha_2)u_1}{u_3}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}+(\alpha_1-\alpha_2)\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.\end{aligned}$$

Once $\alpha_1 < \alpha_2$, we have

$$\begin{aligned}h''(u)\mathbf{A}(u) =& \Big\{\begin{pmatrix} \frac{\beta}{(1-\alpha_1)u_1} & 0 \\ 0 & \frac{\beta}{(1-\alpha_2)u_2} \end{pmatrix}+\frac{1}{u_3}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\Big\}\cdot\begin{pmatrix} \alpha_1+(1-\alpha_1)u_2 & (\alpha_1-1)u_1 \\ (\alpha_2-1)u_2 & \alpha_2+(1-\alpha_2)u_1 \end{pmatrix}\\ =&\beta\begin{pmatrix} \frac{\alpha_1}{(1-\alpha_1)u_1}+\frac{u_2}{u_1} & -1 \\ -1 & \frac{\alpha_2}{(1-\alpha_2)u_2}+\frac{u_1}{u_2} \end{pmatrix}+\frac{\alpha_1+(\alpha_2-\alpha_1)u_2}{u_3}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}+(\alpha_2-\alpha_1)\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.\end{aligned}$$

Let us choose the constant β as

$$\beta=\frac{(1-\alpha_1)(\alpha_2-\alpha_1)}{2\alpha_1},\ \text{and denote } c=\min\Big\{\frac{\alpha_2-\alpha_1}{4},\frac{(1-\alpha_1)(\alpha_2-\alpha_1)\alpha_2}{2(1-\alpha_2)\alpha_1}\Big\},$$

which follows

$$\begin{aligned}&\frac{\beta\alpha_1 z_1^2}{(1-\alpha_1)u_1}+\Big(\frac{\beta\alpha_2}{(1-\alpha_2)u_2}+(\alpha_2-\alpha_1)\Big)z_2^2+(\alpha_2-\alpha_1)z_1z_2\\ =&(\alpha_2-\alpha_1)(\frac{z_1}{2}+z_2)^2+\Big(\frac{\beta\alpha_1}{(1-\alpha_1)u_1}-\frac{\alpha_2-\alpha_1}{4}\Big)z_1^2+\frac{\beta\alpha_2 z_2^2}{(1-\alpha_2)u_2}\\ \geq&\frac{(\alpha_2-\alpha_1)z_1^2}{4}+\frac{(1-\alpha_1)(\alpha_2-\alpha_1)\alpha_2 z_2^2}{2(1-\alpha_2)\alpha_1}\geq c(z_1^2+z_2^2).\end{aligned}$$

For every $u\in\mathcal{D}$, $z=(z_1,z_2)\in\mathbb{R}^2$, we have

$$\begin{aligned}z^{\mathrm{T}}h''(u)\mathbf{A}(u)z=&\frac{\beta\alpha_1 z_1^2}{(1-\alpha_1)u_1}+\Big(\frac{\beta\alpha_2}{(1-\alpha_2)u_2}+(\alpha_2-\alpha_1)\Big)z_2^2+(\alpha_2-\alpha_1)z_1z_2\\ &+\beta(\sqrt{\frac{u_2}{u_1}}z_1-\sqrt{\frac{u_1}{u_2}}z_2)^2+\frac{\alpha_1+(\alpha_2-\alpha_1)u_2}{u_3}\cdot(z_1+z_2)^2\geq c(z_1^2+z_2^2).\end{aligned}$$

Once $\alpha_1 > \alpha_2$, we have

$$h''(u)\mathbf{A}(u)=\beta\begin{pmatrix} \frac{\alpha_1}{(1-\alpha_1)u_1}+\frac{u_2}{u_1} & -1 \\ -1 & \frac{\alpha_2}{(1-\alpha_2)u_2}+\frac{u_1}{u_2} \end{pmatrix}+\frac{\alpha_2+(\alpha_1-\alpha_2)u_1}{u_3}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}+(\alpha_1-\alpha_2)\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

Let us choose the constant β as

$$\beta=\frac{(1-\alpha_2)(\alpha_1-\alpha_2)}{2\alpha_2},\ \text{and denote } c=\min\Big\{\frac{\alpha_1-\alpha_2}{4},\frac{(1-\alpha_2)(\alpha_1-\alpha_2)\alpha_1}{2(1-\alpha_1)\alpha_2}\Big\},$$

which follows

$$\begin{aligned}&\Big(\frac{\beta\alpha_1}{(1-\alpha_1)u_1}+(\alpha_1-\alpha_2)\Big)z_1^2+\frac{\beta\alpha_2 z_2^2}{(1-\alpha_2)u_2}+(\alpha_1-\alpha_2)z_1z_2\\ =&(\alpha_1-\alpha_2)(z_1+\frac{z_2}{2})^2+\frac{\beta\alpha_1 z_1^2}{(1-\alpha_1)u_1}+\Big(\frac{\beta\alpha_2}{(1-\alpha_2)u_2}-\frac{\alpha_1-\alpha_2}{4}\Big)z_2^2\\ \geq&\frac{(1-\alpha_2)(\alpha_1-\alpha_2)\alpha_1 z_1^2}{2(1-\alpha_1)\alpha_2}+\frac{(\alpha_1-\alpha_2)z_2^2}{4}\geq c(z_1^2+z_2^2).\end{aligned}$$

For every $u \in \mathcal{D}$, $z = (z_1, z_2) \in \mathbb{R}^2$, we have

$$z^{\mathrm{T}}h''(u)\mathbf{A}(u)z = \Big(\frac{\beta\alpha_1}{(1-\alpha_1)u_1} + (\alpha_1-\alpha_2)\Big)z_1^2 + \frac{\beta\alpha_2 z_2^2}{(1-\alpha_2)u_2} + (\alpha_1-\alpha_2)z_1z_2 + \beta\Big(\sqrt{\frac{u_2}{u_1}}z_1 - \sqrt{\frac{u_1}{u_2}}z_2\Big)^2 + \frac{\alpha_2 + (\alpha_1-\alpha_2)u_1}{u_3}\cdot(z_1+z_2)^2 \geq c(z_1^2+z_2^2).$$

Once $\alpha_1 = \alpha_2 = \alpha$, we choose $\beta = 1$, and denote $c = \alpha/(1-\alpha)$. For every $u \in \mathcal{D}$, $z = (z_1, z_2) \in \mathbb{R}^2$, we have

$$z^{\mathrm{T}}h''(u)\mathbf{A}(u)z = \frac{\alpha z_1^2}{(1-\alpha)u_1} + \frac{\alpha z_2^2}{(1-\alpha)u_2} + \Big(\sqrt{\frac{u_2}{u_1}}z_1 - \sqrt{\frac{u_1}{u_2}}z_2\Big)^2 + \frac{\alpha}{u_3}\cdot(z_1+z_2)^2 \geq c(z_1^2+z_2^2).$$

Then, we finish the proof of Lemma 2.1. □

2.3. Case 2: $\alpha_1, \alpha_2 > 1$

Let us define the entropy $h(u)$ as

$$h(u) = \frac{\beta u_1(\log u_1 - 1)}{\alpha_1 - 1} + \frac{\beta u_2(\log u_2 - 1)}{\alpha_2 - 1} + u_3(\log u_3 - 1) + \frac{\beta}{\alpha_1 - 1} + \frac{\beta}{\alpha_2 - 1} + 1, \quad \beta > 0, \tag{2.2}$$

when $\alpha_1, \alpha_2 > 1$. Then, we show Lemma 2.2.

Lemma 2.2. *For every $\alpha_1, \alpha_2 > 1$, $z = (z_1, z_2) \in \mathbb{R}^2$, $u \in \mathcal{D}$, $h(u)$ given by* (2.2)*, we have*

$$z^{\mathrm{T}}h''(u)\boldsymbol{A}(u)z \geq c(z_1^2 + z_2^2),$$

with $c > 0$ a constant independent of u and z.

Proof. By (2.2), we have

$$h''(u) = \begin{pmatrix} \frac{\beta}{(\alpha_1-1)u_1} & 0 \\ 0 & \frac{\beta}{(\alpha_2-1)u_2} \end{pmatrix} + \frac{1}{u_3}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Once $\alpha_1 < \alpha_2$, we have

$$\begin{aligned} h''(u)\mathbf{A}(u) &= \Big\{\begin{pmatrix} \frac{\beta}{(\alpha_1-1)u_1} & 0 \\ 0 & \frac{\beta}{(\alpha_2-1)u_2} \end{pmatrix} + \frac{1}{u_3}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\Big\}\cdot\begin{pmatrix} \alpha_1 + (1-\alpha_1)u_2 & (\alpha_1-1)u_1 \\ (\alpha_2-1)u_2 & \alpha_2 + (1-\alpha_2)u_1 \end{pmatrix} \\ &= \beta\begin{pmatrix} \frac{1}{(\alpha_1-1)u_1} + \frac{u_3}{u_1} + 1 & 1 \\ 1 & \frac{1}{(\alpha_2-1)u_2} + \frac{u_3}{u_2} + 1 \end{pmatrix} \\ &\quad + \frac{\alpha_1 + (\alpha_2-\alpha_1)u_2}{u_3}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + (\alpha_2-\alpha_1)\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Let us choose the constant β as

$$\beta = \frac{(\alpha_1-1)(\alpha_2-\alpha_1)}{2}, \text{ and denote } c = \min\Big\{\frac{\alpha_2-\alpha_1}{4}, \frac{(\alpha_1-1)(\alpha_2-\alpha_1)}{2(\alpha_2-1)}\Big\},$$

which follows

$$\begin{aligned}
&\frac{\beta z_1^2}{(\alpha_1-1)u_1}+\Big(\frac{\beta}{(\alpha_2-1)u_2}+(\alpha_2-\alpha_1)\Big)z_2^2+(\alpha_2-\alpha_1)z_1z_2\\
&=(\alpha_2-\alpha_1)(\frac{z_1}{2}+z_2)^2+\Big(\frac{\beta}{(\alpha_1-1)u_1}-\frac{\alpha_2-\alpha_1}{4}\Big)z_1^2+\frac{\beta z_2^2}{(\alpha_2-1)u_2}\\
&\geq\frac{(\alpha_2-\alpha_1)z_1^2}{4}+\frac{(\alpha_1-1)(\alpha_2-\alpha_1)z_2^2}{2(\alpha_2-1)}\geq c(z_1^2+z_2^2).
\end{aligned}$$

For every $u\in\mathcal{D}$, $z=(z_1,z_2)\in\mathbb{R}^2$, we have

$$\begin{aligned}
z^{\mathrm{T}}h''(u)\mathbf{A}(u)z=&\frac{\beta z_1^2}{(\alpha_1-1)u_1}+\Big(\frac{\beta}{(\alpha_2-1)u_2}+(\alpha_2-\alpha_1)\Big)z_2^2+(\alpha_2-\alpha_1)z_1z_2\\
&+\beta u_3(\frac{z_1^2}{u_1}+\frac{z_2^2}{u_2})+(\beta+\frac{\alpha_1+(\alpha_2-\alpha_1)u_2}{u_3})\cdot(z_1+z_2)^2\geq c(z_1^2+z_2^2).
\end{aligned}$$

Once $\alpha_1>\alpha_2$, we have

$$h''(u)\mathbf{A}(u)=\beta\begin{pmatrix}\frac{1}{(\alpha_1-1)u_1}+\frac{u_3}{u_1}+1 & 1\\ 1 & \frac{1}{(\alpha_2-1)u_2}+\frac{u_3}{u_2}+1\end{pmatrix}+\frac{\alpha_2+(\alpha_1-\alpha_2)u_1}{u_3}\begin{pmatrix}1&1\\1&1\end{pmatrix}+(\alpha_1-\alpha_2)\begin{pmatrix}1&0\\1&0\end{pmatrix}.$$

Let us choose the constant β as

$$\beta=\frac{(\alpha_2-1)(\alpha_1-\alpha_2)}{2},\ \text{and denote } c=\min\Big\{\frac{\alpha_1-\alpha_2}{4},\frac{(\alpha_2-1)(\alpha_1-\alpha_2)}{2(\alpha_1-1)}\Big\},$$

which follows

$$\begin{aligned}
&\Big(\frac{\beta}{(\alpha_1-1)u_1}+(\alpha_1-\alpha_2)\Big)z_1^2+\frac{\beta z_2^2}{(\alpha_2-1)u_2}+(\alpha_1-\alpha_2)z_1z_2\\
&=(\alpha_1-\alpha_2)(z_1+\frac{z_2}{2})^2+\frac{\beta z_1^2}{(\alpha_1-1)u_1}+\Big(\frac{\beta}{(\alpha_2-1)u_2}-\frac{\alpha_1-\alpha_2}{4}\Big)z_2^2\\
&\geq\frac{(\alpha_2-1)(\alpha_1-\alpha_2)z_1^2}{2(\alpha_1-1)}+\frac{(\alpha_1-\alpha_2)z_2^2}{4}\geq c(z_1^2+z_2^2).
\end{aligned}$$

For every $u\in\mathcal{D}$, $z=(z_1,z_2)\in\mathbb{R}^2$, we have

$$\begin{aligned}
z^{\mathrm{T}}h''(u)\mathbf{A}(u)z=&\Big(\frac{\beta}{(\alpha_1-1)u_1}+(\alpha_1-\alpha_2)\Big)z_1^2+\frac{\beta z_2^2}{(\alpha_2-1)u_2}+(\alpha_1-\alpha_2)z_1z_2\\
&+\beta u_3(\frac{z_1^2}{u_1}+\frac{z_2^2}{u_2})+(\beta+\frac{\alpha_2+(\alpha_1-\alpha_2)u_1}{u_3})\cdot(z_1+z_2)^2\geq c(z_1^2+z_2^2).
\end{aligned}$$

Once $\alpha_1=\alpha_2=\alpha$, we choose $\beta=1$, and denote $c=1/(\alpha-1)$. For every $u\in\mathcal{D}$, $z=(z_1,z_2)\in\mathbb{R}^2$, we have

$$z^{\mathrm{T}}h''(u)\mathbf{A}(u)z=\frac{z_1^2}{(\alpha-1)u_1}+\frac{z_2^2}{(\alpha-1)u_2}+(\frac{\alpha}{u_3}+1)\cdot(z_1+z_2)^2+u_3(\frac{z_1^2}{u_1}+\frac{z_2^2}{u_2})\geq c(z_1^2+z_2^2).$$

Then, we have shown Lemma 2.2.

□

2.4. Case 3: $0 < \alpha_2 < 1 < \alpha_1$

If $0 < \alpha_2 < 1 < \alpha_1$, we define the entropy $h(u)$ as

$$h(u) = \frac{\beta u_1(\log u_1 - 1)}{\alpha_1 - 1} + \frac{\beta u_2(\log u_2 - 1)}{1 - \alpha_2} + u_3(\log u_3 - 1) + \frac{\beta}{\alpha_1 - 1} + \frac{\beta}{1 - \alpha_2} + 1, \quad \beta > 0. \tag{2.3}$$

Lemma 2.3. *For every* $0 < \alpha_2 < 1 < \alpha_1$, $z = (z_1, z_2) \in \mathbb{R}^2$, $u \in \mathcal{D}$, $h(u)$ *given by* (2.3), *we have*

$$z^{\mathrm{T}} h''(u) \boldsymbol{A}(u) z \geq c(z_1^2 + z_2^2),$$

with $c > 0$ *a constant independent of* u *and* z.

Proof. By (2.3), we have

$$h''(u) = \begin{pmatrix} \frac{\beta}{(\alpha_1 - 1)u_1} & 0 \\ 0 & \frac{\beta}{(1-\alpha_2)u_2} \end{pmatrix} + \frac{1}{u_3} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

which follows

$$\begin{aligned} h''(u)\mathbf{A}(u) &= \left\{ \begin{pmatrix} \frac{\beta}{(\alpha_1 - 1)u_1} & 0 \\ 0 & \frac{\beta}{(1-\alpha_2)u_2} \end{pmatrix} + \frac{1}{u_3} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\} \cdot \begin{pmatrix} \alpha_1 + (1 - \alpha_1)u_2 & (\alpha_1 - 1)u_1 \\ (\alpha_2 - 1)u_2 & \alpha_2 + (1 - \alpha_2)u_1 \end{pmatrix} \\ &= \beta \begin{pmatrix} \frac{1}{(\alpha_1 - 1)u_1} + \frac{u_3}{u_1} + 1 & 1 \\ -1 & \frac{\alpha_2}{(1-\alpha_2)u_2} + \frac{u_1}{u_2} \end{pmatrix} \\ &\quad + \frac{\alpha_2 + (\alpha_1 - \alpha_2)u_1}{u_3} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + (\alpha_1 - \alpha_2) \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Let us choose the constant β as

$$\beta = \frac{(1 - \alpha_2)(\alpha_1 - \alpha_2)}{2\alpha_2}, \text{ and denote } c = \min\left\{ \frac{\alpha_1 - \alpha_2}{4}, \frac{(1 - \alpha_2)(\alpha_1 - \alpha_2)\alpha_1}{2(\alpha_1 - 1)\alpha_2} \right\},$$

which follows

$$\begin{aligned} &\Big(\frac{\beta}{(\alpha_1 - 1)u_1} + (\alpha_1 - \alpha_2 + \beta) \Big) z_1^2 + \frac{\beta \alpha_2 z_2^2}{(1 - \alpha_2)u_2} + (\alpha_1 - \alpha_2) z_1 z_2 \\ &= (\alpha_1 - \alpha_2)(z_1 + \frac{z_2}{2})^2 + \beta z_1^2 + \frac{\beta z_1^2}{(\alpha_1 - 1)u_1} + \Big(\frac{\beta \alpha_2}{(1 - \alpha_2)u_2} - \frac{\alpha_1 - \alpha_2}{4} \Big) z_2^2 \\ &\geq \frac{(1 - \alpha_2)(\alpha_1 - \alpha_2)\alpha_1 z_1^2}{2(\alpha_1 - 1)\alpha_2} + \frac{(\alpha_1 - \alpha_2) z_2^2}{4} \geq c(z_1^2 + z_2^2). \end{aligned}$$

For every $u \in \mathcal{D}$, $z = (z_1, z_2) \in \mathbb{R}^2$, we have

$$\begin{aligned} x z^{\mathrm{T}} h''(u)\mathbf{A}(u) z =& \Big(\frac{\beta}{(\alpha_1 - 1)u_1} + (\alpha_1 - \alpha_2 + \beta) \Big) z_1^2 + \frac{\beta \alpha_2 z_2^2}{(1 - \alpha_2)u_2} + (\alpha_1 - \alpha_2) z_1 z_2 \\ &+ \frac{\beta u_3}{u_1} \cdot z_1^2 + \frac{\beta u_1}{u_2} \cdot z_2^2 + \frac{\alpha_2 + (\alpha_1 - \alpha_2)u_1}{u_3} \cdot (z_1 + z_2)^2 \geq c(z_1^2 + z_2^2). \end{aligned}$$

Then, we have shown Lemma 2.3.

□

2.5. Case 4: $0 < \alpha_1 < 1 < \alpha_2$

If $0 < \alpha_1 < 1 < \alpha_2$, we define the entropy $h(u)$ as

$$h(u) = \frac{\beta u_1(\log u_1 - 1)}{1-\alpha_1} + \frac{\beta u_2(\log u_2 - 1)}{\alpha_2 - 1} + u_3(\log u_3 - 1) + \frac{\beta}{1-\alpha_1} + \frac{\beta}{\alpha_2 - 1} + 1, \quad \beta > 0. \tag{2.4}$$

Lemma 2.4. *For every* $0 < \alpha_1 < 1 < \alpha_2$, $z = (z_1, z_2) \in \mathbb{R}^2$, $u \in \mathcal{D}$, $h(u)$ *given by* (2.4), *we have*

$$z^{\mathrm{T}} h''(u) \boldsymbol{A}(u) z \geq c(z_1^2 + z_2^2),$$

with $c > 0$ *a constant independent of* u *and* z.

Proof. By (2.4), we have

$$h''(u) = \begin{pmatrix} \frac{\beta}{(1-\alpha_1)u_1} & 0 \\ 0 & \frac{\beta}{(\alpha_2-1)u_2} \end{pmatrix} + \frac{1}{u_3}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

which follows

$$\begin{aligned} h''(u)\mathbf{A}(u) &= \left\{ \begin{pmatrix} \frac{\beta}{(1-\alpha_1)u_1} & 0 \\ 0 & \frac{\beta}{(\alpha_2-1)u_2} \end{pmatrix} + \frac{1}{u_3}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\} \cdot \begin{pmatrix} \alpha_1 + (1-\alpha_1)u_2 & (\alpha_1 - 1)u_1 \\ (\alpha_2 - 1)u_2 & \alpha_2 + (1-\alpha_2)u_1 \end{pmatrix} \\ &= \beta \begin{pmatrix} \frac{\alpha_1}{(1-\alpha_1)u_1} + \frac{u_2}{u_1} & -1 \\ 1 & \frac{1}{(\alpha_2-1)u_2} + \frac{u_3}{u_2} + 1 \end{pmatrix} \\ &\quad + \frac{\alpha_1 + (\alpha_2 - \alpha_1)u_2}{u_3}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + (\alpha_2 - \alpha_1)\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Let us choose the constant β as

$$\beta = \frac{(1-\alpha_1)(\alpha_2 - \alpha_1)}{2\alpha_1}, \text{ and denote } c = \min\left\{ \frac{\alpha_2 - \alpha_1}{4}, \frac{(1-\alpha_1)(\alpha_2 - \alpha_1)\alpha_2}{2(\alpha_2 - 1)\alpha_1} \right\},$$

which follows

$$\begin{aligned} &\frac{\beta\alpha_1 z_1^2}{(1-\alpha_1)u_1} + \Big(\frac{\beta}{(\alpha_2 - 1)u_2} + (\alpha_2 - \alpha_1 + \beta)\Big)z_2^2 + (\alpha_2 - \alpha_1)z_1 z_2 \\ &= (\alpha_2 - \alpha_1)(\frac{z_1}{2} + z_2)^2 + \Big(\frac{\beta\alpha_1}{(1-\alpha_1)u_1} - \frac{\alpha_2 - \alpha_1}{4}\Big)z_1^2 + \beta z_2^2 + \frac{\beta z_2^2}{(\alpha_2 - 1)u_2} \\ &\geq \frac{(\alpha_2 - \alpha_1)z_1^2}{4} + \frac{(1-\alpha_1)(\alpha_2 - \alpha_1)\alpha_2 z_2^2}{2(\alpha_2 - 1)\alpha_1} \geq c(z_1^2 + z_2^2). \end{aligned}$$

For every $u \in \mathcal{D}$, $z = (z_1, z_2) \in \mathbb{R}^2$, we have

$$\begin{aligned} z^{\mathrm{T}} h''(u)\mathbf{A}(u)z =& \frac{\beta\alpha_1 z_1^2}{(1-\alpha_1)u_1} + \Big(\frac{\beta}{(\alpha_2 - 1)u_2} + (\alpha_2 - \alpha_1 + \beta)\Big)z_2^2 + (\alpha_2 - \alpha_1)z_1 z_2 \\ &+ \frac{\beta u_2}{u_1} \cdot z_1^2 + \frac{\beta u_3}{u_2} \cdot z_2^2 + \frac{\alpha_1 + (\alpha_2 - \alpha_1)u_2}{u_3} \cdot (z_1 + z_2)^2 \geq c(z_1^2 + z_2^2). \end{aligned}$$

Then, we have shown Lemma 2.4.

□

3. Proof of Theorem 1.1

Let us give the proof of Theorem 1.1.

Proof. **Step 1.** Let $T > 0$, $N \in \mathbb{N}$, $\tau = T/N$, $0 < \varepsilon < 1$, and $m \in \mathbb{N}$ with $m > d/2$. This ensures that the embedding $H^m(\Omega) \hookrightarrow L^\infty(\Omega)$ is compact.

Lemmas 2.1–2.4 indicate that Assumption (H2) can be satisfied. By Lemma 5 of [2], we can show that given $w^{k-1} \in L^\infty(\Omega)$ for $k \in \mathbb{N}$, there exists $w^k \in H^m(\Omega)$, $u(w^k(x)) \in \mathcal{D}$ for $x \in \Omega$, such that

$$\frac{1}{\tau}\int_\Omega (u(w^k) - u(w^{k-1})) \cdot \phi dx + \int_\Omega \nabla\phi : \mathbf{B}(w^k)\nabla w^k dx + \varepsilon \int_\Omega \Big(\sum_{|\alpha|=m} D^\alpha w^k \cdot D^\alpha \phi + w^k \cdot \phi \Big) dx = 0, \quad (3.1)$$

for every $\phi \in H^m(\Omega)$. Here, $\mathbf{A} : \mathbf{B} = \sum_{i,j} a_{ij}b_{ij}$ for matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$.

In (3.1), $u(w^k) = (h')^{-1}(w^k)$, $\mathbf{B}(w^k) = \mathbf{A}(u(w^k))\mathbf{H}(u(w^k))^{-1}$, and $\mathbf{H}(u) = h''(u)$. $|\alpha| = \alpha_1 + \cdots + \alpha_d = m$ is a multi-index, and $D^\alpha = \partial^{|\alpha|}/(\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d})$ is a partial derivative of order m. If $k = 1$, we define $w^0 = h'(u^0)$. Because $u_1(x), u_2(x) \in [a, b]$ for $x \in \Omega$, where $0 < a < b < 1$, then $w^0 = h'(u^0) \in L^\infty(\Omega)$.

Let us denote $u^k = u(w^k)$, and introduce the piecewise-in-time constant functions $u^{(\tau)} = u^k$, $w^{(\tau)} = w^k$ for $x \in \Omega$, $t \in ((k-1)\tau, k\tau]$. At time $t = 0$, we set $u^{(\tau)}(\cdot, 0) = u^0$. Letting $u^{(\tau)} = (u_1^{(\tau)}, u_2^{(\tau)})$, we define the backward shift operator $(\sigma_\tau u^{(\tau)})(x, t) = u^{k-1}$ for $x \in \Omega$, $t \in ((k-1)\tau, k\tau]$. $u^{(\tau)}$ satisfies that

$$\begin{aligned} &\frac{1}{\tau}\int_0^T \int_\Omega (u^{(\tau)} - \sigma_\tau u^{(\tau)}) \cdot \phi dx dt + \int_0^T \int_\Omega \nabla\phi : \mathbf{B}(w^{(\tau)})\nabla w^{(\tau)} dx dt \\ &+ \varepsilon \int_0^T \int_\Omega \Big(\sum_{|\alpha|=m} D^\alpha w^{(\tau)} \cdot D^\alpha \phi + w^{(\tau)} \cdot \phi \Big) dx dt = 0, \end{aligned} \quad (3.2)$$

for piecewise constant functions $\phi : (0, T) \to H^m(\Omega)$.

Lemma 5 in [2] only requires that $h''(u)\mathbf{A}(u)$ be positive semi-definite. The positive semi-definiteness of $h''(u)\mathbf{A}(u)$ is included in (H2).

Step 2. Relying on (3.1) and the inequality

$$h(u(w^k)) - h(u(w^{k-1})) \le h'(u(w^k)) \cdot (u(w^k) - u(w^{k-1})),$$

we deduce that w^k satisfies the discrete entropy inequality

$$\begin{aligned} &\int_\Omega h(u(w^k))dx + \tau \int_\Omega \nabla w^k : \mathbf{B}(w^k)\nabla w^k dx \\ &+ \varepsilon\tau \int_\Omega \Big(\sum_{|\alpha|=m} |D^\alpha w^k|^2 + |w^k|^2 \Big) dx \le \int_\Omega h(u(w^{k-1}))dx. \end{aligned} \quad (3.3)$$

By the generalized Poincaré inequality, we deduce that

$$\int_\Omega \Big(\sum_{|\alpha|=m} |D^\alpha w^k|^2 + |w^k|^2 \Big) dx \ge C_p \|w^k\|^2_{H^m(\Omega)},$$

where $C_p > 0$ is the Poincaré constant. By Lemmas 2.1–2.4, we have

$$\nabla w^k : \mathbf{B}(w^k)\nabla w^k = \nabla u^k : \mathbf{H}(u^k)\mathbf{A}(u^k)\nabla u^k \ge c(|\nabla u_1^k|^2 + |\nabla u_2^k|^2).$$

Combining the above computation results, by (3.3), we conclude that

$$\int_\Omega h(u^k)dx + c\tau \int_\Omega |\nabla u^k|^2 dx + \varepsilon C_p \tau \|w^k\|^2_{H^m(\Omega)} \le \int_\Omega h(u^{k-1})dx. \tag{3.4}$$

Summing these inequalities over $k = 1, ..., j$ in (3.4), we can show that

$$\int_\Omega h(u^j)dx + c\tau \sum_{k=1}^{j} \int_\Omega |\nabla u^k|^2 dx + \varepsilon C_p \tau \sum_{k=1}^{j} \|w^k\|^2_{H^m(\Omega)} \le \int_\Omega h(u^0)dx,$$

so it follows that

$$\|u^{(\tau)}\|_{L^2(0,T;H^1(\Omega))} + \varepsilon^{1/2} \|w^{(\tau)}\|_{L^2(0,T;H^m(\Omega))} \le C, \tag{3.5}$$

with $C > 0$ a generic constant independent of τ, ε.

Step 3. Let us give a uniform estimate for the discrete time derivative of $u^{(\tau)}$. $\mathbf{A}(u^{(\tau)})$ satisfies Assumption (H3); then, by (3.5) and Hölder's inequality,

$$\begin{aligned} &\frac{1}{\tau}\Big| \int_0^T \int_\Omega (u^{(\tau)} - \sigma_\tau u^{(\tau)}) \cdot \phi dx dt \Big| \le \varepsilon \|w^{(\tau)}\|_{L^2(0,T;H^m(\Omega))} \cdot \|\phi\|_{L^2(0,T;H^m(\Omega))} \\ &+ \|\mathbf{A}(u^{(\tau)})\|_{L^\infty(0,T;L^\infty(\Omega))} \cdot \|\nabla u^{(\tau)}\|_{L^2(0,T;L^2(\Omega))} \cdot \|\nabla \phi\|_{L^2(0,T;L^2(\Omega))} \le C. \end{aligned} \tag{3.6}$$

By (3.5) and (3.6), we apply the Aubin-Lions lemma. There exists a subsequence which we do not relabel, such that as $(\tau, \varepsilon) \to 0$,

$$\begin{aligned} u^{(\tau)} &\to u \text{ strongly in } L^2(0,T;L^2(\Omega)), \text{ and a.e. in } \Omega \times (0,T). \\ u^{(\tau)} &\rightharpoonup u \text{ weakly in } L^2(0,T;H^1(\Omega)). \\ \tau^{-1}(u^{(\tau)} - \sigma_\tau u^{(\tau)}) &\rightharpoonup \partial_t u \text{ weakly in } L^2(0,T;H^m(\Omega)'). \\ \varepsilon w^{(\tau)} &\to 0 \text{ strongly in } L^2(0,T;H^m(\Omega)). \end{aligned} \tag{3.7}$$

Let us denote $J = (J_1, J_2)$ with

$$\begin{aligned} J_1 &= \alpha_1 \nabla(u_1^{(\tau)} - u_1) + (\alpha_1 - 1)((u_1^{(\tau)} \nabla u_2^{(\tau)} - u_2^{(\tau)} \nabla u_1^{(\tau)}) - (u_1 \nabla u_2 - u_2 \nabla u_1)), \\ J_2 &= \alpha_2 \nabla(u_2^{(\tau)} - u_2) - (\alpha_2 - 1)((u_1^{(\tau)} \nabla u_2^{(\tau)} - u_2^{(\tau)} \nabla u_1^{(\tau)}) - (u_1 \nabla u_2 - u_2 \nabla u_1)). \end{aligned}$$

Because

$$u_i^{(\tau)} \nabla u_j^{(\tau)} - u_i \nabla u_j = u_i^{(\tau)} \nabla (u_j^{(\tau)} - u_j) + (u_i^{(\tau)} - u_i) \nabla u_j, \quad 1 \le i, j \le 2, \tag{3.8}$$

by (3.7)–(3.8) and the dominated-convergence theorem, we can show that for every $\phi \in L^2(0,T;H^m(\Omega))$, as $(\tau, \varepsilon) \to 0$,

$$\int_0^T \int_\Omega J \cdot \nabla \phi dx dt \to 0. \tag{3.9}$$

(3.9) implies that for every $\phi \in L^2(0,T;H^m(\Omega))$, as $(\tau, \varepsilon) \to 0$,

$$\int_0^T \int_\Omega (\mathbf{A}(u^{(\tau)}) \nabla u^{(\tau)} - \mathbf{A}(u) \nabla u) \cdot \nabla \phi dx dt \to 0. \tag{3.10}$$

By (3.7) and (3.10), we deduce that for every $T > 0$, $\phi \in L^2(0,T;H^m(\Omega))$, we have

$$\int_0^T \int_\Omega \partial_t u \cdot \phi dxdt + \int_0^T \int_\Omega \nabla\phi : \mathbf{A}(u)\nabla u dxdt = 0. \tag{3.11}$$

By (3.6), $\partial_t u \in L^2(0,T;H^1(\Omega)')$, and consequently, the weak formulation (3.11) also holds for every $\phi \in L^2(0,T;H^1(\Omega))$. Finally, $u \in H^1(0,T;H^1(\Omega)') \hookrightarrow C^0([0,T];H^1(\Omega)')$ shows that the initial condition is satisfied in $H^1(\Omega)'$. u satisfies (1.1)–(1.3) in a weak sense, and we finish the proof. □

4. Proof of Theorem 1.2

In the following uniqueness proof, we state an important lemma, which is Lemma 10 of [4]. This lemma also has a detailed explanation in [19], so we omit the proof.

Lemma 4.1. *If μ is an absolutely continuous measure with respect to the Lebesgue measure, and $f, g : \Omega \to [0,\infty)$ are measurable, bounded, positive functions such that $\sqrt{f}$, $\sqrt{g} \in H^1(\Omega; d\mu)$, then we have*

$$\int_\Omega |\nabla\sqrt{f+g}|^2 d\mu \le \int_\Omega |\nabla\sqrt{f}|^2 d\mu + \int_\Omega |\nabla\sqrt{g}|^2 d\mu.$$

Let us give the proof of Theorem 1.2.

Proof. **Step 1.** Let $\zeta(t) \in H^1(\Omega)$ be the unique solution to

$$\Delta\zeta(t) = u_3(t) - v_3(t) \text{ in } \Omega, \quad \nabla\zeta\cdot\nu = 0 \text{ on } \partial\Omega, \quad \int_\Omega \zeta dx = 0, \quad t > 0.$$

Because $u_3 - v_3 \in L^2(0,T;L^2(\Omega))$, we have $\zeta \in L^2(0,T;H^1(\Omega))$. As $\partial_t(u_3 - v_3) \in L^2(0,T;H^1(\Omega)')$, we have $\Delta\partial_t\zeta \in L^2(0,T;H^1(\Omega)')$. We can prove that $t \to \|\nabla\zeta(t)\|^2_{L^2(\Omega)}$ is continuous on $[0,T]$ and that for a.e. $t > 0$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\zeta|^2 dx &= -\langle\Delta\partial_t\zeta,\zeta\rangle = -\langle\partial_t(u_3 - v_3),\zeta\rangle \\ &= \alpha\int_\Omega \nabla(u_3 - v_3)\cdot\nabla\zeta dx = -\alpha\int_\Omega (u_3 - v_3)^2 dx \le 0.\end{aligned}$$

Here, $\langle\cdot,\cdot\rangle$ denotes the duality pairing of $H^1(\Omega)'$ and $H^1(\Omega)$. It follows that

$$\int_\Omega |\nabla\zeta(t)|^2 dx \le \int_\Omega |\nabla\zeta(0)|^2 dx, \quad t > 0.$$

At time $t = 0$, $\Delta\zeta(0) = u_3(0) - v_3(0) = 0$ in Ω, and thus, $\nabla\zeta(0) = 0$. We deduce that $|\nabla\zeta(t)| = 0$ a.e. in Ω, and $u_3(t) - v_3(t) = \Delta\zeta(t) = 0$ in Ω. In the following proof, we denote $q = u_1 + u_2 = v_1 + v_2$.

Step 2. Let us denote

$$\begin{aligned}I_1 = &-\int_\Omega (\alpha\nabla u_1 + (\alpha-1)(u_1\nabla u_2 - u_2\nabla u_1))\cdot\frac{\nabla u_1}{u_1+\varepsilon}dx \\ &-\int_\Omega (\alpha\nabla u_2 - (\alpha-1)(u_1\nabla u_2 - u_2\nabla u_1))\cdot\frac{\nabla u_2}{u_2+\varepsilon}dx,\end{aligned}$$

$$I_2 = -\int_\Omega (\alpha\nabla v_1 + (\alpha-1)(v_1\nabla v_2 - v_2\nabla v_1)) \cdot \frac{\nabla v_1}{v_1+\varepsilon}dx - \int_\Omega (\alpha\nabla v_2 - (\alpha-1)(v_1\nabla v_2 - v_2\nabla v_1)) \cdot \frac{\nabla v_2}{v_2+\varepsilon}dx,$$

and

$$I_3 = \int_\Omega (\alpha\nabla(u_1+v_1) + (\alpha-1)(u_1\nabla u_2 - u_2\nabla u_1 + v_1\nabla v_2 - v_2\nabla v_1)) \cdot \frac{\nabla(u_1+v_1)}{u_1+v_1+2\varepsilon}dx + \int_\Omega (\alpha\nabla(u_2+v_2) - (\alpha-1)(u_1\nabla u_2 - u_2\nabla u_1 + v_1\nabla v_2 - v_2\nabla v_1)) \cdot \frac{\nabla(u_2+v_2)}{u_2+v_2+2\varepsilon}dx.$$

Then, we have

$$\frac{d}{dt}d_\varepsilon(u,v) = \sum_{i=1,2}\Big(\langle\partial_t u_i, \log(u_i+\varepsilon)\rangle + \langle\partial_t v_i, \log(v_i+\varepsilon)\rangle - \langle\partial_t(u_i+v_i), \log(\frac{u_i+v_i}{2}+\varepsilon)\rangle\Big) = I_1 + I_2 + I_3.$$

We then have

$$u_1\nabla u_2 - u_2\nabla u_1 = u_1\nabla(u_1+u_2) - (u_1+u_2)\nabla u_1 = u_1\nabla q - q\nabla u_1 = (u_1+u_2)\nabla u_2 - u_2\nabla(u_1+u_2) = q\nabla u_2 - u_2\nabla q$$

and

$$v_1\nabla v_2 - v_2\nabla v_1 = v_1\nabla(v_1+v_2) - (v_1+v_2)\nabla v_1 = v_1\nabla q - q\nabla v_1 = (v_1+v_2)\nabla v_2 - v_2\nabla(v_1+v_2) = q\nabla v_2 - v_2\nabla q,$$

which follows

$$I_1 = -\int_\Omega (\frac{|\nabla u_1|^2}{u_1+\varepsilon} + \frac{|\nabla u_2|^2}{u_2+\varepsilon}) \cdot (\alpha(1-q)+q)dx - (\alpha-1)\int_\Omega (\frac{u_1\nabla u_1}{u_1+\varepsilon} + \frac{u_2\nabla u_2}{u_2+\varepsilon})\nabla q dx,$$

$$I_2 = -\int_\Omega (\frac{|\nabla v_1|^2}{v_1+\varepsilon} + \frac{|\nabla v_2|^2}{v_2+\varepsilon}) \cdot (\alpha(1-q)+q)dx - (\alpha-1)\int_\Omega (\frac{v_1\nabla v_1}{v_1+\varepsilon} + \frac{v_2\nabla v_2}{v_2+\varepsilon})\nabla q dx,$$

and

$$I_3 = \int_\Omega (\frac{|\nabla(u_1+v_1)|^2}{u_1+v_1+2\varepsilon} + \frac{|\nabla(u_2+v_2)|^2}{u_2+v_2+2\varepsilon}) \cdot (\alpha(1-q)+q)dx + (\alpha-1)\int_\Omega \Big(\frac{(u_1+v_1)\nabla(u_1+v_1)}{u_1+v_1+2\varepsilon} + \frac{(u_2+v_2)\nabla(u_2+v_2)}{u_2+v_2+2\varepsilon}\Big)\nabla q dx.$$

Let us further denote

$$H_1 = -(\alpha-1)\int_0^t\int_\Omega \Big((\frac{u_1\nabla u_1}{u_1+\varepsilon} + \frac{u_2\nabla u_2}{u_2+\varepsilon}) + (\frac{v_1\nabla v_1}{v_1+\varepsilon} + \frac{v_2\nabla v_2}{v_2+\varepsilon}) - \Big(\frac{(u_1+v_1)\nabla(u_1+v_1)}{u_1+v_1+2\varepsilon} + \frac{(u_2+v_2)\nabla(u_2+v_2)}{u_2+v_2+2\varepsilon}\Big)\Big)\nabla q dx ds, \tag{4.1}$$

and

$$H_2 = \sum_{i=1,2}\int_0^t\int_\Omega \Big(\frac{|\nabla(u_i+v_i)|^2}{u_i+v_i+2\varepsilon} - \frac{|\nabla u_i|^2}{u_i+\varepsilon} - \frac{|\nabla v_i|^2}{v_i+\varepsilon}\Big) \cdot (\alpha(1-q)+q)dx ds = 4\sum_{i=1,2}\int_0^t\int_\Omega (|\nabla\sqrt{u_i+v_i+2\varepsilon}|^2 - |\nabla\sqrt{u_i+\varepsilon}|^2 - |\nabla\sqrt{v_i+\varepsilon}|^2) \cdot (\alpha(1-q)+q)dx ds \le 0. \tag{4.2}$$

Then, we can show that

$$\int_0^t (I_1 + I_2 + I_3)ds = H_1 + H_2. \tag{4.3}$$

In (4.2), we rely on Lemma 4.1 to deduce that $H_2 \le 0$. For $i = 1, 2$, we have

$$\nabla u_i, \nabla v_i, \nabla q \in L^2(0, T; L^2(\Omega)), \quad \Big|\frac{u_i}{u_i + \varepsilon}\Big| \le 1, \quad \Big|\frac{v_i}{v_i + \varepsilon}\Big| \le 1, \quad \Big|\frac{u_i + v_i}{u_i + v_i + 2\varepsilon}\Big| \le 1,$$

and then, by the dominated-convergence theorem, we derive that as $\varepsilon \to 0$,

$$H_1 \to 0. \tag{4.4}$$

Step 3. By (4.2)–(4.4), because $d_\varepsilon(u, v) \ge 0$, $d_\varepsilon(u^0, v^0) = 0$, we conclude that as $\varepsilon \to 0$, $d_\varepsilon(u, v) \to 0$. For $i = 1, 2$ and a.e. $(x, t) \in \Omega \times (0, T)$,

$$\xi_\varepsilon(u_i) + \xi_\varepsilon(v_i) - 2\xi_\varepsilon(\frac{u_i + v_i}{2}) \to 0 \quad \text{as } \varepsilon \to 0. \tag{4.5}$$

By Taylor's formula, there are functions $\theta_\varepsilon, \eta_\varepsilon \in [0, 1]$ such that

$$\xi_\varepsilon(u_i) = \xi_\varepsilon(\frac{u_i + v_i}{2}) + \xi'_\varepsilon(\frac{u_i + v_i}{2}) \cdot \frac{u_i - v_i}{2} + \frac{1}{2}\xi''_\varepsilon(\theta_\varepsilon \cdot \frac{u_i + v_i}{2} + (1 - \theta_\varepsilon)u_i) \cdot (\frac{u_i - v_i}{2})^2,$$
$$\xi_\varepsilon(v_i) = \xi_\varepsilon(\frac{u_i + v_i}{2}) - \xi'_\varepsilon(\frac{u_i + v_i}{2}) \cdot \frac{u_i - v_i}{2} + \frac{1}{2}\xi''_\varepsilon(\eta_\varepsilon \cdot \frac{u_i + v_i}{2} + (1 - \eta_\varepsilon)v_i) \cdot (\frac{u_i - v_i}{2})^2.$$

Because $\xi''_\varepsilon(s) = 1/(s + \varepsilon) \ge 1/2$ for $0 \le s \le 1$, $0 < \varepsilon < 1$, we derive that

$$\xi_\varepsilon(u_i) + \xi_\varepsilon(v_i) - 2\xi_\varepsilon(\frac{u_i + v_i}{2}) \ge \frac{(u_i - v_i)^2}{8}. \tag{4.6}$$

We let $\varepsilon \to 0$ in (4.6). By (4.5), we derive that for $i = 1, 2$, $(u_i - v_i)^2/8 \le 0$; thus, $u = v$ a.e. $(x, t) \in \Omega \times (0, T)$. □

5. Conclusions

We have shown that the two-species chemotaxis model (1.1)–(1.3) has a weak solution. In [20], bounded weak solutions to a class of cross-diffusion systems were also discussed. In the proof, a class of entropy functionals were introduced to yield uniform estimates.

For some cross-diffusion systems, how to show their weak solutions exist remains a great challenge. In the global existence analysis of a chemotaxis model with a cross-diffusion phenomenon, we have demonstrated how to find the entropy structure of the model in this work.

For some cross-diffusion systems, their corresponding entropy structures are still hidden. Among these systems, the Shigesada-Kawasaki-Teramoto-type population system is the most typical one. In [7], weak solutions of the population system were shown to exist when diffusion coefficients satisfy certain assumptions.

We still denote $u = (u_1, u_2)$ and constants $d_0, d_1, d_2 > 0$, $d_0 \ne d_1$, $d_0 \ne d_2$, with

$$\delta(u) = d_1 d_2(1 - u_1 - u_2) + d_0(d_1 u_1 + d_2 u_2).$$

Three-component Maxwell-Stefan equations are systems of parabolic partial differential equations in the form of (1.1), with the diffusion matrix given by

$$\mathbf{A}(u) = \frac{1}{\delta(u)} \begin{pmatrix} d_2 + (d_0 - d_2)u_1 & (d_0 - d_1)u_1 \\ (d_0 - d_2)u_2 & d_1 + (d_0 - d_1)u_2 \end{pmatrix}.$$

For every test function $\varphi \in L^2(0, T; H^1(\Omega))$, $u = (u_1, u_2)$ satisfies that $u \in \overline{\mathcal{D}}$ and that

$$\langle \partial_t u_1, \varphi \rangle + \langle \frac{1}{\delta(u)}((d_2 + (d_0 - d_2)u_1)\nabla u_1 + (d_0 - d_1)u_1 \nabla u_2), \nabla\varphi \rangle = 0,$$
$$\langle \partial_t u_2, \varphi \rangle + \langle \frac{1}{\delta(u)}((d_0 - d_2)u_2 \nabla u_1 + (d_1 + (d_0 - d_1)u_2)\nabla u_2), \nabla\varphi \rangle = 0.$$

Once we denote

$$\beta_1 = \frac{d_1}{d_0 - d_1}, \quad \beta_2 = \frac{d_2}{d_0 - d_2},$$

we can show that

$$\langle \partial_t(\beta_1 u_1 + \beta_2 u_2), \varphi \rangle = \langle \Delta(\frac{\beta_1 u_1}{d_1} + \frac{\beta_2 u_2}{d_2}), \varphi \rangle. \tag{5.1}$$

Let us denote δ_{ij} as the Kronecker delta symbol. For one of the most typical cross-diffusion systems with a degeneration phenomenon, the system is in the form of (1.1), with the diffusion matrix given by

$$A_{ij}(u) = \delta_{ij} D_i u_{n+1} + D_i u_i, \quad u_{n+1} = 1 - \sum_{i=1}^{n} u_i, \quad D_i > 0, \quad i, j = 1, ..., n.$$

For every test function $\varphi \in L^2(0, T; H^1(\Omega))$, the weak solution $u = (u_1, ..., u_n)$ satisfies that $u \in \overline{\mathcal{V}}$ and that

$$\langle \partial_t(\frac{u_i}{D_i}), \varphi \rangle + \langle u_{n+1}\nabla u_i - u_i \nabla u_{n+1}, \nabla\varphi \rangle = 0, \quad 1 \le i \le n,$$

which follows

$$\langle \partial_t(\sum_{i=1}^{n} \frac{u_i}{D_i}), \varphi \rangle = \langle \Delta(\sum_{i=1}^{n} u_i), \varphi \rangle. \tag{5.2}$$

For semiconductor model with electron-hole scattering, the model is in the form of (1.1), with the diffusion matrix given by

$$\mathbf{A}(u) = \frac{1}{1 + \mu_2 u_1 + \mu_1 u_2} \begin{pmatrix} \mu_1(1 + \mu_2 u_1) & \mu_1\mu_2 u_1 \\ \mu_1\mu_2 u_2 & \mu_2(1 + \mu_1 u_2) \end{pmatrix}.$$

For every test function $\varphi \in L^2(0, T; H^1(\Omega))$, the weak solution $u = (u_1, u_2)$ satisfies that

$$\langle \partial_t(\frac{u_1}{\mu_1}), \varphi \rangle + \langle \frac{(1 + \mu_2 u_1)\nabla u_1 + \mu_2 u_1 \nabla u_2}{1 + \mu_2 u_1 + \mu_1 u_2}, \nabla\varphi \rangle = 0,$$
$$\langle \partial_t(\frac{u_2}{\mu_2}), \varphi \rangle + \langle \frac{\mu_1 u_2 \nabla u_1 + (1 + \mu_1 u_2)\nabla u_2}{1 + \mu_2 u_1 + \mu_1 u_2}, \nabla\varphi \rangle = 0,$$

and we can show that

$$\langle \partial_t(\frac{u_1}{\mu_1} + \frac{u_2}{\mu_2}), \varphi \rangle = \langle \Delta(u_1 + u_2), \varphi \rangle. \tag{5.3}$$

We notice that (5.1)–(5.3) are similar to each other. So far, uniqueness of weak solutions results for these cross-diffusion systems have been shown when $d_1 = d_2$ in (5.1), $D_i = D_j$, $1 \le i, j \le n$ in (5.2), and $\mu_1 = \mu_2$ in (5.3). How to show that these weak solutions are unique for general-coefficients cases remains a major open topic in the study of parabolic partial differential equations.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author acknowledges support from the National Natural Science Foundation of China (NSFC), Grant no. 12471206.

Conflict of interest

The author declares no competing interests.

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A. Lemma 5 of [2]

For integrity purposes, we give a brief proof to Lemma 5 of [2]. Let $y \in L^\infty(\Omega;\mathbb{R}^n)$, and $\delta \in [0,1]$, and define

$$a(w,\phi) = \int_\Omega \nabla\phi : \mathbf{B}(y)\nabla w dx + \varepsilon \int_\Omega \Big(\sum_{|\alpha|=m} D^\alpha w \cdot D^\alpha \phi + w \cdot \phi \Big) dx,$$

$$F(\phi) = -\frac{\delta}{\tau} \int_\Omega (u(y) - u(w^{k-1})) \cdot \phi dx.$$

Let us consider the equation

$$a(w,\phi) = F(\phi), \text{ for every } \phi \in H^m(\Omega;\mathbb{R}^n). \tag{A.1}$$

The forms a and F are bounded on $H^m(\Omega;\mathbb{R}^n)$. The matrix $\mathbf{B}(y) = \mathbf{A}(u(y))(h''(u(y)))^{-1}$ is positive semi-definite because

$$z^{\mathrm{T}}\mathbf{B}(y)z = ((h''(u))^{-1}z)^{\mathrm{T}} h''(u)\mathbf{A}(u(y))(h''(u))^{-1}z \geq 0 \text{ for every } z \in \mathbb{R}^n.$$

For the bilinear form a, by the Poincaré inequality, we have

$$a(w,w) \geq \varepsilon \int_\Omega \Big(\sum_{|\alpha|=m} |D^\alpha w|^2 + |w|^2 \Big) dx \geq \varepsilon C_p \|w\|^2_{H^m(\Omega)}.$$

We apply the Lax-Milgram lemma to obtain the existence of a unique solution $w \in H^m(\Omega;\mathbb{R}^n) \hookrightarrow L^\infty(\Omega;\mathbb{R}^n)$ to (A.1). This defines the fixed point operator $S : L^\infty(\Omega;\mathbb{R}^n) \times [0,1] \to L^\infty(\Omega;\mathbb{R}^n)$, $S(y,\delta) = w$, where w solves (A.1).

The fixed point operator S is continuous, compact, and bounded. For the continuity and compactness of the operator S, refer to [2] for a detailed explanation. We are left to prove a uniform bound for all fixed points of $S(\cdot,\delta)$ in $L^\infty(\Omega;\mathbb{R}^n)$.

Let $w \in L^\infty(\Omega;\mathbb{R}^n)$ be such a fixed point. Then, w solves (A.1) with y replaced by w. Let us choose the test function as $\phi = w$; then,

$$\frac{\delta}{\tau} \int_\Omega (u(w) - u(w^{k-1})) \cdot w dx + \int_\Omega \nabla w : \mathbf{B}(w) \nabla w dx + \varepsilon \int_\Omega \Big(\sum_{|\alpha|=m} |D^\alpha w|^2 + |w|^2 \Big) dx = 0.$$

The convexity of h implies that $h(x) - h(y) \leq h'(x)(x-y)$ for all $x, y \in \mathcal{D}$. Let us choose $x = u(w)$, $y = u(w^{k-1})$, and $h'(u(w)) = w$, which follows

$$\delta \int_\Omega h(u(w)) dx + \varepsilon\tau \int_\Omega \Big(\sum_{|\alpha|=m} |D^\alpha w|^2 + |w|^2 \Big) dx \leq \delta \int_\Omega h(u(w^{k-1})) dx.$$

This yields an H^m bound for w uniform in δ. The Leray-Schauder fixed-point theorem shows that there exists a solution $w \in H^m(\Omega;\mathbb{R}^n)$ to (A.1), with y replaced by w and $\delta = 1$.



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