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Research article

An overdetermined problem in a weakly star-shaped annular domain with a spherical boundary component in space forms

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Abstract: We obtained symmetry results of the solution to an overdetermined problem for the equation $\triangle u + nKu = -\delta$, where $\delta = 0, 1$, on a weakly star-shaped annular domain Ω with a spherical boundary component in space forms $\mathbb{S}^n$ and $\mathbb{H}^n$. If the corresponding P-function was constant on $\partial\Omega$, then we would get a Serrin-type symmetry. In addition, extending existing results, we established a Serrin-type rigidity theorem under boundary conditions.

Keywords: overdetermined problem; annular domain; space form
Mathematics Subject Classification: 35N25, 35R01, 53C24

1. Introduction

In his seminal 1971 paper [23], Serrin proved that if there exists a solution to the following Poisson equation

$$\triangle u = -1 \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with C^2 boundary $\partial\Omega$ such that

$$u = 0 \quad \text{and} \quad u_\nu = c \quad \text{on} \quad \partial\Omega,$$

then Ω must be a ball and u radially symmetric. Serrin's original proof was based on the method of moving planes. Right after, Weinberger [26] dealt with the above problem with a simple proof, making use of the subharmonic function corresponding to the given partial differential equation (1.1) and the maximum principle. After the work of Serrin [23] and Weinberger [26], there has been a huge amount of work on the overdetermined problem. First, Serrin's overdetermined problem has been generalized in various ways in the Euclidean space. We refer the reader to [2, 6–8, 14, 18] and references therein.

Moreover, it has also been extended to space forms and beyond (see [3, 11, 17, 20] for $\mathbb{S}^n$, [4, 13] for $\mathbb{H}^n$, and [16, 22] for warped product manifolds). Qiu and Xia [20] proved that if there exists a solution to

$$\triangle u + nu = -1 \tag{1.2}$$

in a bounded domain $\Omega \subset \mathbb{S}^n$ such that

$$u = 0 \quad \text{and} \quad u_\nu = c \quad \text{on} \quad \partial\Omega,$$

then Ω is a geodesic ball and u is radially symmetric. Ciraolo and Vezzoni [4] extended this result to the hyperbolic space $\mathbb{H}^n$.

On the other hand, there are not many known results concerning a Serrin-type overdetermined problem on an annular domain Ω. In [19], Philippin proved that if u is a solution of

$$\triangle u = 0$$

in an annular domain $\Omega = \Omega_0 \setminus \overline{\Omega_1} \subset \mathbb{R}^n$ such that

$$\begin{cases} u = 0 \quad \text{on} \quad \partial\Omega_0 \quad \text{and} \quad u = 1 \quad \text{on} \quad \partial\Omega_1, \\ |u_\nu| = c \quad \text{on} \quad \partial\Omega, \end{cases}$$

then Ω is a radially symmetric annular domain and u is radially symmetric, under the assumption that each Ω_i is star-shaped, $i = 0, 1$. Reichel [21] and Sirakov [24] generalized this to the equation (1.1) and further. See [1, 9, 10, 12, 25, 27] for an overdetermined problem on an annular domain $\Omega \subset \mathbb{R}^n$. In an n-dimensional sphere $\mathbb{S}^n$, Fall, Minlend, and Weth [5] showed that there exists a non-radially symmetric annular domain Ω for which (1.1) with $u = 0$ and $u_\nu = c$ on $\partial\Omega$ is solvable. Recently, Lee and Seo [15] obtained Serrin-type symmetry results for Eq (1.2) in annular domains with a spherical boundary component in $\mathbb{S}^n$, and the analogous results in $\mathbb{H}^n$.

In this paper, we consider the Serrin-type overdetermined problem for

$$\triangle u + nKu = -\delta \tag{1.3}$$

in an annular domain, such that one of its boundary components Γ is spherical, in space forms of sectional curvature $K = 1, -1$, where $\delta = 0, 1$. We derive necessary conditions on boundary data for the existence of a solution to the overdetermined problem $\nabla^2 u + Kug = -\frac{\delta}{n}g$. Under these conditions, we prove Serrin-type symmetry result in an annular domain with a spherical boundary component in space forms for (1.3) (see Theorems 3.4 and 3.5). Additionally, we extend several results of Lee and Seo [15], specifically Theorems 2.4, 2.5, and 3.3. Corresponding results are presented in Theorems 3.6 and 3.8.

2. Preliminaries

Let M^n be an n-dimensional complete simply connected space form of sectional curvature $K \neq 0$. We may assume that $K = 1, -1$ via homotheties. The corresponding space forms to $K = 1$ and -1 are

the unit sphere $\mathbb{S}^n$ and the hyperbolic space $\mathbb{H}^n$, respectively.

Fix a point $p \in M$. One can consider a space form M as a warped product space $M = I \times \mathbb{S}^{n-1}$ with the metric $g = dr^2 + h(r)^2 g_{\mathbb{S}^{n-1}}$, where r is the distance from p and $g_{\mathbb{S}^{n-1}}$ is the round metric on $\mathbb{S}^{n-1}$. The warping function h is defined as follows.

$$h(r) = \begin{cases} \sin r \quad \text{on} \quad I = [0, \pi) & \text{if} \quad K = 1, \\ \sinh r \quad \text{on} \quad I = [0, \infty) & \text{if} \quad K = -1. \end{cases}$$

Here, we use an abuse of notation and identify $h(x)$ with $h(r)$ to emphasize that it is radial. From now on, we will employ the same notation when no ambiguity arises, i.e., denote $\varphi(x)$ by $\varphi(r)$ for a radial function φ.

Denote the Laplacian and the gradient on M by $\triangle$ and ∇. Let $\Omega \subset M$ be a bounded domain with C^2 boundary $\partial\Omega$. For a constant $\delta = 0, 1$, let u be a solution of

$$\triangle u + nKu = -\delta$$

in Ω. Define two P-functions

$$\begin{aligned} P(u) &= |\nabla u|^2 + Ku^2 + \frac{2\delta}{n} u, \\ \widetilde{P}(u) &= \langle \nabla u, \nabla h' \rangle + Kuh' + \frac{\delta}{n} h', \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the inner product on M and h' denotes the radial derivative $\frac{d}{dr}h$ of h. The following properties of P-functions are well-known and the proofs can be found in many papers (see [4, 15, 20, 26]). We include a proof for the reader's convenience.

Lemma 2.1 (P-functions, [4, 15, 20, 26])**.**

$$\triangle P \geq 0 \quad \textit{and} \quad \triangle \widetilde{P} = 0.$$

Moreover, $\triangle P = 0$ *if, and only if,* $\nabla^2 u + Kug = -\frac{\delta}{n} g$.

Proof. Applying the Bochner formula, we compute

$$\begin{aligned} \triangle |\nabla u|^2 &= 2\left(|\nabla^2 u|^2 + \langle \nabla \triangle u, \nabla u \rangle + Ric(\nabla u, \nabla u) \right) \\ &= 2\left(|\nabla^2 u|^2 - nK|\nabla u|^2 + (n-1)K|\nabla u|^2 \right) \\ &= 2\left(|\nabla^2 u|^2 - K|\nabla u|^2 \right), \end{aligned}$$

where $\nabla^2 u$ is the Hessian of u and $Ric(\cdot, \cdot)$ is the Ricci curvature of M. Then we have

$$\begin{aligned} \triangle P &= \triangle \left(|\nabla u|^2 + Ku^2 + \frac{2\delta}{n} u \right) \\ &= 2\left(|\nabla^2 u|^2 - K|\nabla u|^2 \right) + 2\left(Ku \triangle u + K|\nabla u|^2 + \frac{\delta}{n} \triangle u \right) \end{aligned}$$

$$
\begin{aligned}
&= 2\left(|\nabla^2 u + Kug|^2 - 2Ku\triangle u - nK^2u^2 + Ku\triangle u + \frac{\delta}{n}\triangle u\right)\\
&= 2\left(|\nabla^2 u + Kug|^2 - \frac{\delta}{n}\right)\\
&\geq \frac{2}{n}\left(\left(\mathrm{tr}\left(\nabla^2 u + Kug\right)\right)^2 - \delta\right) = 0.
\end{aligned}
$$

The fourth equality holds because $\delta^2 = \delta$. The inequality follows from $|A|^2 = \sum_{i,j}^n A_{ij}^2 \geq \sum_i^n A_{ii}^2 \geq \frac{1}{n}(\sum_i^n A_{ii})^2 = \frac{1}{n}(\mathrm{tr}A)^2$ for an $n \times n$ matrix A. The equality holds if, and only if, $A_{ii} = A_{jj}$ and $A_{ij} = 0$ for $1 \leq i \neq j \leq n$. Thus, $\triangle P = 0$ if, and only if, $\nabla^2 u + Kug = \lambda g$ for some $\lambda \in \mathbb{R}$. This implies that $n\lambda = \mathrm{tr}\left(\nabla^2 u + Kug\right) = \triangle u + nKu = -\delta$. We obtain $\lambda = -\frac{\delta}{n}$.

Note that

$$
h'(r) = \begin{cases} \cos r \quad \text{on} \quad I = [0, \pi) & \text{if} \quad K = 1, \\ \cosh r \quad \text{on} \quad I = [0, \infty) & \text{if} \quad K = -1. \end{cases}
$$

It is well established that

$$
\nabla^2 h' + Kh'g = 0 \quad \text{and} \quad \triangle h' + nKh' = 0. \tag{2.1}
$$

Using the Bochner formula, we have

$$
\begin{aligned}
\triangle\langle\nabla u, \nabla h'\rangle &= 2\langle\nabla^2 u, \nabla^2 h'\rangle + \langle\nabla\triangle u, \nabla h'\rangle + \langle\nabla u, \nabla\triangle h'\rangle + 2Ric(\nabla u, \nabla h')\\
&= 2\left(-Kh'\triangle u - nK\langle\nabla u, \nabla h'\rangle + (n-1)K\langle\nabla u, \nabla h'\rangle\right)\\
&= 2\left(-Kh'\triangle u - K\langle\nabla u, \nabla h'\rangle\right).
\end{aligned}
$$

Then,

$$
\begin{aligned}
\triangle\widetilde{P} &= \triangle\left(\langle\nabla u, \nabla h'\rangle + Kuh' + \frac{\delta}{n}h'\right)\\
&= 2\left(-Kh'\triangle u - K\langle\nabla u, \nabla h'\rangle\right) + Kh'\triangle u + Ku\triangle h' + 2K\langle\nabla u, \nabla h'\rangle + \frac{\delta}{n}\triangle h'\\
&= -Kh'\triangle u + Ku\triangle h' + \frac{\delta}{n}\triangle h' = 0.
\end{aligned}
$$

□

Lemma 2.2. *Let $\Omega \subset M$ be a bounded domain with C^2 boundary $\partial\Omega$ such that one of its boundary components Γ is spherical. Let $\Gamma = \partial B_R(p)$ for a fixed point $p \in M$, and let r be the distance from p. Suppose that there exists a solution $u \in C^2(\overline{\Omega})$ of*

$$
\begin{cases} \nabla^2 u + Kug = -\dfrac{\delta}{n}g, \\ u = a \quad \text{and} \quad \frac{\partial}{\partial r}u = c \quad \text{on} \quad \Gamma, \end{cases} \tag{2.2}
$$

where a and c are constants. Then u is a radial function. More precisely,

$$
u(x) = Ah'(r(x)) - \frac{\delta K}{n},
$$

where $A = -\frac{cK}{h(R)}$.

Proof. Let $\gamma_v : I \to M$ be a unit speed maximal geodesic such that $\gamma_v(0) = p$ and $\gamma_v'(0) = v \in T_pM$. For each v, there exists a maximal subinterval J_v of I such that $\gamma_v(J_v) \subset \overline{\Omega}$ and $R \in J_v$. Define a function φ on J_v by $\varphi(t) = u \circ \gamma_v(t)$. Then $\varphi'(t) = \langle \nabla u, \gamma_v'(t) \rangle$ and

$$\varphi''(t) = \nabla^2 u(\gamma_v'(t), \gamma_v'(t)) = -K\varphi - \frac{\delta}{n}$$

since γ_v is a geodesic. Solving the above ordinary differential equation yields

$$\varphi(t) = Ah'(t) + Bh(t) - \frac{\delta K}{n}$$

for some constants A and B. Since $\gamma_v(R) \in \Gamma$, by using the boundary condition on Γ,

$$\begin{cases} a = \varphi(R) = Ah'(R) + Bh(R) - \frac{\delta K}{n}, \\ c = \varphi'(R) = -AKh(R) + Bh'(R), \end{cases}$$

where we used the relation $h'' = -Kh$. Since $h^2 + Kh'^2 = K$, we have

$$B = \left(a + \frac{\delta K}{n}\right) Kh(R) + ch'(R).$$

For any tangent vector fields X, Y of a geodesic sphere $\Gamma = \partial B_R(p)$,

$$\mathrm{II}(X, Y) = \frac{h'(R)}{h(R)} g(X, Y)n,$$

where n is the unit normal to $\Gamma = \partial B_R(p)$ pointing into the geodesic ball $B_R(p)$. The fact that u is constant on Γ implies that

$$\begin{aligned} \nabla^2 u(X, Y) &= X(Y(u)) - (\nabla^\Gamma_X Y + \mathrm{II}(X, Y))(u) \\ &= -\mathrm{II}(X, Y)(u) \\ &= \left\langle \nabla u, \frac{h'(R)}{h(R)} g(X, Y) \frac{\partial}{\partial r} \right\rangle, \end{aligned}$$

where ∇^Γ and II denote the tangential connection and the second fundamental form of Γ in M, respectively. Let X be a unit tangent vector field to $\Gamma = \partial B_R(p)$. Since $u = a$ and $u_r = c$ there,

$$-\frac{\delta}{n} = -\frac{\delta}{n} g(X, X) = (\nabla^2 u + Kug)(X, X) = \frac{h'(R)}{h(R)} c + Ka,$$

which is equivalent, since $K^2 = 1$, to

$$B = \left(a + \frac{\delta K}{n}\right) Kh(R) + ch'(R) = 0.$$

Therefore,

$$A = -\frac{cK}{h(R)} \quad \text{and} \quad \varphi(t) = Ah'(t) - \frac{\delta K}{n},$$

which implies that φ depends only on the distance from p, and not on the choice of v. It follows that $u(x) = \varphi(r(x))$ for all $x \in \bigcup \{\gamma_v(J_v) \,|\, v \in T_pM\} \subset \Omega$. Note that there exists a constant $\epsilon > 0$, independent of v, such that for all $v \in T_pM$, either $(R - \epsilon, R] \subset J_v$ or $[R, R + \epsilon) \subset J_v$ since $\Gamma \subset \partial\Omega$ is compact. The function $w = u - \left(Ah' - \frac{\delta K}{n}\right)$ satisfies $\triangle w + nKw = 0$ and vanishes together with ∇w on an open subset of the connected domain Ω, so $w \equiv 0$ by unique continuation. This completes the proof. □

3. Overdetermined problem on a weakly star-shaped annular domain with a spherical boundary component

From now on, let $p \in M$ be a fixed point, and let r denote the distance from p. For an annular domain $\Omega \subset M$ with boundary $\partial\Omega$ such that one of its boundary components Γ is spherical, we find necessary conditions that a solution of (2.2) with constant boundary conditions must satisfy. Let $\Gamma = \partial B_R(p)$. Then, either $\Omega \subset B_R(p)$ or $\Omega \subset M \setminus B_R(p)$. We consider the first case in the following theorem.

Theorem 3.1. *Let $\Omega \subset B_R(p)$ be an annular domain in M with C^2 boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ such that $\Gamma_2 = \partial B_R(p)$. Suppose that there exists a nonconstant solution $u \in C^2(\overline{\Omega})$ of*

$$\nabla^2 u + Kug = -\frac{\delta}{n} g$$

such that

$$\begin{cases} u = a_1, \quad u_\nu = c_1 \quad on \quad \Gamma_1, \\ u = a_2, \quad u_\nu = c_2 \quad on \quad \Gamma_2 = \partial B_R(p), \end{cases}$$

where ν is the outward unit normal to $\partial\Omega$ and a_1, a_2, c_1, c_2 are constants. Then the following hold.

1. $P|_{\Gamma_1} = P|_{\Gamma_2}$.
2. $c_1 \cdot c_2 < 0$.
3. $\Omega = \Omega^*$, *where* $\Omega^* = \{x \in M \,|\, 0 < r_1 < r(x) < r_2\}$ *such that*

$$\begin{cases} r_1 = \operatorname{arccot}(\frac{b_1}{c_1}), \quad r_2 = \operatorname{arccot}(-\frac{b_2}{c_2}) \quad where \quad b_i = a_i + \frac{\delta}{n} \quad if \quad K = 1, \\ r_1 = \operatorname{arccoth}(-\frac{b_1}{c_1}), \quad r_2 = \operatorname{arccoth}(\frac{b_2}{c_2}) \quad where \quad b_i = a_i - \frac{\delta}{n} \quad if \quad K = -1, \end{cases}$$

   where arccot *is taken to have values in* $(0, \pi)$.

4. $$u(x) = \begin{cases} sgn(c_1)\sqrt{b_1^2 + c_1^2}\, h'(r(x)) - \frac{\delta}{n} = -sgn(c_2)\sqrt{b_2^2 + c_2^2}\, h'(r(x)) - \frac{\delta}{n} & if \quad K = 1, \\ -sgn(c_1)\sqrt{b_1^2 - c_1^2}\, h'(r(x)) + \frac{\delta}{n} = sgn(c_2)\sqrt{b_2^2 - c_2^2}\, h'(r(x)) + \frac{\delta}{n} & if \quad K = -1, \end{cases}$$

   where $sgn(c_i)$ is the sign of c_i, $i = 1, 2$.

Proof. It is enough to consider the case $\delta = 0$. This is because defining $\varphi = u + \frac{K}{n}$ yields $\nabla^2\varphi + K\varphi g = 0$ such that $\varphi = b_i := a_i + \frac{K}{n}$, $\varphi_\nu = c_i$ on Γ_i, $i = 1, 2$, when $\delta = 1$. If we prove Theorem 3.1 for φ, then

$$P(\varphi) - P(u) = \left(|\nabla\varphi|^2 + K\varphi^2\right) - \left(|\nabla u|^2 + Ku^2 + \frac{2}{n}u\right) = \frac{K}{n^2}.$$

This implies that Theorem 3.1 holds for u.

Now we assume $\delta = 0$. Note that $\frac{\partial}{\partial r}u = u_\nu$ on $\Gamma_2 = \partial B_R(p)$ since $\Omega \subset B_R(p)$. It follows from Lemma 2.2 and $h'' = -Kh$ that $u = Ah'$ and $u' = Ah'' = -AKh$. Here, we may assume $A \neq 0$ because u is not a constant function. Then

$$u = Ah'(r(x)) = a_i \quad \text{for} \quad x \in \Gamma_i,$$

which implies that $h'(r)$ is constant on Γ_i, as is r. Denote it by r_i for $i = 1, 2$. In this case, $r_2 = R$. It follows from Γ_i being a geodesic sphere centered at p that $|\nabla u| = |u_\nu| = |u'|$ on Γ_i, which implies that

$$\begin{aligned} P|_{\Gamma_i} &= c_i^2 + Ka_i^2 \\ &= (-AKh(r_i))^2 + K\left(Ah'(r_i)\right)^2 \\ &= A^2(h(r_i)^2 + K(h'(r_i))^2) \end{aligned}$$

because $K^2 = 1$. Since $h^2 + K(h')^2 = K$, we get

$$P|_{\Gamma_1} = P|_{\Gamma_2} = KA^2.$$

Note that $c_i = u_\nu = u'\langle \nabla r, \nu\rangle = -AKh\langle \nabla r, \nu\rangle$. Observe that $\langle \nabla r, \nu\rangle$ has an opposite sign on each Γ_i since each Γ_i is a geodesic sphere centered at p, which implies $c_1 \cdot c_2 < 0$.

On the other hand, $a_1 = Ah'(r_1)$, $a_2 = Ah'(r_2)$, and $\langle \nabla r, \nu\rangle|_{\Gamma_1} = -1$. Then $c_1 = u_\nu|_{\Gamma_1} = -u'|_{\Gamma_1} = AKh(r_1)$ and $c_2 = -AKh(r_2)$.

- For the case of $K = 1$ ($h = \sin r$, $h' = \cos r$),
  $r_1 = \operatorname{arccot}(\frac{a_1}{c_1})$, $r_2 = \operatorname{arccot}(-\frac{a_2}{c_2})$,

$$\begin{aligned} A &= \frac{c_1}{\sin r_1} = \frac{c_1}{\sin(\operatorname{arccot}(\frac{a_1}{c_1}))} = \operatorname{sgn}(c_1)\sqrt{a_1^2 + c_1^2}, \\ &= \frac{-c_2}{\sin r_2} = \frac{-c_2}{\sin(\operatorname{arccot}(-\frac{a_2}{c_2}))} = -\operatorname{sgn}(c_2)\sqrt{a_2^2 + c_2^2}. \end{aligned}$$

- For the case of $K = -1$ ($h = \sinh r$, $h' = \cosh r$),
  $r_1 = \operatorname{arccoth}(-\frac{a_1}{c_1})$, $r_2 = \operatorname{arccoth}(\frac{a_2}{c_2})$,

$$\begin{aligned} A &= \frac{a_1}{\cosh r_1} = \frac{a_1}{\cosh(\operatorname{arccoth}(-\frac{a_1}{c_1}))} = -\operatorname{sgn}(c_1)\sqrt{a_1^2 - c_1^2}, \\ &= \frac{a_2}{\cosh r_2} = \frac{a_2}{\cosh(\operatorname{arccoth}(\frac{a_2}{c_2}))} = \operatorname{sgn}(c_2)\sqrt{a_2^2 - c_2^2}. \end{aligned}$$

This completes the proof. □

Remark 3.2. *The overdetermined problem in Theorem 3.1 does not admit a solution for an arbitrary* $\{a_1, a_2, c_1, c_2\}$. *Necessary conditions on boundary data for the existence of a solution to the overdetermined problem in Theorem 3.1 are listed below:*

- $\left(a_2 + \frac{\delta K}{n}\right) h(R) + c_2 K h'(R) = 0$ *(see the proof of Lemma 2.2).*
- $P|_{\Gamma_1} = P|_{\Gamma_2}$, *equivalently* $Ka_1^2 + \frac{2\delta}{n}a_1 + c_1^2 = Ka_2^2 + \frac{2\delta}{n}a_2 + c_2^2$.
- $c_1 \cdot c_2 < 0$.
- *If* $K = 1$, *then* $\frac{c_2}{\sin R} = -sgn(c_1)\sqrt{b_1^2 + c_1^2}$.
- *If* $K = -1$, *then* $\frac{b_1}{c_1} < 0$, $\frac{b_2}{c_2} > 0$, *and* $\frac{c_2}{\sinh R} = -sgn(c_1)\sqrt{b_1^2 - c_1^2}$.

Remark 3.3. *Similar to Theorem 3.1, we can prove the case $\Omega \subset M \setminus B_R(p)$ as follows: Let $\Omega \subset M \setminus B_R(p)$ be an annular domain in M with C^2 boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ such that $\Gamma_1 = \partial B_R(p)$. Suppose that there exists a nonconstant solution $u \in C^2(\overline{\Omega})$ of*

$$\nabla^2 u + Kug = -\frac{\delta}{n}g$$

such that

$$\begin{cases} u = a_1, \quad u_\nu = c_1 \quad on \quad \Gamma_1 = \partial B_R(p), \\ u = a_2, \quad u_\nu = c_2 \quad on \quad \Gamma_2, \end{cases}$$

where ν is the outward unit normal to $\partial\Omega$ and a_1, a_2, c_1, c_2 are constants. Then the conclusions of Theorem 3.1 remain valid. Necessary conditions on boundary data for the existence of a solution to the overdetermined problem in Remark 3.3 are listed below:

- $\left(a_1 + \frac{\delta K}{n}\right) h(R) - c_1 K h'(R) = 0$ *(see the proof of Lemma 2.2).*
- $P|_{\Gamma_1} = P|_{\Gamma_2}$, *equivalently* $Ka_1^2 + \frac{2\delta}{n}a_1 + c_1^2 = Ka_2^2 + \frac{2\delta}{n}a_2 + c_2^2$.
- $c_1 \cdot c_2 < 0$.
- *If* $K = 1$, *then* $\frac{c_1}{\sin R} = -sgn(c_2)\sqrt{b_2^2 + c_2^2}$.
- *If* $K = -1$, *then* $\frac{b_1}{c_1} < 0$, $\frac{b_2}{c_2} > 0$, *and* $\frac{c_1}{\sinh R} = -sgn(c_2)\sqrt{b_2^2 - c_2^2}$.

The notion of a weakly star-shaped annular domain was defined by Lee and Seo [15].

Definition. *Let Ω be an annular domain. Ω is weakly star-shaped with respect to p if each component of the boundary $\partial\Omega$ can be written as a graph over a geodesic sphere with center p.*

In this case, we simply call Ω a weakly star-shaped annular domain. Let Ω be a weakly star-shaped annular domain in M with C^2 boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ such that one of the boundary components is $\partial B_R(p)$. We deal with the following overdetermined problem in Ω. For $\delta = 0, 1$,

$$\begin{cases} \triangle u + nKu = -\delta & \text{in } \Omega, \\ u = a_1, \quad u_\nu = c_1 & \text{on } \Gamma_1, \\ u = a_2, \quad u_\nu = c_2 & \text{on } \Gamma_2, \end{cases} \tag{3.1}$$

where a_1, a_2, c_1, c_2 are constants.

In the Serrin's overdetermined problem, the conditions $u = 0$ and $u_\nu = c$ on $\partial\Omega$ are assumed, so the corresponding P-function $P(u)$ has a constant value on $\partial\Omega$. On the other hand, in Theorem 3.1, we show that any solution u of $\nabla^2 u + Kug = -\frac{\delta}{n}g$ (and hence to $\triangle u + nKu = -\delta$) in an annular domain with a spherical boundary component must satisfy that $P|_{\Gamma_1} = P|_{\Gamma_2}$ and $c_1 \cdot c_2 < 0$. In this manner, it is natural to consider the following.

Theorem 3.4. *Let $\Omega \subset B_R(p)$ be a weakly star-shaped annular domain in M with C^2 boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ such that $\Gamma_2 = \partial B_R(p)$. Suppose that there exists a solution $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ of* (3.1) *with boundary data $\{a_1, a_2, c_1, c_2\}$ satisfying*

$$\begin{cases} P|_{\Gamma_1} = P|_{\Gamma_2}, \quad (\textit{equivalently, } \ Ka_1^2 + \frac{2\delta}{n}a_1 + c_1^2 = Ka_2^2 + \frac{2\delta}{n}a_2 + c_2^2) \\ c_1 \cdot c_2 < 0. \end{cases} \tag{3.2}$$

Then Ω must be a radially symmetric annular domain and u radially symmetric; moreover, the specific forms of Ω and u are detailed in Theorem 3.1.

Proof. For the case $\delta = 1$, we let $\varphi = u + \frac{\delta K}{n}$. Then φ satisfies the case $\delta = 0$ with (3.2), so it is enough to consider the case $\delta = 0$. Assume $\delta = 0$. Note that $\triangle P \geq 0$, which gives $P|_{\Gamma_1} = P|_{\Gamma_2} = \max_{\overline{\Omega}} P$. We claim that P is constant. Suppose that P is not constant. Since ν denotes the outward unit normal to $\partial\Omega$, the Hopf maximum principle yields $P_\nu > 0$ on $\partial\Omega$. Let $e_1, \ldots, e_{n-1}, e_n = \nu$ be an orthonormal basis near $q \in \partial\Omega$. Then $u_i = u_{\nu i} = 0$ for $i = 1, \ldots, n-1$. We have

$$\begin{aligned} P_\nu &= \left(\sum_{i=1}^{n} u_i^2 + Ku^2 \right)_\nu \\ &= 2\left(u_{\nu\nu} + Ku\right) u_\nu > 0. \end{aligned}$$

From the assumption $c_1 \cdot c_2 < 0$, it follows that

$$\left(u_{\nu\nu} + Ku\right)|_{q_1} \cdot \left(u_{\nu\nu} + Ku\right)|_{q_2} < 0 \tag{3.3}$$

for $q_i \in \Gamma_i$, $i = 1, 2$.

On the other hand, we compute the normal derivative of $\widetilde{P}$ making use of (2.1).

$$\begin{aligned} \widetilde{P}_\nu &= \left(\sum_{i=1}^{n} u_i h_i' + Kuh' \right)_\nu \\ &= \left(\sum_{i=1}^{n} \left(u_{i\nu} h_i' + u_i h_{i\nu}' \right) \right) + Ku_\nu h' + Kuh_\nu' \\ &= \left(\sum_{i=1}^{n} u_{i\nu} h_i' \right) - Ku_\nu h' + Ku_\nu h' + Kuh_\nu' \\ &= \left(u_{\nu\nu} + Ku\right) h_\nu'. \end{aligned} \tag{3.4}$$

Since h is radial, $h_\nu' = \langle \nabla h', \nu \rangle = h'' \langle \nabla r, \nu \rangle$. We compute directly that $h'' = -\sin r < 0$, and $\sinh r > 0$ in Ω for $K = 1$, and -1, respectively.

Note that $-p \notin \Omega$. Because Ω is weakly star-shaped with respect to p,

$$\langle \nabla r, \nu \rangle|_{q_1} \cdot \langle \nabla r, \nu \rangle|_{q_2} < 0,$$

and thus

$$h_\nu'|_{q_1} \cdot h_\nu'|_{q_2} < 0, \tag{3.5}$$

for $q_i \in \Gamma_i$, $i = 1, 2$. Joining (3.3) and (3.5), the equality (3.4) implies that either $\widetilde{P}_\nu > 0$ or $\widetilde{P}_\nu < 0$ on the whole $\partial\Omega$. However, either $\widetilde{P} \equiv$ const. in Ω or $\widetilde{P}_\nu$ does not have a single sign over the entire boundary $\partial\Omega$ by applying the Hopf maximum principle to $\triangle\widetilde{P} = 0$. Then, $\widetilde{P}$ becomes constant in Ω, and hence $u_{\nu\nu} + Ku = 0$ on $\partial\Omega$, which contradicts (3.3). The claim holds. Since P is constant, $\nabla^2 u + Kug = 0$ from Lemma 2.1. Then, Theorem 3.1 completes the proof. □

The proof in the case $\Omega \subset M \setminus B_R(p)$ and $\Gamma_1 = \partial B_R(p)$ is analogous and follows the same argument.

Theorem 3.5. *Let $\Omega \subset M \setminus B_R(p)$ be a weakly star-shaped annular domain in $M = \mathbb{S}^n \setminus \{-p\}$ or $\mathbb{H}^n$ with C^2 boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ such that $\Gamma_1 = \partial B_R(p)$. Suppose that there exists a solution $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ of (3.1) with boundary data $\{a_1, a_2, c_1, c_2\}$ satisfying*

$$\begin{cases} P|_{\Gamma_1} = P|_{\Gamma_2}, & (\textit{equivalently, } Ka_1^2 + \frac{2\delta}{n}a_1 + c_1^2 = Ka_2^2 + \frac{2\delta}{n}a_2 + c_2^2) \\ c_1 \cdot c_2 < 0. \end{cases}$$

Then Ω must be a radially symmetric annular domain and u radially symmetric; moreover, the specific forms of Ω and u are detailed in Theorem 3.1.

Lee and Seo dealt with the overdetermined problem (3.1) with constant boundary data on a weakly star-shaped annular domain in $M = \mathbb{S}^n, \mathbb{H}^n$, under the assumption that either one of the boundary components is spherical (Theorems 2.4, 2.5, and 3.3 in [15]). They considered the case where the boundary condition $u = 0$ is present on a non-spherical boundary component. We extend their results in Theorems 3.6 and 3.8.

Theorem 3.6. *Let $\Omega \subset B_R(p)$ be a weakly star-shaped annular domain in $\mathbb{S}^n$ with C^2 boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ such that $\Gamma_2 = \partial B_R(p)$ and $R > \frac{\pi}{2}$. Suppose that there exists a solution $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ of (3.1) with boundary data $\{a_1, a_2, c_1, c_2\}$ satisfying*

$$\begin{cases} c_1 \cdot c_2 < 0, \\ b_2 \cdot c_2 > 0, \\ b_1^2 + c_1^2 = \left(\dfrac{c_2}{\sin R}\right)^2 \end{cases}$$

where $b_i := a_i + \frac{\delta}{n}$. Then Ω must be a radially symmetric annular domain and u radially symmetric; moreover, the specific forms of Ω and u are detailed in Theorem 3.1.

Proof. Let $\varphi = u + \frac{\delta}{n}$. Then

$$\begin{cases} \triangle\varphi + n\varphi = 0 & \text{in } \Omega, \\ \varphi = b_1, \quad \varphi_\nu = c_1 & \text{on } \Gamma_1, \\ \varphi = b_2, \quad \varphi_\nu = c_2 & \text{on } \Gamma_2. \end{cases}$$

The assumption $R > \frac{\pi}{2}$ implies $h'(R) = \cos R < 0$ on Γ_2.

Case 1: $c_2 > 0$.
We claim that $P(\varphi) = |\nabla\varphi|^2 + \varphi^2$ is constant. Suppose that $P(\varphi)$ is not constant. If $P(\varphi)|_{\Gamma_1} \leq P(\varphi)|_{\Gamma_2}$, then $P(\varphi)_\nu|_{\Gamma_2} > 0$ by the Hopf maximum principle. Because $b_2 \cdot c_2 > 0$ and $c_1 \cdot c_2 < 0$, we have $b_2, c_2 > 0$. Thus,

$$\begin{aligned} c_2^2 + b_2^2 = P(\varphi)|_{\Gamma_2} \geq P(\varphi)|_{\Gamma_1} = c_1^2 + b_1^2 = \left(\frac{c_2}{\sin R}\right)^2 &\implies b_2^2 \geq \left(\frac{\cos R}{\sin R}c_2\right)^2 \\ &\implies b_2 \geq -\frac{\cos R}{\sin R}c_2. \end{aligned}$$

With respect to the spherical coordinates, the Laplacian of φ is given by

$$\triangle\varphi = \varphi'' + (n-1)\frac{h'}{h}\varphi' + \frac{1}{h^2}\triangle_{\mathbb{S}^{n-1}}\varphi, \tag{3.6}$$

where $\triangle_{\mathbb{S}^{n-1}}$ is the Laplacian on $\mathbb{S}^{n-1}$. Since Γ_2 is a geodesic sphere centered at p, on Γ_2,

$$\triangle_{\mathbb{S}^{n-1}}\varphi = 0. \tag{3.7}$$

Meanwhile, $\varphi_\nu = \varphi'$ and $\varphi_{\nu\nu} = \varphi''$ on Γ_2. Using (3.6) and (3.7), we compute

$$\begin{aligned} P(\varphi)_\nu|_{\Gamma_2} &= 2(\varphi_{\nu\nu} + \varphi)\varphi_\nu = 2(\varphi'' + \varphi)\varphi_\nu \\ &= -2(n-1)\left(\frac{\cos R}{\sin R}\varphi' + \varphi\right)\varphi_\nu \\ &= -2(n-1)\left(\frac{\cos R}{\sin R}c_2 + b_2\right)c_2 \le 0, \end{aligned}$$

which is a contradiction. Therefore, we have $P(\varphi)|_{\Gamma_2} < P(\varphi)|_{\Gamma_1}$, and thus $P(\varphi)_\nu|_{\Gamma_1} > 0$. Since $P(\varphi)|_{\Gamma_2} < P(\varphi)|_{\Gamma_1}$,

$$\begin{aligned} c_2^2 + b_2^2 = P(\varphi)|_{\Gamma_2} < P(\varphi)|_{\Gamma_1} = c_1^2 + b_1^2 = \left(\frac{c_2}{\sin R}\right)^2 &\Longrightarrow \quad b_2^2 < \left(\frac{\cos R}{\sin R}c_2\right)^2 \\ &\Longrightarrow \quad b_2 < -\frac{\cos R}{\sin R}c_2. \end{aligned}$$

which yields

$$(\varphi_{\nu\nu} + \varphi)|_{\Gamma_2} = -(n-1)\left(\frac{\cos R}{\sin R}c_2 + b_2\right) > 0.$$

It follows from $P(\varphi)_\nu|_{\Gamma_1} = 2(\varphi_{\nu\nu} + \varphi)\varphi_\nu > 0$ and $\varphi_\nu = c_1 < 0$ on Γ_1 that

$$(\varphi_{\nu\nu} + \varphi)|_{\Gamma_1} < 0.$$

Therefore, we have

$$(\varphi_{\nu\nu} + \varphi)\,|_{q_1} \cdot (\varphi_{\nu\nu} + \varphi)\,|_{q_2} < 0,$$

for $q_i \in \Gamma_i$, $i = 1, 2$. Analogous to the proof of Theorem 3.4, either $\widetilde{P}(\varphi)_\nu > 0$ or $\widetilde{P}(\varphi)_\nu < 0$ on the whole $\partial\Omega$ since Ω is weakly star-shaped with respect to p. By applying the Hopf maximum principle to $\triangle\widetilde{P}(\varphi) = 0$, we get a contradiction and the claim holds. Therefore, Ω and φ are given as in Theorem 3.1, and so is u.

Case 2: $c_2 < 0$.
Define $\tilde{\varphi} = -\varphi$. Then, $\tilde{\varphi}$ satisfies Case 1, which completes the proof. □

Remark 3.7. *Consider the following problem : Let $\Omega \subset \mathbb{S}^n \setminus B_R(p)$ be a weakly star-shaped annular domain in $\mathbb{S}^n \setminus \{-p\}$ with C^2 boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ such that $\Gamma_1 = \partial B_R(p)$ and $R < \frac{\pi}{2}$. Suppose that there exists a solution $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ of (3.1) with boundary data $\{a_1, a_2, c_1, c_2\}$ satisfying*

$$\begin{cases} c_1 \cdot c_2 < 0, \\ b_1 \cdot c_1 > 0, \\ b_2^2 + c_2^2 = \left(\dfrac{c_1}{\sin R}\right)^2 \end{cases}$$

where $b_i := a_i + \frac{\delta}{n}$. By taking $-p$ instead of p and thus replacing r with $\pi - r$, it becomes identical to the problem in Theorem 3.6.

Theorem 3.8. *Let $\Omega \subset B_R(p)$ be a weakly star-shaped annular domain in $\mathbb{H}^n$ with C^2 boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ such that $\Gamma_2 = \partial B_R(p)$. Suppose that there exists a solution $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ of (3.1) with boundary data $\{a_1, a_2, c_1, c_2\}$ satisfying*

$$\begin{cases} c_1 \cdot c_2 < 0, \\ b_2 \cdot c_2 > 0, \\ -b_1^2 + c_1^2 = -\left(\dfrac{c_2}{\sinh R}\right)^2, \end{cases}$$

where $b_i := a_i - \frac{\delta}{n}$. Then Ω must be a radially symmetric annular domain and u radially symmetric; moreover, the specific forms of Ω and u are detailed in Theorem 3.1.

Proof. Note that $h'(r) = \cosh r > 0$ in $\mathbb{H}^n$. Let $\varphi = u - \frac{\delta}{n}$. Then,

$$\begin{cases} \triangle\varphi - n\varphi = 0 & \text{in } \Omega, \\ \varphi = b_1, \quad \varphi_\nu = c_1 & \text{on } \Gamma_1, \\ \varphi = b_2, \quad \varphi_\nu = c_2 & \text{on } \Gamma_2 = \partial B_R(p). \end{cases}$$

Case 1: $c_2 < 0$.
We claim that $P(\varphi) = |\nabla\varphi|^2 - \varphi^2$ is constant. Suppose that $P(\varphi)$ is not constant. If $P(\varphi)|_{\Gamma_1} \le P(\varphi)|_{\Gamma_2}$, then $P(\varphi)_\nu|_{\Gamma_2} > 0$ by the Hopf maximum principle. Because $b_2 \cdot c_2 > 0$, we have $b_2, c_2 < 0$. Thus,

$$\begin{aligned} c_2^2 - b_2^2 = P(\varphi)|_{\Gamma_2} \ge P(\varphi)|_{\Gamma_1} = c_1^2 - b_1^2 = -\left(\frac{c_2}{\sinh R}\right)^2 &\Longrightarrow \quad b_2^2 \le \left(\frac{\cosh R}{\sinh R} c_2\right)^2 \\ &\Longrightarrow \quad b_2 \ge \frac{\cosh R}{\sinh R} c_2. \end{aligned}$$

Analogous to the proof of Theorem 3.6, we have

$$\begin{aligned} P(\varphi)_\nu|_{\Gamma_2} &= 2(\varphi_{\nu\nu} - \varphi)\varphi_\nu = 2(\varphi'' - \varphi)\varphi_\nu \\ &= -2(n-1)\left(\frac{\cosh R}{\sinh R}\varphi' - \varphi\right)\varphi_\nu \\ &= -2(n-1)\left(\frac{\cosh R}{\sinh R} c_2 - b_2\right) c_2 \le 0, \end{aligned}$$

which is a contradiction. Therefore, we have $P(\varphi)|_{\Gamma_1} > P(\varphi)|_{\Gamma_2}$, and thus $P(\varphi)_\nu|_{\Gamma_1} > 0$. Since $P(\varphi)|_{\Gamma_1} > P(\varphi)|_{\Gamma_2}$,

$$\begin{aligned} c_2^2 - b_2^2 = P(\varphi)|_{\Gamma_2} < P(\varphi)|_{\Gamma_1} = c_1^2 - b_1^2 = -\left(\frac{c_2}{\sinh R}\right)^2 &\Longrightarrow \quad b_2^2 > \left(\frac{\cosh R}{\sinh R} c_2\right)^2 \\ &\Longrightarrow \quad b_2 < \frac{\cosh R}{\sinh R} c_2. \end{aligned}$$

which yields

$$(\varphi_{\nu\nu} - \varphi)|_{\Gamma_2} = -(n-1)\left(\frac{\cosh R}{\sinh R}c_2 - b_2\right) < 0.$$

It follows from $P(\varphi)_\nu|_{\Gamma_1} = 2(\varphi_{\nu\nu} - \varphi)\varphi_\nu > 0$ and $\varphi_\nu = c_1 > 0$ on Γ_1 that

$$(\varphi_{\nu\nu} - \varphi)|_{\Gamma_1} > 0.$$

Therefore, we have

$$(\varphi_{\nu\nu} - \varphi)\,|_{q_1} \cdot (\varphi_{\nu\nu} - \varphi)\,|_{q_2} < 0,$$

for $q_i \in \Gamma_i$, $i = 1, 2$. Analogous to the proof of Theorem 3.4, either $\widetilde{P}(\varphi)_\nu > 0$ or $\widetilde{P}(\varphi)_\nu < 0$ on the whole $\partial\Omega$ since Ω is weakly star-shaped with respect to p. By applying the Hopf maximum principle to $\triangle\widetilde{P}(\varphi) = 0$, we get a contradiction and the claim holds. Therefore, Ω and φ are given as in Theorem 3.1, and so is u.

Case 2: $c_2 > 0$.
Define $\tilde{\varphi} = -\varphi$. Then $\tilde{\varphi}$ satisfies Case 1, which completes the proof of this case. □

4. Conclusions

We study overdetermined problems on annular domains with a spherical boundary component in space forms $\mathbb{S}^n$ and $\mathbb{H}^n$, and establish several rigidity results using the P-function method. We first consider overdetermined Hessian equations in space forms $\mathbb{S}^n$, $\mathbb{H}^n$. If the domain has a spherical boundary component, then the existence of a solution to the overdetermined Hessian problem implies that the domain is rotationally symmetric and that the solution is a radial function. Moreover, we derive necessary conditions that the solution must satisfy. For weakly star-shaped annular domains, we prove Serrin-type rigidity results using P-functions. As a consequence, the domain must be radially symmetric and the solution takes the form of a Serrin-type radial function. We also deal with non-homogeneous cases and obtain similar symmetry results. It is significant that known results have been extended to broader classes of boundary conditions.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest.

References

1. V. Agostiniani, S. Borghini, L. Mazzieri, On the Serrin problem for ring-shaped domains, *J. Eur. Math. Soc. (JEMS)*, **27** (2025), 2705–2749. https://doi.org/10.4171/jems/1422
2. B. Brandolini, C. Nitsch, P. Salani, C. Trombetti, Serrin-type overdetermined problems: an alternative proof, *Arch. Ration. Mech. Anal.*, **190** (2008), 267–280. https://doi.org/10.1007/s00205-008-0119-3
3. G. Ciraolo, L. Vezzoni, A rigidity problem on the round sphere, *Commun. Contemp. Math.*, **19** (2017), 1750001. https://doi.org/10.1142/S0219199717500018
4. G. Ciraolo, L. Vezzoni, On Serrin's overdetermined problem in space forms, *Manuscripta Math.*, **159** (2019), 445–452. https://doi.org/10.1007/s00229-018-1079-z
5. M. M. Fall, I. A. Minlend, T. Weth, Serrin's overdetermined problem on the sphere, *Calc. Var. Partial Dif.*, **57** (2018), 3. https://doi.org/10.1007/s00526-017-1275-9
6. A. Farina, B. Kawohl, Remarks on an overdetermined boundary value problem, *Calc. Var. Partial Dif.*, **31**, (2008), 351–357. https://doi.org/10.1007/s00526-007-0115-8
7. I. Fragalà, F. Gazzola, B. Kawohl, Overdetermined problems with possibly degenerate ellipticity, a geometric approach, *Math. Z.*, **254** (2006), 117–132. https://doi.org/10.1007/s00209-006-0937-7
8. N. Garofalo, J. L. Lewis, A symmetry result related to some overdetermined boundary value problems, *Amer. J. Math.*, **111** (1989), 9–33. https://doi.org/10.2307/2374477
9. A. Henrot, A. G. A. Philippin, Approximate radial symmetry for solutions of a class of boundary value problems in ring-shaped domains, Special issue dedicated to Lawrence E. Payne, *Z. Angew. Math. Phys.*, **54** (2003), 784–796. https://doi.org/10.1007/s00033-003-3205-2
10. A. Henrot, G. A. Philippin, H. Prébet, Overdetermined problems on ring shaped domains, *Adv. Math. Sci. Appl.*, **9** (1999), 737–747.
11. M. A. Karlovitz, *Some solutions to overdetermined boundary value problems on subsets of spheres*, Thesis (Ph.D.)–University of Maryland, College Park, ProQuest LLC, Ann Arbor, MI, 1990.
12. D. Khavinson, A. Y. Solynin, D. Vassilev, Overdetermined boundary value problems, quadrature domains and applications, *Comput. Methods Funct. Theory*, **5** (2005), 19–48. https://doi.org/10.1007/BF03321084
13. S. Kumaresan, J. Prajapat, Serrin's result for hyperbolic space and sphere, *Duke Math. J.*, **91** (1998), 17–28. https://doi.org/10.1215/S0012-7094-98-09102-5
14. E. Lee, K. Seo, An overdetermined Steklov eigenvalue problem on Riemannian manifolds with nonnegative Ricci curvature, *Results Math.*, **80** (2025), 102. https://doi.org/10.1007/s00025-025-02422-5
15. J. Lee, K. Seo, Overdetermined problems in annular domains with a spherical-boundary component in space forms, *J. Inequal. Appl.*, 2023, 45. https://doi.org/10.1186/s13660-023-02956-8
16. J. Lee, K. Seo, Overdetermined problems with a nonconstant Neumann boundary condition in a warped product manifold, *Nonlinear Anal.*, **247** (2024), 113603. https://doi.org/10.1016/j.na.2024.113603

17. R. Molzon, Symmetry and overdetermined boundary value problems, *Forum Math.*, **3** (1991), 143–156. https://doi.org/10.1515/form.1991.3.143

18. C. Nitsch, C. Trombetti, The classical overdetermined Serrin problem, *Complex Var. Elliptic Equ.*, **63** (2018), 1107–1122. https://doi.org/10.1080/17476933.2017.1410798

19. G. A. Philippin, On a free boundary problem in electrostatics, *Math. Methods Appl. Sci.*, **12** (1990), 387–392. https://doi.org/10.1002/mma.1670120503

20. G. Qiu, C. Xia, Overdetermined boundary value problems in $\mathbb{S}^n$, *J. Math. Study*, **50** (2017), 165–173. https://doi.org/10.4208/jms.v50n2.17.03

21. W. Reichel, *Radial symmetry by moving planes for semilinear elliptic BVPs on annuli and other non-convex domains*, Elliptic and parabolic problems (Pont-à-Mousson, 1994), Pitman Res. Notes Math. Ser., **325** (1995), 164–182.

22. A. Roncoroni, A Serrin-type symmetry result on model manifolds: An extension of the Weinberger argument, *C. R. Math. Acad. Sci. Paris*, **356** (2018), 648–656. https://doi.org/10.1016/j.crma.2018.04.012

23. J. Serrin, A symmetry problem in potential theory, *Arch. Rational Mech. Anal.*, **43** (1971), 304–318. https://doi.org/10.1007/BF00250468

24. B. Sirakov, Symmetry for exterior elliptic problems and two conjectures in potential theory, *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, **18** (2001), 135–156. https://doi.org/10.1016/S0294-1449(00)00052-4

25. B. Wang, J. Bao, Over-determined problems for k-Hessian equations in ring-shaped domains, *Nonlinear Anal.*, **127** (2015), 143–156. https://doi.org/10.1016/j.na.2015.06.032

26. H. F. Weinberger, Remark on the preceding paper of Serrin, *Arch. Rational Mech. Anal.*, **43** (1971), 319–320. https://doi.org/10.1007/BF00250469

27. N. B. Willms, G. M. L. Gladwell, D. Siegel, Symmetry theorems for some overdetermined boundary value problems on ring domains, *Z. Angew. Math. Phys.*, **45** (1994), 556–579. https://doi.org/10.1007/BF00991897



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