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Research article

Extremal ℓ-cyclic graphs for the Sombor and elliptic Sombor indices

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Abstract: For integers $n \geq 2$ and $0 \leq \ell \leq n-2$, we determined the unique maximizers, up to isomorphism, of the Sombor and elliptic Sombor indices among connected graphs of order n and cyclomatic number ℓ. The extremal graph $H_{n,\ell}$ was obtained from the star S_n by joining one leaf to ℓ other leaves. This proved the conjectures of Réti, Došlić, and Ali for the Sombor index and of Ali, Gutman, Réti, Albalahi, and Hamza for the elliptic Sombor index. We found a discrete formulation of the universal-vertex reduction and combined it with an exact identity on the residual graph, which has ℓ edges. A local inequality then forced this residual graph to be a star. For the classical Sombor weight, the inequality followed by compensating a possible chord-slope loss with the gain on a residual edge. For the elliptic Sombor weight, a convexity certificate sufficed. The method also provided the unique maximizer for $\mathrm{SO}_\beta(G) = \sum_{uv\in E(G)}(d_G(u)^2 + d_G(v)^2)^\beta$ for every $\beta \geq 1/2$, in the same range of n, ℓ. For the family with edge weight $\Psi_{\alpha,\beta}(x,y) = (x+y)^\alpha(x^2+y^2)^\beta$, $\alpha \geq 0, \beta > 0$, we characterized the smooth certificate exactly by $\alpha + 2\beta \geq 2$ and proved sufficient extremal regions beyond it, including $0 \leq \alpha \leq 1$, $\alpha + 2\beta \geq 1$.

Keywords: Sombor index; elliptic Sombor index; BID index; cyclomatic number; extremal graph; degree-based topological index
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Abbreviations: SO: Sombor index; ESO: elliptic Sombor index; BID: bond-incident degree.

1. Introduction

Gutman introduced the Sombor index through a geometric interpretation of degree pairs [1]. The elliptic Sombor index was subsequently introduced by Gutman, Furtula, and Oz [2]. For a graph G, these indices are

$$\mathrm{SO}(G) = \sum_{uv\in E(G)} \sqrt{d_G(u)^2 + d_G(v)^2}, \qquad \mathrm{ESO}(G) = \sum_{uv\in E(G)} (d_G(u) + d_G(v))\sqrt{d_G(u)^2 + d_G(v)^2}.$$

The extremal theory of Sombor indices was surveyed in [3]; for the elliptic and Euler–Sombor variants, see [4], and for the wider BID framework, see [5]. General bounds and relations for the ESO were also developed in [6, 7]. Their study also included explicit network models: Shang [8] computed the Sombor index of growing simplicial networks. Here the question is extremal: Which connected graphs maximize the index when both order and size are fixed?

Let $\mathcal{G}_{n,\ell}$ denote the connected graphs of order n and size $n-1+\ell$, where $n \geq 2$ and $0 \leq \ell \leq n-2$. The candidate $H_{n,\ell}$ is obtained from the star S_n by joining one leaf to ℓ other leaves. No assertion is made for $\ell > n-2$.

Early extremal questions for unicyclic and bicyclic SOs were studied by Cruz and Rada [9]. Réti, Došlić, and Ali [10] established the small-cyclomatic Sombor cases and conjectured the same maximizer for $5 \leq \ell \leq n-2$. Das, Ghalavand, and Ashrafi [11] proved the six-cyclic case and a general result under a bound of $n-\ell-2$ on the number of pendent vertices. The tricyclic SO problem was revisited in [12]; Hamza et al. [13] explained how its maximum already follows from [10]. For the ESO, Tang, Li, and Deng [14, Theorem 4.1] established the unicyclic maximum, and Li and Zuo [15, Theorem 2] stated the tricyclic maximum, which agrees with $H_{n,3}$ for $n \geq 5$. Ali, Gutman, Réti, Albalahi, and Hamza [4, Conjecture 2.1.1] proposed the elliptic Sombor statement for $n \geq \ell + 2 \geq 8$. Table 1 separates these earlier results from the conclusions proved below.

Table 1. Earlier results and the range treated here. The notation $H_{n,\ell}$ always requires $n \geq \ell + 2$.

Source and index	Earlier range or hypothesis	Conclusion of this paper
Réti et al. [10], SO	$\ell = 1, \ldots, 4$, $n \geq 3 + \lceil(\ell+1)/2\rceil$; $\ell = 5$, $n \geq 7$, also verified there	All $0 \leq \ell \leq n-2$
Das et al. [11], SO	$\ell = 6$; general case with at most $n-\ell-2$ pendent vertices	No pendent-vertex restriction
Tang et al. [14]; Qi–Lin [16]; Li–Zuo [15]; survey [4], ESO	Unicyclic maximum; bicyclic case for $n \geq 7$; tricyclic statement for $n \geq 5$ here; $\ell = 3, 4, 5$ in the survey, using [17]	All $0 \leq \ell \leq n-2$, including $\ell \geq 6$
Ali–Dimitrov [18]; Ali et al. [17], BID	Increment and unbalancing conditions; smooth sufficient hypotheses	Discrete formulation and local criterion on residual edges
Khan et al. [19], SO_β	Same power convention, $\beta \geq 1/2$; trees, unicyclic, and bicyclic classes	Uniform theorem for all $0 \leq \ell \leq n-2$

We work with the bond-incident-degree index

$$\mathrm{BID}_\Psi(G) = \sum_{uv \in E(G)} \Psi(d_G(u), d_G(v)),$$

where Ψ is symmetric. The universal-vertex reduction goes back to Ali and Dimitrov [18, Lemma 2.1];

see also [10, Lemma 2.2 and Corollary 2.5], [17, Lemma 12], and [20]. We formulate it on integer degree pairs. After deleting a universal vertex, an exact identity distributes its contribution over the ℓ remaining edges. The resulting weight F_a combines the universal and internal edge contributions; the local criterion (K1)–(K2) forces the residual graph to be a star.

The comparison with multiplicative indices is also relevant. Dehgardi, Azari, and Shang [21] studied $\prod_{v\in V(G)} d_G(v)^\rho$, $\rho \neq 0$, including fixed-maximum-degree classes and low cyclomatic numbers. Its logarithm is the separable vertex sum $\rho \sum_v \log d_G(v)$. Our objective is an additive edge sum: Changing one vertex degree changes the weights on all its incident edges. Thus the common use of edge transformations does not make the objectives interchangeable. The core identity below accounts jointly for the changed edge contributions. Khan et al. [19] used the same convention $\sum_{uv}(d_G(u)^2+d_G(v)^2)^\beta, \beta \geq 1/2$, in their study of tree, unicyclic, and bicyclic graphs. Our power theorem extends the comparison to every $0 \leq \ell \leq n-2$ through one uniform argument. Faheem, Ahmad, and Farooq [22] studied the SO and ESO for tree and unicyclic graphs ($c \in \{0,1\}$) with prescribed maximum degree Δ. Their Theorem 3.2 maximizes the ESO on trees for $\max\{3,\lceil n/2\rceil\} \leq \Delta \leq n-2$, while Theorem 3.5 maximizes both indices on unicyclic graphs for $\max\{3,\lceil (n+1)/2\rceil\} \leq \Delta \leq n-2$. They treated $\Delta = n-1$ separately as the singleton star or star-plus-one-edge class. Their Lemmas 2.6–2.7 use degree-concentrating transformations for these two weights. These are related antecedents of the transfer approach; our general BID reduction is attributed above to the earlier work. The present problem optimizes over all maximum degrees with order and size fixed, uniformly for $0 \leq \ell \leq n-2$; it does not solve their problem with an arbitrary prescribed Δ. Related fixed-maximum-degree bounds for general BID indices were given in [23, 24]; ESO trees under prescribed branching, maximum-degree, or segment constraints were studied in [25].

Other recent work addresses adjacent optimization problems. Liu et al. [26] treated minimum BID indices, while Gao [27] and Su–Deng [28] gave sufficient conditions for extremal vertex-degree-based indices on cyclic graphs. Alex and Imran [29] introduced classes of weights through monotonicity and increment conditions. Their Theorem 6 gives a universal vertex in every maximizer under their F_2 hypotheses; their property P_3 is a coordinate-increment monotonicity condition, related to (D2) below. Thus neither the universal-vertex reduction nor the use of integer-degree conditions is claimed here as a new method. Their explicit applications concern tree, unicyclic, bicyclic, and tricyclic graphs, together with cacti having an arbitrary number of cycles. The cactus result is a constrained problem: for $\ell \geq 2$, the triangles of $H_{n,\ell}$ share an edge, so $H_{n,\ell}$ is not a cactus. Their paper did not supply the residual-edge criterion (K1)–(K2) or the compensation estimates used here. The tricyclic SO case was already covered by [10, 13]. Further constrained settings include the ESO on cacti [30], the SO with the repetition number and cyclomatic number prescribed [31], and generalized Sombor indices with a prescribed degree sequence [32]. The latter family has weight $(x^b + y^b)^a$, whose specialization $b = 2$ is our power weight; it differs from the two-parameter elliptic family below. These additional constraints are distinct from fixing only order and size.

Our principal conclusions are as follows. Theorems 7.2 and 8.1 prove that $H_{n,\ell}$ uniquely maximizes the SO and ESO, respectively, for $0 \leq \ell \leq n-2$. Theorem 9.1 gives the same conclusion for all powers $(x^2 + y^2)^\beta$ with $\beta \geq 1/2$. For

$$\Psi_{\alpha,\beta}(x,y) = (x+y)^\alpha (x^2+y^2)^\beta, \qquad \alpha \geq 0, \quad \beta > 0,$$

Theorem 10.1 identifies the exact region of a specified smooth certificate, while Theorem 11.5 proves

larger sufficient extremal regions by compensation. The five-vertex comparison in Proposition 10.2 shows that a uniform assertion over the whole parameter plane is false. It does not characterize the full extremal region.

This two-parameter weight was introduced as the generalized elliptic Sombor family in [33] and was subsequently studied in [34]. Its specialization $\beta = \alpha/2$ was studied in [35]. The transfer lemmas and the basic convexity and divided-difference facts are used as established tools. The one-variable comparisons in [22, Lemmas 2.1–2.5] provide further antecedents for such SO and ESO estimates; they do not use the residual-edge weight F_a defined here. The contribution is the combined residual-edge weight F_a, the local criterion (K1)–(K2), and its uniform verification, in particular, through compensation for weights outside the smooth certificate. The resulting theorems extend beyond the previously established small-cyclomatic cases. The parameter results concern certificates and extremal regions for this existing index family.

Sections 3–6 establish the reduction and its certificates. Sections 7 and 8 give the two principal applications before the parameter extensions. Appendix A derives the differential identities and Appendix B gives the polynomial coefficient calculations used in the compensation estimates.

2. Setting and the discrete hypotheses

All graphs are finite and simple. Uniqueness is always understood up to isomorphism. The symbol k is reserved for a transfer amount, never for the cyclomatic number. Let $\ell \geq 0$ and $n \geq \ell + 2$ be integers, and let $\mathcal{G}_{n,\ell}$ denote the class of connected graphs of order n and size $n + \ell - 1$. Let $H_{n,\ell}$ be obtained from the star S_n by choosing one leaf v and joining v to ℓ other leaves. Equivalently, $H_{n,\ell}$ has a universal vertex whose deletion leaves

$$K_{1,\ell} \sqcup (n - \ell - 2)K_1.$$

Here $K_{1,0} = K_1$, so the same description includes $\ell = 0$.

For a symmetric function

$$\Psi : \mathbb{N}_{\geq 1}^2 \longrightarrow \mathbb{R},$$

define

$$\mathrm{BID}_\Psi(G) := \sum_{uv \in E(G)} \Psi(d_G(u), d_G(v)).$$

For integers $r, s \geq 1$, we write

$$\Delta_1\Psi(r, s) := \Psi(r + 1, s) - \Psi(r, s).$$

For integers $a \geq 2$ and $1 \leq t \leq a - 1$, we define

$$\sigma_a(t) := \frac{\Psi(a, t + 1) - \Psi(a, 1)}{t}. \tag{2.1}$$

For the same integer a and integers $1 \leq x, y \leq a - 1$, set

$$F_a(x, y) := \sigma_a(x) + \sigma_a(y) + \Psi(x + 1, y + 1). \tag{2.2}$$

These are the domains in the discrete results below; all variables in (D1)–(D4) are integers. Neither $\sigma_a(0)$ nor F_a at a zero argument is used. For a smooth weight, the same formulas extend to real $a > 1$, $0 < t \leq a - 1$, and $0 < x, y \leq a - 1$.

Definition 2.1 (Discrete core-admissibility)**.** A symmetric weight $\Psi : \mathbb{N}_{\geq 1}^2 \to \mathbb{R}$ is called *discretely core-admissible* if the following conditions hold.

(D1) $\Delta_1\Psi(r, s) > 0$ for all $r, s \geq 1$.
(D2) For all $r, s \geq 1$,

$$\Delta_1\Psi(r + 1, s) \geq \Delta_1\Psi(r, s).$$

(D3) If $r \geq s \geq k + 1 \geq 2$, then

$$\Psi(r + k, s - k) > \Psi(r, s).$$

(D4) For every $a \geq 4$, the finite sequence $\sigma_a(1), \ldots, \sigma_a(a - 1)$ is discretely convex, i.e.,

$$\sigma_a(t + 2) - \sigma_a(t + 1) \geq \sigma_a(t + 1) - \sigma_a(t),$$

for every integer t with $1 \leq t \leq a - 3$. For $a \leq 3$, this condition is vacuous.

Conditions (D1)–(D3) are the discrete input for universalization. Condition (D4) is needed only after a universal vertex has been obtained. None of the four conditions requires a real extension of Ψ.

3. Discrete universalization

The following transfer is the one in [18, Lemma 2.1], formulated for integer degrees. Related sufficient conditions appear in [17, Lemma 12] and [29, Theorem 6]. Our strict conditions (D1), (D3) and increment condition (D2) suffice for the reduction; no necessity or containment of all earlier cases is asserted. We first record the discrete convexity comparison used in the transfer; its real counterpart is standard [36, Section 3.1].

Lemma 3.1 (Equal-length increment inequality)**.** *Assume* (D2)*. If x, c, t, y are positive integers with $x \geq c > t \geq 1$, then*

$$\Psi(x + t, y) - \Psi(x, y) + \Psi(c - t, y) - \Psi(c, y) \geq 0.$$

Proof. By telescoping,

$$\Psi(x + t, y) - \Psi(x, y) = \sum_{j=0}^{t-1} \Delta_1\Psi(x + j, y),$$

whereas

$$\Psi(c, y) - \Psi(c - t, y) = \sum_{j=0}^{t-1} \Delta_1\Psi(c - t + j, y).$$

Since $x + j \geq c - t + j$, repeated application of (D2) compares the two summands term by term and proves the claim. □

Lemma 3.2 (Degree-increasing edge transfer)**.** *Let Ψ satisfy* (D1)–(D3)*, and let G be a connected simple graph on $n \geq 3$ vertices. If G has no universal vertex, then there exists a connected simple graph G^* on the same vertex set and with the same number of edges such that*

$$\mathrm{BID}_\Psi(G^*) > \mathrm{BID}_\Psi(G), \qquad \Delta(G^*) > \Delta(G).$$

Proof. Choose u of maximum degree and a shortest path $u = p_0, p_1, p_2, \dots, p_r$ to a vertex distinct from u and not adjacent to u. Then $v = p_1$ is adjacent to u, whereas $p_2 \in N_G(v) \setminus N_G(u)$. Partition the relevant neighbors as

$$\begin{aligned} A &= N_G(u) \setminus (N_G(v) \cup \{v\}), \\ C &= N_G(u) \cap N_G(v), \\ M &= N_G(v) \setminus (N_G(u) \cup \{u\}), \qquad s = |M| \geq 1. \end{aligned}$$

Delete vz and add uz for each $z \in M$. No loop or multiple edge is created, and the number of edges is unchanged. The edge uv remains. Every deleted step zv can be replaced by the walk zuv, so the new graph G^* is connected.

Write $p = d_G(u)$, $q = d_G(v)$, and $d_z = d_G(z)$ for $z \notin \{u, v\}$. Only the degrees of u, v change, to $p + s, q - s$, respectively, and $p \geq q \geq s + 1$. All changed contributions are accounted for by

$$\begin{aligned} \mathrm{BID}_\Psi(G^*) - \mathrm{BID}_\Psi(G) = &\sum_{z \in A} [\Psi(p + s, d_z) - \Psi(p, d_z)] \\ &+ \sum_{z \in C} [\Psi(p + s, d_z) - \Psi(p, d_z) + \Psi(q - s, d_z) - \Psi(q, d_z)] \\ &+ \sum_{z \in M} [\Psi(p + s, d_z) - \Psi(q, d_z)] \\ &+ \Psi(p + s, q - s) - \Psi(p, q). \end{aligned} \tag{3.1}$$

Edges not represented here have unchanged degrees at both endpoints. Each term in the first sum is positive by (D1). For each common neighbor, the bracket in the second sum equals

$$\sum_{j=0}^{s-1} [\Delta_1 \Psi(p + j, d_z) - \Delta_1 \Psi(q - s + j, d_z)] \geq 0,$$

by (D2), since $p + j \geq q - s + j$. The third sum is strictly positive: It has $s \geq 1$ terms, and $p + s > q$ allows repeated use of (D1). The last term is positive by (D3). Strict increase holds also for $p = q$ and $s = 1$. All other degrees are at most p, so $\Delta(G^*) = p + s > \Delta(G)$. □

Lemma 3.3 (Finite universalization)**.** *Let Ψ satisfy* (D1)–(D3) *and let G be a connected simple graph on $n \geq 3$ vertices. Then there exists a connected simple graph $\widehat{G}$ with the same vertex set and number of edges, having a universal vertex, such that*

$$\mathrm{BID}_\Psi(\widehat{G}) \geq \mathrm{BID}_\Psi(G).$$

If G has no universal vertex, the inequality is strict.

Proof. Induct on $n - 1 - \Delta(G)$. If it is zero, take $\widehat{G} = G$. Otherwise Lemma 3.2 gives a graph of the same order and size with a larger BID value and larger maximum degree. Thus $n - 1 - \Delta$ decreases by at least one, so at most $n - 1 - \Delta(G)$ transfers are required. If a transfer is performed, the index increases strictly, and all subsequent transfers also increase it. □

Consequently every maximizer in a fixed connected order-and-size class has a universal vertex. Otherwise the first transfer gives a contradiction. Strict improvement excludes every non-universal maximizer, regardless of the transfer sequence.

4. The core-edge extremal principle

Deleting a universal vertex leaves a graph with exactly ℓ edges. We express the index as a baseline plus one effective weight per residual edge.

We use F_a on the domains specified in Section 2. The identity below is a weighted incidence double count specialized to the universal vertex and its residual graph. Baseline-plus-edge-correction identities for BID indices also occur in [33, Eq (4.15)], [23, Theorem 1], and [28, Lemma 1]; the correction here depends on a and the residual degrees.

Lemma 4.1 (Core-edge identity)**.** *Let $G \in \mathcal{G}_{n,\ell}$ have a universal vertex u, set $a = n - 1$, and let $C = G - u$. Then $|E(C)| = \ell$ and*

$$\mathrm{BID}_\Psi(G) = a\Psi(a, 1) + \sum_{vw \in E(C)} F_a(d_C(v), d_C(w)).$$

For $n = 2$, the sum is empty and the identity reads $\mathrm{BID}_\Psi(K_2) = \Psi(1, 1)$; no value of F_1 is required.

Proof. For every $v \in V(C)$, $d_G(v) = d_C(v) + 1$. Hence the universal-vertex edges contribute

$$\begin{aligned}\sum_{v \in V(C)} \Psi(a, d_C(v) + 1) &= a\Psi(a, 1) + \sum_{\substack{v \in V(C) \\ d_C(v) > 0}} d_C(v)\sigma_a(d_C(v)) \\ &= a\Psi(a, 1) + \sum_{vw \in E(C)} (\sigma_a(d_C(v)) + \sigma_a(d_C(w))).\end{aligned}$$

The restriction $d_C(v) > 0$ avoids evaluating σ_a at zero; isolated vertices contribute only the baseline term $\Psi(a, 1)$. Adding the contributions of the edges of C proves the identity. □

The following incident-edge count gives the degree range needed below; its tree specialization is used in [35, Theorem 9].

Lemma 4.2 (Edge degree bound)**.** *If a simple graph C has exactly ℓ edges and $vw \in E(C)$, then*

$$d_C(v) + d_C(w) - 1 \le \ell.$$

Proof. The edges incident with v or w form a subset of $E(C)$ of cardinality $d_C(v) + d_C(w) - 1$. □

Theorem 4.3 (Core-edge criterion)**.** *Fix $n \ge \ell + 2$ and $\ell \ge 2$, and suppose that every maximizer of BID_Ψ over $\mathcal{G}_{n,\ell}$ has a universal vertex. Set $a = n - 1$. Assume*

(K1) *$F_a(x, 1) < F_a(x + 1, 1)$ for every integer $1 \le x < \ell$;*
(K2) *if integers $x \ge y \ge 2$ satisfy $x + y \le \ell + 1$, then*

$$F_a(x, y) < F_a(x + y - 1, 1).$$

Then $H_{n,\ell}$ is the unique maximizer of BID_Ψ over $\mathcal{G}_{n,\ell}$.

Proof. Let G be a maximizer and u a universal vertex. Set $C = G - u$. For an edge vw, write $x = d_C(v) \geq y = d_C(w)$. By Lemma 4.2, $x + y - 1 \leq \ell$. If $y = 1$, repeated use of (K1) gives $F_a(x, 1) \leq F_a(\ell, 1)$, with equality only for $x = \ell$. If $y \geq 2$, then

$$F_a(x, y) < F_a(x + y - 1, 1) \leq F_a(\ell, 1).$$

Lemma 4.1 therefore gives an upper bound $\ell F_a(\ell, 1)$ for the core sum. The graph $H_{n,\ell}$ belongs to $\mathcal{G}_{n,\ell}$ and attains the bound $a\Psi(a, 1) + \ell F_a(\ell, 1)$. Equality for a maximizer therefore holds in every edge comparison and forces every core edge to have the endpoint-degree pair $\{\ell, 1\}$. Choose any core edge. One endpoint has degree ℓ and is incident with ℓ distinct edges. These are all the edges of C, since $|E(C)| = \ell$. Their other endpoints have degree one, and every remaining vertex is isolated. Since C has $n - 1$ vertices, it follows that

$$C \cong K_{1,\ell} \sqcup (n - \ell - 2)K_1.$$

Restoring u gives exactly $H_{n,\ell}$, up to isomorphism. For $\ell \geq 1$, its value can also be read directly from its edge types:

$$\begin{aligned} \mathrm{BID}_\Psi(H_{n,\ell}) = (n - \ell - 2)\Psi(n - 1, 1) + \Psi(n - 1, \ell + 1) \\ + \ell\Psi(n - 1, 2) + \ell\Psi(\ell + 1, 2). \end{aligned} \tag{4.1}$$

For $\ell = 0$, the value is $(n - 1)\Psi(n - 1, 1)$. □

Proposition 4.4 (The cases $\ell = 0, 1$)**.** *Assume* Ψ *satisfies* (D1)–(D3). *Then* $H_{n,0}$ *is the unique maximizer over* $\mathcal{G}_{n,0}$ *for every* $n \geq 2$*, and* $H_{n,1}$ *is the unique maximizer over* $\mathcal{G}_{n,1}$ *for every* $n \geq 3$.

Proof. For $n = 2, \ell = 0$, there is only one connected graph. For $n \geq 3$, Lemma 3.3 gives a universal vertex in every maximizer. The remaining core has respectively zero or one edge and is therefore uniquely determined. □

5. A discrete chord-slope certificate

The core criterion is deliberately stated in terms of (K1)–(K2), since these inequalities can sometimes be proved by compensation even when convexity fails. We now give a simple sufficient certificate in terms of (D1)–(D4). The first lemma is the elementary running-average form of monotonicity of secant slopes of a convex function [36, Section 3.1]; it yields (K1). The second yields (K2).

Lemma 5.1 (Monotonicity of discrete chord slopes)**.** *Assume* (D1) *and* (D2). *Then for every* $a \geq 2$*, the sequence* $\sigma_a(1), \ldots, \sigma_a(a - 1)$ *is nondecreasing.*

Proof. By symmetry, set

$$q_j := \Psi(a, j + 1) - \Psi(a, j) = \Delta_1\Psi(j, a).$$

Then $0 < q_1 \leq q_2 \leq \cdots$ and

$$\sigma_a(t) = \frac{q_1 + \cdots + q_t}{t}.$$

Since $q_{t+1} \geq \sigma_a(t)$,

$$\sigma_a(t + 1) - \sigma_a(t) = \frac{q_{t+1} - \sigma_a(t)}{t + 1} \geq 0.$$

□

The next slope comparison is the two-coordinate case of the convex-sum majorization inequality [37, Proposition 3.C.1]: Linearly interpolate the convex sequence and use $(x, y) \prec (x + y - 1, 1)$, where $\prec$ denotes majorization. The proof below gives the integer telescoping argument and adds the strict edge-weight comparison.

Lemma 5.2 (Discrete chord unbalancing)**.** *Assume* (D3) *and* (D4). *Let* $a \geq 4$ *and let* $x \geq y \geq 2$ *be integers with* $x + y - 1 \leq a - 1$. *Then*

$$F_a(x, y) < F_a(x + y - 1, 1).$$

Proof. Set $\delta_a(t) = \sigma_a(t + 1) - \sigma_a(t)$ for integers $1 \leq t \leq a - 2$. Condition (D4) makes these increments nondecreasing. The largest index below is $x + y - 2 \leq a - 2$, so every term is in range. Since $x + j \geq 1 + j$ for $0 \leq j \leq y - 2$, comparison term by term gives

$$\begin{aligned}\sigma_a(x + y - 1) - \sigma_a(x) &= \sum_{j=0}^{y-2} \delta_a(x + j) \\ &\geq \sum_{j=0}^{y-2} \delta_a(1 + j) = \sigma_a(y) - \sigma_a(1).\end{aligned}$$

Thus

$$\sigma_a(x) + \sigma_a(y) \leq \sigma_a(x + y - 1) + \sigma_a(1).$$

Applying (D3) with $r = x + 1$, $s = y + 1$, $k = y - 1$ gives

$$\Psi(x + 1, y + 1) < \Psi(x + y, 2).$$

The chord-slope comparison is non-strict, but this last comparison is strict by (D3). Adding therefore proves the strict inequality for F_a. □

Theorem 5.3 (Discrete core-edge extremal theorem)**.** *Let* $\Psi : \mathbb{N}_{\geq 1}^2 \to \mathbb{R}$ *be discretely core-admissible. Then, for every* $0 \leq \ell \leq n - 2$, *the graph* $H_{n,\ell}$ *is the unique maximizer of* BID_Ψ *over* $\mathcal{G}_{n,\ell}$.

Proof. The cases $\ell = 0, 1$ are Proposition 4.4. Let $\ell \geq 2$. Lemma 3.3 gives a universal vertex in every maximizer. By Lemma 5.1, σ_a is nondecreasing, while (D1) makes $x \mapsto \Psi(x + 1, 2)$ strictly increasing. Hence (K1) holds. Condition (K2) is vacuous when $\ell = 2$ because $x, y \geq 2$ would imply $x + y \geq 4 > \ell + 1$. When $\ell \geq 3$, it follows from Lemma 5.2, since $x + y - 1 \leq \ell \leq a - 1$ and $a = n - 1 \geq 4$. Thus universalization and both hypotheses of Theorem 4.3 have been checked in the full remaining range. □

6. A smooth certificate

The preceding theorem is purely discrete. For the smooth indices considered below, however, it is convenient to verify its hypotheses through derivatives. This section records an analytic certificate whose purpose is to imply the discrete conditions of Section 5. The integral representation of a first divided difference is classical; see [38, pp. 63–64]. The next lemma is its standard application to a convex derivative, written in the present notation.

Definition 6.1 (Chord-admissible real extension)**.** A symmetric function $\Psi : [1,\infty)^2 \to \mathbb{R}$ is *chord-admissible* if:

(A0) the first partial derivatives exist, one-sidedly on the boundary, and for every $a > 1$, $b \mapsto \Psi(a,b)$ is C^3 on $[1,a]$;
(A1) Ψ is strictly increasing in each variable;
(A2) Ψ_{x_i} is nondecreasing in x_i, for $i = 1,2$;
(A3) $\Psi(x_1+k, x_2-k) > \Psi(x_1,x_2)$ whenever $x_1 \geq x_2 \geq k+1 \geq 2$;
(A4) $\partial^3_{x_2}\Psi(a,b) \geq 0$ for $a > 1$ and $1 \leq b \leq a$.

Lemma 6.2 (Convexity of the real chord slope)**.** *Let Ψ be chord-admissible. For fixed $a > 1$, extend*

$$\sigma_a(t) = \frac{\Psi(a,1+t) - \Psi(a,1)}{t}$$

to $0 < t \leq a-1$. Then σ_a is nondecreasing and convex.

Proof. Set $g_a(t) = \Psi(a,1+t)$. By (A2), g_a' is nondecreasing, so the chord slope from zero is nondecreasing. Moreover

$$\sigma_a(t) = \int_0^1 g_a'(st)\,ds,$$

and differentiation twice gives

$$\sigma_a''(t) = \int_0^1 s^2 g_a'''(st)\,ds \geq 0$$

by (A4). □

Proposition 6.3 (Smooth-to-discrete certificate)**.** *If Ψ is chord-admissible, then its restriction to $\mathbb{N}^2_{\geq 1}$ is discretely core-admissible.*

Proof. (A1) gives (D1). Integrating the monotone derivative in (A2) over unit intervals gives (D2). Condition (A3) gives (D3). Finally, Lemma 6.2 makes the unit increments of σ_a nondecreasing, which is (D4). □

Theorem 6.4 (Chord-slope criterion)**.** *Let Ψ be chord-admissible. Then, for every $0 \leq \ell \leq n-2$, $H_{n,\ell}$ is the unique maximizer of* BID_Ψ *over $\mathcal{G}_{n,\ell}$.*

Proof. Proposition 6.3 gives (D1)–(D4). For $\ell = 0,1$, use Proposition 4.4; its $n = 2$ case needs no smooth argument. For $\ell \geq 2$, (D1)–(D3) give universalization, Lemma 5.1 and (D1) give (K1), and (D3)–(D4) give (K2), which is vacuous for $\ell = 2$. These are exactly the steps of Theorem 5.3. □

7. The classical Sombor index

For the classical Sombor weight, the chord slope is concave, so (D4) fails. We instead verify (K2) by compensating the chord-slope loss with the internal-edge gain.

For the classical Sombor weight

$$\Psi_{\mathrm{SO}}(x,y) = \sqrt{x^2+y^2},$$

fix $a > 1$ and set

$$g_a(t) = \sqrt{a^2 + (1+t)^2}, \qquad \sigma_a(t) = \frac{g_a(t) - g_a(0)}{t}.$$

Then

$$g_a''(t) = \frac{a^2}{(a^2 + (1+t)^2)^{3/2}}, \qquad 0 < g_a''(t) < \frac{1}{a},$$

and

$$\sigma_a(t) = \int_0^1 g_a'(st)\, ds.$$

Consequently

$$0 < \sigma_a'(t) = \int_0^1 s\, g_a''(st)\, ds < \frac{1}{2a}. \tag{7.1}$$

In particular, $x \mapsto F_a(x, 1)$ is strictly increasing.

Lemma 7.1 (Sombor compensation inequality)**.** *Let $a \geq \ell + 1$ and let $x \geq y \geq 2$ satisfy $x + y \leq \ell + 1$. Then*

$$F_a(x, y) < F_a(x + y - 1, 1).$$

Proof. Set $s = x + y$. The assumptions give $4 \leq s \leq \ell + 1 \leq a$. Separate the two contributions as

$$\Delta_\sigma = \sigma_a(s-1) + \sigma_a(1) - \sigma_a(x) - \sigma_a(y),$$
$$\Delta_E = \sqrt{s^2 + 4} - \sqrt{(x+1)^2 + (y+1)^2}.$$

Their sum is $F_a(s-1, 1) - F_a(x, y)$. Since $s - 1 > x$ and σ_a is strictly increasing,

$$\Delta_\sigma > \sigma_a(1) - \sigma_a(y) = -\int_1^y \sigma_a'(t)\, dt > -\frac{y-1}{2a}$$

by (7.1). For the internal edge, the difference of the squared radicands is $2(x-1)(y-1) > 0$, so rationalization yields

$$\Delta_E = \frac{2(x-1)(y-1)}{\sqrt{s^2+4} + \sqrt{(x+1)^2 + (y+1)^2}} > \frac{(x-1)(y-1)}{\sqrt{s^2+4}}.$$

Here $s \geq 4$ implies $\sqrt{s^2+4} < 2s$ and $s \leq a$ implies $2s \leq 2a$. Since $x - 1 \geq 1$,

$$\Delta_E > \frac{(x-1)(y-1)}{\sqrt{s^2+4}} > \frac{y-1}{2a}.$$

Adding the last bound to the bound for Δ_σ gives a strictly positive total. All strict inequalities remain valid at $x = y = 2$. □

Theorem 7.2 (Uniform Sombor extremal theorem)**.** *For every*

$$0 \leq \ell \leq n - 2,$$

the graph $H_{n,\ell}$ is the unique maximizer of the Sombor index over $\mathcal{G}_{n,\ell}$.

Proof. The Sombor weight is symmetric and strictly increasing in each variable. Its second partial derivatives are

$$\frac{\partial^2 \Psi_{\mathrm{SO}}}{\partial x^2}(x,y) = \frac{y^2}{(x^2+y^2)^{3/2}} > 0, \qquad \frac{\partial^2 \Psi_{\mathrm{SO}}}{\partial y^2}(x,y) = \frac{x^2}{(x^2+y^2)^{3/2}} > 0,$$

so the first partial derivatives are increasing in their own variables. If $x_1 \geq x_2 \geq k+1$, then

$$(x_1+k)^2 + (x_2-k)^2 - (x_1^2+x_2^2) = 2k(x_1-x_2+k) > 0,$$

which gives (D3). Strict coordinate monotonicity gives (D1), and coordinate convexity gives (D2) on integer degree pairs. Hence Lemma 3.3 shows that every Sombor maximizer has a universal vertex.

If $\ell = 0$ or $\ell = 1$, Proposition 4.4 completes the proof. Assume $\ell \geq 2$. Condition (K1) follows from (7.1), and (K2) is Lemma 7.1. Apply Theorem 4.3. □

Corollary 7.3 (Réti–Došlić–Ali conjecture)**.** *The cyclic Sombor conjecture of Réti, Došlić, and Ali [10] holds: For every*

$$5 \leq \ell \leq n-2,$$

the graph $H_{n,\ell}$ is the unique graph in $\mathcal{G}_{n,\ell}$ with the maximum Sombor index.

Proof. This is the corresponding range of Theorem 7.2. □

Remark 7.4 (Relation with previous work). Réti, Došlić, and Ali [10] proved the case where $\ell \leq 4$, stated Conjecture 2.1 for $5 \leq \ell \leq n-2$, and observed that their method also verifies $\ell = 5$. Das, Ghalavand, and Ashrafi [11] proved the six-cyclic case without an additional hypothesis and obtained a general result when the number of pendent vertices is at most $n-\ell-2$. Theorem 7.2 removes that restriction entirely. The theorem gives one uniform proof for every $\ell \geq 2$, including $\ell \geq 7$, beyond the published cases described above.

8. The elliptic Sombor conjecture

The elliptic Sombor result follows directly from the smooth certificate, without the two-parameter calculations of the later sections. Write

$$\Psi_{\mathrm{ESO}}(x,y) = (x+y)\sqrt{x^2+y^2}.$$

Theorem 8.1 (Uniform elliptic Sombor extremal theorem)**.** *For every $n \geq 2$ and $0 \leq \ell \leq n-2$, the graph $H_{n,\ell}$ is the unique maximizer of the* ESO *over $\mathcal{G}_{n,\ell}$.*

Proof. The weight is symmetric and smooth on the positive quadrant. With $v = a^2+b^2$, differentiation of $\Psi_{\mathrm{ESO}}(a,b) = (a+b)\sqrt{v}$ gives

$$\begin{aligned}
\partial_b \Psi_{\mathrm{ESO}}(a,b) &= \sqrt{v} + \frac{b(a+b)}{\sqrt{v}} > 0, \\
\partial_b^2 \Psi_{\mathrm{ESO}}(a,b) &= \frac{2b}{\sqrt{v}} + \frac{a^2(a+b)}{v^{3/2}} > 0, \\
\partial_b^3 \Psi_{\mathrm{ESO}}(a,b) &= \frac{3a^3(a-b)}{v^{5/2}} \geq 0 \qquad (0 < b \leq a).
\end{aligned}$$

The first two inequalities and symmetry verify (A1)–(A2), and the last verifies (A4); smoothness gives (A0). A transfer $(x, y) \mapsto (x+k, y-k)$ preserves $x+y$ and strictly increases x^2+y^2, so (A3) holds. Thus Ψ_{ESO} is chord-admissible. Theorem 6.4 applies, including $\ell = 0, 1$ as specified in its proof. □

Corollary 8.2 (Elliptic Sombor conjecture)**.** *Conjecture 2.1.1 of [4] holds: for every $\ell \geq 6$ and $n \geq \ell + 2$, the graph $H_{n,\ell}$ uniquely maximizes the* ESO *over $\mathcal{G}_{n,\ell}$.*

Proof. This is the corresponding range of Theorem 8.1. □

Remark 8.3 (Previously established elliptic cases). The unicyclic maximum is given in [14, Theorem 4.1]. For $n \geq 7$, the bicyclic case was proved by Qi and Lin [16]; the present theorem also includes the admissible orders $n = 4, 5, 6$. The cases $\ell = 3, 4, 5$ are recorded in [4, Corollary 2.1.2], based on [17]. The tricyclic maximum also appears in [15, Theorem 2]; the comparison here is restricted to $n \geq 5$, where $H_{n,3}$ is defined. Our proof is independent of those small-cyclomatic arguments. The uniform theorem also covers the tree and unicyclic cases. Its extremal values follow by substituting Ψ_{ESO} into (4.1) (or the stated star value for $\ell = 0$).

9. Beyond chord-admissibility: General Sombor powers

The same compensation mechanism extends the classical Sombor endpoint to every power $\beta \geq 1/2$. This power is the specialization $KA_{2,\beta}$ of the general Sombor family studied in [39]. The proof treats $1/2 \leq \beta < 1$ directly. For $\beta \geq 1$, it invokes the smooth certificate verified independently in Theorem 10.1 below; that theorem does not use the present power result.

For $\beta > 0$, define the general Sombor index

$$\mathrm{SO}_\beta(G) := \sum_{uv \in E(G)} (d_G(u)^2 + d_G(v)^2)^\beta.$$

The classical Sombor index is the endpoint $\beta = \frac{1}{2}$.

Theorem 9.1 (General Sombor powers)**.** *Let $\beta \geq \frac{1}{2}$. For every pair of integers*

$$\ell \geq 0, \qquad n \geq \ell + 2,$$

the graph $H_{n,\ell}$ is the unique maximizer of SO_β *over $\mathcal{G}_{n,\ell}$.*

Proof. Set

$$\Psi_\beta(x, y) = (x^2 + y^2)^\beta.$$

For every $\beta \geq \frac{1}{2}$, this weight is symmetric and strictly increasing in each variable, and

$$\frac{\partial^2 \Psi_\beta}{\partial y^2}(x, y) = 2\beta(x^2 + y^2)^{\beta-2}(x^2 + (2\beta - 1)y^2) > 0,$$

with the symmetric formula in the first variable. At the endpoint $\beta = 1/2$, the bracket is $x^2 > 0$, so strict convexity persists. Consequently its first partial derivatives are increasing in their own variables, and their integrals over unit intervals verify (D2) throughout $\beta \geq 1/2$. The transfer $(x_1, x_2) \mapsto (x_1 +$

$k, x_2 - k)$ strictly increases $x_1^2 + x_2^2$ whenever $x_1 \geq x_2 \geq k+1$. This gives (D3). Hence Lemma 3.3 applies for every $\beta \geq \frac{1}{2}$.

The cases $\ell = 0, 1$ now follow from Proposition 4.4. Assume $\ell \geq 2$ and set $a = n - 1$. If $\beta \geq 1$, then Ψ_β is the specialization $(\alpha, \beta) = (0, \beta)$ of Theorem 10.1, since $2\beta \geq 2$. Thus it remains only to treat

$$\frac{1}{2} \leq \beta < 1.$$

For this weight, define

$$g_a(t) = \left(a^2 + (1+t)^2\right)^\beta, \qquad \sigma_a(t) = \frac{g_a(t) - g_a(0)}{t}.$$

For $b = 1 + t$, differentiation first gives $g_a'(t) = 2\beta b(a^2 + b^2)^{\beta-1}$. Differentiating this product gives

$$g_a''(t) = 2\beta(a^2 + (1+t)^2)^{\beta-2}(a^2 + (2\beta - 1)(1+t)^2).$$

Since $0 \leq 2\beta - 1 < 1$ and $\beta - 1 < 0$,

$$0 < g_a''(t) < 2\beta a^{2\beta-2}.$$

Using

$$\sigma_a'(t) = \int_0^1 r\, g_a''(rt)\, dr,$$

we obtain

$$0 < \sigma_a'(t) < \beta a^{2\beta-2}. \tag{9.1}$$

Thus σ_a is strictly increasing, and consequently $x \mapsto F_a(x, 1)$ is strictly increasing. This is (K1).

For (K2), let $x \geq y \geq 2$, set $s = x + y$, and assume $s \leq \ell + 1 \leq a$. Since $s - 1 > x$,

$$\begin{aligned} &\sigma_a(s-1) + \sigma_a(1) - \sigma_a(x) - \sigma_a(y) \\ &> \sigma_a(1) - \sigma_a(y) = -\int_1^y \sigma_a'(t)\, dt \\ &> -\beta(y-1)a^{2\beta-2} \geq -\beta(y-1)s^{2\beta-2}, \end{aligned}$$

where the last inequality uses $2\beta - 2 \leq 0$ and $a \geq s$.

For the internal edge term, set

$$A = s^2 + 4, \qquad B = (x+1)^2 + (y+1)^2.$$

Then

$$A - B = 2(x-1)(y-1) > 0.$$

Since $0 < \beta < 1$,

$$A^\beta - B^\beta = \beta \int_B^A t^{\beta-1}\, dt > 2\beta(x-1)(y-1)A^{\beta-1}.$$

Furthermore $s \geq 4$, $x - 1 \geq 1$, and $\beta - 1 \geq -\frac{1}{2}$, so

$$\begin{aligned} 2(x-1)A^{\beta-1} &= 2(x-1)s^{2\beta-2}\left(1+\frac{4}{s^2}\right)^{\beta-1} \\ &\geq 2s^{2\beta-2}\left(1+\frac{4}{s^2}\right)^{-1/2} \\ &\geq \frac{4}{\sqrt{5}}\, s^{2\beta-2} > s^{2\beta-2}. \end{aligned}$$

Hence

$$A^\beta - B^\beta > \beta(y-1)s^{2\beta-2}.$$

The uniform margin $4/\sqrt{5} > 1$ remains strict at $\beta = 1/2$; thus there is no equality exception at that endpoint. This gain strictly dominates the possible chord-slope loss, and therefore

$$F_a(x,y) < F_a(x+y-1,1).$$

Thus (K2) holds, and Theorem 4.3 completes the proof. □

10. A two-parameter elliptic Sombor family

We now pass from the Sombor weight to the homogeneous family that contains the elliptic Sombor index. There are two distinct questions: When does the smooth chord-slope certificate apply, and when is $H_{n,\ell}$ actually extremal? This section answers the first question exactly; later compensation estimates enlarge the extremal region beyond that certificate.

We use the generalized elliptic Sombor family introduced in [33, p. 6789]; further bounds for the same family are given in [34]. For $\alpha \geq 0$ and $\beta > 0$, write

$$\Psi_{\alpha,\beta}(x,y) := (x+y)^\alpha (x^2+y^2)^\beta.$$

The subfamily $\beta = \alpha/2$ is the general elliptic Sombor index studied in [35], whose Theorem 9 gives the star as the unique tree maximizer for every $\alpha > 0$. The family includes the elliptic Sombor weight at

$$(\alpha,\beta) = \left(1, \frac{1}{2}\right).$$

The relevant parameter is the degree of homogeneity

$$m := \alpha + 2\beta.$$

Theorem 10.1 (Exact chord-admissible region)**.** *Let $\alpha \geq 0$ and $\beta > 0$. Then*

$$\Psi_{\alpha,\beta}$$

is chord-admissible if and only if

$$\alpha + 2\beta \geq 2.$$

Consequently, if $\alpha + 2\beta \geq 2$, then for every

$$0 \leq \ell \leq n-2,$$

the graph $H_{n,\ell}$ is the unique maximizer of $\mathrm{BID}_{\Psi_{\alpha,\beta}}$ *over $\mathcal{G}_{n,\ell}$.*

Proof. We first prove sufficiency. Set

$$m = \alpha + 2\beta \geq 2, \qquad p = \frac{m}{2} \geq 1, \qquad \gamma = \frac{2\beta}{m} \in (0, 1].$$

Define the homogeneous degree-two weight

$$\Phi_\gamma(x, y) := (x + y)^{2-2\gamma}(x^2 + y^2)^\gamma.$$

Since

$$p(2 - 2\gamma) = \alpha, \qquad p\gamma = \beta,$$

we have

$$\Psi_{\alpha,\beta} = \Phi_\gamma^p. \tag{10.1}$$

Fix $a > 0$, and write

$$\phi(b) := \Phi_\gamma(a, b), \qquad b > 0.$$

Appendix A derives the next two factorizations from the logarithmic derivative of ϕ, with all denominators displayed. The first is

$$\begin{aligned} 2\phi\phi'' - (\phi')^2 = {} & 4a^2\gamma(a + b)^{2-4\gamma}(a^2 + b^2)^{2\gamma-2} \\ & \times (\gamma(a - b)^2 + 4ab). \end{aligned} \tag{10.2}$$

Hence

$$2\phi\phi'' > (\phi')^2,$$

and in particular,

$$\phi'' > 0 \qquad (b > 0).$$

Thus the first derivative is strictly increasing on the whole positive half-line, as required in (A2).

For $0 < b \leq a$, the second factorization is

$$\phi'''(b) = 8a^3\gamma(1 - \gamma)(a - b)(a + b)^{-2\gamma-1}(a^2 + b^2)^{\gamma-3} \times (\gamma(a - b)^2 + a^2 + b^2 + 4ab). \tag{10.3}$$

Since $0 < \gamma \leq 1$ and $0 < b \leq a$,

$$\phi'''(b) \geq 0.$$

We now pass from ϕ to ϕ^p. Differentiating,

$$(\phi^p)'' = p\phi^{p-2}(\phi\phi'' + (p - 1)(\phi')^2) > 0.$$

Furthermore,

$$(\phi^p)''' = p\phi^{p-3}\Big[\phi^2\phi''' + (p - 1)\phi'(3\phi\phi'' + (p - 2)(\phi')^2)\Big]. \tag{10.4}$$

By (10.2),

$$\phi\phi'' > \frac{1}{2}(\phi')^2,$$

so

$$3\phi\phi'' + (p - 2)(\phi')^2 > \left(p - \frac{1}{2}\right)(\phi')^2 > 0.$$

As $p \geq 1$, $\phi' > 0$, and $\phi''' \geq 0$, Eq (10.4) gives

$$(\phi^p)''' \geq 0.$$

This includes $p = 1$, where the term with $p - 1$ vanishes, and $\gamma = 1$, where $\phi(b) = a^2 + b^2$ has a third derivative of zero. Non-strict positivity is exactly what (A4) requires; strict unbalancing will be supplied by (A3). Using (10.1) and symmetry between the two variables, this proves (A2) on $(0, \infty)^2$ and (A4) on $0 < b \leq a$. The weight is smooth on the positive quadrant, so (A0) also holds.

Condition (A1) is immediate because $\beta > 0$ and both factors are nondecreasing in each positive variable, with strict increase supplied by $(x^2 + y^2)^\beta$. For (A3), the transfer

$$(x, y) \mapsto (x + k, y - k)$$

preserves $x + y$, while

$$(x + k)^2 + (y - k)^2 - (x^2 + y^2) = 2k(x - y + k) > 0.$$

Hence the transfer is strict. Thus $\Psi_{\alpha,\beta}$ is chord-admissible whenever $m \geq 2$.

For necessity, failure of either (A2) or (A4) suffices to exclude chord-admissibility. Suppose first that

$$1 \leq m < 2.$$

At any diagonal point $b = a > 1$, Appendix A gives

$$\frac{\partial^3}{\partial b^3}\Psi_{\alpha,\beta}(a, b)\Big|_{b=a} = 2^{\alpha+\beta-3}a^{m-3}(m - 2)(m^2 - m + 6\beta). \tag{10.5}$$

Since $m \geq 1$ and $\beta > 0$,

$$m^2 - m + 6\beta > 0,$$

whereas $m - 2 < 0$. Thus (A4) fails.

Now suppose $0 < m < 1$. Fix $a > 1$. The explicit second-derivative formula in Appendix A, whose polynomial has the leading term $m(m - 1)b^4$, gives

$$\lim_{b\to\infty} b^{2-m}\partial_b^2\Psi_{\alpha,\beta}(a, b) = m(m - 1) < 0.$$

Thus a coordinate section has a negative second derivative for sufficiently large b, contradicting (A2). Condition (A2) is required on the whole quadrant $[1, \infty)^2$; the restriction $b \leq a$ applies only to (A4).

Therefore chord-admissibility is possible exactly for

$$m = \alpha + 2\beta \geq 2.$$

The extremal conclusion follows from Theorem 6.4. □

Proposition 10.2 (A genuine obstruction below the chord region)**.** *For every $\alpha \in (1, 2)$, there exists $\varepsilon_\alpha > 0$ such that, whenever*

$$0 < \beta < \varepsilon_\alpha,$$

the graph $H_{5,3}$ does not maximize $\mathrm{BID}_{\Psi_{\alpha,\beta}}$ over $\mathcal{G}_{5,3}$. The parameter can be chosen so that $1 < \alpha + 2\beta < 2$.

Proof. Let

$$J := K_1 \vee (K_3 \sqcup K_1).$$

Both J and $H_{5,3} = K_1 \vee K_{1,3}$ belong to $\mathcal{G}_{5,3}$. The edge types are as follows and the multiplicities in each row sum to seven.

Graph	Endpoint degrees	Multiplicities
$H_{5,3}$	$(4,4)$, $(4,2)$	1, 6
J	$(4,3)$, $(4,1)$, $(3,3)$	3, 1, 3

Both graphs have five vertices and seven edges, and hence they are of cyclomatic number three. Substitution of these edge types gives

$$\begin{aligned}
\mathrm{BID}_{\Psi_{\alpha,\beta}}(H_{5,3}) &= 8^{\alpha}32^{\beta} + 6\,6^{\alpha}20^{\beta},\\
\mathrm{BID}_{\Psi_{\alpha,\beta}}(J) &= 3\,7^{\alpha}25^{\beta} + 5^{\alpha}17^{\beta} + 3\,6^{\alpha}18^{\beta}.
\end{aligned}$$

Extend the difference continuously to $\beta = 0$. The tricyclic sum-connectivity comparison at this boundary already appears in [18]. At $\beta = 0$,

$$\begin{aligned}
D_{\alpha} &:= \mathrm{BID}_{\Psi_{\alpha,0}}(H_{5,3}) - \mathrm{BID}_{\Psi_{\alpha,0}}(J)\\
&= 8^{\alpha} - 3\,7^{\alpha} + 3\,6^{\alpha} - 5^{\alpha}.
\end{aligned}$$

For $f(t) = t^{\alpha}$, this is the third forward difference $\Delta^3 f(5)$. The classical integral representation of finite differences [38, p. 64] gives

$$D_{\alpha} = \int_0^1 \int_0^1 \int_0^1 f'''(5 + r + s + t)\, dr\, ds\, dt.$$

When $1 < \alpha < 2$,

$$f'''(t) = \alpha(\alpha - 1)(\alpha - 2)t^{\alpha - 3} < 0,$$

so $D_{\alpha} < 0$. The extension to the present positive-β family uses continuity: The same strict inequality persists for all sufficiently small positive β. Shrinking ε_{α}, if necessary, so that $\beta < (2 - \alpha)/2$ also ensures $1 < \alpha + 2\beta < 2$. □

11. A larger extremal region for the two-parameter family

We now enlarge the sufficient extremal region by compensation. The polynomial derivations are given below and in the appendices. The secant-slope and convex-spreading steps use the facts cited in Sections 5–6. Throughout this section, set

$$m := \alpha + 2\beta.$$

For the core comparisons, take $a \geq 4$ and write

$$f_a(t) := \Psi_{\alpha,\beta}(a, 1 + t), \qquad \sigma_a(t) = \frac{f_a(t) - f_a(0)}{t}.$$

For integers $x \geq y \geq 2$ with $x + y \leq a$, set

$$\mathcal{L}_a(x, y) := \sigma_a(x) + \sigma_a(y) - \sigma_a(x + y - 1) - \sigma_a(1)$$

and

$$\mathcal{G}(x, y) := \Psi_{\alpha,\beta}(x + y, 2) - \Psi_{\alpha,\beta}(x + 1, y + 1).$$

Then

$$F_a(x + y - 1, 1) - F_a(x, y) = \mathcal{G}(x, y) - \mathcal{L}_a(x, y). \tag{11.1}$$

The gain is always strictly positive because the transfer preserves the sum of the two arguments and strictly increases the sum of their squares.

Lemma 11.1 (Convex sections and pendant monotonicity)**.** *Let $\alpha \geq 0$, $\beta > 0$, and set $m = \alpha + 2\beta$. If $m \geq 1$, then every section*

$$b \longmapsto \Psi_{\alpha,\beta}(a, b), \qquad a > 0,$$

is strictly convex on $(0, \infty)$. For $a > 1$ and arguments in the real domains specified in Section 2, the chord slope σ_a is nondecreasing and $x \mapsto F_a(x, 1)$ is strictly increasing.

Proof. Appendix A derives

$$\partial_b^2 \Psi_{\alpha,\beta}(a, b) = (a + b)^{\alpha-2}(a^2 + b^2)^{\beta-2} N_{\alpha,\beta}(a, b),$$

where

$$\begin{aligned} N_{\alpha,\beta}(a, b) = {} & (\alpha^2 - \alpha + 2\beta)a^4 + 4\beta(\alpha + 1)a^3 b \\ & + (4\beta^2 + 2\alpha(m - 1))a^2 b^2 + 4\beta(m - 1)ab^3 + m(m - 1)b^4. \end{aligned}$$

All coefficients are nonnegative when $m \geq 1$: indeed

$$\alpha^2 - \alpha + 2\beta = (\alpha - 1)^2 + (m - 1) \geq 0,$$

and the coefficient of $a^3 b$ is strictly positive. Hence $N_{\alpha,\beta}(a, b) > 0$ for $a, b > 0$, proving strict convexity.

For a convex section $f(t) = \Psi_{\alpha,\beta}(a, 1 + t)$, the secant slope $(f(t) - f(0))/t = \sigma_a(t)$ is nondecreasing for $t > 0$. Finally

$$F_a(x, 1) = \sigma_a(x) + \sigma_a(1) + \Psi_{\alpha,\beta}(x + 1, 2).$$

The first term is nondecreasing, while the last term is strictly increasing because $\beta > 0$. Therefore $x \mapsto F_a(x, 1)$ is strictly increasing. □

Lemma 11.2 (Compensation for $0 \leq \alpha \leq 1$)**.** *Assume $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, and $x \geq y \geq 2$, $x + y \leq a$. Then*

$$\mathcal{L}_a(x, y) \leq 4\beta a^{m-3}(x - 1)(y - 1),$$

whereas

$$\mathcal{G}(x, y) \geq 2\beta(x - 1)(y - 1)(x + y + 2)^{\alpha}((x + y)^2 + 4)^{\beta-1}.$$

Moreover, throughout this parameter range (without assuming $m \geq 1$),

$$\mathcal{L}_a(x, y) < \mathcal{G}(x, y).$$

Proof. Set $u = a + b$, $v = a^2 + b^2$, and $c = 1 - \beta \in [0, 1]$. Appendix A derives $\partial_b^3 \Psi_{\alpha,\beta} = u^{\alpha-3} v^{\beta-3} M_{\alpha,\beta}$ and specifies every term of $M_{\alpha,\beta}$. For $0 \le \alpha \le 1$, its negative terms are controlled by the identity

$$\begin{aligned} M_{\alpha,\beta} + 12\beta u^2 v^2 &= \alpha(\alpha-1)(\alpha-2)v^3 + 6\alpha\beta u v^2 (a + \alpha b) \\ &\quad + 12\beta c(1-\alpha) b^2 u^2 v + \beta u^2 R, \\ R &= 12v^2 + 8c(1+c) b^3 u - 12cbv(a + 2b). \end{aligned} \tag{11.2}$$

The first three terms on the right are nonnegative. To control R, set $r = b/a \in [0, 1]$ and $\mathcal{R}(r, c) = R/a^4$. Then

$$\mathcal{R}(r, c) = 12(1 + r^2)^2 + 8c(1 + c) r^3 (1 + r) - 12cr(1 + r^2)(1 + 2r).$$

For $0 \le c \le 1$,

$$\begin{aligned} \partial_c \mathcal{R}(r, c) &= 8(1 + 2c) r^3 (1 + r) - 12r(1 + r^2)(1 + 2r) \\ &\le 12r(r^2 - 2r - 1) \le 0 \end{aligned}$$

because $r^2 - 2r - 1 \le -1$ on $[0, 1]$. Consequently

$$\mathcal{R}(r, c) \ge \mathcal{R}(r, 1) = 12(1 - r) + 4r^3 + 4r^4 \ge 0.$$

This proves $M_{\alpha,\beta} \ge -12\beta u^2 v^2$. As $\alpha - 1 \le 0$, $\beta - 1 \le 0$, $u \ge a$, and $v \ge a^2$, we obtain

$$\partial_b^3 \Psi_{\alpha,\beta}(a, b) \ge -12\beta u^{\alpha-1} v^{\beta-1} \ge -12\beta a^{m-3} \qquad (0 < b \le a).$$

Thus $W(t) = f_a(t) + 2\beta a^{m-3} t^3$ has a nonnegative third derivative on $[0, a - 1]$. The integral proof in Lemma 6.2 shows that its chord slope is convex; no monotonicity of W is needed. Its chord slope is $\sigma_a(t) + 2\beta a^{m-3} t^2$. Writing $s = x + y$ and $P = (x - 1)(y - 1) > 0$, the convex spreading inequality and

$$(s - 1)^2 + 1 - x^2 - y^2 = 2P$$

give $\mathcal{L}_a(x, y) \le 4\beta a^{m-3} P$.

For the gain, let $A = s^2 + 4$ and $B = (x + 1)^2 + (y + 1)^2$, so $A - B = 2P$. The two edge arguments have the common sum $s + 2$. Since $0 < \beta \le 1$,

$$\mathcal{G}(x, y) = (s + 2)^\alpha \beta \int_B^A z^{\beta-1}\, dz \ge 2\beta P (s + 2)^\alpha A^{\beta-1}.$$

Finally $m = \alpha + 2\beta \le 3$, $a \ge s \ge 4$, and therefore

$$a^{m-3} \le s^{m-3}, \qquad (s + 2)^\alpha (s^2 + 4)^{\beta-1} \ge \frac{s^m}{s^2 + 4}.$$

It follows that

$$\mathcal{L}_a(x, y) \le 4\beta P s^{m-3} < \frac{2\beta P s^m}{s^2 + 4} \le \mathcal{G}(x, y),$$

where strictness is equivalent to $2(s^2 + 4) < s^3$. For $s \ge 4$, $s^2(s - 2) \ge 2s^2 \ge 32 > 8$, which proves this inequality. In particular, the local conclusion does not require $m \ge 1$. □

The local compensation holds throughout the stated range, including $m < 1$. To turn it into a global extremal result by the present method, we will also require $m \ge 1$ for universalization. The next bound treats $1 \le \alpha \le 2$.

Lemma 11.3 (Uniform third-derivative bound for $1 \le \alpha \le 2$)**.** *Assume* $1 \le \alpha \le 2$, $0 \le \beta \le 1$, *and* $0 < b \le a$. *Set*

$$K_{\alpha,\beta} := \alpha(\alpha-1)(2-\alpha) + 8\beta, \qquad m = \alpha + 2\beta.$$

At $\beta = 0$, *the weight is defined by the same formula,* $\Psi_{\alpha,0}(a,b) = (a+b)^\alpha$. *Then*

$$\partial_b^3 \Psi_{\alpha,\beta}(a,b) \ge -K_{\alpha,\beta} a^{m-3}.$$

Proof. Set $u = a + b$, $v = a^2 + b^2$. The derivative formula is obtained in Appendix A:

$$\partial_b^3 \Psi_{\alpha,\beta}(a,b) = u^{\alpha-3} v^{\beta-3} M_{\alpha,\beta}(a,b),$$

with

$$\begin{aligned} M_{\alpha,\beta} = {} & \alpha(\alpha-1)(\alpha-2)v^3 + 6\alpha\beta(\alpha-1)buv^2 + 6\alpha\beta u^2 v^2 \\ & + 12\alpha\beta(\beta-1)b^2u^2v + 12\beta(\beta-1)bu^3v + 8\beta(\beta-1)(\beta-2)b^3u^3. \end{aligned}$$

The required polynomial inequality is

$$M_{\alpha,\beta} + K_{\alpha,\beta} a u v^2 \ge 0. \tag{11.3}$$

Collect the terms in powers of β. Appendix B shows the coefficient calculation and its regrouping; the result is

$$T_0 + \beta A + \beta^2 B + \beta^3 C,$$

where

$$\begin{aligned} T_0 &= \alpha(\alpha-1)(2-\alpha)v^2 b(a-b), \\ A &= (a-b)Q(a,b) + (\alpha-1)G(a,b), \\ Q(a,b) &= 14a^5 + 22a^4b + 8a^3b^2 - 8a^2b^3 - 6ab^4 + 2b^5, \\ G(a,b) &= 6\alpha buv^2 + 6u^2v^2 - 12b^2u^2v, \\ B &= 12\alpha b^2u^2v + 12bu^3v - 24b^3u^3, \qquad C = 8b^3u^3. \end{aligned}$$

Here $T_0, C \ge 0$. Moreover

$$a(a^2 + 2ab - b^2) \ge 0 \qquad (0 \le b \le a),$$

and the identities

$$\begin{aligned} G &= 6uv\, a(a^2 + 2ab - b^2) + 6(\alpha-1)buv^2, \\ B &= 12bu^2\, a(a^2 + 2ab - b^2) + 12(\alpha-1)b^2u^2v \end{aligned}$$

show that $G, B \ge 0$. Finally,

$$Q(a,b) \ge 14a^5 - 8a^2b^3 - 6ab^4 \ge 0$$

because $b^3 \le a^3$ and $b^4 \le a^4$ give the lower bound $(14 - 8 - 6)a^5 = 0$. All terms discarded from Q are nonnegative. Thus $A \ge 0$, including at $a = b$. This proves (11.3).

Multiplying (11.3) by the positive factor $u^{\alpha-3}v^{\beta-3}$ gives

$$\partial_b^3 \Psi_{\alpha,\beta}(a,b) \ge -K_{\alpha,\beta}\, a\, u^{\alpha-2} v^{\beta-1}.$$

Since $\alpha - 2 \le 0$, $\beta - 1 \le 0$, $u \ge a$, and $v \ge a^2$,

$$a\, u^{\alpha-2} v^{\beta-1} \le a\, a^{\alpha-2} (a^2)^{\beta-1} = a^{m-3},$$

which proves the claim. □

Lemma 11.4 (Compensation for $1 \le \alpha \le 2$)**.** *Assume* $1 \le \alpha \le 2$, $0 < \beta \le \frac{1}{2}$, *and*

$$5\alpha(\alpha-1)(2-\alpha) < (56+48\alpha)\beta.$$

Then, for all $x \ge y \ge 2$ *and* $x + y \le a$,

$$F_a(x,y) < F_a(x+y-1,1).$$

Proof. Set $K = K_{\alpha,\beta}$ and $s = x + y$. By Lemma 11.3, the corrected section

$$t \longmapsto f_a(t) + \frac{K}{6}a^{m-3}t^3$$

has a nonnegative third derivative on $[0, a-1]$. Its chord slope is $\sigma_a(t) + (K/6)a^{m-3}t^2$. Using

$$(s-1)^2 + 1 - x^2 - y^2 = 2(x-1)(y-1)$$

in the convex spreading inequality gives

$$\mathcal{L}_a(x,y) \le \frac{K}{3}a^{m-3}(x-1)(y-1). \tag{11.4}$$

The gain estimate used in Lemma 11.2 remains valid, so

$$\mathcal{G}(x,y) \ge 2\beta(x-1)(y-1)(s+2)^{\alpha}(s^2+4)^{\beta-1}. \tag{11.5}$$

It remains to compare the coefficients in (11.4) and (11.5). Since $1 \le \alpha \le 2$, $0 < \beta \le 1/2$, we have $m - 3 \le 0$, and hence $a^{m-3} \le s^{m-3}$. Bernoulli's inequality and the monotonicity of real powers give

$$(s+2)^{\alpha} \ge s^{\alpha}\frac{s+2\alpha}{s}, \qquad (s^2+4)^{\beta-1} \ge \frac{s^{2\beta}}{s^2+4}.$$

After cancelling the positive factor $P = (x-1)(y-1)$, the required coefficient comparison is

$$\frac{K}{3}a^{m-3} < 2\beta(s+2)^{\alpha}(s^2+4)^{\beta-1}.$$

Using the preceding bounds, a sufficient condition is

$$\frac{K}{3}s^{m-3} < \frac{2\beta s^{m-1}(s+2\alpha)}{s^2+4},$$

which, after multiplying by $3(s^2+4)s^{3-m} > 0$, becomes

$$K(s^2+4) < 6\beta s^2(s+2\alpha).$$

For $s \ge 4$,

$$s^2 + 4 \le \frac{5}{4}s^2, \qquad 4 + 2\alpha \le s + 2\alpha,$$

so the last inequality follows from

$$5K < (96+48\alpha)\beta.$$

Substituting $K = \alpha(\alpha-1)(2-\alpha) + 8\beta$ and subtracting 40β from both sides shows that this condition is exactly

$$5\alpha(\alpha-1)(2-\alpha) < (56+48\alpha)\beta.$$

Hence $\mathcal{L}_a(x,y) < \mathcal{G}(x,y)$, and (11.1) proves the desired strict inequality. □

Theorem 11.5 (Enlarged two-parameter extremal region)**.** *Let $\alpha \geq 0$, $\beta > 0$. The graph $H_{n,\ell}$ is the unique maximizer of* $\mathrm{BID}_{\Psi_{\alpha,\beta}}$ *over $\mathcal{G}_{n,\ell}$, for every $0 \leq \ell \leq n-2$, whenever at least one of the following holds:*

(1) $0 \leq \alpha \leq 1$ *and* $\alpha + 2\beta \geq 1$*;*
(2) $\alpha + 2\beta \geq 2$*;*
(3) $1 \leq \alpha \leq 2$, $0 < \beta \leq \frac{1}{2}$, *and*

$$5\alpha(\alpha-1)(2-\alpha) < (56+48\alpha)\beta.$$

In particular, if $\alpha \geq 0$, $\beta \geq 3/200$, and $\alpha + 2\beta \geq 1$, then the same conclusion holds.

Proof. Assume first that $m = \alpha + 2\beta \geq 1$. By Lemma 11.1, every coordinate section is strictly convex and (K1) holds. Strict coordinate monotonicity follows directly from $\beta > 0$. Moreover, if $r \geq s \geq k+1$, then the transfer $(r, s) \mapsto (r + k, s - k)$ preserves $r + s$ and strictly increases $r^2 + s^2$; hence it strictly increases $\Psi_{\alpha,\beta}$. Thus the hypotheses of the universalization lemma are satisfied.

For $\ell = 0, 1$, Proposition 4.4 now gives the conclusion, including $n = 2$. In the remaining argument, assume $\ell \geq 2$.

In case (1), if $\beta \leq 1$, (K2) follows from Lemma 11.2; if $\beta > 1$, then $m > 2$ and Theorem 10.1 applies. Case (2) is exactly the extremal conclusion of Theorem 10.1. In case (3), (K2) follows from Lemma 11.4; since $\alpha \geq 1$, one has $m > 1$, so universalization and (K1) are available. The core-edge criterion then proves uniqueness in every case.

Finally,

$$\frac{5\alpha(\alpha-1)(2-\alpha)}{56+48\alpha} < \frac{3}{200} \qquad (1 \leq \alpha \leq 2),$$

as proved in Appendix B. If $m \geq 1$ but $m < 2$, the part with $\alpha \leq 1$ is case (1). In the remaining part, $1 < \alpha < 2$ and $\beta < (2-\alpha)/2 < 1/2$; the displayed bound and $\beta \geq 3/200$ imply the strict condition in case (3). This proves the final sufficient region. The constant $3/200$ is a convenient bound from these estimates, not an asserted optimal threshold. □

Table 2 summarizes the sufficient extremal regions. The first row is the exact region for the specified smooth certificate. The union of the rows is a sufficient region for extremality, not a necessary-and-sufficient classification of the extremal phenomenon.

Table 2. Sufficient extremal regions for $H_{n,\ell}$, always with $0 \le \ell \le n-2$, $\alpha \ge 0$, $\beta > 0$, and $m = \alpha + 2\beta$. Overlap between rows is intentional.

Parameter conditions	Certificate	Reference
$m \ge 2$	Smooth chord certificate	Theorem 10.1
$0 \le \alpha \le 1$, $m \ge 1$	Compensation if $\beta \le 1$; smooth certificate otherwise	Lemma 11.2, Theorem 11.5
$1 \le \alpha \le 2$, $0 < \beta \le 1/2$, $5\alpha(\alpha-1)(2-\alpha) < (56+48\alpha)\beta$	Quantitative compensation	Lemmas 11.3–11.4
$\alpha = 0$, $\beta \ge 1/2$	Power compensation for $\beta < 1$; smooth certificate for $\beta \ge 1$	Theorem 9.1
$(\alpha,\beta) = (1, 1/2)$	Direct smooth certificate	Theorem 8.1
$\beta \ge 3/200$, $m \ge 1$	Consequence of the preceding regions	Theorem 11.5

Proposition 11.6 (Quantitative localization of the five-vertex obstruction)**.** *Define*

$$\Phi(\alpha,\beta) := 8^{\alpha}32^{\beta} + 6\,6^{\alpha}20^{\beta} - 3\,7^{\alpha}25^{\beta} - 5^{\alpha}17^{\beta} - 3\,6^{\alpha}18^{\beta}.$$

If $\alpha \ge 0$, $\beta > 0$, and $\Phi(\alpha,\beta) \le 0$, then

$$1 < \alpha < 2, \qquad \alpha + 2\beta < 2,$$

and

$$(56+48\alpha)\beta \le 5\alpha(\alpha-1)(2-\alpha).$$

In particular, every such obstruction satisfies $\beta < 3/200$.

Proof. The edge-type calculation in Proposition 10.2 gives

$$\Phi = \mathrm{BID}_{\Psi_{\alpha,\beta}}(H_{5,3}) - \mathrm{BID}_{\Psi_{\alpha,\beta}}(J), \qquad J = K_1 \vee (K_3 \sqcup K_1).$$

Deleting a universal vertex gives, respectively, the cores $K_{1,3}$ and $K_3 \sqcup K_1$. Their three edges have core degrees $(3,1)$ and $(2,2)$, so the common baseline cancels and

$$\Phi = 3(F_4(3,1) - F_4(2,2)).$$

If $m = \alpha + 2\beta \ge 2$, Theorem 10.1 implies $\Phi > 0$, since the two graphs are nonisomorphic. Thus $\Phi \le 0$ forces $m < 2$ and, in particular, $\alpha < 2$. If $0 \le \alpha \le 1$, $m < 2$ gives $0 < \beta < 1$; Lemma 11.2, applied to $(a,x,y) = (4,2,2)$, then gives $\Phi > 0$. This use is valid even when $m < 1$, since that lemma's local conclusion has no such lower bound. Hence $1 < \alpha < 2$. Now $m < 2$ implies $\beta < (2-\alpha)/2 < 1/2$. If the strict condition of Lemma 11.4 held, the same substitution would again give $\Phi > 0$. Its failure is precisely

$$(56+48\alpha)\beta \le 5\alpha(\alpha-1)(2-\alpha).$$

Appendix B bounds the right-hand ratio strictly by $3/200$, proving the last assertion. If $\Phi < 0$, $H_{5,3}$ is not a maximizer. If $\Phi = 0$, it cannot be a unique maximizer, since J has the same value. The latter statement does not assert that either graph is maximal among all competitors. □

Remark 11.7 (Scope of the remaining parameter problem). There are three different questions: Validity of the smooth certificate, extremality for a fixed pair (n, ℓ), and extremality for every admissible (n, ℓ). The line $m = 2$ is the exact boundary only for the first question. For $0 \le \alpha < 1$ and $0 < \beta < (1 - \alpha)/2$, Lemma 11.2 still proves the local core inequality, but a coordinate section fails convexity for sufficiently large arguments, as in the necessity proof of Theorem 10.1. Hence our universalization certificate does not resolve the global problem there; this failure alone is not a counterexample to extremality.

For $1 < \alpha < 2$, the part not covered by the sufficient regions lies in

$$0 < \beta \le \frac{5\alpha(\alpha - 1)(2 - \alpha)}{56 + 48\alpha} < \frac{3}{200}.$$

Within it, $\Phi(\alpha, \beta) > 0$ is a necessary condition for a uniform unique-maximizer theorem: it excludes this particular five-vertex competitor. No sufficiency is established here, since other cores and other orders remain to be compared in the unresolved region. Proposition 11.6 gives a necessary localization of this obstruction, not an exact phase boundary.

12. Conclusions

For every $0 \le \ell \le n - 2$, the graph $H_{n,\ell}$ is the unique maximizer, up to isomorphism, of both the Sombor and elliptic Sombor indices in $\mathcal{G}_{n,\ell}$. The discrete universal-vertex reduction and the core-edge identity also establish the same conclusion for general Sombor powers with $\beta \ge 1/2$. For the two-parameter family, Table 2 records the proved sufficient regions. The exact boundary of the smooth certificate and the unresolved extremal parameter ranges are distinguished in Remark 11.7.

Use of Generative-AI tools declaration

The author declares that ChatGPT (OpenAI) was used for language and LaTeX editing. The author is fully responsible for the mathematical content and the final manuscript.

Conflict of interest

The author declares that there is no conflict of interest.

References

1. I. Gutman, Geometric approach to degree-based topological indices: Sombor indices, *MATCH Commun. Math. Comput. Chem.*, **86** (2021), 11–16.
2. I. Gutman, B. Furtula, M. S. Oz, Geometric approach to vertex-degree-based topological indices–Elliptic Sombor index, theory and application, *Int. J. Quantum Chem.*, **124** (2024), e27346. https://doi.org/10.1002/qua.27346
3. H. Liu, I. Gutman, L. You, Y. Huang, Sombor index: Review of extremal results and bounds, *J. Math. Chem.*, **60** (2022), 771–798. https://doi.org/10.1007/s10910-022-01333-y

4. A. Ali, I. Gutman, T. Réti, A. M. Albalahi, A. E. Hamza, Survey on extremal results and bounds for elliptic Sombor and Euler–Sombor indices, *Discrete Math. Lett.*, **17** (2026), 101–112. https://doi.org/10.47443/dml.2026.056
5. A. Ali, I. Gutman, B. Furtula, Extremal results on bond incident degree indices of graphs: A survey, *Aequat. Math.*, **100** (2026), 66. https://doi.org/10.1007/s00010-026-01306-5
6. F. Movahedi, Results on the elliptic Sombor index, *RAIRO Oper. Res.*, **59** (2025), 2201–2212. https://doi.org/10.1051/ro/2025086
7. A. Granados, E. D. Molina, Y. Quintana, E. Tourís, Bounds for the elliptic Sombor index and applications, *J. Math. Chem.*, **63** (2025), 2051–2067. https://doi.org/10.1007/s10910-025-01757-2
8. Y. Shang, Sombor index and degree-related properties of simplicial networks, *Appl. Math. Comput.*, **419** (2022), 126881. https://doi.org/10.1016/j.amc.2021.126881
9. R. Cruz, J. Rada, Extremal values of the Sombor index in unicyclic and bicyclic graphs, *J. Math. Chem.*, **59** (2021), 1098–1116. https://doi.org/10.1007/s10910-021-01232-8
10. T. Réti, T. Došlić, A. Ali, On the Sombor index of graphs, *Contrib. Math.*, **3** (2021), 11–18. https://doi.org/10.47443/cm.2021.0006
11. K. C. Das, A. Ghalavand, A. R. Ashrafi, On a conjecture about the Sombor index of graphs, *Symmetry*, **13** (2021), 1830. https://doi.org/10.3390/sym13101830
12. M. Zhang, B. Zhao, Extremal values of the Sombor index in tricyclic graphs, *MATCH Commun. Math. Comput. Chem.*, **89** (2023), 741–758. https://doi.org/10.46793/match.89-3.741Z
13. A. E. Hamza, Z. Raza, A. Ali, Z. Alsheekhhussain, On Sombor indices of tricyclic graphs, *MATCH Commun. Math. Comput. Chem.*, **90** (2023), 223–234. https://doi.org/10.46793/match.90-1.223H
14. Z. Tang, Y. Li, H. Deng, Elliptic Sombor index of trees and unicyclic graphs, *Electron. J. Math.*, **7** (2024), 19–34. https://doi.org/10.47443/ejm.2024.009
15. J. Li, X. Zuo, Extremal elliptic Sombor index in tricyclic graphs, *MATCH Commun. Math. Comput. Chem.*, **95** (2026), 919–934. https://doi.org/10.46793/match.95-3.17425
16. F. Qi, Z. Lin, Maximal elliptic Sombor index of bicyclic graphs, *Contrib. Math.*, **10** (2024), 25–29. https://doi.org/10.47443/cm.2024.030
17. A. Ali, D. Dimitrov, T. Réti, A. M. Alotaibi, A. M. Alanazi, T. S. Hassan, Degree-based graphical indices of k-cyclic graphs, *AIMS Mathematics*, **10** (2025), 13540–13554. https://doi.org/10.3934/math.2025609
18. A. Ali, D. Dimitrov, On the extremal graphs with respect to bond incident degree indices, *Discrete Appl. Math.*, **238** (2018), 32–40. https://doi.org/10.1016/j.dam.2017.12.007
19. M. Khan, M. Y. Khan, G. Ali, I. -L. Popa, Investigation of general Sombor index for optimal values in bicyclic graphs, trees, and unicyclic graphs using well-known transformations, *Symmetry*, **17** (2025), 968. https://doi.org/10.3390/sym17060968
20. A. Ali, D. Dimitrov, Z. Du, F. Ishfaq, On the extremal graphs for general sum-connectivity index (χ_α) with given cyclomatic number when $\alpha > 1$, *Discrete Appl. Math.*, **257** (2019), 19–30. https://doi.org/10.1016/j.dam.2018.10.009

21. N. Dehgardi, M. Azari, Y. Shang, Extremal c-cyclic graphs with respect to the general multiplicative first Zagreb index, *Filomat*, **39** (2025), 11597–11605.

22. H. Faheem, S. Ahmad, R. Farooq, Extremal c-cyclic graphs with given maximum degree for Sombor and elliptic Sombor indices, *Int. J. Comput. Math. Comput. Syst. Theory*, **11** (2026), 187–210. https://doi.org/10.1080/23799927.2026.2665699

23. A. M. Albalahi, D. Dimitrov, T. Réti, A. Ali, S. Hussain, Bond incident degree indices of connected (n, m)-graphs with fixed maximum degree, *MATCH Commun. Math. Comput. Chem.*, **92** (2024), 607–630. https://doi.org/10.46793/match.92-2.607A

24. I. Tomescu, Maximum bond incident degree indices for trees and unicyclic graphs with given maximum degree, *Discrete Appl. Math.*, **382** (2026), 272–276. https://doi.org/10.1016/j.dam.2025.12.021

25. S. Ahmad, K. C. Das, R. Farooq, On elliptic Sombor index with applications, *Bull. Malays. Math. Sci. Soc.*, **48** (2025), 108. https://doi.org/10.1007/s40840-025-01894-6

26. H. Liu, Z. Du, Y. Huang, H. Chen, S. Elumalai, Note on the minimum bond incident degree indices of k-cyclic graphs, *MATCH Commun. Math. Comput. Chem.*, **91** (2024), 255–266. https://doi.org/10.46793/match.91-1.255L

27. W. Gao, Extremal graphs with respect to vertex-degree-based topological indices for c-cyclic graphs, *MATCH Commun. Math. Comput. Chem.*, **93** (2025), 549–566. https://doi.org/10.46793/match.93-2.549G

28. Z. Su, H. Deng, A note on the extremal graphs with respect to vertex-degree-based topological indices for c-cyclic graphs, *MATCH Commun. Math. Comput. Chem.*, **95** (2026), 163–178. https://doi.org/10.46793/match.95-1.12725

29. L. Alex, M. Imran, On graphs having maximum bond incident degree based indices with given cyclomatic number, *Kuwait J. Sci.*, **54** (2027), 100676. https://doi.org/10.1016/j.kjs.2026.100676

30. Sabeel-e-Hafi, S. Ahmad, R. Farooq, Bounds for maximum, second maximum and minimum elliptic Sombor index of cacti, *Kuwait J. Sci.*, **54** (2027), 100686. https://doi.org/10.1016/j.kjs.2026.100686

31. H. Hua, The Sombor index of connected graphs with given repetition number and cyclomatic number, *Axioms*, **15** (2026), 757. https://doi.org/10.3390/axioms15100757

32. S. Ahmad, R. Farooq, K. C. Das, Generalized Sombor index of graphs with given degree sequences: Extremal results and applications, *Discrete Appl. Math.*, **391** (2026), 83–100. https://doi.org/10.1016/j.dam.2026.04.034

33. A. Ali, I. Gutman, B. Furtula, A. M. Albalahi, A. E. Hamza, On chemical and mathematical characteristics of generalized degree-based molecular descriptors, *AIMS Mathematics*, **10** (2025), 6788–6804. https://doi.org/10.3934/math.2025311

34. N. Vijendra, S. H. Manohar, H. S. Boregowda, On the general elliptic Sombor index and its applications, *Adv. Appl. Discrete Math.*, **43** (2026), 543–560. https://doi.org/10.17654/0974165826035

35. J. Rada, J. M. Rodríguez, J. M. Sigarreta, Optimization problems for general elliptic Sombor index, *MATCH Commun. Math. Comput. Chem.*, **93** (2025), 819–838. https://doi.org/10.46793/match.93-3.819R

36. S. Boyd, L. Vandenberghe, *Convex optimization*, Cambridge: Cambridge University Press, 2004. https://doi.org/10.1017/CBO9780511804441

37. A. W. Marshall, I. Olkin, B. C. Arnold, *Inequalities: Theory of majorization and its applications*, 2 Eds., New York: Springer, 2011. https://doi.org/10.1007/978-0-387-68276-1

38. C. de Boor, Divided differences, *Surv. Approx. Theory*, **1** (2005), 1–23.

39. J. C. Hernández, J. M. Rodríguez, O. Rosario, J. M. Sigarreta, Extremal problems on the general Sombor index of a graph, *AIMS Mathematics*, **7** (2022), 8330–8343. https://doi.org/10.3934/math.2022464

Appendix A. Derivation of the differential identities

This appendix records the derivative calculations used in Theorem 10.1 and Lemmas 11.1–11.3. All arguments a, b in this appendix are positive real numbers.

A.1. The two-parameter weight

Set $u = a + b$, $v = a^2 + b^2$, and $h(b) = u^\alpha v^\beta$. For the factor v^β, differentiation gives

$$\begin{aligned}(v^\beta)' &= 2\beta b v^{\beta-1},\\ (v^\beta)'' &= 2\beta v^{\beta-1} + 4\beta(\beta-1)b^2 v^{\beta-2},\\ (v^\beta)''' &= 12\beta(\beta-1)b v^{\beta-2} + 8\beta(\beta-1)(\beta-2)b^3 v^{\beta-3}.\end{aligned}$$

Applying the product rule to $u^\alpha v^\beta$ twice yields

$$h'' = u^{\alpha-2}v^{\beta-2}\left[\alpha(\alpha-1)v^2 + 4\alpha\beta buv + 2\beta u^2 v + 4\beta(\beta-1)b^2u^2\right]. \tag{A.1}$$

The four terms in brackets come, respectively, from $(u^\alpha)''v^\beta$, $2(u^\alpha)'(v^\beta)'$, and the two terms of $u^\alpha(v^\beta)''$. Expanding their a, b powers gives

$$\begin{aligned}N_{\alpha,\beta}(a,b) = {}& (\alpha^2 - \alpha + 2\beta)a^4 + 4\beta(\alpha+1)a^3 b\\ &+ (4\beta^2 + 2\alpha(m-1))a^2b^2 + 4\beta(m-1)ab^3 + m(m-1)b^4, \qquad m = \alpha + 2\beta.\end{aligned} \tag{A.2}$$

Thus $h'' = u^{\alpha-2}v^{\beta-2}N_{\alpha,\beta}$. In particular, for fixed $a > 0$, division by b^4 in this polynomial and the leading powers of u, v give

$$\lim_{b\to\infty} b^{2-m}h''(b) = m(m-1).$$

This proves the limiting sign test used in the necessity argument.

Applying the product rule three times, in the order $(u^\alpha)'''v^\beta + 3(u^\alpha)''(v^\beta)' + 3(u^\alpha)'(v^\beta)'' + u^\alpha(v^\beta)'''$, gives

$$\begin{aligned}h''' &= u^{\alpha-3}v^{\beta-3}M_{\alpha,\beta},\\ M_{\alpha,\beta} &= \alpha(\alpha-1)(\alpha-2)v^3 + 6\alpha\beta(\alpha-1)buv^2 + 6\alpha\beta u^2v^2\\ &\quad + 12\alpha\beta(\beta-1)b^2u^2v + 12\beta(\beta-1)bu^3v + 8\beta(\beta-1)(\beta-2)b^3u^3.\end{aligned} \tag{A.3}$$

Each summand is obtained from one of the displayed product-rule terms; no sign inference is required for this identity.

For the diagonal test, it is shorter to use the logarithmic derivative

$$L = \frac{h'}{h} = \frac{\alpha}{u} + \frac{2\beta b}{v}, \quad L' = -\frac{\alpha}{u^2} + \frac{2\beta}{v} - \frac{4\beta b^2}{v^2}, \quad L'' = \frac{2\alpha}{u^3} - \frac{12\beta b}{v^2} + \frac{16\beta b^3}{v^3}.$$

At $b = a$, these become $m/(2a)$, $-\alpha/(4a^2)$, and $(\alpha/4 - \beta)/a^3$. Since $h''' = h(L^3 + 3LL' + L'')$,

$$h'''(a) = 2^{\alpha+\beta-3} a^{m-3}[m^3 - 3m\alpha + 2\alpha - 8\beta].$$

Substitution of $\alpha = m - 2\beta$ factors the bracket as $(m - 2)(m^2 - m + 6\beta)$, proving (10.5).

A.2. The degree-two factor

Let $\phi = u^{2-2\gamma} v^{\gamma}$, $0 < \gamma \leq 1$, and set

$$P = \frac{\phi'}{\phi} = \frac{2(1-\gamma)}{u} + \frac{2\gamma b}{v}.$$

Then

$$P' = -\frac{2(1-\gamma)}{u^2} + \frac{2\gamma(a^2 - b^2)}{v^2}, \qquad P'' = \frac{4(1-\gamma)}{u^3} + \frac{4\gamma b(b^2 - 3a^2)}{v^3}.$$

Set

$$\begin{aligned} T &= 2(1-\gamma)v + 2\gamma b u, \\ V &= -2(1-\gamma)v^2 + 2\gamma(a^2 - b^2)u^2, \\ W &= 4(1-\gamma)v^3 + 4\gamma b(b^2 - 3a^2)u^3. \end{aligned}$$

These definitions give $P = T/(uv)$, $P' = V/(u^2 v^2)$, and $P'' = W/(u^3 v^3)$. Since $2\phi\phi'' - (\phi')^2 = \phi^2(P^2 + 2P')$, the numerator to compute is

$$T^2 + 2V = 4a^2\gamma[\gamma(a - b)^2 + 4ab].$$

Indeed, its constant coefficient in γ is zero, its linear coefficient is $16a^3 b$, and its quadratic coefficient is $4a^2(a - b)^2$. Multiplication by $\phi^2/(u^2 v^2) = u^{2-4\gamma} v^{2\gamma-2}$ gives (10.2).

Likewise $\phi''' = \phi(P^3 + 3PP' + P'')$, whose numerator is $T^3 + 3TV + W$. Its nonzero coefficients in powers of γ are

γ	$8a^3(a - b)(v + 4ab)$
γ^2	$-48a^4 b(a - b)$
γ^3	$-8a^3(a - b)^3$.

The identity $(a - b)^2 - (v + 4ab) = -6ab$ therefore gives the factorization

$$T^3 + 3TV + W = 8a^3\gamma(1-\gamma)(a - b)[\gamma(a - b)^2 + v + 4ab].$$

Multiplying by $\phi/(u^3 v^3) = u^{-1-2\gamma} v^{\gamma-3}$ proves (10.3).

Appendix B. Polynomial regrouping and the uniform numerical bound

B.1. Coefficients in the upper compensation bound

We derive the decomposition used in Lemma 11.3. Write $H = \alpha(\alpha - 1)(2 - \alpha)$, so $K = H + 8\beta$. Using (A.3), collect $M_{\alpha,\beta} + Kauv^2$ in powers of β. The constant coefficient is

$$-Hv^3 + Hauv^2 = Hv^2(au - v) = Hv^2b(a - b) = T_0.$$

The coefficients of β, β^2, and β^3 are, respectively,

$$\begin{aligned} D_1 &= 6\alpha(\alpha - 1)buv^2 + 6\alpha u^2v^2 \\ &\qquad - 12\alpha b^2u^2v - 12bu^3v + 16b^3u^3 + 8auv^2, \\ D_2 &= 12\alpha b^2u^2v + 12bu^3v - 24b^3u^3, \\ D_3 &= 8b^3u^3. \end{aligned}$$

The last two are B, C in that lemma. For the first, subtraction at $\alpha = 1$ gives

$$D_1(\alpha) - D_1(1) = (\alpha - 1)[6\alpha buv^2 + 6u^2v^2 - 12b^2u^2v] = (\alpha - 1)G.$$

Expanding $D_1(1)$ in a, b gives

$$D_1(1) = 14a^6 + 8a^5b - 14a^4b^2 - 16a^3b^3 + 2a^2b^4 + 8ab^5 - 2b^6.$$

Multiplication of the polynomial Q in the lemma by $a - b$ gives these same coefficients. Hence $D_1 = (a - b)Q + (\alpha - 1)G = A$. The other regroupings used there follow by factoring out $6uv$ and $12bu^2$:

$$\begin{aligned} G/(6uv) &= \alpha bv + uv - 2b^2u = a(a^2 + 2ab - b^2) + (\alpha - 1)bv, \\ B/(12bu^2) &= \alpha bv + uv - 2b^2u = a(a^2 + 2ab - b^2) + (\alpha - 1)bv. \end{aligned}$$

All these expressions are polynomial identities; at $b = 0$, the identity for B is read after clearing the displayed factor. This proves each regrouping used in the sign argument.

B.2. A uniform rational bound

To prove

$$\frac{5\alpha(\alpha - 1)(2 - \alpha)}{56 + 48\alpha} < \frac{3}{200} \quad (1 \le \alpha \le 2),$$

it suffices to prove the positivity of

$$p = \frac{3}{200}(56 + 48\alpha) - 5\alpha(\alpha - 1)(2 - \alpha).$$

Set $z = \alpha - 3/2 \in [-1/2, 1/2]$. Algebraic calculation gives

$$p = 5z^3 + \frac{15}{2}z^2 - \frac{53}{100}z + \frac{9}{200}.$$

Since $5z^3 \geq -(5/2)z^2$ on this interval,

$$p \geq 5z^2 - \frac{53}{100}z + \frac{9}{200} = 5\left(z - \frac{53}{1000}\right)^2 + \frac{6191}{200{,}000} > 0.$$

The denominator $56+48\alpha$ is positive, so the desired strict ratio bound follows. It is a sufficient uniform estimate, not an optimization of the compensation constants.



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