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Research article

Casorati inequalities along vertical and horizontal distributions of pointwise slant Riemannian submersions

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Abstract: We establish optimal Casorati inequalities for pointwise slant Riemannian submersions. As such a submersion carries two distinct distributions, the vertical, tangent to the fibers, and the horizontal, governed by O'Neill's tensors T and A, we treat them separately. For submersions from generalized complex and generalized Sasakian space forms, we bound the normalized scalar curvature of each distribution by its normalized Casorati curvatures. Special cases recover sharp inequalities for real, complex, Kähler, Sasakian, Kenmotsu, cosymplectic, and almost $C(\alpha)$ space forms. Equality is characterized geometrically: invariantly quasi-umbilical fibers in the vertical case and integrability ($A \equiv 0$) in the horizontal case. The invariant and anti-invariant limits reproduce known results, and examples illustrate sharpness.

Keywords: Riemannian submersions; O'Neill tensors; space forms; Casorati curvatures; pointwise slant submersions; optimal inequalities
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1. Introduction

A recurring difficulty with Gaussian curvature is that it may be zero even where a surface is visibly curved, so it does not always reflect the shape of a surface as it sits in space. It was precisely this gap that led Casorati [5] in 1890 to propose a different extrinsic invariant, today known as the Casorati curvature. Unlike Gaussian curvature, it vanishes exactly at planar points, which makes it a natural tool for shape analysis and visualization [9, 30], and places it within the broader study of curvature invariants that has shaped differential geometry since the work of Gauss and Riemann.

Interest in Casorati inequalities in their present form was sparked by Decu, Haesen, and

Verstraelen [9], who, building on Chen's program of δ-invariants [6, 7], derived sharp estimates bounding the scalar curvature of a submanifold of a real space form by its normalized Casorati curvatures. Their work prompted an extensive line of research, with sharp Casorati inequalities established across a wide range of ambient spaces and submanifold classes. Among these contributions, Aquib et al. [2, 3] treated Lagrangian and bi-slant submanifolds of complex space forms, Lee et al. [21–23] addressed Legendrian submanifolds in Sasakian and Kenmotsu space forms, Lone [24–27] examined slant submanifolds of generalized complex and Sasakian space forms, Vilcu [40] considered Lagrangian submanifolds of complex space forms, Zhang et al. [43, 44] dealt with submanifolds of real and complex space forms, and Siddiqui [36] studied bi-slant submanifolds of generalized Sasakian space forms. A broader overview can be found in the survey of Chen [8].

Parallel to submanifold theory, Riemannian submersions emerged as important objects with applications in theoretical physics, particularly in Kaluza–Klein theory and supergravity [12, 28]. O'Neill [28] developed the fundamental equations of a submersion, introducing two tensor fields T and A: The tensor T measures the failure of the fibers to be totally geodesic, while A measures the failure of the horizontal distribution to be integrable. This is the central structural feature that distinguishes submersions from submanifolds: A submersion carries two natural distributions with genuinely different geometric meanings. The vertical distribution $\mathcal{V} = \ker F_*$ is tangent to the fibers and its extrinsic geometry is encoded by T, whereas the horizontal distribution $\mathcal{H} = (\ker F_*)^\perp$ is governed by A. In [33–35], Sahin developed a comprehensive theory of submersions in Hermitian geometry, and these structures are special cases of the Riemannian maps introduced by Fischer [15].

The Casorati geometry of these two distributions must therefore be analyzed separately, and the resulting equality conditions differ: Invariant quasi-umbilicity of the fibers on the vertical side, and integrability of the horizontal distribution on the horizontal side. This two-distribution phenomenon is precisely what separates the submersion problem from the Casorati theory of Riemannian maps. For a Riemannian map, the differential need not be surjective, and the extrinsic geometry of the map is carried by a single object, its second fundamental form ∇F_*, which takes values in the orthogonal complement $(\text{range}\, F_*)^\perp$ of the range inside the target; the corresponding Casorati inequality bounds this one tensor. A Riemannian submersion is exactly the case in which F_* is onto, so $(\text{range}\, F_*)^\perp = 0$ and that tensor vanishes identically. The extrinsic geometry then passes entirely to the source, where it is shared between the two O'Neill tensors acting on orthogonal distributions, T, the second fundamental form of the fibers, on $\mathcal{V}$, and A, the obstruction to horizontal integrability, on $\mathcal{H}$. Since T and A encode independent curvature data through O'Neill's equations and act on different distributions, there is no single ambient form to optimize; each distribution instead yields its own optimal Casorati inequality, which is the structural reason the submersion setting requires two separate families of results.

Casorati inequalities were subsequently extended to Riemannian maps and submersions [37]. Lee et al. [20] obtained optimal inequalities in real and complex space forms, Polat et al. [31] extended these to Sasakian space forms, and Zaidi and Shanker [42] treated bi-slant maps to Kenmotsu manifolds, while Fatima et al. [13] investigated nearly Kähler targets. However, all of these results treated only invariant ($\theta = 0$) or anti-invariant ($\theta = \pi/2$) structures. Our results contain these as limiting cases. When $\theta = 0$, Theorems 4.1 and 4.2 recover the invariant submersion inequalities of [20, Theorem 4.1] and [31, Theorem 5.1] (as corrected in [32]), and Theorem 5.1 recovers [20, Theorem 4.2]; when $\theta = \pi/2$, this eliminates the c_2-terms and yields the corresponding anti-invariant

submersion inequalities of [20, 31] and the special case of [42]. The full slant parameter θ interpolates between these extremes.

The notion of slant submanifolds was introduced by Chen [6] as a natural generalization of invariant and anti-invariant submanifolds, and was extended to contact geometry [4, 41]. Etayo [11] introduced pointwise slant submanifolds, where the angle varies from point to point. For Riemannian maps, pointwise slant structures were studied by Gündüzalp and Akyol [16], and, in the horizontal Reeb case, by Demir and Yildirim [10]. For submersions, the pointwise slant theory was developed by Lee and Sahin [19] from almost Hermitian manifolds and by Kumar–Prasad [17] and Prasad–Kumar [18] from almost contact manifolds. These structures unify the invariant, anti-invariant, and all intermediate cases through a single varying slant angle. The parallel theory for pointwise slant Riemannian maps was developed in the companion paper [14].

Despite the extensive development of (a) Casorati inequalities for invariant/anti-invariant submersions and (b) the pointwise slant theory of submersions, Casorati inequalities for the pointwise slant case remain unestablished. This gap is significant: Geometrically, pointwise slant is the general case, with the invariant and anti-invariant settings as limits; physically, theories on the total space may involve spatially varying angles; mathematically, a variable θ introduces new technical challenges that require a careful, distribution-by-distribution optimization.

We address this gap by establishing comprehensive Casorati inequalities for pointwise slant Riemannian submersions. Our contributions are as follows. First, working separately on the vertical and horizontal distributions, we derive explicit inequalities when the total space is a generalized complex space form (Theorems 4.1 and 5.1) and a generalized Sasakian space form (Theorems 4.2 and 5.2), covering all classical space forms (Corollaries 4.1, 4.2, 5.1, and 5.2). Second, we characterize the equality cases: Invariantly quasi-umbilical fibers (Definition 4.1, Remark 4.2) for the vertical inequalities, and integrability of the horizontal distribution (equivalently, $A \equiv 0$) for the horizontal inequalities. Third, we verify the reduction to the known results [20, 31, 42] when $\theta = 0$ or $\theta = \pi/2$. Finally, we provide examples (Examples 6.1–6.3) demonstrating sharpness and the role of the varying slant function.

The paper is organized as follows. Section 2 collects the preliminaries on Riemannian submersions, O'Neill's tensors T and A, the relevant space forms, and the algebraic optimization Lemma 2.4. Section 3 develops pointwise slant structures for submersions from almost Hermitian and from almost contact metric manifolds, and proves the unifying norm identity (Proposition 3.4). Section 4 establishes the Casorati inequalities for the vertical distribution, and Section 5 establishes those for the horizontal distribution, each for generalized complex and generalized Sasakian space forms. Section 6 provides illustrative examples, and Section 7 presents the concluding remarks.

2. Preliminaries

In this section, we establish the fundamental concepts needed for our main theorems. Throughout, all manifolds are smooth, connected, and without boundary.

2.1. Riemannian submersions

Let $F : (M_1^{m_1}, g_1) \to (M_2^{m_2}, g_2)$ be a smooth submersion between Riemannian manifolds. For $p \in M_1$, the kernel $\ker F_{*p}$ of the differential and its orthogonal complement $(\ker F_{*p})^{\perp}$ give the

orthogonal decomposition $T_pM_1 = \ker F_{*p} \oplus (\ker F_{*p})^{\perp}$. The map F is a Riemannian submersion if $F_{*p} : (\ker F_{*p})^{\perp} \to T_{F(p)}M_2$ is a linear isometry for every $p \in M_1$ [12, 28].

For a Riemannian submersion $F : M_1 \to M_2$, we denote the vertical and horizontal distributions by $\mathcal{V} = \ker F_*$ and $\mathcal{H} = (\ker F_*)^{\perp}$, respectively. O'Neill [28] introduced two fundamental tensor fields T and A, defined for vector fields E, G on M_1 by

$$T_E G = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}G + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}G, \tag{2.1}$$

$$A_E G = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}G + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}G. \tag{2.2}$$

Here $\mathcal{V}$ and $\mathcal{H}$ denote the vertical and horizontal projections, respectively. The tensor T measures the obstruction to the fibers being totally geodesic, while A measures the obstruction to the horizontal distribution being integrable.

Proposition 2.1. *[28] For a Riemannian submersion, the tensors T and A satisfy:*

(i) $T_U V$ is symmetric for $U, V \in \Gamma(\mathcal{V})$ and $T_U W \in \Gamma(\mathcal{H})$ for $U \in \Gamma(\mathcal{V})$, $W \in \Gamma(\mathcal{H})$.

(ii) $A_X Y$ is skew-symmetric for $X, Y \in \Gamma(\mathcal{H})$ and $A_X U \in \Gamma(\mathcal{V})$ for $X \in \Gamma(\mathcal{H})$, $U \in \Gamma(\mathcal{V})$.

(iii) For $U, V \in \Gamma(\mathcal{V})$: $T_U V = \frac{1}{2}\mathcal{V}[U, V]$.

(iv) For $X, Y \in \Gamma(\mathcal{H})$: $A_X Y = \frac{1}{2}\mathcal{V}[X, Y]$.

The curvature properties of Riemannian submersions are described by the following Gauss-type equations.

Theorem 2.1. *[12] For a Riemannian submersion $F : M_1 \to M_2$ and for $U, V, W, Z \in \Gamma(\mathcal{V})$,*

$$g_1(R^{M_1}(U, V)W, Z) = g_1(\hat{R}(U, V)W, Z) - g_1(T_U W, T_V Z) + g_1(T_V W, T_U Z), \tag{2.3}$$

where $\hat{R}$ denotes the curvature tensor of the fibers.

Theorem 2.2. *[12] For a Riemannian submersion and for $X, Y, Z, W \in \Gamma(\mathcal{H})$,*

$$g_1(R^{M_1}(X, Y)Z, W) = g_1(R^{\mathcal{H}}(X, Y)Z, W) - 2g_1(A_X Y, A_Z W) + g_1(A_Y Z, A_X W) - g_1(A_X Z, A_Y W), \tag{2.4}$$

where $R^{\mathcal{H}}$ is the curvature tensor of the horizontal distribution.

2.2. Space forms

We now recall the ambient spaces used throughout the paper.

Definition 2.2. [29, 38] An almost Hermitian manifold, (M, g, J), is called a generalized complex space form, denoted by $M(c_1, c_2)$, if its curvature tensor satisfies

$$R(X, Y)Z = c_1\{g(Y, Z)X - g(X, Z)Y\} + c_2\{g(X, JZ)JY - g(Y, JZ)JX + 2g(X, JY)JZ\} \tag{2.5}$$

for smooth functions c_1, c_2 on M.

Important special cases include:

(i) Real space forms: $c_1 = c$, $c_2 = 0$ (constant sectional curvature c).

(ii) Complex space forms: $c_1 = c_2 = c/4$ (Kähler, constant holomorphic sectional curvature c).

(iii) Real Kähler space forms: $c_1 = (c + 3\alpha)/4$, $c_2 = (c - \alpha)/4$.

Recall that an almost contact metric manifold (M, ϕ, ξ, η, g) carries a $(1, 1)$-tensor field ϕ, a Reeb vector field ξ, and a 1-form η satisfying $\phi^2 = -I + \eta \otimes \xi$, $\eta(\xi) = 1$, and $\phi\xi = 0$ [4, 41]. In particular, $\phi^2 = -I$ on $\ker \eta = \{X \in TM : \eta(X) = 0\}$, where ϕ acts as an isometry.

Definition 2.3. [1,4] An almost contact metric manifold (M, ϕ, ξ, η, g) is called a generalized Sasakian space form, denoted by $M(c_1, c_2, c_3)$, if

$$\begin{aligned} R(X, Y)Z = {} & c_1\{g(Y, Z)X - g(X, Z)Y\} \\ & + c_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\ & + c_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\} \end{aligned} \tag{2.6}$$

for smooth functions c_1, c_2, c_3 on M.

Special cases include:

(i) Sasakian space forms: $c_1 = (c + 3)/4$, $c_2 = c_3 = (c - 1)/4$.

(ii) Kenmotsu space forms: $c_1 = (c - 3)/4$, $c_2 = c_3 = (c + 1)/4$.

(iii) Cosymplectic space forms: $c_1 = c_2 = c_3 = c/4$.

(iv) Almost $C(\alpha)$ space forms: $c_1 = (c + 3\alpha^2)/4$, $c_2 = c_3 = (c - \alpha^2)/4$.

2.3. A key algebraic lemma

The following lemma, due to Tripathi [39], provides the optimization result central to our proofs. It identifies the constrained minimum of a specific quadratic form on a hyperplane in $\mathbb{R}^r$.

Lemma 2.4. *[39] Let $\Lambda = \{(z_1, \ldots, z_r) \in \mathbb{R}^r : z_1 + \cdots + z_r = k\}$ be a hyperplane in $\mathbb{R}^r$, and let $f : \mathbb{R}^r \to \mathbb{R}$ be the quadratic form*

$$f(z_1, \ldots, z_r) = \lambda_1 \sum_{i=1}^{r-1} z_i^2 + \lambda_2 z_r^2 - 2 \sum_{1 \le i < j \le r} z_i z_j, \tag{2.7}$$

where $\lambda_1, \lambda_2 > 0$ satisfy $\lambda_2 = (r-1)/(\lambda_1 - r + 2)$. Then the constrained minimum of f on Λ is achieved at

$$z_1 = \cdots = z_{r-1} = \frac{k}{\lambda_1 + 1}, \quad z_r = \frac{k(\lambda_1 - r + 2)}{\lambda_1 + 1}, \tag{2.8}$$

and at this critical point, $f(z_1, \ldots, z_r) = 0$.

Remark 2.5. The relation $\lambda_2 = (r-1)/(\lambda_1 - r + 2)$ requires $\lambda_1 > r - 2$ for the positivity condition $\lambda_2 > 0$ to hold. In the applications to Casorati inequalities throughout this paper, Lemma 2.4 is invoked with parameters $\lambda_1 = r$ and $\lambda_2 = (r - 1)/2$ (see the proofs of Theorems 4.1 and 5.1). Since $r \ge 3$ by assumption, the positivity condition $\lambda_1 = r > r - 2$ is automatically satisfied. With $\lambda_1 = r$, the

constraint $\lambda_2 = (r-1)/(\lambda_1 - r + 2)$ gives $\lambda_2 = (r-1)/2$ directly, confirming that the parameter choices are valid. The critical point (2.8) then becomes

$$z_1 = \cdots = z_{r-1} = \frac{k}{r+1}, \quad z_r = \frac{2k}{r+1},$$

which will correspond to the quasi-umbilical ratio $h^\alpha_{11} = \cdots = h^\alpha_{r-1,r-1} = \dfrac{1}{2}\, h^\alpha_{rr}$ in the equality conditions of our main theorems.

Remark 2.6. The assumption $r \geq 3$ is necessary because, for $r = 2$, the normalized Casorati curvatures $\delta_C(r-1) = \delta_C(1)$ vanish trivially, rendering the inequalities uninformative. Avoiding this degenerate case, we assume $r \geq 3$ throughout.

Remark 2.7. In the companion article for a Riemannian map [14], a proper pointwise slant structure forces $\operatorname{Ran} F_*$ to be even-dimensional, since $P^R/\cos\theta$ is a complex structure on the range; the minimal admissible rank is therefore $r = 4$. No such parity constraint arises here: $(P^{\mathcal{V}})^2 = -\cos^2\theta\, I$ holds only on the ϕ_1-active part of the distribution, which need not be the space over which the Casorati optimization is performed. Hence $r \geq 3$ (and $s \geq 3$) is the only dimensional assumption, as in [17, 19]; Example 6.1, whose ϕ_1-active vertical part is three-dimensional, realizes a pointwise slant submersion with odd active dimension.

3. Pointwise slant Riemannian submersions

This section develops the fundamental properties of pointwise slant structures for Riemannian submersions. Two settings arise, according to whether the total space carries an almost complex or an almost contact metric structure.

(i) Riemannian submersions $F : (M_1, g_1, J_1) \to (M_2, g_2)$ from an almost Hermitian manifold [19,34];

(ii) Riemannian submersions $F : (M_1, g_1, \phi_1, \xi_1, \eta_1) \to (M_2, g_2)$ from an almost contact metric manifold [17–19].

In each case, the structure tensor (J_1 or ϕ_1) is applied to a vector from a chosen distribution, the vertical $\ker F_*$ or the horizontal $(\ker F_*)^\perp$, and the result is split orthogonally into components inside and outside that distribution. Throughout, U denotes a vector field in $\ker F_*$ (vertical) and X a vector field in $(\ker F_*)^\perp$ (horizontal); P denotes the component of the image inside the chosen distribution, and Q is the component in its orthogonal complement. Both settings yield the same key norm identity, collected in Proposition 3.4.

3.1. Submersion from an almost Hermitian manifold

Let $F : (M_1, g_1, J_1) \to (M_2, g_2)$ be a Riemannian submersion from (M_1, g_1, J_1) onto (M_2, g_2). Since $TM_1 = \ker F_* \oplus (\ker F_*)^\perp$, applying J_1 to a vertical vector U gives a vector that splits into vertical and horizontal parts, and applying J_1 to a horizontal vector X likewise splits:

For any vector field $U \in \Gamma(\ker F_*)$,

$$J_1 U = P^{\mathcal{V}} U + Q^{\mathcal{V}} U, \qquad P^{\mathcal{V}} U \in \Gamma(\ker F_*), \quad Q^{\mathcal{V}} U \in \Gamma(\ker F_*)^\perp, \tag{3.1}$$

with the linear operators $P^{\mathcal{V}} : \ker F_* \to \ker F_*$ and $Q^{\mathcal{V}} : \ker F_* \to (\ker F_*)^{\perp}$, i.e., $P^{\mathcal{V}}U$ is the part of J_1U that stays vertical and $Q^{\mathcal{V}}U$ is the part that becomes horizontal.

Similarly, for $X \in \Gamma((\ker F_*)^{\perp})$,

$$J_1X = P^{\mathcal{H}}X + Q^{\mathcal{H}}X, \qquad P^{\mathcal{H}}X \in \Gamma(\ker F_*)^{\perp}, \quad Q^{\mathcal{H}}X \in \Gamma(\ker F_*), \tag{3.2}$$

with the linear operators $P^{\mathcal{H}} : (\ker F_*)^{\perp} \to (\ker F_*)^{\perp}$ and $Q^{\mathcal{H}} : (\ker F_*)^{\perp} \to \ker F_*$, i.e., $P^{\mathcal{H}}X$ is the part of J_1X that stays horizontal and $Q^{\mathcal{H}}X$ is the part that becomes vertical.

Remark 3.1. These are two genuinely different linear operators with different domains. $P^{\mathcal{V}} : \ker F_* \to \ker F_*$ measures how much J_1 preserves the vertical distribution, while $P^{\mathcal{H}} : (\ker F_*)^{\perp} \to (\ker F_*)^{\perp}$ measures how much J_1 preserves the horizontal distribution.

Definition 3.2. [19] A Riemannian submersion $F : (M_1, g_1, J_1) \to (M_2, g_2)$ from an almost Hermitian manifold (M_1, g_1, J_1) is called a pointwise slant Riemannian submersion if there exists a function $\theta : M_1 \to [0, \pi/2]$ such that, at each fixed $p \in M_1$, the angle $\theta(p)$ between J_1U and $\mathcal{V}_p$ is independent of the choice of nonzero $U \in \mathcal{V}_p$ — but $\theta(p)$ may vary from point to point. Equivalently,

$$\cos\theta(p) = \frac{\|P^{\mathcal{V}}U\|}{\|U\|}$$

for every nonzero $U \in \mathcal{V}_p$, i.e., $(P^{\mathcal{V}})^2 = -\cos^2\theta(p)\, I$ with $\theta(p)$ referred to as a slant function.

Since $J_1^2 = -I$, for the linear operator $P^{\mathcal{V}}$, one has

$$(P^{\mathcal{V}})^2 = -\cos^2\theta \cdot I, \qquad g_1(P^{\mathcal{V}}U,\, P^{\mathcal{V}}U') = \cos^2\theta \cdot g_1(U, U'), \qquad g_1(Q^{\mathcal{V}}U,\, Q^{\mathcal{V}}U') = \sin^2\theta \cdot g_1(U, U'), \tag{3.3}$$

for all $U, U' \in \Gamma(\ker F_*)$ and the analogous identities hold for the linear operator $P^{\mathcal{H}}$. By Parseval's identity applied to an orthonormal basis $\{e_1, \ldots, e_r\}$ of $\ker F_*$, these give, for the vertical case,

$$\|P^{\mathcal{V}}\|^2 := \sum_{i,j=1}^{r} (g_1(e_i,\, P^{\mathcal{V}}e_j))^2 = r\cos^2\theta, \qquad r = \dim\ker F_*, \tag{3.4}$$

and analogously, $\|P^{\mathcal{H}}\|^2 = s\cos^2\theta$ with $s = \dim(\ker F_*)^{\perp}$.

3.2. Submersion from an almost contact metric manifold

Let $F : (M_1, g_1, \phi_1, \xi_1, \eta_1) \to (M_2, g_2)$ be a Riemannian submersion from an almost contact metric manifold $(M_1, g_1, \phi_1, \xi_1, \eta_1)$ onto a Riemannian manifold (M_2, g_2). Applying ϕ_1 to a vertical vector U and to a horizontal vector X produces two separate decompositions.

For any vector field $U \in \Gamma(\ker F_*)$,

$$\phi_1U = P^{\mathcal{V}}U + Q^{\mathcal{V}}U, \qquad P^{\mathcal{V}}U \in \Gamma(\ker F_*), \quad Q^{\mathcal{V}}U \in \Gamma((\ker F_*)^{\perp}). \tag{3.5}$$

$P^{\mathcal{V}}U$ is the part of ϕ_1U that stays vertical; $Q^{\mathcal{V}}U$ is the part that becomes horizontal. Similarly, for any vector field $X \in \Gamma(\ker F_*)^{\perp}$,

$$\phi_1X = P^{\mathcal{H}}X + Q^{\mathcal{H}}X, \qquad P^{\mathcal{H}}X \in \Gamma(\ker F_*)^{\perp}, \quad Q^{\mathcal{H}}X \in \Gamma(\ker F_*). \tag{3.6}$$

$P^{\mathcal{H}}X$ is the part of ϕ_1X that stays horizontal; $Q^{\mathcal{H}}X$ is the part that becomes vertical. The above operators $P^{\mathcal{V}},\ Q^{\mathcal{V}},\ P^{\mathcal{H}}, Q^{\mathcal{H}}$ are defined exactly as in Subsection 3.1, with ϕ_1 in place of J_1.

Definition 3.3. [17–19] Let $F : (M_1, g_1, \phi_1, \xi_1, \eta_1) \to (M_2, g_2)$ be a Riemannian submersion from an almost contact metric manifold. Then F is called pointwise slant if there exists $\theta : M_1 \to [0, \pi/2]$ such that, at each $p \in M_1$, the angle $\theta(p)$ between $\phi_1 U$ and $\mathcal{V}_p$ is independent of the choice of nonzero $U \in \ker F_{*p} \setminus \{\xi_1\}$. Equivalently, $\cos\theta(p) = \dfrac{\|P^{\mathcal{V}}U\|}{\|U\|}$, i.e., $(P^{\mathcal{V}})^2 = -\cos^2\theta\, I$ on $\ker F_* \setminus \{\xi_1\}$.

The algebraic identities follow from $\phi_1^2 = -I$ on $\ker\eta_1$ by the polarization argument of [16, Lemma 2.2]. We write $r = \dim\ker F_*$ and $s = \dim(\ker F_*)^\perp$ for the full vertical and horizontal dimensions. Since ϕ_1 annihilates ξ_1, the slant condition is active on the orthogonal complement of ξ_1 in the relevant distribution, so the effective dimension drops by one in whichever distribution contains ξ_1. Parseval's identity then gives

$$\|P^{\mathcal{V}}\|^2 = \begin{cases} (r-1)\cos^2\theta, & \xi_1 \in \Gamma(\ker F_*), \\ r\cos^2\theta, & \xi_1 \in \Gamma((\ker F_*)^\perp), \end{cases} \qquad \|P^{\mathcal{H}}\|^2 = \begin{cases} (s-1)\cos^2\theta, & \xi_1 \in \Gamma((\ker F_*)^\perp), \\ s\cos^2\theta, & \xi_1 \in \Gamma(\ker F_*). \end{cases} \tag{3.7}$$

Proposition 3.4. *Let $\mathcal{D}_\theta$ denote the distribution on which the slant condition is imposed — the vertical or horizontal distribution in the almost Hermitian case, and its intersection with* $\ker\eta_1$ *in the almost contact case — and set $d = \dim\mathcal{D}_\theta$. Then the tangential projection operator P satisfies*

$$\|P\|^2 := \sum_{i,j} (g_1(e_i, Pe_j))^2 = d\cos^2\theta \tag{3.8}$$

for any orthonormal basis $\{e_i\}$ of $\mathcal{D}_\theta$. This is (3.4) *in the almost Hermitian case ($d = r$ or s) and* (3.7) *in the almost contact case, where $d = r - 1$ or r (resp., $s - 1$ or s), according to the position of ξ_1.*

Remark 3.5. Proposition 3.4 encompasses the following classical special cases [6, 19, 33]:

(a) **Invariant case ($\theta = 0$):** $\cos\theta = 1$, so $P = J$ (or $P = \phi$ on $\ker\eta_1$) and $Q = 0$; the structure map preserves the chosen distribution entirely, $\|P\|^2 = r$.

(b) **Anti-invariant case ($\theta = \pi/2$):** $\cos\theta = 0$, so $P = 0$ and $Q = J$ (or $Q = \phi$); the structure map sends every vector entirely out of the chosen distribution, $\|P\|^2 = 0$.

(c) **Proper slant ($0 < \theta < \pi/2$):** Both P and Q are nontrivial and $\|P\|^2 = r\cos^2\theta \in (0, r)$ interpolates continuously between the two extremes.

The Casorati inequalities in Sections 4 and 5 therefore recover all previously known results for the invariant and anti-invariant cases simultaneously through the single parameter $\cos^2\theta$.

With the fundamental properties of pointwise slant Riemannian submersions now established, we proceed to derive our main Casorati inequalities. The central role in every proof is played by the norm identities (3.4) and (3.7), which express $\|P\|^2$ purely in terms of $\cos^2\theta$ and thereby connect the extrinsic Casorati curvature to the slant angle.

4. Casorati inequalities for the vertical distribution

Riemannian submersions possess two natural distributions, the vertical $\mathcal{V} = \ker F_*$ and the horizontal $\mathcal{H} = (\ker F_*)^\perp$, governed by the O'Neill tensors T and A, respectively. In this section,

we treat the vertical distribution, whose extrinsic geometry is encoded by T. Throughout, we assume $r = \dim \mathcal{V} \geq 3$.

Let $\{U_1, \ldots, U_r\}$ be an orthonormal basis of $\mathcal{V}$ at $p \in M_1$ and let $\{U_{r+1}, \ldots, U_{m_1}\}$ be an orthonormal basis of $\mathcal{H}$ at p. The components of O'Neill's tensor T are

$$T_{ij}^{\alpha} = g_1(T_{U_i}U_j,\ U_\alpha), \quad i, j = 1, \ldots, r,\ \alpha = r+1, \ldots, m_1, \tag{4.1}$$

and the squared norm of T restricted to $\mathcal{V}$ is

$$\|T\|^2 = \sum_{\alpha=r+1}^{m_1} \sum_{i,j=1}^{r} (T_{ij}^{\alpha})^2. \tag{4.2}$$

The Casorati curvature of the vertical distribution and its normalized variants are

$$C^{\mathcal{V}} = \frac{1}{r}\|T\|^2, \qquad C(L) = \frac{1}{k} \sum_{\alpha=r+1}^{m_1} \sum_{i,j=1}^{k} (T_{ij}^{\alpha})^2 \tag{4.3}$$

for a k-dimensional subspace $L \subset \mathcal{V}$ ($k \geq 2$) with orthonormal basis $\{U_1, \ldots, U_k\}$,

$$\delta_C^{\mathcal{V}}(r-1) = \frac{1}{2}C^{\mathcal{V}} + \frac{r+1}{2r} \inf\{C(L) : L \text{ is a hyperplane of } \mathcal{V}\}, \tag{4.4}$$

$$\hat{\delta}_C^{\mathcal{V}}(r-1) = 2C^{\mathcal{V}} - \frac{2r-1}{2r} \sup\{C(L) : L \text{ is a hyperplane of } \mathcal{V}\}. \tag{4.5}$$

The fiber scalar curvature $\tau^{\mathcal{V}}$, computed from the curvature tensor $\hat{R}$ of the fibers, and the ambient scalar curvature $\tau_{M_1}^{\mathcal{V}}$, computed from R^{M_1}, have normalized forms

$$\rho^{\mathcal{V}} = \frac{2\tau^{\mathcal{V}}}{r(r-1)}, \qquad \rho_{M_1}^{\mathcal{V}} = \frac{2\tau_{M_1}^{\mathcal{V}}}{r(r-1)}. \tag{4.6}$$

The equality condition for the case of vertical distribution is characterized by invariant quasi-umbilicity of the fibers: There exist orthonormal bases such that $T_{ij}^{\alpha} = 0$ for $i \neq j$ and $T_{11}^{\alpha} = \cdots = T_{r-1,r-1}^{\alpha} = \frac{1}{2}T_{rr}^{\alpha}$ for all α, in complete analogy with Definition 4.1.

The equality cases of our vertical inequalities are characterized by the following notion, which generalizes classical quasi-umbilical submanifolds to the fibers of a submersion.

Definition 4.1. The fibers of a Riemannian submersion $F : (M_1, g_1) \to (M_2, g_2)$ are called invariantly quasi-umbilical at $p \in M_1$ if there exist orthonormal bases $\{U_1, \ldots, U_r\}$ of $\mathcal{V} = \ker F_{*p}$ and $\{U_{r+1}, \ldots, U_{m_1}\}$ of $\mathcal{H} = (\ker F_{*p})^{\perp}$ such that the components $T_{ij}^{\alpha} = g_1(T_{U_i}U_j, U_\alpha)$ of O'Neill's tensor satisfy

(i) $T_{ij}^{\alpha} = 0$ for all $i \neq j$ and all $\alpha = r+1, \ldots, m_1$;

(ii) $T_{11}^{\alpha} = T_{22}^{\alpha} = \cdots = T_{r-1,r-1}^{\alpha}$ for all $\alpha = r+1, \ldots, m_1$.

Remark 4.2. Condition (i) ensures that T is diagonal in suitable bases, while condition (ii) requires the first $r-1$ principal components to be equal. The equality case in Theorem 4.1 below corresponds to the sharper requirement

$$T^{\alpha}_{11} = T^{\alpha}_{22} = \cdots = T^{\alpha}_{r-1,r-1} = \frac{1}{2}\,T^{\alpha}_{rr}, \quad T^{\alpha}_{ij} = 0 \text{ for } i \neq j,$$

for all $\alpha = r+1, \ldots, m_1$. We stress that these two notions are distinct: Invariant quasi-umbilicity (Definition 4.1) requires only that the first $r-1$ principal components coincide, $T^{\alpha}_{11} = \cdots = T^{\alpha}_{r-1,r-1}$, with no constraint relating them to T^{α}_{rr}, whereas equality in Theorem 4.1 imposes the additional relation $T^{\alpha}_{r-1,r-1} = \frac{1}{2}T^{\alpha}_{rr}$. Thus every equality configuration is invariantly quasi-umbilical, but not conversely; the equality case is the strictly stronger condition.

4.1. Generalized complex space forms

Our first result for submersions gives an explicit bound when the total space is a generalized complex space form.

Theorem 4.1. *Let $F : (M_1^{m_1}(c_1, c_2), g_1, J_1) \to (M_2^{m_2}, g_2)$ be a pointwise slant Riemannian submersion from a generalized complex space form onto a Riemannian manifold with vertical space of dimension $r \geq 3$ and slant function θ. Then*

$$\rho^{\mathcal{V}} \leq \delta_C^{\mathcal{V}}(r-1) + c_1 + \frac{3c_2}{r(r-1)}\|P^{\mathcal{V}}\|^2 = \delta_C^{\mathcal{V}}(r-1) + c_1 + \frac{3c_2}{r-1}\cos^2\theta \tag{4.7}$$

and

$$\rho^{\mathcal{V}} \leq \hat{\delta}_C^{\mathcal{V}}(r-1) + c_1 + \frac{3c_2}{r(r-1)}\|P^{\mathcal{V}}\|^2 = \hat{\delta}_C^{\mathcal{V}}(r-1) + c_1 + \frac{3c_2}{r-1}\cos^2\theta. \tag{4.8}$$

Equality holds at $p \in M_1$ if and only if, for suitable orthonormal bases, the components of the tensor T satisfy

$$T^{\alpha}_{11} = T^{\alpha}_{22} = \cdots = T^{\alpha}_{r-1,r-1} = \frac{1}{2}T^{\alpha}_{rr}, \quad T^{\alpha}_{ij} = 0 \text{ for } i \neq j,$$

for all $\alpha = r+1, \ldots, m_1$. Geometrically, this means that the fibers of F are invariantly quasi-umbilical in the sense of Definition 4.1.

Proof. Using the curvature formula (2.5) for $M_1(c_1, c_2)$ and an orthonormal basis $\{U_1, \ldots, U_r\}$ of $\ker F_*$, we compute

$$\begin{aligned}
2\tau^{\mathcal{V}}_{M_1} &= \sum_{i,j=1}^{r} g_1(R^{M_1}(U_i, U_j)U_j, U_i) \\
&= \sum_{i,j=1}^{r} c_1\{g_1(U_j, U_j)g_1(U_i, U_i) - g_1(U_i, U_j)g_1(U_j, U_i)\} \\
&\quad + \sum_{i,j=1}^{r} c_2\{g_1(U_i, J_1U_j)g_1(J_1U_j, U_i) - g_1(U_j, J_1U_j)g_1(J_1U_i, U_i) \\
&\quad + 2g_1(U_i, J_1U_j)g_1(J_1U_j, U_i)\}.
\end{aligned}$$

The first sum gives $r(r-1)c_1$ by orthonormality. For the second sum, we use the decomposition (3.1) to write $J_1U_j = P^{\mathcal{V}}U_j + Q^{\mathcal{V}}U_j$, where $P^{\mathcal{V}}U_j \in \ker F_*$ and $Q^{\mathcal{V}}U_j \in (\ker F_*)^{\perp}$. Since J_1 is an almost complex structure, $g_1(U_j, J_1U_j) = 0$, and we obtain

$$\sum_{i,j=1}^{r} c_2\{g_1(U_i, P^{\mathcal{V}}U_j)g_1(P^{\mathcal{V}}U_j, U_i) + 2g_1(U_i, P^{\mathcal{V}}U_j)g_1(P^{\mathcal{V}}U_j, U_i)\} = 3c_2\|P^{\mathcal{V}}\|^2,$$

where $\|P^{\mathcal{V}}\|^2 = \sum_{i,j=1}^{r}(g_1(U_i, P^{\mathcal{V}}U_j))^2$ as in Eq (3.4).

Therefore, $2\tau^{\mathcal{V}}_{M_1} = r(r-1)c_1 + 3c_2\|P^{\mathcal{V}}\|^2$, which gives

$$\rho^{\mathcal{V}}_{M_1} = c_1 + \frac{3c_2}{r(r-1)}\|P^{\mathcal{V}}\|^2.$$

Applying the algebraic optimization of Lemma 2.4 to the vertical Gauss Eq (2.3), we obtain

$$\rho^{\mathcal{V}} \leq \delta^{\mathcal{V}}_C(r-1) + \rho^{\mathcal{V}}_{M_1}.$$

Substituting $\rho^{\mathcal{V}}_{M_1}$ and using $\|P^{\mathcal{V}}\|^2 = r\cos^2\theta$ from Eq (3.4), we obtain the desired inequalities (4.7) and (4.8).

Equality holds if and only if the components of the tensor T satisfy the condition stated in the theorem, which follows from Lemma 2.4 with parameters $\lambda_1 = r$ and $\lambda_2 = \dfrac{r-1}{2}$. □

Corollary 4.1. *Let $F : (M_1^{m_1}, g_1) \to (M_2^{m_2}, g_2)$ be a pointwise slant Riemannian submersion with vertical space of dimension $r \geq 3$ and slant function θ.*

1. *If $M_1(c)$ is a real space form, then*

$$\rho^{\mathcal{V}} \leq \delta^{\mathcal{V}}_C(r-1) + c, \quad \rho^{\mathcal{V}} \leq \hat{\delta}^{\mathcal{V}}_C(r-1) + c. \tag{4.9}$$

2. *If $M_1(c)$ is a complex space form, then*

$$\rho^{\mathcal{V}} \leq \delta^{\mathcal{V}}_C(r-1) + \frac{c}{4}\left(1 + \frac{3}{r-1}\cos^2\theta\right), \tag{4.10}$$

$$\rho^{\mathcal{V}} \leq \hat{\delta}^{\mathcal{V}}_C(r-1) + \frac{c}{4}\left(1 + \frac{3}{r-1}\cos^2\theta\right). \tag{4.11}$$

3. *If $M_1(c)$ is a real Kähler space form, then*

$$\rho^{\mathcal{V}} \leq \delta^{\mathcal{V}}_C(r-1) + \frac{c+3\alpha}{4} + \frac{3(c-\alpha)}{4(r-1)}\cos^2\theta, \tag{4.12}$$

$$\rho^{\mathcal{V}} \leq \hat{\delta}^{\mathcal{V}}_C(r-1) + \frac{c+3\alpha}{4} + \frac{3(c-\alpha)}{4(r-1)}\cos^2\theta. \tag{4.13}$$

The proofs are immediate by substitution.

4.2. Generalized Sasakian space forms

When the total space is a generalized Sasakian space form, we must again account for the Reeb vector field, leading to two cases depending on whether $\xi_1 \in \ker F_*$ or $\xi_1 \in (\ker F_*)^\perp$.

Theorem 4.2. *Let $F : (M_1^{m_1}(c_1, c_2, c_3), \phi_1, \xi_1, \eta_1, g_1) \to (M_2^{m_2}, g_2)$ be a pointwise slant Riemannian submersion from a generalized Sasakian space form onto a Riemannian manifold with vertical space of dimension $r \geq 3$ and slant function θ. Then*

$$\rho^{\mathcal{V}} \leq \begin{cases} \delta_C^{\mathcal{V}}(r-1) + c_1 + \dfrac{3c_2}{r}\cos^2\theta - \dfrac{2}{r}c_3, & \text{if } \xi_1 \in \Gamma(\ker F_*), \\ \delta_C^{\mathcal{V}}(r-1) + c_1 + \dfrac{3c_2}{r-1}\cos^2\theta, & \text{if } \xi_1 \in \Gamma(\ker F_*)^\perp, \end{cases} \tag{4.14}$$

and similarly for $\hat{\delta}_C^{\mathcal{V}}$. Equality holds if and only if the fibers are invariantly quasi-umbilical, which is the same as in Theorem 4.1.

Proof. The computation follows Theorem 4.1 for the c_1, c_2 terms, with the additional c_3 (Reeb) term in (2.6) contributing according to whether ξ_1 is vertical or horizontal. By (3.7), $\|P^{\mathcal{V}}\|^2 = (r-1)\cos^2\theta$ when $\xi_1 \in \Gamma(\ker F_*)$ and $\|P^{\mathcal{V}}\|^2 = r\cos^2\theta$ when $\xi_1 \in \Gamma((\ker F_*)^\perp)$. Hence

$$\rho_{M_1}^{\mathcal{V}} = c_1 + \frac{3c_2}{r(r-1)}(r-1)\cos^2\theta - \frac{2c_3}{r} = c_1 + \frac{3c_2}{r}\cos^2\theta - \frac{2c_3}{r}, \qquad \xi_1 \in \Gamma(\ker F_*),$$

$$\rho_{M_1}^{\mathcal{V}} = c_1 + \frac{3c_2}{r(r-1)}r\cos^2\theta = c_1 + \frac{3c_2}{r-1}\cos^2\theta, \qquad \xi_1 \in \Gamma((\ker F_*)^\perp).$$

Substituting into $\rho^{\mathcal{V}} \leq \delta_C^{\mathcal{V}}(r-1) + \rho_{M_1}^{\mathcal{V}}$ obtains the result. □

As in the complex case, specializing Theorem 4.2 to the standard contact space forms gives the following explicit inequalities.

Corollary 4.2. *Let $F : (M_1^{m_1}, g_1) \to (M_2^{m_2}, g_2)$ be a pointwise slant Riemannian submersion with vertical space of dimension $r \geq 3$ and slant function θ.*

1. *If $M_1(c)$ is a Sasakian space form ($c_1 = \dfrac{c+3}{4}$, $c_2 = c_3 = \dfrac{c-1}{4}$), then*

$$\rho^{\mathcal{V}} \leq \begin{cases} \delta_C^{\mathcal{V}}(r-1) + \dfrac{c+3}{4} + \dfrac{3(c-1)}{4r}\cos^2\theta - \dfrac{c-1}{2r}, & \xi_1 \in \Gamma(\ker F_*), \\ \delta_C^{\mathcal{V}}(r-1) + \dfrac{c+3}{4} + \dfrac{3(c-1)}{4(r-1)}\cos^2\theta, & \xi_1 \in \Gamma((\ker F_*)^\perp), \end{cases}$$

   and similarly for $\hat{\delta}_C^{\mathcal{V}}(r-1)$.

2. *If $M_1(c)$ is a Kenmotsu space form ($c_1 = \dfrac{c-3}{4}$, $c_2 = c_3 = \dfrac{c+1}{4}$), then*

$$\rho^{\mathcal{V}} \leq \begin{cases} \delta_C^{\mathcal{V}}(r-1) + \dfrac{c-3}{4} + \dfrac{3(c+1)}{4r}\cos^2\theta - \dfrac{c+1}{2r}, & \xi_1 \in \Gamma(\ker F_*), \\ \delta_C^{\mathcal{V}}(r-1) + \dfrac{c-3}{4} + \dfrac{3(c+1)}{4(r-1)}\cos^2\theta, & \xi_1 \in \Gamma((\ker F_*)^\perp), \end{cases}$$

   and similarly for $\hat{\delta}_C^{\mathcal{V}}(r-1)$.

3. If $M_1(c)$ is a cosymplectic space form ($c_1 = c_2 = c_3 = \dfrac{c}{4}$), then

$$\rho^{\mathcal{V}} \leq \begin{cases} \delta_C^{\mathcal{V}}(r-1) + \dfrac{c}{4} + \dfrac{3c}{4r}\cos^2\theta - \dfrac{c}{2r}, & \xi_1 \in \Gamma(\ker F_*), \\ \delta_C^{\mathcal{V}}(r-1) + \dfrac{c}{4} + \dfrac{3c}{4(r-1)}\cos^2\theta, & \xi_1 \in \Gamma((\ker F_*)^{\perp}), \end{cases}$$

and similarly for $\hat{\delta}_C^{\mathcal{V}}(r-1)$.

4. If $M_1(c)$ is an almost $C(\alpha)$ space form ($c_1 = \dfrac{c+3\alpha^2}{4}$, $c_2 = c_3 = \dfrac{c-\alpha^2}{4}$), then

$$\rho^{\mathcal{V}} \leq \begin{cases} \delta_C^{\mathcal{V}}(r-1) + \dfrac{c+3\alpha^2}{4} + \dfrac{3(c-\alpha^2)}{4r}\cos^2\theta - \dfrac{c-\alpha^2}{2r}, & \xi_1 \in \Gamma(\ker F_*), \\ \delta_C^{\mathcal{V}}(r-1) + \dfrac{c+3\alpha^2}{4} + \dfrac{3(c-\alpha^2)}{4(r-1)}\cos^2\theta, & \xi_1 \in \Gamma((\ker F_*)^{\perp}), \end{cases}$$

and similarly for $\hat{\delta}_C^{\mathcal{V}}(r-1)$.

In all cases, the equality conditions are the same as in Theorem 4.1.

5. Casorati inequalities for the horizontal distribution

We now turn to the horizontal distribution $\mathcal{H} = (\ker F_*)^{\perp}$, where the geometric picture is quite different. For horizontal distributions, O'Neill's tensor A plays the central role, and the equality condition in the Casorati inequality corresponds to integrability of the horizontal distribution rather than to quasi-umbilicity. Recall from (2.2) that A measures the failure of $\mathcal{H}$ to be integrable; when $A = 0$, the horizontal distribution is integrable by Frobenius' theorem. Throughout, we assume $\dim \mathcal{H} \geq 3$.

Let $\{X_1, \ldots, X_s\}$ be an orthonormal basis of $\mathcal{H}$ at p, where $s = \dim \mathcal{H} \geq 3$. The components of O'Neill's tensor A are

$$A_{ij}^k = g_1(A_{X_i}X_j,\, U_k), \quad i, j = 1, \ldots, s,\ k = 1, \ldots, r, \tag{5.1}$$

with the squared norm

$$\|A\|^2 = \sum_{k=1}^{r}\sum_{i,j=1}^{s}(A_{ij}^k)^2. \tag{5.2}$$

The Casorati curvature $C^{\mathcal{H}}$ and normalized variants $\delta_C^{\mathcal{H}}(s-1)$, $\hat{\delta}_C^{\mathcal{H}}(s-1)$ are defined in the same manner with (4.3)–(4.5), replacing T_{ij}^{α} with A_{ij}^k and r with s. The scalar curvature of the horizontal distribution and its normalized form are

$$\tau^{\mathcal{H}}(p) = \frac{1}{2}\sum_{i,j=1}^{s} g_1(R^{M_1}(X_i, X_j)X_j, X_i), \qquad \rho^{\mathcal{H}} = \frac{2\tau^{\mathcal{H}}}{s(s-1)}, \tag{5.3}$$

where $\tau^{\mathcal{H}}$ and $\rho^{\mathcal{H}}$ are computed from the ambient curvature R^{M_1}. Analogously, we write

$$\tau^{\mathcal{H}}_{\mathcal{H}}(p) = \frac{1}{2}\sum_{i,j=1}^{s} g_1(R^{\mathcal{H}}(X_i, X_j)X_j, X_i), \qquad \rho^{\mathcal{H}}_{\mathcal{H}} = \frac{2\tau^{\mathcal{H}}_{\mathcal{H}}}{s(s-1)},$$

for the intrinsic normalized scalar curvature computed from the curvature tensor $R^{\mathcal{H}}$ of the horizontal distribution itself. Throughout Section 5, a superscript $\mathcal{H}$ refers to the horizontal distribution and the subscript $\mathcal{H}$ indicates that the quantity is intrinsic to it; thus $\rho^{\mathcal{H}}$ is the ambient horizontal scalar curvature and $\rho^{\mathcal{H}}_{\mathcal{H}}$ its intrinsic counterpart. The equality condition for the case of horizontal distribution is $A \equiv 0$, which by Proposition 2.1(iv) is equivalent to complete integrability of $\mathcal{H}$.

5.1. Generalized complex space forms

Theorem 5.1. *Let $F : (M_1^{m_1}(c_1, c_2), g_1, J_1) \to (M_2^{m_2}, g_2)$ be a pointwise slant Riemannian submersion from a generalized complex space form onto a Riemannian manifold with horizontal space of dimension $s \geq 3$ and slant function θ. Then*

$$\rho^{\mathcal{H}}_{\mathcal{H}} \leq \delta^{\mathcal{H}}_C(s-1) + c_1 + \frac{3c_2}{s-1}\cos^2\theta, \tag{5.4}$$

and similarly for $\hat{\delta}^{\mathcal{H}}_C$. Equality holds if and only if the tensor A vanishes identically, i.e., $A_X Y = 0$ for all $X, Y \in \Gamma((\ker F_*)^{\perp})$, which is equivalent to the horizontal distribution being completely integrable.

Proof. Using the curvature formula (2.5) for $M_1(c_1, c_2)$ and an orthonormal basis $\{X_1, \ldots, X_s\}$ of $(\ker F_*)^{\perp}$, the same computation as in Theorem 4.1 with decomposition (3.2) and $g_1(X_j, J_1X_j) = 0$ gives

$$\rho^{\mathcal{H}} = c_1 + \frac{3c_2}{s(s-1)}\|P^{\mathcal{H}}\|^2 = c_1 + \frac{3c_2}{s-1}\cos^2\theta, \tag{5.5}$$

where the second equality uses $\|P^{\mathcal{H}}\|^2 = s\cos^2\theta$ from Eq (3.4). This identity is a direct consequence of the pointwise slant condition on the horizontal distribution: The polarization identity $(P^{\mathcal{H}})^2 = -\cos^2\theta\, I$ holds on the whole of $\mathcal{H} = (\ker F_*)^{\perp}$, and summing $g_1(P^{\mathcal{H}}X_j, P^{\mathcal{H}}X_j) = \cos^2\theta$ over an orthonormal basis $\{X_1, \ldots, X_s\}$ of $\mathcal{H}$ gives $\|P^{\mathcal{H}}\|^2 = s\cos^2\theta$. In the almost Hermitian (complex) case, no vector is annihilated by J_1, so the full horizontal distribution is active and no additional assumption on $\mathcal{H}$ is needed.

Setting $X_1 = X_4 = X_i$ and $X_2 = X_3 = X_j$ in the horizontal Gauss Eq (2.4) and summing over $i, j = 1, \ldots, s$, we obtain

$$2\tau^{\mathcal{H}} = 2\tau^{\mathcal{H}}_{\mathcal{H}} + 3s\,C^{\mathcal{H}} - \|\operatorname{tr} A^{\mathcal{H}}\|^2,$$

where $\operatorname{tr} A^{\mathcal{H}} = \sum_i A_{X_i}X_i$. Since A is skew-symmetric by Proposition 2.1(ii), we have $A_{X_i}X_i = 0$ for all i, so $\operatorname{tr} A^{\mathcal{H}} = 0$ and the equation simplifies to

$$2\tau^{\mathcal{H}}_{\mathcal{H}} = 2\tau^{\mathcal{H}} - 3s\,C^{\mathcal{H}}.$$

Constructing the polynomial $\mathcal{P}^{\mathcal{H}} = \frac{1}{2}s(s-1)C^{\mathcal{H}} + \frac{1}{2}(s^2-1)C(L^{\mathcal{H}}) + 2\tau^{\mathcal{H}} - 2\tau^{\mathcal{H}}_{\mathcal{H}}$ and substituting $2\tau^{\mathcal{H}}_{\mathcal{H}} = 2\tau^{\mathcal{H}} - 3s\,C^{\mathcal{H}}$, a direct computation gives

$$\mathcal{P}^{\mathcal{H}} = \frac{1}{2}s(s+5)\,C^{\mathcal{H}} + \frac{1}{2}(s^2-1)\,C(L^{\mathcal{H}}).$$

Since A is skew-symmetric, $A^k_{ij} = 0$ whenever $i = j$, so both $C^{\mathcal{H}}$ and $C(L^{\mathcal{H}})$ involve only off-diagonal components and are non-negative. Hence $\mathcal{P}^{\mathcal{H}} \geq 0$ with equality if and only if every $A^k_{ij} = 0$. Rearranging and substituting (5.5) gives (5.4).

Equality holds if and only if $\mathcal{P}^{\mathcal{H}} = 0$, which requires $A^k_{ij} = 0$ for all $i \neq j$, i.e., $A \equiv 0$. By Proposition 2.1(iv), this is equivalent to complete integrability of $(\ker F_*)^{\perp}$. □

Corollary 5.1. *Let $F : (M_1^{m_1}, g_1) \to (M_2^{m_2}, g_2)$ be a pointwise slant Riemannian submersion with horizontal space of dimension $s \geq 3$ and slant function θ.*

1. *If $M_1(c)$ is a real space form, then*

$$\rho^{\mathcal{H}}_{\mathcal{H}} \leq \delta^{\mathcal{H}}_C(s-1) + c, \quad \rho^{\mathcal{H}}_{\mathcal{H}} \leq \hat{\delta}^{\mathcal{H}}_C(s-1) + c.$$

2. *If $M_1(c)$ is a complex space form, then*

$$\rho^{\mathcal{H}}_{\mathcal{H}} \leq \delta^{\mathcal{H}}_C(s-1) + \frac{c}{4}\left(1 + \frac{3}{s-1}\cos^2\theta\right),$$

and similarly for $\hat{\delta}^{\mathcal{H}}_C$.

3. *If $M_1(c)$ is a real Kähler space form, then*

$$\rho^{\mathcal{H}}_{\mathcal{H}} \leq \delta^{\mathcal{H}}_C(s-1) + \frac{c+3\alpha}{4} + \frac{3(c-\alpha)}{4(s-1)}\cos^2\theta,$$

and similarly for $\hat{\delta}^{\mathcal{H}}_C$.

5.2. Generalized Sasakian space forms

Finally, for generalized Sasakian space forms:

Theorem 5.2. *Let $F : (M_1^{m_1}(c_1, c_2, c_3), \phi_1, \xi_1, \eta_1, g_1) \to (M_2^{m_2}, g_2)$ be a pointwise slant Riemannian submersion from a generalized Sasakian space form onto a Riemannian manifold with horizontal space of dimension $s \geq 3$ and slant function θ. Then*

$$\rho^{\mathcal{H}}_{\mathcal{H}} \leq \begin{cases} \delta^{\mathcal{H}}_C(s-1) + c_1 + \dfrac{3c_2}{s}\cos^2\theta - \dfrac{2}{s}c_3, & \text{if } \xi_1 \in \Gamma((\ker F_*)^{\perp}), \\ \delta^{\mathcal{H}}_C(s-1) + c_1 + \dfrac{3c_2}{s-1}\cos^2\theta, & \text{if } \xi_1 \in \Gamma(\ker F_*), \end{cases} \tag{5.6}$$

and similarly for $\hat{\delta}^{\mathcal{H}}_C$. Equality holds if and only if the tensor A vanishes identically, i.e., $A_X Y = 0$ for all $X, Y \in \Gamma((\ker F_*)^{\perp})$, which is equivalent to the horizontal distribution being completely integrable.

Proof. The argument parallels Theorem 5.1 for the c_1, c_2 terms and the optimization of the tensor A, with the additional c_3 (Reeb) term in (2.6) contributing exactly as in Theorem 4.2, according to whether ξ_1 is horizontal or vertical. By (3.7), $\|P^{\mathcal{H}}\|^2 = (s-1)\cos^2\theta$ when $\xi_1 \in \Gamma((\ker F_*)^{\perp})$ and $\|P^{\mathcal{H}}\|^2 = s\cos^2\theta$ when $\xi_1 \in \Gamma(\ker F_*)$, so

$$\rho^{\mathcal{H}} = c_1 + \frac{3c_2}{s(s-1)}(s-1)\cos^2\theta - \frac{2c_3}{s} = c_1 + \frac{3c_2}{s}\cos^2\theta - \frac{2c_3}{s}, \qquad \xi_1 \in \Gamma((\ker F_*)^{\perp}),$$

$$\rho^{\mathcal{H}} = c_1 + \frac{3c_2}{s(s-1)}\, s\cos^2\theta = c_1 + \frac{3c_2}{s-1}\cos^2\theta, \qquad \xi_1 \in \Gamma(\ker F_*).$$

Substituting into the horizontal Casorati estimate $\rho^{\mathcal{H}}_{\mathcal{H}} \le \delta^{\mathcal{H}}_C(s-1) + \rho^{\mathcal{H}}$ obtained in Theorem 5.1 yields the two cases; the $A \equiv 0$ equality condition is unchanged from that theorem. □

Specializing Theorem 5.2 to the classical contact space forms yields explicit bounds in each case.

Corollary 5.2. *Let $F : (M_1^{m_1}, g_1) \to (M_2^{m_2}, g_2)$ be a pointwise slant Riemannian submersion with horizontal space of dimension $s \ge 3$ and slant function θ.*

1. *If $M_1(c)$ is a Sasakian space form ($c_1 = \dfrac{c+3}{4}$, $c_2 = c_3 = \dfrac{c-1}{4}$), then*

$$\rho^{\mathcal{H}}_{\mathcal{H}} \le \begin{cases} \delta^{\mathcal{H}}_C(s-1) + \dfrac{c+3}{4} + \dfrac{3(c-1)}{4s}\cos^2\theta - \dfrac{c-1}{2s}, & \xi_1 \in \Gamma((\ker F_*)^{\perp}), \\ \delta^{\mathcal{H}}_C(s-1) + \dfrac{c+3}{4} + \dfrac{3(c-1)}{4(s-1)}\cos^2\theta, & \xi_1 \in \Gamma(\ker F_*), \end{cases}$$

   and similarly for $\hat{\delta}^{\mathcal{H}}_C(s-1)$.

2. *If $M_1(c)$ is a Kenmotsu space form ($c_1 = \dfrac{c-3}{4}$, $c_2 = c_3 = \dfrac{c+1}{4}$), then*

$$\rho^{\mathcal{H}}_{\mathcal{H}} \le \begin{cases} \delta^{\mathcal{H}}_C(s-1) + \dfrac{c-3}{4} + \dfrac{3(c+1)}{4s}\cos^2\theta - \dfrac{c+1}{2s}, & \xi_1 \in \Gamma((\ker F_*)^{\perp}), \\ \delta^{\mathcal{H}}_C(s-1) + \dfrac{c-3}{4} + \dfrac{3(c+1)}{4(s-1)}\cos^2\theta, & \xi_1 \in \Gamma(\ker F_*), \end{cases}$$

   and similarly for $\hat{\delta}^{\mathcal{H}}_C(s-1)$.

3. *If $M_1(c)$ is a cosymplectic space form ($c_1 = c_2 = c_3 = \dfrac{c}{4}$), then*

$$\rho^{\mathcal{H}}_{\mathcal{H}} \le \begin{cases} \delta^{\mathcal{H}}_C(s-1) + \dfrac{c}{4} + \dfrac{3c}{4s}\cos^2\theta - \dfrac{c}{2s}, & \xi_1 \in \Gamma((\ker F_*)^{\perp}), \\ \delta^{\mathcal{H}}_C(s-1) + \dfrac{c}{4} + \dfrac{3c}{4(s-1)}\cos^2\theta, & \xi_1 \in \Gamma(\ker F_*), \end{cases}$$

   and similarly for $\hat{\delta}^{\mathcal{H}}_C(s-1)$.

4. *If $M_1(c)$ is an almost $C(\alpha)$ space form ($c_1 = \dfrac{c+3\alpha^2}{4}$, $c_2 = c_3 = \dfrac{c-\alpha^2}{4}$), then*

$$\rho^{\mathcal{H}}_{\mathcal{H}} \le \begin{cases} \delta^{\mathcal{H}}_C(s-1) + \dfrac{c+3\alpha^2}{4} + \dfrac{3(c-\alpha^2)}{4s}\cos^2\theta - \dfrac{c-\alpha^2}{2s}, & \xi_1 \in \Gamma((\ker F_*)^{\perp}), \\ \delta^{\mathcal{H}}_C(s-1) + \dfrac{c+3\alpha^2}{4} + \dfrac{3(c-\alpha^2)}{4(s-1)}\cos^2\theta, & \xi_1 \in \Gamma(\ker F_*), \end{cases}$$

   and similarly for $\hat{\delta}^{\mathcal{H}}_C(s-1)$.

In all cases, the equality conditions are the same as in Theorem 5.2.

6. Examples

We provide explicit examples illustrating the main theorems and the sharpness of our inequalities. Example 6.1 gives a properly slant Riemannian submersion from a flat cosymplectic total space, Example 6.2 treats a warped-product submersion with integrable horizontal distribution, and Example 6.3 exhibits a genuinely varying slant function.

Example 6.1. We construct a pointwise slant Riemannian submersion from $\mathbb{R}^7$, equipped with a flat cosymplectic structure.

Let $M_1 = \mathbb{R}^7$ with coordinates $(t, x_1, y_1, x_2, y_2, x_3, y_3)$, Reeb vector field $\xi = \partial/\partial t$, and almost contact structure

$$\phi\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \quad \phi\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i}, \quad \phi(\xi) = 0, \qquad i = 1, 2, 3.$$

Fix $\alpha \in (0, \pi/2)$ and define $F : \mathbb{R}^7 \to \mathbb{R}^3$ by

$$F(t, x_1, y_1, x_2, y_2, x_3, y_3) = (\sin\alpha\, y_1 - \cos\alpha\, x_3,\ -\cos\alpha\, x_1 + \sin\alpha\, y_2,\ -\cos\alpha\, x_2 + \sin\alpha\, y_3). \tag{6.1}$$

The pushforward is

$$\begin{aligned}
F_*\left(\frac{\partial}{\partial t}\right) &= 0, & F_*\left(\frac{\partial}{\partial x_1}\right) &= -\cos\alpha\, e_2,\\
F_*\left(\frac{\partial}{\partial y_1}\right) &= \sin\alpha\, e_1, & F_*\left(\frac{\partial}{\partial x_2}\right) &= -\cos\alpha\, e_3,\\
F_*\left(\frac{\partial}{\partial y_2}\right) &= \sin\alpha\, e_2, & F_*\left(\frac{\partial}{\partial x_3}\right) &= -\cos\alpha\, e_1,\\
F_*\left(\frac{\partial}{\partial y_3}\right) &= \sin\alpha\, e_3,
\end{aligned}$$

where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$. The kernel is $\ker F_* = \operatorname{span}\{\xi,\ V_1,\ V_2,\ V_3\}$, where

$$V_1 = \cos\alpha\,\frac{\partial}{\partial y_1} + \sin\alpha\,\frac{\partial}{\partial x_3}, \quad V_2 = \cos\alpha\,\frac{\partial}{\partial y_2} + \sin\alpha\,\frac{\partial}{\partial x_1}, \quad V_3 = \cos\alpha\,\frac{\partial}{\partial y_3} + \sin\alpha\,\frac{\partial}{\partial x_2}. \tag{6.2}$$

One verifies $g_1(\xi, V_i) = 0$ and $g_1(V_i, V_j) = \delta_{ij}$, so $\dim\ker F_* = r = 4$. The horizontal space is $(\ker F_*)^\perp = \operatorname{span}\{H_1,\ H_2,\ H_3\}$, where

$$H_1 = \sin\alpha\,\frac{\partial}{\partial y_1} - \cos\alpha\,\frac{\partial}{\partial x_3}, \quad H_2 = -\cos\alpha\,\frac{\partial}{\partial x_1} + \sin\alpha\,\frac{\partial}{\partial y_2}, \quad H_3 = -\cos\alpha\,\frac{\partial}{\partial x_2} + \sin\alpha\,\frac{\partial}{\partial y_3}. \tag{6.3}$$

One checks $g_1(H_i, H_j) = \delta_{ij}$ and $F_*(H_i) = e_i$, and confirms that $F^* g_{\mathbb{R}^3} = g_1|_{(\ker F_*)^\perp}$, so F is a Riemannian submersion.

Since $\phi(\xi) = 0$, we compute the slant angle on $\mathcal{V}' = \operatorname{span}\{V_1, V_2, V_3\}$. We have

$$\phi(V_1) = -\cos\alpha\,\frac{\partial}{\partial x_1} + \sin\alpha\,\frac{\partial}{\partial y_3}, \quad \phi(V_2) = -\cos\alpha\,\frac{\partial}{\partial x_2} + \sin\alpha\,\frac{\partial}{\partial y_1}, \quad \phi(V_3) = -\cos\alpha\,\frac{\partial}{\partial x_3} + \sin\alpha\,\frac{\partial}{\partial y_2}.$$

Projecting $\phi(V_1)$ onto $\ker F_*$ and using $\phi(V_1) \perp \xi$:

$$
\begin{aligned}
&g_1(\phi(V_1), V_1) = 0,\\
&g_1(\phi(V_1), V_2) = g_1\left(-\cos\alpha\,\frac{\partial}{\partial x_1} + \sin\alpha\,\frac{\partial}{\partial y_3},\ \cos\alpha\,\frac{\partial}{\partial y_2} + \sin\alpha\,\frac{\partial}{\partial x_1}\right) = -\sin\alpha\cos\alpha,\\
&g_1(\phi(V_1), V_3) = g_1\left(-\cos\alpha\,\frac{\partial}{\partial x_1} + \sin\alpha\,\frac{\partial}{\partial y_3},\ \cos\alpha\,\frac{\partial}{\partial y_3} + \sin\alpha\,\frac{\partial}{\partial x_2}\right) = \sin\alpha\cos\alpha.
\end{aligned}
$$

Hence

$$(\phi V_1)^{\ker F_*} = \sin\alpha\cos\alpha(-V_2 + V_3)$$

and

$$\|(\phi V_1)^{\ker F_*}\|^2 = 2\sin^2\alpha\cos^2\alpha.$$

By the cyclic symmetry of (6.2),

$$\|(\phi V_i)^{\ker F_*}\|^2 = 2\sin^2\alpha\cos^2\alpha \quad \text{for } i = 1, 2, 3.$$

Moreover, the slant condition holds for an *arbitrary* vertical vector, not only for the basis vectors. For $V = \sum_{i=1}^{3} a_i V_i$ with $\sum_i a_i^2 = 1$, linearity of ϕ together with the cyclic structure of (6.2) gives $\|(\phi V)^{\ker F_*}\|^2 = 2\sin^2\alpha\cos^2\alpha$, independent of the coefficients a_i. Hence the angle between ϕV and $\ker F_*$ is the same for every nonzero $V \in \operatorname{span}\{V_1, V_2, V_3\}$, so F is pointwise slant in the sense of Definition 3.3. Since $\|\phi(V_i)\| = 1$, the slant angle θ satisfies

$$\cos^2\theta = 2\sin^2\alpha\cos^2\alpha = \frac{1}{2}\sin^2 2\alpha. \tag{6.4}$$

For $\alpha \in (0, \pi/2)$, we have $\cos^2\theta \in (0, \frac{1}{2}]$, so $\theta \in [\pi/4, \pi/2)$. Thus F is a properly pointwise slant Riemannian submersion with constant slant angle $\theta \in [\pi/4, \pi/2)$.

With the Euclidean metric and $\eta = dt$, one has $d\eta = 0$, so $(\mathbb{R}^7, \phi, \xi, \eta, g)$ is a flat cosymplectic $(C(0))$ space form, a special case of a generalized Sasakian space form with $c_1 = c_2 = c_3 = 0$. Theorem 4.2 then gives

$$\rho^{\mathcal{V}} \le \delta_C^{\mathcal{V}}(r-1) + c_1 + \frac{3c_2}{r(r-1)}\|P^{\mathcal{V}}\|^2 = \delta_C^{\mathcal{V}}(3).$$

Since ξ, V_1, V_2, V_3 all have constant coefficients with respect to the Euclidean metric, every Christoffel symbol vanishes, so $\nabla^{\mathbb{R}^7}_{V_i} V_j = 0$. Hence $T_{V_i} V_j = 0$, and therefore $T^{\alpha}_{ij} = 0$ for all i, j, α; the fibers are totally geodesic and in particular invariantly quasi-umbilical. As the metric is flat, $\rho^{\mathcal{V}} = 0$ and $\delta_C^{\mathcal{V}}(3) = 0$, so both sides vanish and equality holds, confirming the sharpness of the vertical inequality.

Example 6.2. Consider the warped product $M_1 = \mathbb{R} \times_f \mathbb{C}^n$ with metric

$$g_1 = dt^2 + f^2(t)\, g_{\mathbb{C}^n},$$

where $f : \mathbb{R} \to (0, \infty)$ is smooth. By [1], M_1 is a generalized Sasakian space form with

$$c_1 = -\frac{f''}{f}, \quad c_2 = 0, \quad c_3 = -\frac{f''}{f} + \frac{(f')^2}{f^2}.$$

Define $F : M_1 \to (\mathbb{C}^n, f^2 g_{\mathbb{C}^n})$ by projection onto the $\mathbb{C}^n$ factor: $F(t, z_1, \ldots, z_n) = (z_1, \ldots, z_n)$, where the target carries the leaf metric $f^2(t)\, g_{\mathbb{C}^n}$. Then $\ker F_* = \operatorname{span}\{\partial/\partial t\}$, the Reeb vector field $\xi = \partial/\partial t$ is vertical, and the horizontal distribution $\mathcal{H}$ has dimension $s = 2n \geq 3$ for $n \geq 2$. For horizontal X, Y, one has $g_2(F_*X, F_*Y) = f^2(t)\, g_{\mathbb{C}^n}(X, Y) = g_1(X, Y)$, so F_* restricts to an isometry on $\mathcal{H}$ and F is a Riemannian submersion. Since ϕ preserves $\mathcal{H}$, the slant angle is $\theta = 0$ (invariant case).

For warped products, O'Neill's tensor $A \equiv 0$ since the horizontal leaves $\{t\} \times \mathbb{C}^n$ are totally geodesic product factors [12]. Hence equality holds in Theorem 5.2:

$$\rho_{\mathcal{H}}^{\mathcal{H}} = \delta_C^{\mathcal{H}}(2n-1) + \rho^{\mathcal{H}}.$$

For the explicit choice $f(t) = e^t$: $c_1 = -1$, $c_2 = c_3 = 0$, and since $\xi \in \ker F_*$, Theorem 5.2 gives

$$\rho_{\mathcal{H}}^{\mathcal{H}} = \delta_C^{\mathcal{H}}(2n-1) - 1,$$

which ensures the sharpness of the horizontal inequality.

Example 6.3. Let $M_1 = \mathbb{R}^7$ carry the Kenmotsu structure with coordinates $(t, x_1, y_1, x_2, y_2, x_3, y_3)$, metric

$$g = dt^2 + e^{2t} \sum_{i=1}^{3} (dx_i^2 + dy_i^2),$$

Reeb vector field $\xi = \partial/\partial t$, and almost contact structure

$$\phi\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \quad \phi\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i}, \quad \phi(\xi) = 0.$$

This is a Kenmotsu space form with $c = -1$, giving $c_1 = -1$, $c_2 = c_3 = 0$.

Define $F_\psi : M_1 \to \mathbb{R}^3$ by

$$F_\psi(t, x_i, y_i) = e^t(\sin\psi(t)\, y_1 - \cos\psi(t)\, x_3,\ -\cos\psi(t)\, x_1 + \sin\psi(t)\, y_2,\ -\cos\psi(t)\, x_2 + \sin\psi(t)\, y_3),$$

where $\psi : \mathbb{R} \to (0, \pi/2)$ is smooth. The factor e^t compensates for the warped metric. When $\psi \equiv \alpha$ is constant, F_ψ reduces to the construction of Example 6.1 with $\dim\ker(F_\psi)_* = 4$ and $\cos^2\theta = \frac{1}{2}\sin^2 2\alpha$. A direct computation gives $(F_\psi)_*(\partial/\partial t) = 0$, so $\xi = \partial/\partial t \in \ker(F_\psi)_*$ is vertical. At $\psi \equiv \alpha$, the restriction $(F_\psi)_*|_{\mathcal{H}}$ is a linear isometry onto its image (Example 6.1); the rank and isometry condition are governed by the Gram matrix of $(F_\psi)_*$ on $\{\partial_{x_i}, \partial_{y_i}\}$, whose entries depend continuously on $\psi(t)$ and $\psi'(t)$. For non-constant ψ with $\|\psi'\|_{C^0} < \varepsilon$ (for sufficiently small $\varepsilon > 0$), this Gram matrix is a small perturbation of its value at $\psi' \equiv 0$ and hence remains nondegenerate; thus F_ψ remains a Riemannian submersion with $\dim\ker(F_\psi)_* = 4$, and the slant angle varies as

$$\cos^2\theta(t) = \frac{1}{2}\sin^2 2\psi(t), \quad t \in \mathbb{R}.$$

Since $c_2 = 0$, Corollary 4.2 (Kenmotsu case, $\xi \in \ker(F_\psi)_*$) gives

$$\rho^{\mathcal{V}} \leq \delta_C^{\mathcal{V}}(3) - 1,$$

independently of $\theta(t)$. This example illustrates Corollary 4.2 in the Kenmotsu case and exhibits a genuinely varying slant function $\theta(t)$; since $c_2 = 0$, the bound does not depend on $\theta(t)$, and the example does not realize the equality case (sharpness being demonstrated by Examples 6.1 and 6.2).

Remark 6.1. Examples 6.1 and 6.2 exhibit equality, confirming the sharpness of the vertical and horizontal inequalities, respectively: Example 6.1 provides a properly slant Riemannian submersion with $\theta \in [\pi/4, \pi/2)$ and vertical dimension $r = 4 \geq 3$, while Example 6.2 attains equality on the horizontal side. Example 6.3 illustrates a genuinely varying slant function $\theta(t)$.

7. Conclusions

We have established optimal Casorati inequalities for pointwise slant Riemannian submersions from generalized complex and generalized Sasakian space forms, treating the vertical and horizontal distributions separately, and characterizing equality by invariant quasi-umbilicity of the fibers and by integrability of the horizontal distribution, respectively. The slant function enters solely through $\cos^2\theta$, so the invariant and anti-invariant cases arise as the limits $\theta = 0$ and $\theta = \pi/2$.

Authors contributions

All authors have equal contribution and have agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflict of interest in this paper.

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