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Research article

Linearized asymptotic stability for Caputo-tempered fractional differential equations

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Abstract: We developed Lyapunov's first method for nonlinear fractional differential equations with the Caputo-tempered derivative of order $\alpha \in (0, 1)$ and tempering parameter $\lambda \geq 0$. If every eigenvalue μ of the linearization at the zero equilibrium satisfies $\left|\arg \mu\right| > \alpha\pi/2$, we proved that the trivial solution of the nonlinear tempered equation is asymptotically stable, with an explicit local envelope $\|x(t)\| \leq Me^{-\lambda t} \|x(0)\|$ and convergence of the weighted state $e^{\lambda t}x(t)$ to zero. The proof rests on a tempered Lyapunov-Perron operator, built from the tempered variation of constants formula, which is a contraction on a small ball of an exponentially weighted space in which the tempering factors cancel exactly; for $\lambda = 0$, the argument recovers the classical linearization theorem for Caputo equations. Using subcritical exponential weights, we then extended the result to inner-sector eigenvalues with $\mathrm{Re}(\mu^{1/\alpha}) < \lambda$; together, the two theorems show that, locally and under a smallness condition on the nonlinearity, the trivial solution is asymptotically stable whenever the spectrum lies in the open region of the complex plane in which the linear tempered modes decay. As an application, the zero equilibrium of a tempered fractional Lotka-Volterra model with intrinsic growth rate h was shown to be unstable for $h > \lambda^\alpha$, asymptotically stable for $h < \lambda^\alpha$, and stabilizable by linear state feedback for every h; the critical case $h = \lambda^\alpha$ was left open. Numerical experiments with a validated predictor-corrector scheme were consistent with the guaranteed decay rates.

Keywords: tempered fractional derivative; linearized stability; Mittag-Leffler function; Lyapunov-Perron operator; asymptotic stability; feedback stabilization
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1. Introduction

Fractional differential equations are widely used to model processes with memory and hereditary effects [1–3], and high-order numerical methods have been developed for fractional diffusion-wave,

Burgers-type, and convection-diffusion-reaction models [4–6]. Tempered fractional operators multiply the power-law kernel by an exponential factor $e^{-\lambda(t-s)}$, $\lambda \geq 0$, so that the memory fades after a characteristic time [7, 8]; they appear, for instance, in tempered fractional Brownian motion [9], turbulence modeling [10], and anomalous sub-diffusion [11]. For the basic theory and system-theoretic properties of tempered fractional differential equations, we refer to [12, 13].

Two classical routes lead to stability statements for such systems. Lyapunov's *second* (direct) method and integral-inequality techniques construct energy-like functionals or Gronwall-type bounds; they are the tools used in most of the tempered literature: see [14] for tempered Mittag-Leffler stability, [15] for weakly singular inequalities and Ulam-type stability, [16] for finite-time stability with delays, [17] for tempered ρ-Caputo systems, [18–20] for practical stabilization, and [21] for practical stability, a notion in which the state is only guaranteed to converge to a small neighborhood of the equilibrium, in contrast to the asymptotic stability studied here; related theories for other generalized operators appear in [22–24].

Lyapunov's *first* (indirect) method, in contrast, decides the local stability of an equilibrium from the spectrum of the linearization alone. If the right-hand side F of an autonomous equation is differentiable at the zero equilibrium with $F(0) = 0$, one writes $F(x) = Ax+f(x)$ with $A = DF(0)$ and $f(x) = F(x)-Ax$, so that f vanishes to first order at the origin; the question is then whether the spectrum of A decides the local behavior of the nonlinear equation. For classical Caputo equations of order $\alpha \in (0, 1)$, the corresponding linearization theorem was proved by Cong, Doan, Siegmund, and Tuan [25]: if the spectrum of the linearization lies in the sector $\{\mu \in \mathbb{C} : \left|\arg \mu\right| > \alpha\pi/2\}$, then the trivial solution of the nonlinear equation is asymptotically stable.

Three differences between the classical and the tempered setting should be kept in mind. First, for $\lambda > 0$, the linear tempered modes associated with eigenvalues in the sector $\left|\arg \mu\right| > \alpha\pi/2$ decay exponentially, at least at the rate λ, whereas for $\lambda = 0$, the decay is only algebraic, of order $t^{-\alpha}$; exponential stability with an explicit envelope, rather than asymptotic stability, is therefore the natural conclusion in the tempered case. Second, the region of the complex plane in which the linear tempered modes decay is strictly larger than the classical sector: eigenvalues inside the sector, including positive real ones, are admissible as long as $\mathrm{Re}(\mu^{1/\alpha}) < \lambda$. Third, the two parts of this region are described by two different kinds of spectral conditions, an angular condition on $\arg \mu$ in the classical sector and a condition on $\mathrm{Re}(\mu^{1/\alpha})$, the growth exponent of the Mittag-Leffler function, in the inner sector. We are not aware of a linearization theorem of this type for the Caputo tempered derivative; the present paper provides one and, in doing so, also covers the inner part of the tempered stability region.

In this paper, we consider the nonlinear tempered Caputo equation

$$ {}^{C}D^{\alpha,\lambda}_{0+}x(t) = Ax(t) + f(x(t)), \qquad t > 0, \tag{1.1} $$

with $\alpha \in (0, 1)$, $\lambda \geq 0$, $A \in \mathbb{R}^{d\times d}$, and a locally Lipschitz continuous nonlinearity f vanishing at the origin together with its Lipschitz constant. The conjugation $y(t) = e^{\lambda t}x(t)$ turns (1.1) into the nonautonomous Caputo equation ${}^{C}D^{\alpha}_{0+}y = Ay + e^{\lambda t}f(e^{-\lambda t}y)$, and our results can be read as a Lyapunov-Perron analysis, in the spirit of [25], of this conjugated problem in exponentially weighted norms $\sup_{t\geq 0} e^{\vartheta t}\,\|\cdot\|$; the precise definition of the operator ${}^{C}D^{\alpha,\lambda}_{0+}$, the rigorous justification of this conjugation, and its consequences for the nonlinear problem are given in Definition 2.3, Lemma 2.5, and Remark 2.1. Our contributions are the following.

- Under the sector condition $\left|\arg \mu\right| > \alpha\pi/2$ for all eigenvalues μ of A, the trivial solution of (1.1)

is asymptotically stable, with the explicit local estimate $\|x(t)\| \le Me^{-\lambda t}\|x(0)\|$ and the weighted convergence $e^{\lambda t}x(t) \to 0$ (Theorem 3.1); for $\lambda = 0$, this recovers [25]. The new ingredients with respect to [25] are the tempered Lyapunov-Perron operator (3.8), built from the tempered variation of constants formula, and the exponentially weighted space C_λ of (3.7), in which the tempering factors of this operator cancel exactly, so that the classical Mittag-Leffler estimates can be used without change. Here $e^{\lambda t}x(t) \to 0$ means $\|x(t)\| = o(e^{-\lambda t})$, and the improvement over the envelope is generically only the algebraic factor $t^{-\alpha}$.

- Theorem 3.2 goes beyond the classical sector, a situation without a counterpart for $\lambda = 0$: it admits inner-sector eigenvalues with $\mathrm{Re}(\mu^{1/\alpha}) < \lambda$ and yields the envelope $e^{-\beta t}$ for every rate $\beta < \lambda - \mathrm{Re}(\mu^{1/\alpha})$, but not necessarily for the limiting value itself; together, the two theorems cover, locally for the nonlinear equation and under the smallness condition (3.2) on f, the full open asymptotic-stability region of the tempered linear part (Corollary 3.3, Figure 1, and Remark 3.6), whose outer boundary is excluded because the corresponding linear modes are not attractive (Remark 3.7). The key new ingredient is a set of exponentially damped Mittag-Leffler kernel estimates outside the sector (Lemma 3.5).

- For a tempered Lotka-Volterra system, we obtain an essentially complete picture of the zero equilibrium: instability for $h > \lambda^\alpha$, asymptotic stability for $h < \lambda^\alpha$, and stabilization by linear state feedback for every h (Proposition 3.2); the critical case $h = \lambda^\alpha$, in which the linearization is not attractive, is not covered.

- The predictions are illustrated numerically with a predictor-corrector scheme [26] applied to the conjugated problem, including a mesh-refinement study against the exact Mittag-Leffler solution, a step-size sensitivity test on the long fitting windows, and experiments for several orders α; observed decay slopes are consistently distinguished from guaranteed rates.

The paper is organized as follows: Section 2 collects the definitions and known results that we use, together with a discussion of the stability terminology. Section 3 contains the two linearization theorems, their complete proofs, a description of the spectral stability region, and the application to the tempered Lotka-Volterra system. Section 4 reports the numerical simulations, including the pseudocode of the scheme. Section 5 concludes the work. Proofs of auxiliary results are collected in Appendix A.

Throughout the paper, $\mathbb{R}_{\ge 0}$ denotes the set of nonnegative real numbers, and $\mathbb{C}^d$ is equipped with the maximum norm $\|x\| = \max(|x_1|, \dots, |x_d|)$; matrix norms are the induced operator norms. The argument of a nonzero complex number is taken in $(-\pi, \pi]$, and for $\mu \in \mathbb{C} \setminus \{0\}$ and $\alpha \in (0, 1)$, the power $\mu^{1/\alpha}$ always denotes the principal branch

$$\mu^{1/\alpha} := |\mu|^{1/\alpha} e^{i \arg(\mu)/\alpha}, \qquad \text{so that} \qquad \mathrm{Re}(\mu^{1/\alpha}) = |\mu|^{1/\alpha} \cos(\arg(\mu)/\alpha). \tag{1.2}$$

For $T > 0$, $C([0, T], \mathbb{R}^d)$ is the space of continuous functions from $[0, T]$ to $\mathbb{R}^d$, and

$$C_\infty(\mathbb{R}_{\ge 0}, \mathbb{C}^d) := \Big\{\xi : \mathbb{R}_{\ge 0} \to \mathbb{C}^d \text{ continuous} : \sup_{t \ge 0} \|\xi(t)\| < \infty\Big\}, \qquad \|\xi\|_\infty := \sup_{t \ge 0} \|\xi(t)\|.$$

The Euler Gamma function is denoted by Γ, and $E_{\beta,\gamma}$ denotes the two-parameter Mittag-Leffler function of Definition 2.4, with the convention $E_\beta := E_{\beta,1}$; only the functions E_α and $E_{\alpha,\alpha}$, with the order α of (1.1), enter the solution formulas, whereas $E_{1,\theta}$ with $\theta > 0$ appears in the auxiliary Lemma 2.2, in Remark 2.2, and in the proof of Lemma 3.5.

2. Preliminaries

This section collects the definitions and known results used in the paper. Throughout, $\alpha \in (0,1)$ is the fractional order and $\lambda \geq 0$ is the tempering parameter. We first recall the tempered fractional operators, following [7, 8, 15].

Definition 2.1 ([8, 15]). *Let $u \in L^1([0,T],\mathbb{R})$ and $\beta > 0$. The Riemann-Liouville tempered fractional integral of order β of u is*

$$I_{0+}^{\beta,\lambda}u(t) := e^{-\lambda t}\, I_{0+}^{\beta}(e^{\lambda t}u(t)) = \frac{1}{\Gamma(\beta)}\int_0^t e^{-\lambda(t-s)}(t-s)^{\beta-1}u(s)\,ds,$$

where I_{0+}^{β} denotes the classical Riemann-Liouville integral of order β [1].

Definition 2.2 ([8, 15]). *The Riemann-Liouville tempered fractional derivative of order α of u is*

$$D_{0+}^{\alpha,\lambda}u(t) := e^{-\lambda t}\, D_{0+}^{\alpha}(e^{\lambda t}u(t)) = \frac{e^{-\lambda t}}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}e^{\lambda s}u(s)\,ds,$$

where D_{0+}^{α} denotes the classical Riemann-Liouville derivative of order α [1].

Definition 2.3 ([8, 15]). *The Caputo tempered fractional derivative of order α of u is*

$${}^{C}D_{0+}^{\alpha,\lambda}u(t) := D_{0+}^{\alpha,\lambda}(u(t) - e^{-\lambda t}u(0)).$$

If $t \mapsto e^{\lambda t}u(t)$ is absolutely continuous, then

$${}^{C}D_{0+}^{\alpha,\lambda}u(t) = \frac{e^{-\lambda t}}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{d}{ds}(e^{\lambda s}u(s))\,ds.$$

For a vector-valued function $u = (u_1,\ldots,u_d)^T$, the operator acts component-wise. For $\lambda = 0$, all three operators reduce to the classical Riemann-Liouville and Caputo operators [1, 2].

Definition 2.4 ([1, 2]). *The two-parameter Mittag-Leffler function is*

$$E_{\beta,\gamma}(z) := \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\beta+\gamma)}, \qquad z \in \mathbb{C},\ \beta > 0,\ \gamma > 0,$$

and $E_\beta := E_{\beta,1}$. Both are entire functions of z.

The next four lemmas, concerning the functions of Definition 2.4, are stated without proof; they are known results and are used exactly in the cited form.

Lemma 2.1 ([1]). *Let $\beta \in (0,2]$ and $\gamma > 0$. Then,*

$$E_{\beta,\gamma}(x^\beta) \sim \frac{1}{\beta}\,x^{1-\gamma}e^{x}, \qquad x \to +\infty.$$

Lemma 2.2 ([1]). *For every $\theta > 0$, $\rho \in \mathbb{R}$, and $t \geq 0$,*

$$\frac{1}{\Gamma(\theta)}\int_0^t (t-w)^{\theta-1}e^{\rho w}\,dw = t^{\theta}E_{1,\theta+1}(\rho t).$$

Lemma 2.3 ([14]). *Let $\mu \in \mathbb{C}$ and let $g : [0, T) \to \mathbb{C}^k$ be continuous, $0 < T \le \infty$. The solution of*

$$ {}^{C}D^{\alpha,\lambda}_{0+} v(t) = \mu\, v(t) + g(t), \qquad t \in (0, T), $$

is given by

$$ v(t) = e^{-\lambda t} E_\alpha(\mu t^\alpha) v(0) + \int_0^t (t-w)^{\alpha-1} E_{\alpha,\alpha}(\mu (t-w)^\alpha) e^{-\lambda(t-w)} g(w)\, dw. $$

Lemma 2.4 ([27]). *Let $\beta \in (0, 1]$ and $\gamma \ge \beta$. Then $s \mapsto E_{\beta,\gamma}(-s)$ is completely monotone on $(0, \infty)$; in particular, $s \mapsto E_\beta(-s)$ is nonincreasing and nonnegative on $[0, \infty)$. Consequently $E_{\beta,\gamma}(x) \ge 0$ for every $x \in \mathbb{R}$ because for $x \ge 0$, all terms of the defining series are nonnegative.*

Lemma 2.5 ([8, 14]). *Let $u : [0, T) \to \mathbb{C}^d$ be continuous and set $y(t) := e^{\lambda t} u(t)$. Then ${}^{C}D^{\alpha,\lambda}_{0+} u(t)$ exists for some $t \in (0, T)$ if and only if the classical Caputo derivative ${}^{C}D^{\alpha}_{0+} y(t)$ exists, and in this case,*

$$ {}^{C}D^{\alpha,\lambda}_{0+} u(t) = e^{-\lambda t}\, {}^{C}D^{\alpha}_{0+} y(t). $$

Proof. Set $w := u - e^{-\lambda t} u(0)$, so that $e^{\lambda t} w = y - y(0)$. By Definitions 2.2, and 2.3 and the identity ${}^{C}D^{\alpha}_{0+} y = D^{\alpha}_{0+}(y - y(0))$ [2, 3],

$$ {}^{C}D^{\alpha,\lambda}_{0+} u(t) = D^{\alpha,\lambda}_{0+} w(t) = e^{-\lambda t}\, D^{\alpha}_{0+}(y - y(0))(t) = e^{-\lambda t}\, {}^{C}D^{\alpha}_{0+} y(t), $$

each side existing precisely when the other does. □

We now fix the notion of solution. Consider the initial value problem

$$ {}^{C}D^{\alpha,\lambda}_{0+} x(t) = F(t, x(t)), \qquad x(0) = x_0, \tag{2.1} $$

where $F : [0, T_0) \times \mathbb{C}^d \to \mathbb{C}^d$, with $0 < T_0 \le \infty$, is continuous and locally Lipschitz continuous in the state, uniformly on compact time intervals: for all $T \in (0, T_0)$ and $R > 0$, there is $L(T, R) \ge 0$ such that $\|F(t, z_1) - F(t, z_2)\| \le L(T, R)\, \|z_1 - z_2\|$ for all $t \in [0, T]$ and $z_1, z_2 \in \bar{B}(0, R)$. Autonomous right-hand sides $F(x)$ with F locally Lipschitz continuous are included, with $T_0 = \infty$. Following the mild formulation customary for tempered Caputo problems [8, 15], a *solution* of (2.1) on $[0, T)$, $0 < T \le T_0$, is a continuous function $x : [0, T) \to \mathbb{C}^d$ satisfying the integral equation

$$ x(t) = e^{-\lambda t} x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t e^{-\lambda(t-s)} (t-s)^{\alpha-1} F(s, x(s))\, ds, \qquad t \in [0, T). \tag{2.2} $$

Note that the integral term is exactly the tempered integral $I^{\alpha,\lambda}_{0+}$ of Definition 2.1 applied to $F(\cdot, x(\cdot))$. The next proposition, which is the tempered form of classical Caputo well-posedness theory, shows that this concept agrees with the pointwise differential formulation and collects the facts used later.

Proposition 2.1 ([3, 8, 15]). *Let F be as above. Then the following statements hold.*

(i) A continuous function $x : [0, T) \to \mathbb{C}^d$ with $x(0) = x_0$ and $T \le T_0$ satisfies (2.2) if and only if the derivative ${}^{C}D^{\alpha,\lambda}_{0+} x(t)$ exists and equals $F(t, x(t))$ for every $t \in (0, T)$. In this case, the natural fractional regularity holds automatically: $e^{\lambda \cdot} x(\cdot) - x_0 = I^{\alpha}_{0+}[e^{\lambda s} F(s, x(s))]$ belongs to $I^{\alpha}_{0+}(C([0, T), \mathbb{C}^d))$, so the derivative in Definition 2.3 is applied within its proper domain.

(ii) Problem (2.1) *has a unique solution* $\varphi(\cdot, x_0)$ *on a maximal interval of existence* $[0, t_{\max}(x_0))$ *with* $0 < t_{\max}(x_0) \le T_0$*, and any two solutions of* (2.1) *coincide on the intersection of their intervals of existence.*

Proof. Set $y := e^{\lambda t}x$ and $g(t, y) := e^{\lambda t}F(t, e^{-\lambda t}y)$; then g inherits the Lipschitz hypothesis of F with the same constants, and multiplying (2.2) by $e^{\lambda t}$ shows that x satisfies (2.2) if and only if y satisfies the classical Volterra equation $y = x_0 + I^{\alpha}_{0+}[g(\cdot, y(\cdot))]$. For the classical Caputo problem, the equivalence with the pointwise formulation, local existence and uniqueness, and maximal continuation are well–known [3, Lemma 6.2, Theorems 6.5 and 6.8]; [2, Chapter 3]. Lemma 2.5 transports all statements to the tempered problem; see also [8, 15]. □

Remark 2.1 (Consequences of the conjugation for the nonlinear problem)**.** *Lemma 2.5 and Proposition 2.1 show that the tempered problem* (2.1) *is equivalent, through* $y = e^{\lambda t}x$*, to the classical Caputo problem* ${}^{C}D^{\alpha}_{0+}y = e^{\lambda t}F(t, e^{-\lambda t}y)$*,* $y(0) = x_0$*, with the same regularity, existence, and uniqueness properties; the operator* ${}^{C}D^{\alpha,\lambda}_{0+}$ *is always understood in the sense of Definition 2.3, on the domain described in Proposition 2.1(i). For the nonlinear Eq* (1.1)*, this equivalence has the following consequences, which motivate the approach of Section 3.*

(a) The conjugated equation ${}^{C}D^{\alpha}_{0+}y = Ay + g(t, y)$*,* $g(t, y) := e^{\lambda t}f(e^{-\lambda t}y)$*, is nonautonomous, so the linearization theorem of [25], formulated for autonomous equations, cannot be invoked directly; the analysis is therefore carried out for the tempered equation itself, with the tempered variation of constants formula of Lemma 2.3.*

(b) The smallness of f *near the origin is transferred to* g *uniformly in time: for* $\|y\|, \|\hat{y}\| \le r$*, one has* $\left\|e^{-\lambda t}y\right\|, \left\|e^{-\lambda t}\hat{y}\right\| \le r$*, hence*

$$\|g(t, y) - g(t, \hat{y})\| \le e^{\lambda t}\,\ell_f(r)\,\left\|e^{-\lambda t}(y - \hat{y})\right\| = \ell_f(r)\,\|y - \hat{y}\| \qquad \text{for all } t \ge 0,$$

where $\ell_f(r)$ *denotes the Lipschitz constant of* f *on* $\bar{B}(0, r)$ *introduced at the beginning of Section 3. This exact cancellation of the tempering factors is what makes a contraction argument in exponentially weighted norms possible (Proposition 3.1).*

(c) Bounds on the weighted state $y(t) = e^{\lambda t}x(t)$ *translate into exponential envelopes for* x*; this is why, for* $\lambda > 0$*, exponential stability with rate* λ*, rather than mere asymptotic stability, is the natural conclusion (Theorem 3.1).*

(d) For simulations, the equivalence allows the tempering to be treated exactly, by integrating the conjugated equation with a classical solver and multiplying by $e^{-\lambda t}$ *(Section 4).*

The following comparison result for scalar tempered differential inequalities is needed for the instability analysis in Section 3.

Lemma 2.6. *Let* $c \in \mathbb{R}$*,* $0 < T \le \infty$*, and let* $v : [0, T) \to \mathbb{R}$ *be continuous such that* ${}^{C}D^{\alpha,\lambda}_{0+}v$ *exists on* $(0, T)$*, extends to a continuous function on* $[0, T)$*, and satisfies* ${}^{C}D^{\alpha,\lambda}_{0+}v(t) \ge c\,v(t)$ *for* $t \in (0, T)$*. Then*

$$v(t) \ge e^{-\lambda t}E_{\alpha}(c\,t^{\alpha})v(0), \qquad t \in [0, T).$$

Proof. See Appendix A. □

Remark 2.2 (Equilibria of tempered equations)**.** *For $\lambda > 0$, the trivial solution is the only constant solution that an autonomous tempered equation ${}^{C}D^{\alpha,\lambda}_{0+}x = F(x)$ can have. Indeed, let $x^* \in \mathbb{C}^d$ be a constant. By Definitions 2.2 and 2.3, ${}^{C}D^{\alpha,\lambda}_{0+}x^* = e^{-\lambda t}D^{\alpha}_{0+}[(e^{\lambda t}-1)x^*] = e^{-\lambda t}\,{}^{C}D^{\alpha}_{0+}(e^{\lambda t})\,x^*$, and term-by-term integration of the exponential series gives ${}^{C}D^{\alpha}_{0+}e^{\lambda t} = I^{1-\alpha}_{0+}(\lambda e^{\lambda t}) = \lambda t^{1-\alpha}E_{1,2-\alpha}(\lambda t)$; hence*

$$ {}^{C}D^{\alpha,\lambda}_{0+}x^* = \lambda\, e^{-\lambda t}\, t^{1-\alpha}E_{1,2-\alpha}(\lambda t)\, x^*, \qquad t > 0, $$

which tends to zero as $t \to 0^+$ and is nonzero for $t > 0$ unless $x^* = 0$, whereas $F(x^*)$ is constant. Consequently, a nonzero constant is never an equilibrium of a tempered equation with $\lambda > 0$, and the change of variables $x \mapsto x - x^*$, which reduces the study of an arbitrary equilibrium to that of the trivial one when $\lambda = 0$, has no tempered analogue. The stability theory of this paper therefore concerns the trivial solution $x \equiv 0$ throughout; we use the terms "trivial solution" and "zero equilibrium" interchangeably.

Next, we recall the stability notions, adapted from [3, Definition 7.2] and [25]. Suppose that $F(t, 0) = 0$ for all $t \geq 0$, so that $x \equiv 0$ solves (2.1).

Definition 2.5. *The trivial solution of* (2.1) *is called:*

(i) stable *if for every $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that $\|x_0\| < \delta$ implies $t_{\max}(x_0) = \infty$ and $\|\varphi(t, x_0)\| \leq \varepsilon$ for all $t \geq 0$; and* unstable *if it is not stable;*

(ii) attractive *if there is $\delta_0 > 0$ such that $\lim_{t\to\infty}\varphi(t, x_0) = 0$ whenever $\|x_0\| < \delta_0$;*

(iii) asymptotically stable *if it is stable and attractive;*

(iv) exponentially asymptotically stable with rate $\vartheta \geq 0$ *if there exist $\rho > 0$ and $M \geq 1$ such that $\|x_0\| \leq \rho$ implies $t_{\max}(x_0) = \infty$,*

$$ \|\varphi(t, x_0)\| \leq M\, e^{-\vartheta t}\, \|x_0\| \quad \text{for all } t \geq 0, \qquad \text{and} \qquad \lim_{t\to\infty} e^{\vartheta t}\varphi(t, x_0) = 0. $$

The properties (i)–(iv) are related as follows. Property (iv) implies asymptotic stability for every $\vartheta \geq 0$: given $\varepsilon > 0$, take $\delta(\varepsilon) = \min(\rho, \varepsilon/M)$. The following remark relates Definition 2.5(iv) to the standard terminology.

Remark 2.3. *(Terminology: envelopes, rates, and the case $\vartheta = 0$)*

(a) For $\vartheta > 0$, the envelope $\|\varphi(t, x_0)\| \leq Me^{-\vartheta t}\|x_0\|$ in (iv) is the usual local exponential stability with rate ϑ, as used for fractional and tempered systems in [14, 21]; for linear evolution families and cocycles, exponential stability and its counterpart, exponential instability, are characterized by Datko-type integral conditions [28]. The additional requirement $\lim_{t\to\infty} e^{\vartheta t}\varphi(t, x_0) = 0$ is genuinely stronger: it states that $\|\varphi(t, x_0)\| = o(e^{-\vartheta t})$, i.e., that the weighted state $e^{\vartheta t}\varphi(t, x_0)$ is itself attracted to the origin, which is exactly the conclusion delivered by the weighted fixed point argument of Section 3. It does not *assert an exponential rate larger than ϑ: for the linear tempered equation, the improvement over the envelope is only the algebraic factor $t^{-\alpha}$ of the Mittag-Leffler asymptotics [1, Theorem 1.4], Corollary 2.1, and Example 4.1, and no envelope $e^{-\vartheta' t}$ with $\vartheta' > \vartheta$ holds in general.*

(b) For $\vartheta = 0$, no exponential decay is claimed at all. Property (iv) then states that the trivial solution is attractive and that $\|\varphi(t, x_0)\| \le M \|x_0\|$ for all $t \ge 0$, i.e., asymptotic stability with a uniform (Lipschitz) bound on the solution in terms of the initial value. This is in particular the situation for $\lambda = 0$, where the modes of the linearization decay only algebraically, like $t^{-\alpha}$, and no exponential rate is available; for $\lambda > 0$, the rate 0 also occurs as the case where $\beta = 0$ of Theorem 3.2, where it simply expresses asymptotic stability without a guaranteed exponential envelope. The expression "exponentially asymptotically stable with rate 0" is used only for the sake of a uniform formulation of Theorems 3.1 and 3.2; whenever no rate is involved, we simply say "asymptotically stable".

(c) In this paper, "rate" always refers to the exponent ϑ of a guaranteed envelope. The relation $e^{\vartheta t}\varphi(t, x_0) \to 0$ gives $\|\varphi(t, x_0)\| = o(e^{-\vartheta t})$ but does not imply decay with a strictly larger exponential rate. Decay slopes measured in numerical experiments are observed quantities of particular trajectories; they are compared with, but never identified with, guaranteed rates (Section 4).

Finally, we recall the Mittag-Leffler estimates of [25] in the stability sector.

Lemma 2.7 ([1, 25]). *Let $\alpha \in (0, 1)$ and $\mu \in \mathbb{C}$ with $\frac{\alpha\pi}{2} < \left|\arg \mu\right| \le \pi$. Then the following statements hold.*

(i) There are positive numbers $M(\alpha, \mu)$ and $t_0(\alpha, \mu)$ such that

$$\left|t^{\alpha-1} E_{\alpha,\alpha}(\mu t^\alpha)\right| \le \frac{M(\alpha, \mu)}{t^{\alpha+1}} \qquad \text{for all } t \ge t_0(\alpha, \mu).$$

(ii) The quantity

$$C(\alpha, \mu) := \sup_{t\ge 0} \int_0^t u^{\alpha-1} \left|E_{\alpha,\alpha}(\mu u^\alpha)\right| \, du$$

is finite.

(iii) $E^*(\alpha, \mu) := \sup_{t\ge 0} \left|E_\alpha(\mu t^\alpha)\right|$ is finite, $E^*(\alpha, \mu) \ge 1$, and $\lim_{t\to\infty} E_\alpha(\mu t^\alpha) = 0$.

Items (i) and (ii) are from [25, Proposition 4]; item (iii) follows from the algebraic asymptotic expansion $E_\alpha(z) = -\sum_{k=1}^{p} z^{-k}/\Gamma(1 - \alpha k) + O(|z|^{-p-1})$ valid for $\left|\arg z\right| > \alpha\pi/2$ [1, Theorem 1.4], together with the continuity of E_α and $E_\alpha(0) = 1$.

We close this section with the stability of the linear tempered equation.

Corollary 2.1. *Let $A \in \mathbb{R}^{d\times d}$ and assume that every eigenvalue μ of A satisfies $\left|\arg \mu\right| > \frac{\alpha\pi}{2}$. Then every solution of ${}^C D^{\alpha,\lambda}_{0+} x(t) = Ax(t)$ has the form $x(t) = e^{-\lambda t} E_\alpha(t^\alpha A) x_0$, where $E_\alpha(t^\alpha A) = \sum_{k\ge 0} t^{\alpha k} A^k/\Gamma(\alpha k + 1)$ [14], and the trivial solution is exponentially asymptotically stable with rate λ; in fact, $\|x(t)\| \le K e^{-\lambda t} \|x_0\|$ for all $x_0 \in \mathbb{R}^d$ with $K := \sup_{t\ge 0} \|E_\alpha(t^\alpha A)\| < \infty$, and $e^{\lambda t} x(t) \to 0$.*

Proof. See Appendix A. □

3. Main results

This section contains two linearization theorems for (1.1)—Theorem 3.1 for the classical spectral sector and Theorem 3.2 for the remaining part of the tempered stability region—and their application to

a tempered Lotka-Volterra system. Throughout, (1.1) is regarded as the linearization of an autonomous equation ${}^{C}D^{\alpha,\lambda}_{0+}x = F(x)$ at its zero equilibrium: if F is differentiable at 0 with $F(0) = 0$, then $A = DF(0)$ and $f(x) = F(x) - Ax$; the precise relation between differentiability and assumption (3.2) below is discussed in Remark 3.1. Following [25], solutions of (1.1) are represented as fixed points of an integral operator built from Lemma 2.3, which is shown to be a contraction on a small ball of an exponentially weighted space. For a locally Lipschitz continuous function g, we write

$$\ell_g(r) := \sup_{\substack{x,y\in\bar{B}(0,r)\\ x\neq y}} \frac{\|g(x) - g(y)\|}{\|x - y\|}, \qquad r > 0,$$

for its Lipschitz constant on the closed ball $\bar{B}(0, r)$ of radius r centered at the origin. Note that $r \mapsto \ell_g(r)$ is nondecreasing.

Theorem 3.1. *Let $\alpha \in (0, 1)$, $\lambda \geq 0$, $A \in \mathbb{R}^{d\times d}$, and let $f : \mathbb{R}^d \to \mathbb{R}^d$ be locally Lipschitz continuous. Let $\hat{\mu}_1, \ldots, \hat{\mu}_m$ denote the distinct eigenvalues of A ($m \leq d$) and assume that*

$$\left|\arg \hat{\mu}_i\right| > \frac{\alpha\pi}{2}, \qquad i = 1, \ldots, m, \tag{3.1}$$

and that

$$f(0) = 0, \qquad \lim_{r\to 0^+} \ell_f(r) = 0. \tag{3.2}$$

Then there exist $\rho > 0$ and $M \geq 1$ such that for every $x_0 \in \mathbb{R}^d$ with $\|x_0\| \leq \rho$, the solution of (1.1) *with $x(0) = x_0$ is global and satisfies*

$$\|x(t; x_0)\| \leq M\, e^{-\lambda t}\, \|x_0\| \quad \text{for all } t \geq 0 \qquad \text{and} \qquad \lim_{t\to\infty} e^{\lambda t} x(t; x_0) = 0. \tag{3.3}$$

In other words, the trivial solution of (1.1) *is exponentially asymptotically stable with rate λ in the sense of Definition 2.5(iv). In particular, it is asymptotically stable, and for $\lambda > 0$, it is locally exponentially stable with a rate of at least λ.*

For $\lambda = 0$, Theorem 3.1 recovers the theorem of [25], strengthened by the explicit estimate (3.3). The proof rests on a scaled Jordan transformation, a tempered Lyapunov-Perron operator on an exponentially weighted space, and the estimates of Lemma 2.7.

Remark 3.1 (On assumption (3.2))**.** *Local Lipschitz continuity of f alone cannot suffice for a linearization theorem because it does not distinguish the linear part A from the remainder f: for instance, $f(x) = Bx$ is locally Lipschitz continuous for every matrix B, and the decomposition $Ax + f(x)$ would then carry no information. Condition* (3.2) *gives the decomposition* (1.1) *its meaning by requiring that f vanish to first order at the origin* together with its Lipschitz constant*; this is exactly what is needed in the contraction estimates of Proposition 3.1 and Lemma 3.4, where the Lipschitz constant of the nonlinearity on a small ball must be dominated by the reciprocal of a Mittag-Leffler constant. It relates to differentiability as follows.*

(a) If (3.2) *holds, then f is (Fréchet) differentiable at 0 with $Df(0) = 0$, since $\|f(x)\| = \|f(x) - f(0)\| \leq \ell_f(\|x\|)\, \|x\| = o(\|x\|)$ as $x \to 0$.*

(b) If f is continuously differentiable on a neighborhood of 0 *with* $f(0) = 0$ *and* $Df(0) = 0$*, then* (3.2) *holds: by the mean value inequality,* $\ell_f(r) \le \sup_{\|x\|\le r} \|Df(x)\| \to 0$ *as* $r \to 0^+$*. Thus, for the important class of* C^1 *right-hand sides* F*, assumption* (3.2) *is equivalent to the standard requirement that* $A = DF(0)$ *be the Jacobian of* F *at the zero equilibrium and* $f = F - A(\cdot)$ *its remainder.*

(c) Differentiability at the origin alone is not sufficient: the scalar function $f(x) = x^2 \sin(1/x)$ *for* $x \neq 0$*,* $f(0) = 0$*, is Lipschitz continuous near* 0 *with* $f'(0) = 0$*, but* $f'(x_k) = -1$ *at* $x_k = 1/(2k\pi)$*, so* $\ell_f(r) \ge 1$ *for every* $r > 0$ *and* (3.2) *fails. For merely locally Lipschitz continuous* f*, condition* (3.2) *is therefore the natural "Lipschitz" form of the requirement* $Df(0) = 0$*; it is also the hypothesis under which the classical theorem of [25] is formulated.*

Remark 3.2 (Complexification)**.** *Since the Jordan transformation below is complex in general, we regard* (1.1) *as an equation in* $\mathbb{C}^d$*. To this end,* f *is extended to* $\mathbb{C}^d$ *by* $\tilde f(z) := f(\operatorname{Re} z)$*, where* $\operatorname{Re} z$ *denotes the vector of real parts. Three facts have to be verified.*

(a) The Lipschitz properties are preserved. *We have* $\tilde f = f$ *on* $\mathbb{R}^d$*, and* $\tilde f(0) = 0$*. For* $z_1, z_2 \in \bar B(0, r) \subset \mathbb{C}^d$ *the real parts satisfy* $\operatorname{Re} z_1, \operatorname{Re} z_2 \in \bar B(0, r) \subset \mathbb{R}^d$ *and* $\|\operatorname{Re} z_1 - \operatorname{Re} z_2\| \le \|z_1 - z_2\|$ *in the maximum norm because* $|\operatorname{Re} w| \le |w|$ *for every complex number* w*; hence* $\big\|\tilde f(z_1) - \tilde f(z_2)\big\| \le \ell_f(r)\,\|z_1 - z_2\|$*, that is,* $\ell_{\tilde f}(r) \le \ell_f(r)$ *for all* $r > 0$*. In particular,* $\tilde f$ *is locally Lipschitz continuous, and it satisfies* (3.2) *whenever* f *does.*

(b) Real initial values produce real solutions. *Let* $x : [0, t_{\max}) \to \mathbb{C}^d$ *be the maximal solution of the complexified equation* ${}^C D^{\alpha,\lambda}_{0+} x = Ax + \tilde f(x)$ *with* $x(0) = x_0 \in \mathbb{R}^d$*, and write* $x = u + iv$ *with* u, v *being real. Since* A *is real and* $\tilde f(x) = f(u)$ *is real, taking real and imaginary parts in the mild formulation* (2.2)*, whose kernel is real, shows that* u *solves* ${}^C D^{\alpha,\lambda}_{0+} u = Au + f(u)$*,* $u(0) = x_0$*, and that* v *solves the linear equation* ${}^C D^{\alpha,\lambda}_{0+} v = Av$*,* $v(0) = 0$*. By the uniqueness statement of Proposition 2.1(ii),* $v \equiv 0$*; hence* $x = u$ *is real and solves the original real Eq* (1.1)*. Conversely, every solution of* (1.1) *solves the complexified equation.*

(c) Stability is transferred. *Suppose that the trivial solution of the complexified equation is exponentially asymptotically stable with rate* ϑ *and constants* (ρ, M)*. For real* x_0 *with* $\|x_0\| \le \rho$*, the solution of* (1.1) *coincides, by (b), with the solution of the complexified equation; it is therefore global and satisfies* $\|x(t; x_0)\| \le M e^{-\vartheta t}\,\|x_0\|$ *and* $e^{\vartheta t} x(t; x_0) \to 0$*. Thus the trivial solution of* (1.1) *inherits the property with the same constants. Only this implication, from the complexified to the real equation, is used.*

In the sequel, we write f *for* $\tilde f$ *and* ℓ_f *for* $\ell_{\tilde f}$*; by (a), all estimates stated in terms of* ℓ_f *remain valid.*

By the Jordan normal form theorem [29, Chapter 3], there is a nonsingular $T \in \mathbb{C}^{d\times d}$ with

$$T^{-1} A T = \operatorname{diag}(A_1, \dots, A_n), \qquad A_i = \mu_i I_{d_i} + \eta_i N_{d_i}, \qquad i = 1, \dots, n,$$

where $n \ge m$ is the number of Jordan blocks, $\eta_i \in \{0, 1\}$, $\mu_i \in \{\hat\mu_1, \dots, \hat\mu_m\}$ is the eigenvalue of the i-th block (the same eigenvalue may occur in several blocks, and every $\hat\mu_j$ occurs in at least one), $d_i \ge 1$ is the size of the i-th block, $d_1 + \cdots + d_n = d$, $N_k \in \mathbb{R}^{k\times k}$ is the nilpotent matrix with ones on the superdiagonal and zeros elsewhere, and I_k denotes the $k \times k$ identity matrix. Throughout, the index

range $i = 1, \dots, m$ refers to the distinct eigenvalues $\hat{\mu}_i$, whereas $i = 1, \dots, n$ refers to the Jordan blocks and to the quantities μ_i, d_i, η_i, δ_i, A_i, and J_i attached to them; the sector condition (3.1) for $\hat{\mu}_1, \dots, \hat{\mu}_m$ is equivalent to $\left|\arg \mu_i\right| > \alpha\pi/2$ for $i = 1, \dots, n$. Let $\delta > 0$ be a constant, which will be fixed in (3.11) below. With $P_i := \operatorname{diag}(1, \delta, \dots, \delta^{d_i-1})$ and $P := \operatorname{diag}(P_1, \dots, P_n)$, we obtain $P_i^{-1} A_i P_i = \mu_i I_{d_i} + \delta_i N_{d_i}$ with $\delta_i := \delta\eta_i \in \{0, \delta\}$. Since ${}^{C}D_{0+}^{\alpha,\lambda}$ acts component-wise and commutes with constant matrices, the change of variables $y := (TP)^{-1}x$ transforms the complexified Eq (1.1) into

$$ {}^{C}D_{0+}^{\alpha,\lambda} y(t) = Jy(t) + h(y(t)), \qquad J := \operatorname{diag}(J_1, \dots, J_n), \quad J_i := \mu_i I_{d_i}, \tag{3.4} $$

where

$$ h(y) := \operatorname{diag}(\delta_1 N_{d_1}, \dots, \delta_n N_{d_n})\, y + (TP)^{-1} f(TPy). \tag{3.5} $$

Lemma 3.1. *The function $h : \mathbb{C}^d \to \mathbb{C}^d$ defined in* (3.5) *is locally Lipschitz continuous, $h(0) = 0$, and*

$$ \ell_h(r) \le \delta + \left\|(TP)^{-1}\right\| \, \|TP\| \, \ell_f(\|TP\|\, r) \qquad \text{for all } r > 0. $$

In particular, $\limsup_{r\to 0^+} \ell_h(r) \le \delta$.

Proof. See Appendix A. □

Remark 3.3. *Since $\|TP\|^{-1} \|x\| \le \|y\| \le \left\|(TP)^{-1}\right\| \|x\|$ along corresponding solutions, the trivial solutions of the complexified Eq* (1.1) *and of* (3.4) *share all properties of Definition 2.5; quantitatively, if* (3.4) *is exponentially asymptotically stable with rate ϑ and constants (ρ_y, M_y), then* (1.1) *has the same property with*

$$ \rho_x = \frac{\rho_y}{\left\|(TP)^{-1}\right\|}, \qquad M_x = \kappa(TP)\, M_y, \qquad \kappa(TP) := \|TP\| \left\|(TP)^{-1}\right\| \ge 1. \tag{3.6} $$

The condition number $\kappa(TP)$ may be large ($\left\|P^{-1}\right\|$ grows like $\delta^{1-\max_i d_i}$ as δ decreases), so the constants produced by the proof are conservative for strongly non-normal A. All radii and constants stated below for (3.4) *are understood in the transformed coordinates. Some comments on the practical significance of the constants M and ρ of Theorem 3.1 are in order.*

- *(a) The proof yields $M = \kappa(TP)\, E^*/(1-q)$ and $\rho = r^*/\left\|(TP)^{-1}\right\|$, see* (3.16) *and the end of the proof of Theorem 3.1, where E^* and $C(\alpha, \boldsymbol{\mu})$ are the maxima over the Jordan blocks of the Mittag-Leffler constants of Lemma 2.7, defined before Proposition 3.1, and r, $q = C(\alpha, \boldsymbol{\mu})\, \ell_h(r)$ and $r^* = r(1-q)/E^*$ are determined by the Lipschitz function ℓ_h of the transformed nonlinearity through* (3.11) *and Lemma 3.4. None of these quantities depends on the tempering parameter λ: the tempering enters the conclusion only through the factor $e^{-\lambda t}$, a consequence of the exact cancellation of the tempering factors in the weighted norm (Proposition 3.1).*

- *(b) If A is diagonalizable, all Jordan blocks have a size of one, so $P = I_d$ and the scaling parameter δ plays no role. T is a matrix of eigenvectors and $\kappa(TP) = \kappa(T)$ is its condition number in the maximum norm. If, in addition, A is normal (for instance symmetric), T can be chosen unitary, and then $\kappa(T) \le d$. The constants are explicit and moderate, $M \le d\, E^*/(1-q)$, with $E^* = 1$ when the spectrum is real (Corollary 3.2).*

(c) *If A has nontrivial Jordan blocks, the factor $\kappa(P) = \|P\| \left\|P^{-1}\right\|$ enters; with $d_{\max} := \max_i d_i$, one has $\|P\| = \max\{1, \delta^{d_{\max}-1}\}$ and $\left\|P^{-1}\right\| = \max\{1, \delta^{1-d_{\max}}\}$ in the maximum norm, so that $\kappa(P) = \max\{\delta, \delta^{-1}\}^{d_{\max}-1}$. This factor is large when δ is small, i.e., by* (3.11)*, when the constants $C(\alpha, \mu_i)$ are large, which happens when eigenvalues approach the rays $\left|\arg \mu\right| = \alpha\pi/2$ or the origin. It is also large when δ is large, i.e., for eigenvalues of large modulus. For strongly non-normal matrices M, may thus be very large and ρ very small, and the values produced by the proof should be regarded as qualitative guarantees rather than sharp estimates; this is the same limitation as in the classical theory [25]. Sharper constants for a given matrix can be obtained by estimating $\sup_{t\geq 0} \|E_\alpha(t^\alpha A)\|$ and $\sup_{t\geq 0} \int_0^t \left\|u^{\alpha-1} E_{\alpha,\alpha}(u^\alpha A)\right\| du$ directly in the original coordinates, since the fixed point argument uses only these two quantities together with the decay of the kernel and of $E_\alpha(t^\alpha A)$ at infinity.*

For $\lambda \geq 0$, define

$$C_\lambda := \Big\{\xi : \mathbb{R}_{\geq 0} \to \mathbb{C}^d \text{ continuous} : \ \|\xi\|_\lambda := \sup_{t\geq 0} e^{\lambda t} \|\xi(t)\| < \infty\Big\}. \tag{3.7}$$

For $\lambda = 0$, this is the space C_∞ used in [25], and in general, $C_\lambda \subset C_\infty$ with $\|\xi\|_\infty \leq \|\xi\|_\lambda$ because $e^{\lambda t} \geq 1$.

Lemma 3.2. *For every $\vartheta \geq 0$, the space C_ϑ obtained from* (3.7) *by replacing λ with ϑ is a Banach space with the norm $\|\xi\|_\vartheta = \sup_{t\geq 0} e^{\vartheta t} \|\xi(t)\|$. In particular, $(C_\lambda, \|\cdot\|_\lambda)$ is a Banach space.*

Proof. The map $\xi \mapsto e^{\vartheta \cdot} \xi(\cdot)$ is an isometric isomorphism onto the Banach space of bounded continuous functions with the supremum norm. □

We write vectors $z \in \mathbb{C}^d = \mathbb{C}^{d_1} \times \cdots \times \mathbb{C}^{d_n}$ block-wise as $z = (z^1, \ldots, z^n)$, so that $\left\|z^i\right\| \leq \|z\|$ in the maximum norm. For $x \in \mathbb{C}^d$, the tempered Lyapunov-Perron operator $\mathcal{T}_x : C_\lambda \to C_\lambda$ associated with (3.4) is defined block-wise by

$$(\mathcal{T}_x \xi)^i(t) := e^{-\lambda t} E_\alpha(\mu_i t^\alpha) x^i + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\mu_i (t-\tau)^\alpha) e^{-\lambda(t-\tau)} h^i(\xi(\tau))\, d\tau, \tag{3.8}$$

for $i = 1, \ldots, n$ and $t \geq 0$, where $h^i(\xi(\tau))$ denotes the i-th block of $h(\xi(\tau))$. That $\mathcal{T}_x$ maps C_λ into itself is part of Proposition 3.1.

Lemma 3.3. *Let $x \in \mathbb{C}^d$ and let $\xi \in C_\lambda$ satisfy $\xi(0) = x$. Then ξ is a solution of* (3.4) *on $[0, \infty)$ with $\xi(0) = x$ if and only if $\mathcal{T}_x \xi = \xi$.*

Proof. See Appendix A. □

The next proposition is the quantitative heart of the paper: in the weighted norm $\|\cdot\|_\lambda$, the tempering factors cancel identically, so the operator (3.8) satisfies exactly the classical estimates of [25]. Recall the constants $C(\alpha, \mu)$ and $E^*(\alpha, \mu)$ from Lemma 2.7 and set

$$C(\alpha, \boldsymbol{\mu}) := \max_{1\leq i\leq n} C(\alpha, \mu_i), \qquad E^* := \max_{1\leq i\leq n} E^*(\alpha, \mu_i) \ \in [1, \infty),$$

which are finite by (3.1) and Lemma 2.7.

Proposition 3.1. *Assume* (3.1)*. For all $x, \hat{x} \in \mathbb{C}^d$ and $\xi, \hat{\xi} \in C_\lambda$, the function $\mathcal{T}_x \xi$ belongs to C_λ and*

$$\left\|\mathcal{T}_x \xi - \mathcal{T}_{\hat{x}} \hat{\xi}\right\|_\lambda \leq E^* \|x - \hat{x}\| + C(\alpha, \boldsymbol{\mu})\, \ell_h(\max(\|\xi\|_\lambda, \left\|\hat{\xi}\right\|_\lambda))\, \left\|\xi - \hat{\xi}\right\|_\lambda. \tag{3.9}$$

Proof. Fix $x, \hat{x} \in \mathbb{C}^d$, $\xi, \hat{\xi} \in C_\lambda$, and abbreviate $L := \ell_h(\max(\|\xi\|_\lambda, \|\hat{\xi}\|_\lambda))$. For every $\tau \geq 0$, we have $\|\xi(\tau)\| \leq e^{-\lambda\tau} \|\xi\|_\lambda \leq \|\xi\|_\lambda$ and likewise for $\hat{\xi}$, so both values lie in the ball of radius $\max(\|\xi\|_\lambda, \|\hat{\xi}\|_\lambda)$ and the definition of ℓ_h yields

$$\left\|h(\xi(\tau)) - h(\hat{\xi}(\tau))\right\| \leq L \left\|\xi(\tau) - \hat{\xi}(\tau)\right\| \leq L\, e^{-\lambda\tau} \left\|\xi - \hat{\xi}\right\|_\lambda. \tag{3.10}$$

Fix $i \in \{1, \ldots, n\}$ and $t \geq 0$. Since $J_i = \mu_i I_{d_i}$, the definition (3.8) and the triangle inequality give

$$\begin{aligned} &\left\|(\mathcal{T}_x\xi)^i(t) - (\mathcal{T}_{\hat{x}}\hat{\xi})^i(t)\right\| \\ &\leq e^{-\lambda t} \left|E_\alpha(\mu_i t^\alpha)\right| \|x - \hat{x}\| + \int_0^t (t-\tau)^{\alpha-1} \left|E_{\alpha,\alpha}(\mu_i(t-\tau)^\alpha)\right| e^{-\lambda(t-\tau)} \left\|h(\xi(\tau)) - h(\hat{\xi}(\tau))\right\| d\tau, \end{aligned}$$

where we use $\left\|h^i(z) - h^i(w)\right\| \leq \|h(z) - h(w)\|$ and $\left\|x^i - \hat{x}^i\right\| \leq \|x - \hat{x}\|$. Multiplying by $e^{\lambda t}$, inserting (3.10), and using the identity $e^{\lambda t} e^{-\lambda(t-\tau)} e^{-\lambda\tau} = 1$, we obtain

$$e^{\lambda t} \left\|(\mathcal{T}_x\xi)^i(t) - (\mathcal{T}_{\hat{x}}\hat{\xi})^i(t)\right\| \leq \left|E_\alpha(\mu_i t^\alpha)\right| \|x - \hat{x}\| + L \left\|\xi - \hat{\xi}\right\|_\lambda \int_0^t (t-\tau)^{\alpha-1} \left|E_{\alpha,\alpha}(\mu_i(t-\tau)^\alpha)\right| d\tau.$$

The substitution $u := t - \tau$ and Lemma 2.7(ii) give

$$\int_0^t (t-\tau)^{\alpha-1} \left|E_{\alpha,\alpha}(\mu_i(t-\tau)^\alpha)\right| d\tau = \int_0^t u^{\alpha-1} \left|E_{\alpha,\alpha}(\mu_i u^\alpha)\right| du \leq C(\alpha, \mu_i) \leq C(\alpha, \boldsymbol{\mu}),$$

and Lemma 2.7(iii) gives $|E_\alpha(\mu_i t^\alpha)| \leq E^*$. Since $\|z\| = \max_{1 \leq i \leq n} \left\|z^i\right\|$, taking the maximum over i and the supremum over $t \geq 0$ yields (3.9). It remains to show that $\mathcal{T}_x\xi \in C_\lambda$. From $h(0) = 0$, we get $\mathcal{T}_0 0 = 0$, so (3.9) with $\hat{x} = 0$, $\hat{\xi} = 0$ gives

$$\sup_{t \geq 0} e^{\lambda t} \|(\mathcal{T}_x\xi)(t)\| \leq E^* \|x\| + C(\alpha, \boldsymbol{\mu})\, \ell_h(\|\xi\|_\lambda)\, \|\xi\|_\lambda < \infty.$$

Finally, $t \mapsto (\mathcal{T}_x\xi)(t)$ is continuous: the first term in (3.8) is continuous because E_α is entire, and the second term is the convolution of the kernel $u \mapsto u^{\alpha-1} E_{\alpha,\alpha}(\mu_i u^\alpha) e^{-\lambda u}$, which on every compact interval $[0, T]$ admits the integrable majorant $C_T u^{\alpha-1}$ with $C_T := \sup_{u \in [0,T]} \left|E_{\alpha,\alpha}(\mu_i u^\alpha)\right|$, with the bounded continuous function $\tau \mapsto h^i(\xi(\tau))$. Continuity follows from the dominated convergence theorem, as in [25]. The proof is complete. □

From now on, we fix the constant δ in (3.4) and (3.5) as

$$\delta := \frac{1}{2\, C(\alpha, \boldsymbol{\mu})}, \tag{3.11}$$

which is well-defined because $C(\alpha, \boldsymbol{\mu})$ does not depend on δ.

Lemma 3.4. *Assume* (3.1) *and* (3.2)*, and let δ be given by* (3.11)*. Then the following statements hold.*

(i) There exists $r > 0$ such that

$$q := C(\alpha, \boldsymbol{\mu})\, \ell_h(r) \leq \frac{3}{4} < 1. \tag{3.12}$$

(ii) Choose and fix $r > 0$ as in (i) and define

$$r^* := \frac{r(1-q)}{E^*}. \tag{3.13}$$

Let $B_{C_\lambda}(0,r) := \{\xi \in C_\lambda : \|\xi\|_\lambda \le r\}$. Then for every $x \in \mathbb{C}^d$ with $\|x\| \le r^*$, the operator $\mathcal{T}_x$ maps $B_{C_\lambda}(0,r)$ into itself and

$$\left\|\mathcal{T}_x\xi - \mathcal{T}_x\hat{\xi}\right\|_\lambda \le q \left\|\xi - \hat{\xi}\right\|_\lambda \qquad \text{for all } \xi, \hat{\xi} \in B_{C_\lambda}(0,r).$$

Proof. By Lemma 3.1 and (3.2), we have $\limsup_{\rho\to 0^+} \ell_h(\rho) \le \delta$. Hence there exists $r > 0$ with

$$\ell_h(r) \le \frac{3}{2}\,\delta = \frac{3}{4\,C(\alpha,\boldsymbol{\mu})},$$

which gives $q = C(\alpha,\boldsymbol{\mu})\,\ell_h(r) \le \frac{3}{4}$ and proves (i). For (ii), let $\|x\| \le r^*$ and $\xi \in B_{C_\lambda}(0,r)$. Applying (3.9) with $\hat{x} = 0$ and $\hat{\xi} = 0$, and using $\mathcal{T}_0 0 = 0$ together with the monotonicity of ℓ_h, we get

$$\|\mathcal{T}_x\xi\|_\lambda \le E^*\,\|x\| + C(\alpha,\boldsymbol{\mu})\,\ell_h(r)\,\|\xi\|_\lambda \le E^* r^* + q\,r = (1-q)r + q\,r = r,$$

so $\mathcal{T}_x$ maps $B_{C_\lambda}(0,r)$ into itself. Applying (3.9) with $\hat{x} = x$ gives, for $\xi, \hat{\xi} \in B_{C_\lambda}(0,r)$,

$$\left\|\mathcal{T}_x\xi - \mathcal{T}_x\hat{\xi}\right\|_\lambda \le C(\alpha,\boldsymbol{\mu})\,\ell_h(r)\,\left\|\xi - \hat{\xi}\right\|_\lambda = q\,\left\|\xi - \hat{\xi}\right\|_\lambda,$$

which completes the proof. □

Proof of Theorem 3.1. By Remarks 3.2 and 3.3, it suffices to prove that the trivial solution of the transformed equation (3.4) is exponentially asymptotically stable with rate of λ; the constants are then transported to (1.1) through (3.6). Fix δ as in (3.11), and then fix r and q as in Lemma 3.4, and let r^* be given by (3.13). Let $x \in \mathbb{C}^d$ with $\|x\| \le r^*$. By Lemma 3.4(ii), Lemma 3.2, and the contraction mapping principle, $\mathcal{T}_x$ has a unique fixed point $\xi_x \in B_{C_\lambda}(0,r)$. Evaluating (3.8) at $t = 0$ shows $\xi_x(0) = E_\alpha(0)x = x$. By Lemma 3.3, ξ_x is a global solution of (3.4) with initial value x. By Proposition 2.1(ii), the maximal solution $\varphi(\cdot,x)$ of (3.4) coincides with ξ_x on their common interval $[0, t_{\max}(x))$; if $t_{\max}(x)$ were finite, then ξ_x would be a solution extending $\varphi(\cdot,x)$ beyond $t_{\max}(x)$, contradicting maximality. Hence $t_{\max}(x) = \infty$ and $\varphi(\cdot,x) = \xi_x$. We next derive the quantitative estimate of Definition 2.5(iv). Since $\xi_x \in B_{C_\lambda}(0,r)$ and $\mathcal{T}_0 0 = 0$, the estimate (3.9) with $(\hat{x},\hat{\xi}) = (0,0)$ and the monotonicity of ℓ_h give

$$\|\xi_x\|_\lambda = \|\mathcal{T}_x\xi_x\|_\lambda \le E^*\,\|x\| + C(\alpha,\boldsymbol{\mu})\,\ell_h(r)\,\|\xi_x\|_\lambda = E^*\,\|x\| + q\,\|\xi_x\|_\lambda,$$

and therefore

$$\sup_{t\ge 0} e^{\lambda t}\,\|\varphi(t,x)\| = \|\xi_x\|_\lambda \le \frac{E^*}{1-q}\,\|x\| \qquad \text{whenever } \|x\| \le r^*. \tag{3.14}$$

The same manipulation applied to two initial values $\|x\|, \|\hat{x}\| \le r^*$ gives, by (3.9),

$$\|\xi_x - \xi_{\hat{x}}\|_\lambda = \|\mathcal{T}_x\xi_x - \mathcal{T}_{\hat{x}}\xi_{\hat{x}}\|_\lambda \le E^*\,\|x - \hat{x}\| + q\,\|\xi_x - \xi_{\hat{x}}\|_\lambda,$$

hence we have the Lipschitz dependence on the initial value

$$\|\xi_x - \xi_{\hat{x}}\|_\lambda \le \frac{E^*}{1-q}\,\|x - \hat{x}\|, \qquad \|x\|, \|\hat{x}\| \le r^*. \tag{3.15}$$

Thus, by (3.14), the trivial solution of (3.4) satisfies the estimate in Definition 2.5(iv) with

$$\rho_y := r^*, \qquad M_y := \frac{E^*}{1-q} \in [1, \infty). \tag{3.16}$$

We now prove the attractivity part. Let $\|x\| \le r^*$ and let $\xi := \varphi(\cdot, x) = \xi_x \in B_{C_\lambda}(0, r)$ be the corresponding global solution. Define

$$z(t) := e^{\lambda t}\xi(t), \qquad a := \limsup_{t\to\infty} \|z(t)\| \in [0, r].$$

Let $\varepsilon' > 0$ be arbitrary and choose $T' = T'(\varepsilon') > 0$ such that $\|z(t)\| \le a + \varepsilon'$ for all $t \ge T'$. Multiplying the fixed point identity $\xi = \mathcal{T}_x\xi$ by $e^{\lambda t}$ and using $e^{\lambda t}e^{-\lambda(t-\tau)} = e^{\lambda\tau}$, we obtain, for each block $i \in \{1, \dots, n\}$,

$$z^i(t) = E_\alpha(\mu_i t^\alpha)x^i + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\mu_i(t-\tau)^\alpha)\, e^{\lambda\tau} h^i(\xi(\tau))\, d\tau. \tag{3.17}$$

Since $h(0) = 0$ and $\|\xi(\tau)\| \le e^{-\lambda\tau} r \le r$, the definition of ℓ_h gives $\|h(\xi(\tau))\| \le \ell_h(r)\, \|\xi(\tau)\|$, and therefore

$$e^{\lambda\tau} \left\|h^i(\xi(\tau))\right\| \le \ell_h(r)\ \|z(\tau)\| \le \begin{cases} \ell_h(r)\, r, & \tau \ge 0, \\ \ell_h(r)\,(a+\varepsilon'), & \tau \ge T'. \end{cases} \tag{3.18}$$

Fix i and consider $t \ge T' + t_0(\alpha, \mu_i)$, with $t_0(\alpha, \mu_i)$ from Lemma 2.7(i). We split the integral in (3.17) at $\tau = T'$. For $\tau \in [0, T']$, we have $t - \tau \ge t - T' \ge t_0(\alpha, \mu_i)$, so Lemma 2.7(i) and (3.18) yield

$$\left\| \int_0^{T'} (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\mu_i(t-\tau)^\alpha) e^{\lambda\tau} h^i(\xi(\tau))\, d\tau \right\| \le \ell_h(r)\, r\, \frac{T'\, M(\alpha, \mu_i)}{(t-T')^{\alpha+1}} \ \xrightarrow[t\to\infty]{}\ 0.$$

For the remaining part, the substitution $u := t - \tau$, the bound (3.18), and Lemma 2.7(ii) yield

$$\begin{aligned} \left\| \int_{T'}^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\mu_i(t-\tau)^\alpha) e^{\lambda\tau} h^i(\xi(\tau))\, d\tau \right\| &\le \ell_h(r)\,(a+\varepsilon') \int_0^{t-T'} u^{\alpha-1} \left|E_{\alpha,\alpha}(\mu_i u^\alpha)\right| du \\ &\le C(\alpha, \mu_i)\, \ell_h(r)\,(a+\varepsilon'). \end{aligned}$$

Moreover $E_\alpha(\mu_i t^\alpha)x^i \to 0$ as $t \to \infty$ by Lemma 2.7(iii). Combining the three assertions with (3.17) gives

$$\limsup_{t\to\infty} \left\|z^i(t)\right\| \le C(\alpha, \mu_i)\, \ell_h(r)\,(a+\varepsilon') \le q\,(a+\varepsilon'), \qquad i = 1, \dots, n.$$

Since $\|z(t)\| = \max_{1\le i\le n} \left\|z^i(t)\right\|$ and, for finitely many functions, the limit superior of the maximum equals the maximum of the limit superiors, we obtain

$$a = \max_{1\le i\le n} \limsup_{t\to\infty} \left\|z^i(t)\right\| \le q\,(a+\varepsilon').$$

Letting $\varepsilon' \to 0^+$ gives $a \le qa$, and $q < 1$ forces $a = 0$. Hence $\lim_{t\to\infty} e^{\lambda t}\varphi(t, x) = 0$ for every $\|x\| \le r^*$, which is the attractivity part of Definition 2.5(iv). The trivial solution of (3.4) is therefore exponentially asymptotically stable with rate λ and constants (3.16), and by Remark 3.2 and Remark 3.3, the trivial solution of (1.1) has the same property with

$$\rho = \frac{r^*}{\left\|(TP)^{-1}\right\|}, \qquad M = \kappa(TP)\, \frac{E^*}{1-q},$$

which is exactly (3.3). The proof is complete. □

The estimate (3.15) yields continuous dependence on the initial value, with the same envelope.

Corollary 3.1. *Under the hypotheses of Theorem 3.1, with ρ and M as in* (3.3)*, solutions depend Lipschitz continuously on their initial values inside $\bar{B}(0,\rho)$:*

$$\|x(t;x_0)-x(t;\hat{x}_0)\| \le M\,e^{-\lambda t}\,\|x_0-\hat{x}_0\| \qquad \text{for all } t\ge 0 \text{ and } \|x_0\|,\|\hat{x}_0\|\le\rho.$$

Equivalently, the local solution map $x_0 \mapsto x(\cdot\,;x_0)$ is Lipschitz continuous, with constant M, from $\bar{B}(0,\rho)\cap\mathbb{R}^d$ into the weighted space C_λ.

Proof. For $\|x_0\|,\|\hat{x}_0\| \le \rho$, the transformed initial values $y_0 = (TP)^{-1}x_0$ and $\hat{y}_0 = (TP)^{-1}\hat{x}_0$ satisfy $\|y_0\|,\|\hat{y}_0\| \le r^*$. By (3.15),

$$\|x(t;x_0)-x(t;\hat{x}_0)\| \le \|TP\|\,\left\|\xi_{y_0}(t)-\xi_{\hat{y}_0}(t)\right\| \le \|TP\|\,e^{-\lambda t}\,\frac{E^*}{1-q}\,\|y_0-\hat{y}_0\| \le \kappa(TP)\,\frac{E^*}{1-q}\,e^{-\lambda t}\,\|x_0-\hat{x}_0\|,$$

which is the claim with the constant M of (3.3). The proof is complete. □

When all eigenvalues of A are real, the constant E^* can be computed exactly.

Corollary 3.2. *Assume, in addition to the hypotheses of Theorem 3.1, that all eigenvalues of A are real. Then $E^* = 1$; consequently, in the transformed coordinates $y = (TP)^{-1}x$, the constants* (3.16) *become $M_y = 1/(1-q)$ and $\rho_y = r^* = r(1-q)$. The constants for the original variables follow from* (3.6)*.*

Proof. Let μ_i be an eigenvalue of A. Since μ_i is real and nonzero by (3.1), and $\left|\arg\mu_i\right| > \alpha\pi/2$ excludes $\arg\mu_i = 0$, we have $\arg\mu_i = \pi$, that is, $\mu_i < 0$. Hence $E_\alpha(\mu_i t^\alpha) = E_\alpha(-|\mu_i|\,t^\alpha)$ for $t \ge 0$. By Lemma 2.4, $s \mapsto E_\alpha(-s)$ is nonincreasing and nonnegative on $[0,\infty)$, so

$$0 \le E_\alpha(-|\mu_i|\,t^\alpha) \le E_\alpha(0) = 1 \qquad \text{for all } t\ge 0,$$

with equality at $t = 0$. Therefore, $E^*(\alpha,\mu_i) = \sup_{t\ge0}|E_\alpha(\mu_i t^\alpha)| = 1$ for every i, and $E^* = 1$. The remaining assertions follow from (3.13) and (3.16). □

Remark 3.4. *The sector condition* (3.1) *is not necessary for stability when $\lambda > 0$: for real $\mu > 0$, Lemmas 2.1 and 2.3 give $e^{-\lambda t}E_\alpha(\mu t^\alpha) \sim \frac{1}{\alpha}e^{(\mu^{1/\alpha}-\lambda)t}$, so the scalar equation ${}^{C}D^{\alpha,\lambda}_{0+}x = \mu x$ is asymptotically stable if $\mu < \lambda^\alpha$ and unstable if $\mu > \lambda^\alpha$, although* $\arg\mu = 0$ *violates* (3.1)*. Reaching this part of the spectrum requires exponentially damped Mittag-Leffler estimates in place of Lemma 2.7. We establish them now and use them with the subcritical weights $e^{\beta t}$, $0 \le \beta < \lambda$.*

Lemma 3.5. *Let $\alpha \in (0,1)$ and $\nu > 0$. Let $\mu \in \mathbb{C}$ satisfy $\mu = 0$ or $\left|\arg\mu\right| \le \frac{\alpha\pi}{2}$, and set $\omega := \mathrm{Re}(\mu^{1/\alpha})$ for $\mu \ne 0$ (principal branch, see* (1.2)*) and $\omega := 0$ for $\mu = 0$. Then $\omega \ge 0$. Assume $\omega < \nu$ and put $k(u) := u^{\alpha-1}\left|E_{\alpha,\alpha}(\mu u^\alpha)\right| e^{-\nu u}$ for $u > 0$. Then the following statements hold:*

(i) $\bar{C}_\nu(\alpha,\mu) := \sup\limits_{t\ge0}\int_0^t k(u)\,du < \infty$;

(ii) $k(u) \to 0$ *as* $u \to \infty$;

(iii) $\bar{E}_\nu(\alpha,\mu) := \sup\limits_{t\ge0} e^{-\nu t}\,|E_\alpha(\mu t^\alpha)| \in [1,\infty)$ *and* $e^{-\nu t}E_\alpha(\mu t^\alpha) \to 0$ *as* $t \to \infty$.

Proof. We first justify $\omega \geq 0$: for $\mu = 0$, this holds by definition, and for $\mu \neq 0$, the principal branch gives $\arg(\mu^{1/\alpha}) = \arg(\mu)/\alpha \in [-\pi/2, \pi/2]$ because $\left|\arg \mu\right| \leq \alpha\pi/2$, so

$$\omega = \mathrm{Re}(\mu^{1/\alpha}) = |\mu|^{1/\alpha} \cos(\arg(\mu)/\alpha) \geq 0,$$

with $\omega = 0$ precisely when $\mu = 0$ or $\left|\arg \mu\right| = \alpha\pi/2$. Consider now $\mu = 0$. Then $E_{\alpha,\alpha}(0) = 1/\Gamma(\alpha)$ and $E_\alpha(0) = 1$, so $k(u) = u^{\alpha-1}e^{-\nu u}/\Gamma(\alpha)$ and (ii), (iii) are immediate. Substituting $w \mapsto t - w$ in Lemma 2.2, with $\theta = \alpha$ and $\rho = \nu$, gives

$$\int_0^t k(u)\, du = \frac{1}{\Gamma(\alpha)} \int_0^t u^{\alpha-1} e^{-\nu u}\, du = e^{-\nu t}\, t^\alpha E_{1,\alpha+1}(\nu t),$$

and by Lemma 2.1, applied with $\beta = 1$, $\gamma = \alpha + 1$, and $x = \nu t$, the right-hand side converges to $\nu^{-\alpha}$ as $t \to \infty$. Being continuous, nondecreasing, and convergent, it is bounded, which proves (i). Now let $\mu \neq 0$ with $\left|\arg \mu\right| \leq \alpha\pi/2$, and fix ρ_0 with $\alpha\pi/2 < \rho_0 < \min\{\pi, \alpha\pi\}$. Since $\alpha \in (0, 1)$, this means $\rho_0 \in (\alpha\pi/2, \alpha\pi)$. The asymptotic expansion [1, Theorem 1.3] with $p = 1$ states that

$$E_{\alpha,\gamma}(z) = \frac{1}{\alpha}\, z^{(1-\gamma)/\alpha} \exp(z^{1/\alpha}) - \frac{z^{-1}}{\Gamma(\gamma - \alpha)} + O(|z|^{-2}), \qquad |z| \to \infty, \tag{3.19}$$

uniformly in the closed sector $\left|\arg z\right| \leq \rho_0$. For $z = \mu u^\alpha$ with $u > 0$, we have $\left|\arg z\right| = \left|\arg \mu\right| \leq \alpha\pi/2 < \rho_0$, so the ray on which (3.19) is invoked lies strictly inside its sector of validity; this includes the rays $\left|\arg \mu\right| = \alpha\pi/2$. Note that $z^{1/\alpha} = \mu^{1/\alpha} u$ on the principal branch, so $\left|\exp(z^{1/\alpha})\right| = e^{\omega u}$ and $\left|z^{(1-\alpha)/\alpha}\right| = |\mu|^{(1-\alpha)/\alpha}\, u^{1-\alpha}$. Take $\gamma = \alpha$ in (3.19); the middle term vanishes because the reciprocal Gamma function $1/\Gamma$ is entire with a zero at the origin, so the coefficient $1/\Gamma(\gamma - \alpha)$ equals zero for $\gamma = \alpha$. Since (3.19) holds uniformly on the closed sector $\left|\arg z\right| \leq \rho_0$, there are constants $R_0 = R_0(\alpha, \rho_0) \geq 1$ and $K_0 = K_0(\alpha, \rho_0) > 0$ such that the remainder term in (3.19) with $\gamma = \alpha$ is bounded by $K_0\, |z|^{-2}$ for all $|z| \geq R_0$ with $\left|\arg z\right| \leq \rho_0$, irrespective of the ray on which z lies. Multiplying by $u^{\alpha-1}$ and estimating all terms using modulus, we conclude that

$$u^{\alpha-1} \left|E_{\alpha,\alpha}(\mu u^\alpha)\right| \leq K_1\, e^{\omega u} + K_2\, u^{-\alpha-1}, \qquad u \geq u_1, \tag{3.20}$$

with the explicit constants

$$K_1 := \frac{|\mu|^{(1-\alpha)/\alpha}}{\alpha}, \qquad K_2 := K_0\, |\mu|^{-2}, \qquad u_1 := \max\{1, (R_0/\,|\mu|)^{1/\alpha}\}. \tag{3.21}$$

Only the upper bound (3.20) is used in the sequel. Note that K_1 depends on $|\mu|$ but not on $\arg \mu$, and that (3.20) holds with the same constants on the rays $\left|\arg \mu\right| = \alpha\pi/2$, where $\omega = 0$; see Remark 3.5. Consequently, for $u \geq u_1$,

$$k(u) \leq K_1\, e^{-(\nu-\omega)u} + K_2\, u^{-\alpha-1} e^{-\nu u},$$

and since $\nu - \omega > 0$, both summands tend to zero as $u \to \infty$ and belong to $L^1([u_1, \infty))$; this proves (ii) and the finiteness of $\int_{u_1}^\infty k(u)\, du$. On $(0, u_1]$, the triangle inequality in the defining series gives $\left|E_{\alpha,\alpha}(\mu u^\alpha)\right| \leq E_{\alpha,\alpha}(|\mu|\, u^\alpha) \leq E_{\alpha,\alpha}(|\mu|\, u_1^\alpha) =: c_2$ because $E_{\alpha,\alpha}$ has positive Taylor coefficients and is therefore increasing on $[0, \infty)$; hence

$$\int_0^{u_1} k(u)\, du \leq c_2 \int_0^{u_1} u^{\alpha-1}\, du = \frac{c_2\, u_1^\alpha}{\alpha} < \infty,$$

and (i) follows by combining the two ranges. For (iii), take $\gamma = 1$ in (3.19) and estimate using modulus as before: there are $t_1 \geq 1$ and constants $K_3, K_4 > 0$ (explicitly, $K_3 = 1/\alpha$, again independent of $\arg\mu$) with

$$|E_\alpha(\mu t^\alpha)| \leq K_3\, e^{\omega t} + K_4\, t^{-\alpha}, \qquad t \geq t_1,$$

so that $e^{-\nu t}\,|E_\alpha(\mu t^\alpha)| \leq K_3\, e^{-(\nu-\omega)t} + K_4\, t^{-\alpha} e^{-\nu t} \to 0$ as $t \to \infty$. On $[0, t_1]$, the function $t \mapsto e^{-\nu t}\,|E_\alpha(\mu t^\alpha)|$ is continuous, hence bounded, and its value at $t = 0$ is $E_\alpha(0) = 1$, so $\bar{E}_\nu(\alpha, \mu) \in [1, \infty)$. The proof is complete. □

Remark 3.5 (The critical rays $\left|\arg\mu\right| = \alpha\pi/2$)**.** *Lemma 3.5 is applied, in particular, to eigenvalues on the rays* $\left|\arg\mu\right| = \alpha\pi/2$*, where* $\omega = 0$ *and* (3.20) *reads* $u^{\alpha-1}\left|E_{\alpha,\alpha}(\mu u^\alpha)\right| \leq K_1 + K_2 u^{-\alpha-1}$*. No Stokes-type difficulty arises there for the following reasons. First, the expansion* (3.19) *is uniform on the closed sector* $\left|\arg z\right| \leq \rho_0$ *[1, Theorem 1.3], and the rays* $\left|\arg z\right| = \alpha\pi/2$ *lie in the interior of this sector because* $\rho_0 > \alpha\pi/2$*; the constants* R_0 *and* K_0 *of the proof are therefore the same for all rays of the sector. The Stokes phenomenon of* $E_{\alpha,\gamma}$*, i.e., the switching on and off of the exponential term, takes place across the rays* $\left|\arg z\right| = \alpha\pi$*, on which the term* $\exp(z^{1/\alpha})$ *is maximally recessive; these rays lie outside the sector used here, and beyond them, the expansion without an exponential term [1, Theorem 1.4] is the relevant one. The rays* $\left|\arg z\right| = \alpha\pi/2$ *are instead the anti-Stokes lines, on which* $\left|\exp(z^{1/\alpha})\right| = 1$*: the exponential term has the modulus* $\alpha^{-1}\,|z|^{(1-\gamma)/\alpha}$*, which neither grows nor decays exponentially (for* $\gamma = 1$*, it is the constant* $1/\alpha$*, and for* $\gamma = \alpha$*, it is exactly compensated by the factor* $u^{\alpha-1}$ *in* (3.20)*), oscillates, and is described exactly by* (3.19)*. Second, only the modulus bound* (3.20) *enters the proofs, and its constants* (3.21) *do not depend on* $\arg\mu$*. As a consistency check, for* $\left|\arg\mu\right| = \alpha\pi/2$*, the principal branch gives* $\mu^{1/\alpha} = \pm i\,|\mu|^{1/\alpha}$*, so* (3.19) *with* $\gamma = 1$ *yields* $E_\alpha(\mu t^\alpha) = \alpha^{-1} e^{\pm i|\mu|^{1/\alpha} t} + O(t^{-\alpha})$*, a bounded oscillation: this is the well-known fact that the classical linear Caputo equation is stable but not asymptotically stable on the boundary of its stability sector, and it explains why these rays belong to the interior of the tempered spectral set* $\mathcal{S}_{\alpha,\lambda,\beta}$ *of Theorem 3.2 as soon as* $\beta < \lambda$*: the tempering factor* $e^{-\lambda t}$ *turns the bounded oscillation into a decaying one.*

Theorem 3.2. *Let* $\alpha \in (0, 1)$*,* $\lambda \geq 0$*,* $\beta \in [0, \lambda]$*,* $A \in \mathbb{R}^{d\times d}$*, and* $f : \mathbb{R}^d \to \mathbb{R}^d$ *be locally Lipschitz continuous and satisfy* (3.2)*. With* $\omega(\mu) := \mathrm{Re}(\mu^{1/\alpha})$ *for* $\mu \neq 0$ *(principal branch, see* (1.2)*) and* $\omega(0) := 0$*, so that* $\omega(\mu) \geq 0$ *on the inner sector by Lemma 3.5, define the spectral set*

$$\mathcal{S}_{\alpha,\lambda,\beta} := \Big\{\mu \in \mathbb{C} \setminus \{0\} : \left|\arg\mu\right| > \frac{\alpha\pi}{2}\Big\} \cup \Big\{\mu \in \mathbb{C} : \Big(\mu = 0 \text{ or } \left|\arg\mu\right| \leq \frac{\alpha\pi}{2}\Big) \text{ and } \omega(\mu) < \lambda - \beta\Big\}. \quad (3.22)$$

Assume that every eigenvalue of A *belongs to* $\mathcal{S}_{\alpha,\lambda,\beta}$*. Then the trivial solution of* (1.1) *is exponentially asymptotically stable with rate* β*: there exist* $\rho_\beta > 0$ *and* $M_\beta \geq 1$ *such that* $\|x_0\| \leq \rho_\beta$ *implies that the solution is global,* $\|x(t; x_0)\| \leq M_\beta\, e^{-\beta t}\,\|x_0\|$ *for all* $t \geq 0$*, and* $e^{\beta t} x(t; x_0) \to 0$ *as* $t \to \infty$*.*

Proof. Set $\nu := \lambda - \beta \geq 0$. The complexification and the transformation to (3.4) are unchanged, and Remarks 3.2 and 3.3 hold for every rate, so it suffices to prove the claim for (3.4), whose diagonal entries $\mu_1, \ldots, \mu_n$ belong to $\mathcal{S}_{\alpha,\lambda,\beta}$. The functional frame is the space C_β defined by (3.7) with λ replaced by β, a Banach space by Lemma 3.2, and the operator $\mathcal{T}_x$ is defined by the same formula (3.8). Lemma 3.3 holds verbatim with C_λ replaced by C_β because its proof uses only the continuity and boundedness of ξ. For $i = 1, \ldots, n$, put $k_i(u) := u^{\alpha-1}\left|E_{\alpha,\alpha}(\mu_i u^\alpha)\right| e^{-\nu u}$ and

$$\bar{C}_i := \sup_{t\geq 0} \int_0^t k_i(u)\, du, \qquad \bar{E}_i := \sup_{t\geq 0} e^{-\nu t}\,|E_\alpha(\mu_i t^\alpha)|\,.$$

We claim that, for every i,

$$\bar{C}_i < \infty, \qquad \lim_{u\to\infty} k_i(u) = 0, \qquad \bar{E}_i < \infty, \qquad \lim_{t\to\infty} e^{-\nu t} E_\alpha(\mu_i t^\alpha) = 0. \tag{3.23}$$

If $\left|\arg \mu_i\right| > \alpha\pi/2$, then (3.23) follows from Lemma 2.7 and $e^{-\nu u} \le 1$: indeed, $\bar{C}_i \le C(\alpha, \mu_i)$ and $\bar{E}_i \le E^*(\alpha, \mu_i)$, the kernel decay follows from $k_i(u) \le M(\alpha, \mu_i) u^{-\alpha-1}$ for $u \ge t_0(\alpha, \mu_i)$, and the last limit follows from $E_\alpha(\mu_i t^\alpha) \to 0$. In the remaining case, μ_i belongs to the second set in (3.22), so $0 \le \omega(\mu_i) < \nu$, in particular, $\nu > 0$, and (3.23) is exactly Lemma 3.5. With (3.23) in place, the proofs of Proposition 3.1, Lemma 3.4, and Theorem 3.1 carry over to C_β with the three modifications indicated below, with no other change. First, multiplying the block-wise difference of operator values by $e^{\beta t}$ and using

$$e^{\beta t}\, e^{-\lambda(t-\tau)}\, e^{-\beta\tau} = e^{-\nu(t-\tau)}$$

in place of the exact cancellation $e^{\lambda t} e^{-\lambda(t-\tau)} e^{-\lambda\tau} = 1$, the substitution $u = t - \tau$ produces the damped kernel k_i, and (3.9) becomes

$$\left\|\mathcal{T}_x\xi - \mathcal{T}_{\hat{x}}\hat{\xi}\right\|_\beta \le \bar{E}\, \|x - \hat{x}\| + \bar{C}\, \ell_h(\max(\|\xi\|_\beta, \left\|\hat{\xi}\right\|_\beta))\, \left\|\xi - \hat{\xi}\right\|_\beta, \quad \bar{E} := \max_i \bar{E}_i, \quad \bar{C} := \max_i \bar{C}_i, \tag{3.24}$$

together with $\mathcal{T}_x\xi \in C_\beta$. Second, the scaling constant δ of the transformation (3.4)–(3.5), which was fixed by (3.11) in the proof of Theorem 3.1, is now fixed as $\bar{\delta} := 1/(2\bar{C})$ instead. This is legitimate for the same reason that made (3.11) well–defined: the constants $\bar{C}_i$ and $\bar{E}_i$ depend only on α, ν and the eigenvalues μ_i, and not on δ, whereas the transformed nonlinearity h in (3.5), and hence ℓ_h, do depend on δ. Lemma 3.1 holds for every $\delta > 0$ and, with $\delta = \bar{\delta}$, gives $\limsup_{r\to 0^+} \ell_h(r) \le \bar{\delta} = 1/(2\bar{C})$. Therefore the proof of Lemma 3.4(i), with $\bar{C}$ in place of $C(\alpha, \boldsymbol{\mu})$, yields $r > 0$ with $\bar{q} := \bar{C}\, \ell_h(r) \le 3/4$, and the proof of Lemma 3.4(ii), with (3.24) in place of (3.9) and $\bar{q}$ in place of q, shows that for every $\|x\| \le r^*_\beta := r(1 - \bar{q})/\bar{E}$, the operator $\mathcal{T}_x$ maps $B_{C_\beta}(0, r)$ into itself and is a $\bar{q}$-contraction there; hence it has a unique fixed point $\xi_x \in B_{C_\beta}(0, r)$. As in the proof of Theorem 3.1, ξ_x is the global solution $\varphi(\cdot, x)$ of (3.4) and satisfies $\|\xi_x\|_\beta \le \bar{E}(1 - \bar{q})^{-1}\, \|x\|$. Third, in the attractivity argument, $z(t) := e^{\beta t}\xi_x(t)$ satisfies the weighted fixed point identity

$$z^i(t) = e^{-\nu t} E_\alpha(\mu_i t^\alpha) x^i + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\mu_i (t-\tau)^\alpha) e^{-\nu(t-\tau)}\, e^{\beta\tau} h^i(\xi_x(\tau))\, d\tau,$$

with $e^{\beta\tau} \left\|h^i(\xi_x(\tau))\right\| \le \ell_h(r)\, \|z(\tau)\|$ as in (3.18). Given $\varepsilon' > 0$, choose $T' > 0$ with $\|z(t)\| \le a + \varepsilon'$ for $t \ge T'$, where $a := \limsup_{t\to\infty} \|z(t)\| \in [0, r]$, and split the integral at $\tau = T'$: the piece over $[0, T']$ is bounded in norm by $\ell_h(r)\, r\, T'\, \sup_{u\ge t-T'} k_i(u)$, which tends to zero as $t \to \infty$ by the second property in (3.23); the piece over $[T', t]$ is bounded by $\ell_h(r)\, (a + \varepsilon')\, \bar{C}_i$ after the substitution $u = t - \tau$; and the first summand tends to zero by the fourth property in (3.23). Exactly as in the proof of Theorem 3.1, this yields $a \le \bar{q}\, (a + \varepsilon')$ for every $\varepsilon' > 0$, hence $a = 0$ and $e^{\beta t}\varphi(t, x) \to 0$. The trivial solution of (3.4) is therefore exponentially asymptotically stable with rate β and constants $(r^*_\beta, \bar{E}(1 - \bar{q})^{-1})$, and transporting the constants through Remark 3.2 and (3.6) completes the proof. □

Corollary 3.3. *Let $\alpha \in (0, 1)$ and $\lambda > 0$, and let $f : \mathbb{R}^d \to \mathbb{R}^d$ be locally Lipschitz continuous and satisfy* (3.2). *Assume that the spectrum $\sigma(A)$ of A satisfies*

$$\sigma(A) \subset \left\{\mu \in \mathbb{C}\setminus\{0\} : \left|\arg \mu\right| > \frac{\alpha\pi}{2}\right\} \cup \left\{\mu \in \mathbb{C} : \mu = 0 \text{ or } \left(\left|\arg \mu\right| \le \frac{\alpha\pi}{2} \text{ and } \operatorname{Re}(\mu^{1/\alpha}) < \lambda\right)\right\} = \mathcal{S}_{\alpha,\lambda,0}. \tag{3.25}$$

Then the trivial solution of (1.1) *is asymptotically stable. Moreover, let* ω *denote the maximum of* $\omega(\mu)$ *over the inner-sector eigenvalues of A (those with* $\mu = 0$ *or* $\left|\arg\mu\right| \le \alpha\pi/2$*), with* $\omega := 0$ *if there are none. Then the trivial solution is exponentially asymptotically stable with every rate* $\beta \in [0, \lambda - \omega)$*, and if every eigenvalue lies in the classical sector* (3.1)*, the critical rate* $\beta = \lambda$ *is also admissible.*

Proof. If all eigenvalues satisfy (3.1), the claims follow from Theorem 3.1. Otherwise, (3.25) gives $\omega \in [0, \lambda)$, and for every $\beta \in [0, \lambda - \omega)$, every eigenvalue of A belongs to $\mathcal{S}_{\alpha,\lambda,\beta}$, so Theorem 3.2 applies. The choice $\beta = 0$ gives asymptotic stability. The proof is complete. □

Remark 3.6 (Relation between the two theorems)**.** *For* $\beta = \lambda$*, the second set in* (3.22) *is empty because* $\omega(\mu) \ge 0$ *excludes* $\omega(\mu) < 0$*; hence* $\mathcal{S}_{\alpha,\lambda,\lambda}$ *is the classical sector* (3.1)*, and Theorem 3.2 with* $\beta = \lambda$ *is precisely Theorem 3.1. Conversely, whenever A possesses an inner-sector eigenvalue (including* $\mu = 0$*), the critical rate* $\beta = \lambda$ *is not attainable by Theorem 3.2, and only the subcritical rates* $\beta < \lambda - \omega$ *are covered, where* ω *is the largest value of* $\omega(\mu)$ *over the inner-sector eigenvalues (Corollary 3.3). Thus Theorem 3.2 extends Theorem 3.1 in the admissible spectrum, at the price of a smaller guaranteed rate as soon as inner-sector eigenvalues occur. Theorem 3.1 is the special case of Theorem 3.2 in which no such eigenvalue is present and the critical rate is attained. Note that* $\mathcal{S}_{\alpha,\lambda,\beta}$ *contains* $\mu = 0$ *and the critical rays* $\left|\arg\mu\right| = \alpha\pi/2$ *whenever* $\beta < \lambda$*, which the classical theory (*$\lambda = 0$*) never covers.*

Remark 3.7 (Geometry of the spectral set and the boundary cases)**.** *The two parts of* (3.22) *are governed by two different spectral conditions, an angular one and one on* $\mathrm{Re}(\mu^{1/\alpha})$*; Figure 1 shows them for* $\alpha = 0.7$*. The origin of the second condition is the asymptotic behavior of the inner-sector modes: by* (3.19) *with* $\gamma = 1$*, for* $\mu \ne 0$ *with* $\left|\arg\mu\right| \le \alpha\pi/2$*,*

$$e^{-\lambda t}E_\alpha(\mu t^\alpha) = \frac{1}{\alpha}\, e^{(\mu^{1/\alpha}-\lambda)t} + O(t^{-\alpha}e^{-\lambda t}) \qquad \text{as } t \to \infty, \tag{3.26}$$

so the mode behaves like the exponential $e^{(z-\lambda)t}$ *with the complex exponent* $z = \mu^{1/\alpha}$*: it tends to zero if* $\mathrm{Re}\, z < \lambda$*, is unbounded if* $\mathrm{Re}\, z > \lambda$*, and has a bounded, non-decaying leading term if* $\mathrm{Re}\, z = \lambda$*. With the weight* $e^{\beta t}$*, the same trichotomy holds with* λ *replaced by* $\lambda - \beta$*. Geometrically, the principal branch* $z = \mu^{1/\alpha}$ *of* (1.2) *maps the closed inner sector* $\{\mu \ne 0 : \left|\arg\mu\right| \le \alpha\pi/2\}$ *bijectively onto the closed right half-plane* $\{\mathrm{Re}\, z \ge 0\}$ *minus the origin, the rays* $\left|\arg\mu\right| = \alpha\pi/2$ *going to the imaginary axis. The condition* $\mathrm{Re}(\mu^{1/\alpha}) < \lambda - \beta$ *is therefore the preimage of the half-plane* $\{\mathrm{Re}\, z < \lambda - \beta\}$*, and the inner part of* $\mathcal{S}_{\alpha,\lambda,\beta}$ *is the image of the vertical strip* $\{z : 0 \le \mathrm{Re}\, z < \lambda - \beta\}$ *under the principal power* $z \mapsto z^\alpha$*, which contains* $\mu = 0$ *as the image of* $z = 0$*. For* $\beta < \lambda$*, the boundary of* $\mathcal{S}_{\alpha,\lambda,\beta}$ *is the curve* $\{(\lambda - \beta + is)^\alpha : s \in \mathbb{R}\}$ *(the rays* $\left|\arg\mu\right| = \alpha\pi/2$ *being interior points, see Remark 3.5), which meets the real axis at* $\mu = (\lambda - \beta)^\alpha$ *and is asymptotic to the rays as* $|s| \to \infty$ *(Figure 1(a)). In the z-plane, it is simply the vertical line* $\mathrm{Re}\, z = \lambda - \beta$ *(Figure 1(b)). The threshold* λ^α *of the Lotka-Volterra application (Proposition 3.2) is the point at which this curve, for* $\beta = 0$*, meets the positive real axis.*

Condition (3.25) *is an exact spectral description for the* linear *tempered equation: it characterizes the set of* $\mu \in \mathbb{C}$ *for which every solution of the scalar equation* ${}^{C}D_{0+}^{\alpha,\lambda}x = \mu x$ *tends to zero, and it is the interior of the linear stability region. The boundary is excluded for the following reasons:*

(i) At $\mathrm{Re}(\mu^{1/\alpha}) = \lambda$*, the mode* $e^{-\lambda t}E_\alpha(\mu t^\alpha)$ *is not attractive: for real* $\mu = \lambda^\alpha$*, it converges to* $1/\alpha$ *(Lemma 2.1), and for complex* μ*, its leading term* $\alpha^{-1}e^{i\,\mathrm{Im}(\mu^{1/\alpha})t}$ *has the constant modulus* $1/\alpha$ *and oscillates. The scalar linear equation is then stable but not asymptotically stable, and the linear*

system is unstable if such a μ belongs to a nontrivial Jordan block because differentiating the solution formula with respect to μ produces polynomial prefactors.

(ii) For $\lambda = 0$, the rays $|\arg\mu| = \alpha\pi/2$ form the boundary of the classical sector, and $E_\alpha(\mu t^\alpha)$ is bounded but does not tend to zero there, since $\mu^{1/\alpha}$ is purely imaginary (Remark 3.5). For $\lambda > 0$, these rays and the origin lie in the interior of the region since $\omega(\mu) = 0 < \lambda$, and the corresponding modes decay like $e^{-\lambda t}$.

(iii) Being local, the nonlinear results say nothing at the boundary: as in the classical theory, the nonlinear term may then either stabilize or destabilize the trivial solution, and its behavior must be studied by other means (compare Remark 3.8 for the Lotka-Volterra system). More generally, for the nonlinear Eq (1.1)*, the conclusions of Theorems 3.1 and 3.2 are local, require assumption* (3.2)*, and are of the form "the linear part decides", in the spirit of Lyapunov's first method; the phrase "full open asymptotic-stability region" always refers to the spectrum of the linear part.*

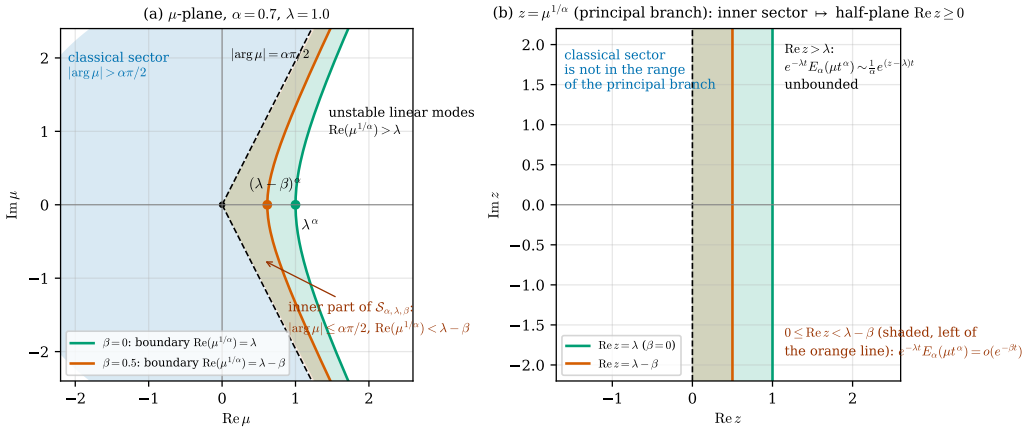


Figure 1. The spectral set $\mathcal{S}_{\alpha,\lambda,\beta}$ of Theorem 3.2 for $\alpha = 0.7$, $\lambda = 1$, and $\beta \in \{0, 0.5\}$. (a) In the μ-plane: the classical sector $|\arg\mu| > \alpha\pi/2$ (blue, Theorem 3.1), bounded by the dashed rays; the inner part is $\{|\arg\mu| \le \alpha\pi/2,\ \mathrm{Re}(\mu^{1/\alpha}) < \lambda - \beta\}$ (shaded, Theorem 3.2), bounded by the curve $\mathrm{Re}(\mu^{1/\alpha}) = \lambda - \beta$ (green for $\beta = 0$, orange for $\beta = 0.5$), which meets the real axis at $(\lambda - \beta)^\alpha$ and is asymptotic to the rays; to the right of the green, curve the linear modes $e^{-\lambda t}E_\alpha(\mu t^\alpha)$ are unbounded. (b) The same sets in the plane of $z = \mu^{1/\alpha}$ (principal branch): the inner sector becomes the closed half-plane $\mathrm{Re}\, z \ge 0$, its bounding rays become the imaginary axis, and the stability boundaries become the vertical lines $\mathrm{Re}\, z = \lambda - \beta$.

We conclude this section with an application extending the example of [25] to the tempered setting. Consider the tempered fractional Lotka-Volterra system

$$\begin{cases} {}^{C}D^{\alpha,\lambda}_{0+}x_1(t) = x_1(t)(h + a\,x_1(t) + b\,x_2(t)), \\ {}^{C}D^{\alpha,\lambda}_{0+}x_2(t) = x_2(t)(-\varrho + c\,x_1(t)), \end{cases} \tag{3.27}$$

where $h, \varrho > 0$ and $a, b, c \in \mathbb{R}$. System (3.27) has the form (1.1) with

$$A = \begin{bmatrix} h & 0 \\ 0 & -\varrho \end{bmatrix}, \qquad f(x) = \begin{bmatrix} a\,x_1^2 + b\,x_1x_2 \\ c\,x_1x_2 \end{bmatrix}, \tag{3.28}$$

and f satisfies (3.2) because it is a quadratic polynomial vanishing to second order at the origin; indeed, (3.28) is exactly the decomposition $F(x) = DF(0)\,x + [F(x) - DF(0)\,x]$ of the right-hand side F of

(3.27), with $A = DF(0)$ the Jacobian at the zero equilibrium, so that (3.27) is the setting of Lyapunov's first method described at the beginning of this section (see also Remark 3.1(b)). The connection with the general theory is as follows. The eigenvalues of A are $h > 0$ and $-\varrho < 0$. The eigenvalue $-\varrho$ lies on the negative real axis, $\left|\arg(-\varrho)\right| = \pi > \alpha\pi/2$, hence it is in the classical sector for every α. The eigenvalue h lies on the positive real axis, $\arg h = 0 \le \alpha\pi/2$, so it belongs to the inner sector and is never admissible in the classical theory: for $\lambda = 0$, the zero equilibrium is unstable for every $h > 0$ (Proposition 3.2(i) with $\lambda = 0$). In the tempered case, its admissibility is decided by Theorem 3.2 through $\omega(h) = \mathrm{Re}(h^{1/\alpha}) = h^{1/\alpha}$: the eigenvalue h belongs to $\mathcal{S}_{\alpha,\lambda,\beta}$ if and only if $h^{1/\alpha} < \lambda - \beta$, that is, $h < (\lambda - \beta)^\alpha$, and the largest admissible growth rate, obtained for $\beta = 0$, is λ^α, the threshold of Remark 3.4: below it, the linear mode $e^{-\lambda t}E_\alpha(ht^\alpha) \sim \alpha^{-1}e^{(h^{1/\alpha}-\lambda)t}$ decays, and above it, the mode grows (Figure 1). The following proposition shows that this linear threshold is also the nonlinear one, except in the critical case where $h = \lambda^\alpha$, which is not covered.

Proposition 3.2. *Consider system* (3.27) *with* $\alpha \in (0,1)$, $\lambda \ge 0$, $h, \varrho > 0$, *and* $a, b, c \in \mathbb{R}$. *Then the following statements hold.*

(i) If $h > \lambda^\alpha$, *then the trivial solution of* (3.27) *is unstable.*

(ii) If $h < \lambda^\alpha$, *then the trivial solution of* (3.27) *is asymptotically stable; more precisely, it is exponentially asymptotically stable with every rate* $\beta \in [0, \lambda - h^{1/\alpha})$, *but not necessarily with the limiting rate* $\lambda - h^{1/\alpha}$ *itself.*

(iii) Let $B = (1,1)^T$ *and* $K = (-2h, 0)$. *Then, for every* $h > 0$, *the trivial solution of the closed-loop system*

$$ {}^{C}D^{\alpha,\lambda}_{0+}x(t) = Ax(t) + f(x(t)) + Bu(t), \qquad u(t) = Kx(t), \tag{3.29} $$

is exponentially asymptotically stable with rate λ, *for every* $\alpha \in (0,1)$ *and every* $\lambda \ge 0$; *in particular, the feedback* $u = Kx$ *locally stabilizes the origin.*

The critical case where $h = \lambda^\alpha$ *is not covered by (i) and (ii); see Remark 3.8.*

Proof. We first prove (i). Since $h > \lambda^\alpha$, we may fix a number $c_0 \in (\lambda^\alpha, h)$, for instance, $c_0 := (\lambda^\alpha + h)/2$, so that c_0 depends only on α, λ, and h, and set

$$ \varepsilon := \frac{h - c_0}{|a| + 1} > 0, $$

so that $h - |a|\,\varepsilon \ge c_0$. Suppose, to the contrary, that the trivial solution of (3.27) is stable. Then there exists $\delta \in (0, \varepsilon)$ such that every solution with $\|x(0)\| < \delta$ is global and satisfies $\|x(t)\| \le \varepsilon$ for all $t \ge 0$. Consider the solution $x(\cdot) = (x_1(\cdot), x_2(\cdot))$ with $x_1(0) = \delta/2$ and $x_2(0) = 0$. We claim that $x_2 \equiv 0$. Indeed, the second component of the mild Eq (2.2) reads

$$ x_2(t) = \frac{1}{\Gamma(\alpha)} \int_0^t e^{-\lambda(t-s)}(t-s)^{\alpha-1}\, x_2(s)\big(-\varrho + c\, x_1(s)\big)\, ds, $$

so, with $K_0 := \varrho + |c|\,\varepsilon$ and $e^{-\lambda(t-s)} \le 1$,

$$ |x_2(t)| \le K_0\, (I^\alpha_{0+}\, |x_2|)(t), \qquad t \ge 0. $$

Fix $T > 0$. Since the operator I_{0+}^{α} has a nonnegative kernel, it preserves pointwise inequalities between nonnegative functions, so the last bound may be iterated. Iterating it n times and using the semigroup property of the Riemann-Liouville integrals together with $I_{0+}^{n\alpha}1 = t^{n\alpha}/\Gamma(n\alpha+1)$ [1,2] gives

$$|x_2(t)| \le K_0^n \left(I_{0+}^{n\alpha}|x_2|\right)(t) \le \Big(\sup_{[0,T]}|x_2|\Big)\frac{(K_0T^{\alpha})^n\,\Gamma(1)}{\Gamma(n\alpha+1)}, \qquad t \in [0,T],$$

where $K_0^n T^{n\alpha}/\Gamma(n\alpha+1)$ is the general term of the convergent series defining $E_{\alpha}(K_0T^{\alpha})$ and therefore tends to zero as $n \to \infty$. Hence $x_2 \equiv 0$ on $[0,T]$ for every $T > 0$, which proves the claim, and the first component satisfies

$$^{C}D_{0+}^{\alpha,\lambda}x_1(t) = x_1(t)(h + a\,x_1(t)), \qquad x_1(0) = \frac{\delta}{2} > 0, \tag{3.30}$$

where the derivative exists by Proposition 2.1(i) and, being equal to the continuous function $x_1(h + a\,x_1)$, extends continuously to $[0,\infty)$. Let $T^* := \sup\{t \ge 0 : x_1 \ge 0 \text{ on } [0,t]\} \in (0,\infty]$. For $t \in [0,T^*)$, we have $0 \le x_1(t) \le \varepsilon$, hence

$$h + a\,x_1(t) \ge h - |a|\,\varepsilon \ge c_0,$$

and therefore, using $x_1(t) \ge 0$,

$$^{C}D_{0+}^{\alpha,\lambda}x_1(t) = x_1(t)(h + a\,x_1(t)) \ge c_0\,x_1(t) \qquad \text{on } (0,T^*).$$

The right-hand side of (3.30) is continuous along the solution, so Lemma 2.6 applies on $[0,T^*)$ and yields

$$x_1(t) \ge e^{-\lambda t}E_{\alpha}(c_0t^{\alpha})\,\frac{\delta}{2} > 0, \qquad t \in [0,T^*), \tag{3.31}$$

where positivity follows from Lemma 2.4 and $E_{\alpha}(c_0t^{\alpha}) \ge E_{\alpha}(0) = 1 > 0$ for $c_0 > 0$. Suppose that T^* is finite. Then, by continuity, $x_1(T^*) \ge e^{-\lambda T^*}E_{\alpha}(c_0(T^*)^{\alpha})\,\delta/2 > 0$, so x_1 would remain positive on a neighborhood of T^*, contradicting the definition of T^*. Hence $T^* = \infty$ and (3.31) holds for all $t \ge 0$. We now apply Lemma 2.1 with $\beta = \alpha$ and $\gamma = 1$ to the argument $x = c_0^{1/\alpha}t$, for which $x^{\beta} = c_0t^{\alpha}$: this gives $E_{\alpha}(c_0t^{\alpha}) \sim \alpha^{-1}\exp(c_0^{1/\alpha}t)$ as $t \to \infty$, and therefore

$$e^{-\lambda t}E_{\alpha}(c_0t^{\alpha}) \sim \frac{1}{\alpha}\,e^{(c_0^{1/\alpha}-\lambda)t} \qquad \text{as } t \to \infty,$$

where $c_0^{1/\alpha} > \lambda$ because $c_0 > \lambda^{\alpha}$. Hence the right-hand side of (3.31) tends to infinity, contradicting $x_1(t) \le \varepsilon$ for all $t \ge 0$. This contradiction proves that the trivial solution of (3.27) is unstable. We next prove (ii). The matrix A in (3.28) has the eigenvalues $-\varrho$ and h. The eigenvalue $-\varrho$ satisfies $\left|\arg(-\varrho)\right| = \pi > \alpha\pi/2$. The eigenvalue h is positive, so $\arg h = 0 \le \alpha\pi/2$ and $\omega(h) = h^{1/\alpha}$, and the assumption $h < \lambda^{\alpha}$ is equivalent to $h^{1/\alpha} < \lambda$. Hence, for every $\beta \in [0, \lambda - h^{1/\alpha})$, both eigenvalues belong to the spectral set $\mathcal{S}_{\alpha,\lambda,\beta}$ of (3.22). The nonlinearity f is a quadratic polynomial with $f(0) = 0$ and a vanishing Jacobian at 0; on the ball $\bar{B}(0,\rho)$, the induced maximum norm of its Jacobian (the maximal row sum) is bounded by $2(|a| + |b| + |c|)\rho$, so $\ell_f(\rho) \le 2(|a| + |b| + |c|)\rho \to 0$ as $\rho \to 0^+$ and (3.2) holds. Theorem 3.2 therefore gives exponential asymptotic stability with every rate $\beta \in [0, \lambda - h^{1/\alpha})$, and the choice $\beta = 0$ gives asymptotic stability. We finally prove (iii). The closed-loop system (3.29) has the form of (1.1) with the same f as in (3.28) and with the linear part

$$A + BK = \begin{bmatrix} h & 0 \\ 0 & -\varrho \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} -2h & 0 \end{bmatrix} = \begin{bmatrix} -h & 0 \\ -2h & -\varrho \end{bmatrix}.$$

This matrix is lower triangular with eigenvalues $-h < 0$ and $-\varrho < 0$, so

$$\left|\arg(-h)\right| = \left|\arg(-\varrho)\right| = \pi > \frac{\alpha\pi}{2} \qquad \text{for every } \alpha \in (0,1),$$

and (3.1) holds. (3.2) was verified above. Theorem 3.1 now gives exponential asymptotic stability with rate λ for the trivial solution of (3.29). The proof is complete. □

Remark 3.8. *Proposition 3.2 settles the stability of the zero equilibrium of* (3.27) *for every $h \neq \lambda^\alpha$, and the critical case where $h = \lambda^\alpha$ is the only one left open: in the critical case, the linearization is not attractive ($e^{-\lambda t}E_\alpha(ht^\alpha) \to 1/\alpha$ by Lemma 2.1), so neither Theorem 3.2 nor the comparison argument of (i) applies, and the behavior of the nonlinear system, which may depend on the coefficients a, b, c, is not decided by the linearization (compare Remark 3.7(iii)). In (iii), the closed-loop spectrum is real and negative, so Corollary 3.2 gives $M_y = 1/(1-q)$ and radius $r(1-q)$ in the transformed coordinates, transported through* (3.6)*. The pair (B, K) is a mathematical benchmark in the spirit of [25]; positivity of the populations, saturation, and robustness are not addressed.*

4. Numerical simulations

By Lemma 2.5 and Proposition 2.1, x solves (1.1) if and only if $y(t) := e^{\lambda t}x(t)$ solves the classical Caputo problem

$${}^{C}D^{\alpha}_{0+}y(t) = g(t, y(t)) := Ay(t) + e^{\lambda t}f(e^{-\lambda t}y(t)), \qquad y(0) = x(0), \tag{4.1}$$

so the tempering can be treated exactly. We integrate (4.1) with the Adams-Bashforth-Moulton predictor-corrector scheme of [26] and recover $x(t) = e^{-\lambda t}y(t)$; dedicated tempered solvers are also available [30–32], and high-order schemes for fractional partial differential equations are developed in [4–6]. On the uniform grid $t_j = jh_{\mathrm{s}}$ with step size $h_{\mathrm{s}} = 10^{-2}$, used in every experiment below (the mesh-refinement study of Table 1 and the step-size sensitivity study of Table 2 vary the step deliberately), one time step consists of the predictor and a single corrector evaluation (PECE mode),

$$y^{P}_{n+1} = y_0 + \frac{h_{\mathrm{s}}^{\alpha}}{\Gamma(\alpha+1)}\sum_{j=0}^{n}\left[(n+1-j)^{\alpha} - (n-j)^{\alpha}\right] g(t_j, y_j),$$

$$y_{n+1} = y_0 + \frac{h_{\mathrm{s}}^{\alpha}}{\Gamma(\alpha+2)}\Big[\sum_{j=0}^{n} a_{j,n+1}\, g(t_j, y_j) + g\big(t_{n+1}, y^{P}_{n+1}\big)\Big],$$

with the standard corrector weights $a_{0,n+1} = n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}$ and $a_{j,n+1} = (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}$ for $1 \le j \le n$ [26]. The factor $e^{\lambda t}$ in g is evaluated exactly at the grid points. The complete procedure is summarized in Algorithm 1. The scheme is explicit: no nonlinear system is solved and no iteration tolerance is involved, so the step size h_{s} is the only discretization parameter. The two memory sums are evaluated as full convolutions in double precision with precomputed weights, at a cost of $O(N^2)$ operations and $O(Nd)$ memory for $N = T/h_{\mathrm{s}}$ steps. The theoretical order of convergence is $\min\{2, 1+\alpha\}$ [26]. The exponential growth of $y = e^{\lambda t}x$ on long intervals is harmless in floating-point arithmetic because all rounding errors scale with y and are removed by the final multiplication with $e^{-\lambda t}$. All norms are maximum norms, every observed slope reported below is the least-squares slope of

$\log \|x(t)\|$ over the stated window, and the Python code and the numerical data underlying every figure and table are available from the author upon request.

Table 1. Mesh-refinement study for the linear problem of Example 4.1 ($\alpha = 0.7$, $\lambda = 0.3$): maximum error against the exact Mittag-Leffler solution over $t \in \{0.4, 0.8, \dots, 10\}$ for $T = 10$ and over $t \in \{1, 2, \dots, 40\}$ for $T = 40$, and the empirical order of convergence (EOC).

T	N	Step size	Max. error	EOC
10	250	0.0400	2.99×10^{-3}	—
10	500	0.0200	8.64×10^{-4}	1.79
10	1000	0.0100	2.57×10^{-4}	1.75
10	2000	0.0050	7.80×10^{-5}	1.72
10	4000	0.0025	2.39×10^{-5}	1.71
40	1000	0.0400	1.36×10^{-3}	—
40	2000	0.0200	4.05×10^{-4}	1.75
40	4000	0.0100	1.22×10^{-4}	1.73
40	8000	0.0050	3.73×10^{-5}	1.71

Table 2. Step-size sensitivity of the observed least-squares slopes of $\log \|x(t)\|$ on the long fitting windows (log-log slope for Example 4.1 with $\lambda = 0$ and semi-logarithmic slopes otherwise); all other parameters as in the respective examples.

Experiment	Window	$h_s = 0.04$	$h_s = 0.02$	$h_s = 0.01$	$h_s = 0.005$
Ex. 4.1, $\lambda = 0$	$[20, 40]$	−0.6939	−0.6938	−0.6937	−0.6937
Ex. 4.1, $\lambda = 0.3$	$[20, 40]$	−0.3237	−0.3237	−0.3237	−0.3237
Ex. 4.1, $\lambda = 0.8$	$[20, 40]$	−0.8237	−0.8237	−0.8237	−0.8237
Ex. 4.4, $h = 0.25$	$[30, 60]$	−0.0861	−0.0861	−0.0861	−0.0861
Ex. 4.4, closed loop	$[10, 20]$	−0.4278	−0.4278	−0.4278	−0.4278

Algorithm 1 PECE scheme for the tempered problem (1.1), applied to the conjugated problem (4.1)

Require: order $\alpha \in (0, 1)$, tempering parameter $\lambda \geq 0$, matrix A, nonlinearity f, initial value x_0, final time T, step size h_s

$N \leftarrow T/h_s$; $t_j \leftarrow jh_s$ for $j = 0, \dots, N$; $g(t, y) \leftarrow Ay + e^{\lambda t} f(e^{-\lambda t} y)$

precompute $b_k \leftarrow (k+1)^\alpha - k^\alpha$ and $c_k \leftarrow (k+2)^{\alpha+1} + k^{\alpha+1} - 2(k+1)^{\alpha+1}$ for $k = 0, \dots, N$

$y_0 \leftarrow x_0$; $g_0 \leftarrow g(t_0, y_0)$

for $n = 0, \dots, N-1$ **do**

$$y^P_{n+1} \leftarrow y_0 + \frac{h_s^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{n} b_{n-j}\, g_j \qquad \triangleright \text{ predictor (fractional Adams-Bashforth)}$$

$a_{0,n+1} \leftarrow n^{\alpha+1} - (n-\alpha)(n+1)^\alpha$; $a_{j,n+1} \leftarrow c_{n-j}$ for $j = 1, \dots, n$

$$y_{n+1} \leftarrow y_0 + \frac{h_s^\alpha}{\Gamma(\alpha+2)} \Big[\sum_{j=0}^{n} a_{j,n+1}\, g_j + g(t_{n+1}, y^P_{n+1}) \Big] \qquad \triangleright \text{ corrector (fractional Adams-Moulton)}$$

$g_{n+1} \leftarrow g(t_{n+1}, y_{n+1})$

end for

return $x_j \leftarrow e^{-\lambda t_j} y_j$ for $j = 0, \dots, N$ ▷ tempered solution on the grid

Guaranteed versus observed quantities. Throughout this section, we distinguish between two kinds of statements. *Guaranteed* quantities are those provided by the theory: the envelope $Me^{-\lambda t}$ and the limit $e^{\lambda t} x(t) \to 0$ of Theorem 3.1, the rates $\beta < \lambda - \omega$ of Theorem 3.2, and the instability statement of

Proposition 3.2(i). *Observed* quantities are features of the particular computed trajectories: least-squares slopes of $\log \|x(t)\|$ on a stated window, maxima of weighted norms, monotonicity, or the ordering of several trajectories. An observed slope steeper than a guaranteed rate is consistent with the theory but does not establish a stronger theoretical rate, and features such as monotone decay, the value $M = 1$ of an envelope constant, or the ordering of trajectories are properties of the examples shown and not consequences of the theorems.

To quantify the discretization error, the scheme was compared, on the linear problem of Example 4.1 with $\lambda = 0.3$, on the intervals $[0, 10]$ and $[0, 40]$, against the exact solution $x(t) = e^{-\lambda t} E_\alpha(t^\alpha A)x_0$, evaluated through the eigendecomposition of the normal matrix A and high-precision summation (80 significant digits) of the Mittag-Leffler series. Table 1 shows the convergence of empirical order about 1.7, matching the theoretical order $\min\{2, 1 + \alpha\} = 1.7$ of the fractional Adams method [26]. At $h_s = 10^{-2}$, the error is below 2.6×10^{-4} on $[0, 10]$ and below 1.3×10^{-4} on $[0, 40]$, negligible on the scale of every quantity reported below. Since several observed slopes below are measured on windows extending to $t = 40$ or $t = 60$, Table 2 additionally reports the observed slopes of the long-time experiments for four step sizes: they differ by at most 2×10^{-4}, so the fitted slopes are insensitive to the discretization.

Example 4.1. *Consider the linear equation* ${}^C D^{\alpha,\lambda}_{0+} x(t) = Ax(t)$ *with*

$$A = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}, \qquad \alpha = 0.7, \qquad x(0) = (1,\, 0.5)^T.$$

The eigenvalues of A are $-1 \pm 2i$ with $\left|\arg(-1 \pm 2i)\right| \approx 2.034 > \alpha\pi/2 \approx 1.100$, so Corollary 2.1 guarantees exponential asymptotic stability with rate λ for every $\lambda \geq 0$. Figure 2 shows $\|x(t)\|$ on a logarithmic scale ($A = \left[\begin{smallmatrix} -1 & 2 \\ -2 & -1 \end{smallmatrix}\right]$, $\alpha = 0.7$, $x(0) = (1, 0.5)^T$, and $\lambda \in \{0, 0.3, 0.8\}$) for $\lambda \in \{0,\, 0.3,\, 0.8\}$. For $\lambda = 0$, the decay is algebraic: the observed log-log slope on $[20, 40]$ is $-0.694 \approx -\alpha$, in agreement with $E_\alpha(\mu t^\alpha) = O(t^{-\alpha})$; this is the algebraic decay of the case "rate 0" discussed in Remark 2.3(b). For $\lambda = 0.3$ and $\lambda = 0.8$, the observed semi-logarithmic slopes on $[20, 40]$ are -0.324 and -0.824: the decay is slightly faster than the guaranteed envelope $e^{-\lambda t}$ because of the algebraic factor $t^{-\alpha}$, whose logarithmic derivative $-\alpha/t \approx -0.0233$ at the window midpoint accounts for the measured excess -0.0237, illustrating that the gain over the envelope is algebraic and not a larger exponential rate [1, Theorem 1.4] (Remark 2.3(a)).

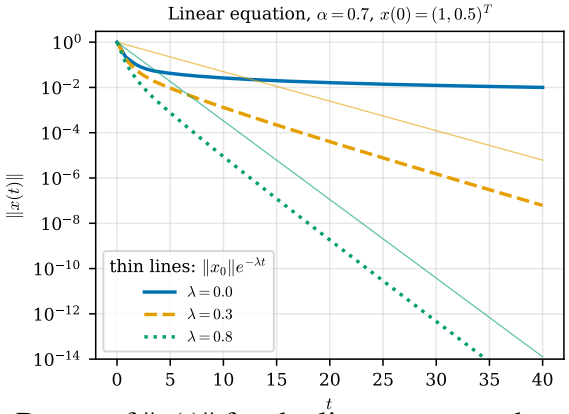


Figure 2. Example 4.1: Decay of $\|x(t)\|$ for the linear tempered equation ${}^C D^{\alpha,\lambda}_{0+} x = Ax$, this lines show the guaranteed envelopes $\|x_0\| e^{-\lambda t}$ for $\lambda = 0.3$ and $\lambda = 0.8$.

Example 4.2 (Dependence on the order α)**.** *To illustrate that the conclusions of Corollary 2.1 and the explanation of the excess over the envelope given in Example 4.1 hold uniformly in the order, the linear equation of Example 4.1 was integrated for* $\alpha \in \{0.3, 0.5, 0.7, 0.9\}$ *with* $\lambda = 0$ *and* $\lambda = 0.3$*. The sector condition holds for every such* α *because* $\left|\arg(-1 \pm 2i)\right| \approx 2.034 > 0.9\pi/2 \approx 1.414$*. Figure 3 shows* $\|x(t)\|$ *together with the observed slopes on* $[20, 40]$*. For* $\lambda = 0$*, the log-log slopes are* -0.303*,* -0.504*,* -0.694*, and* -0.888*, close to* $-\alpha$ *as predicted by the algebraic asymptotics* $E_\alpha(\mu t^\alpha) = O(t^{-\alpha})$*. For* $\lambda = 0.3$*, the semi-logarithmic slopes are* -0.310*,* -0.317*,* -0.324*, and* -0.330*, i.e., the guaranteed exponent* $-\lambda$ *plus an excess of* -0.010*,* -0.017*,* -0.024*, and* -0.030*, which agrees to within* 10^{-3} *with the logarithmic derivative* $-\alpha/t$ *of the factor* $t^{-\alpha}$ *at the window midpoint* $t = 30$ *(*-0.010*,* -0.017*,* -0.023*, and* -0.030*). The order* α *thus changes only the algebraic correction, whereas the exponential envelope* $e^{-\lambda t}$ *is the same for every* α*, as stated in Corollary 2.1.*

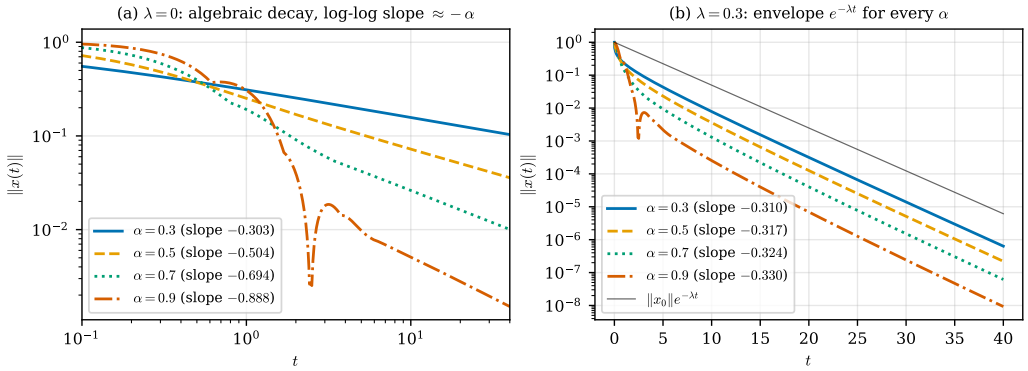


Figure 3. Example 4.2: $\|x(t)\|$ for the linear equation of Example 4.1 with $\alpha \in \{0.3, 0.5, 0.7, 0.9\}$ and $x(0) = (1, 0.5)^T$: (a) $\lambda = 0$ (log-log scale), where the decay is algebraic with a slope close to $-\alpha$; (b) $\lambda = 0.3$ (semi-logarithmic scale), where all curves lie below the guaranteed envelope $\|x_0\|\, e^{-\lambda t}$ (thin black line). The legends report the observed least-squares slopes on $[20, 40]$.

Example 4.3. *Consider* (1.1) *with* A *and* α *as in Example 4.1,* $\lambda = 0.5$*, and*

$$f(x) = (x_2^2,\ -x_1 x_2)^T,$$

which satisfies (3.2) *because it is quadratic and vanishes to second order at the origin. Figure 4(a) shows the two states for* $A = \left[\begin{smallmatrix} -1 & 2 \\ -2 & -1 \end{smallmatrix}\right]$*,* $f(x) = (x_2^2, -x_1 x_2)^T$*,* $\alpha = 0.7$*,* $\lambda = 0.5$*, and* $x(0) = (0.6, -0.4)^T$*; both decay rapidly to zero. Figure 4(b) shows the weighted norm* $e^{\lambda t}\, \|x(t)\|$ *for three initial conditions: it is bounded by a value attained at or very near* $t = 0$ *(respectively,* 0.600*,* 0.513*,* 0.257*) and decreases to* 0.0114*,* 0.0084*,* 0.0046 *at* $t = 25$*, still decaying, as predicted by Theorem 3.1 in its weighted form. The theorem guarantees boundedness by* $M\, \|x_0\|$ *and convergence to zero of the weighted norm, whereas the value* $M \approx 1$ *and the monotone decay observed here are features of these trajectories. The three curves happen to remain ordered by their initial norms on the whole window; this, too, is a property of the particular trajectories shown: Corollary 3.1 bounds the* difference *of two solutions by* $Me^{-\lambda t}\, \|x_0 - \hat{x}_0\|$ *and does not imply any ordering of their norms.*

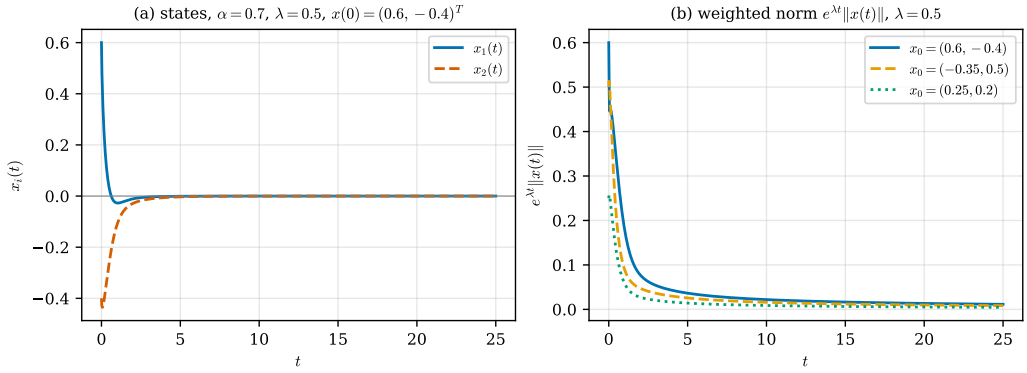


Figure 4. Example 4.3: (a) States of the nonlinear tempered system ${}^{C}D_{0+}^{\alpha,\lambda}x = Ax + f(x)$. (b) weighted norm $e^{\lambda t}\,\|x(t)\|$ for the three initial conditions $x(0) = (0.6, -0.4)^T$, $(-0.35, 0.5)^T$, and $(0.25, 0.2)^T$.

Example 4.4. *Consider the tempered Lotka-Volterra system* (3.27) *with*

$$\alpha = 0.9, \quad \lambda = 0.3, \quad \varrho = 0.4, \quad a = -0.2, \quad b = -0.4, \quad c = 0.3, \quad x(0) = (0.10,\, 0.08)^T,$$

so that $\lambda^\alpha = 0.3^{0.9} \approx 0.338$. *We test the three statements of Proposition 3.2. For* $h = 0.6 > \lambda^\alpha$, *statement (i) predicts instability of the open loop, and Figure 5(a) shows* x_1 *growing away from the zero equilibrium (reaching* 0.88 *at* $t = 12$*) while* x_2 *decays. For* $h = 0.25 < \lambda^\alpha$, *Theorem 3.2 guarantees every exponential rate strictly below* $\lambda - h^{1/\alpha} \approx 0.0857$, *but makes no statement about the limiting value itself; the scalar asymptotics of Remark 3.4 suggest the limiting dominant exponent* $\lambda - h^{1/\alpha}$ *for the linear mode. The observed slope* -0.0861 *on* $[30, 60]$ *in Figure 5(b) agrees with this value to within* 5×10^{-4}, *and differs by about the same amount from the slope* -0.0857 *of the exact linear mode* $e^{-\lambda t}E_\alpha(ht^\alpha)$ *on the same window, a difference compatible with the size of the quadratic terms of* (3.27) *along the trajectory. This is consistent with Theorem 3.2, whose hypothesis is satisfied although the eigenvalue* $h = 0.25$ *lies outside the sector* (3.1)*; the agreement with the limiting exponent is an observation about this trajectory and not a consequence of the theorem, which guarantees only the rates* $\beta < \lambda - h^{1/\alpha}$. *Table 3 repeats this experiment for* $\alpha \in \{0.3, 0.5, 0.7, 0.9\}$, *keeping* $\lambda = 0.3$ *and* $h = 0.25$, *which is subcritical for every such* α *since* $\lambda^\alpha \geq 0.338$. *In every case, the observed slope on* $[30, 60]$ *agrees with the slope of the exact linear mode* $e^{-\lambda t}E_\alpha(ht^\alpha)$ *on the same window to within* 5×10^{-4} *(to within* 1.5×10^{-4} *for* $\alpha \leq 0.7$*). For* $\alpha = 0.9$ *and* $\alpha = 0.7$, *the mode has essentially reached its asymptotic regime, and both slopes coincide with the limiting exponent* $-(\lambda - h^{1/\alpha})$ *to three decimals; for* $\alpha = 0.5$ *and* $\alpha = 0.3$, *the algebraic terms* $-\sum_k (ht^\alpha)^{-k}/\Gamma(1 - \alpha k)$ *of the expansion* (3.19), *which decay only like* $t^{-\alpha}$, *are still visible on the window, so the observed slope is slightly less steep than the limiting exponent, exactly as for the linear mode. In all cases, the observations are consistent with Theorem 3.2, which concerns the asymptotic envelopes* $M_\beta e^{-\beta t}$ *for* $\beta < \lambda - h^{1/\alpha}$ *and not the slope on a finite window. For* $h = 0.6$ *with the feedback* $u = Kx$, $B = (1, 1)^T$, $K = (-1.2, 0)$, *the closed-loop matrix has eigenvalues* -0.6 *and* -0.4, *and statement (iii) guarantees exponential asymptotic stability with rate* $\lambda = 0.3$. *Figure 5(c) shows the decay, with observed slope* -0.428 *on* $[10, 20]$. *The excess over* $-\lambda$ *reflects the still pre-asymptotic Mittag-Leffler decay: the same closed-loop system with* $\lambda = 0$ *gives the slope* -0.129, *and* $-0.129 - 0.3 = -0.429$ *is numerically close to the tempered measurement (an empirical observation, not an exact identity, since the conjugated dynamics* (4.1) *depend on* λ*). The weighted norm* $e^{\lambda t}\,\|x(t)\|$ *never exceeds its initial*

value 0.100 *and reaches* 4.7×10^{-3} *at* $t = 20$*, after a small transient rebound (local minimum* 0.039 *at* $t \approx 1.7$*, local maximum* 0.049 *at* $t \approx 3.0$*) caused by the overshoot of* x_2*; the observed envelope* $\|x(t)\| \le \|x_0\| e^{-\lambda t}$ *is an empirical feature of this trajectory, since the theory guarantees* $Me^{-\lambda t}\|x_0\|$ *with* $M \ge 1$ *(Remark 3.8) and asserts neither* $M = 1$ *nor monotonicity.*

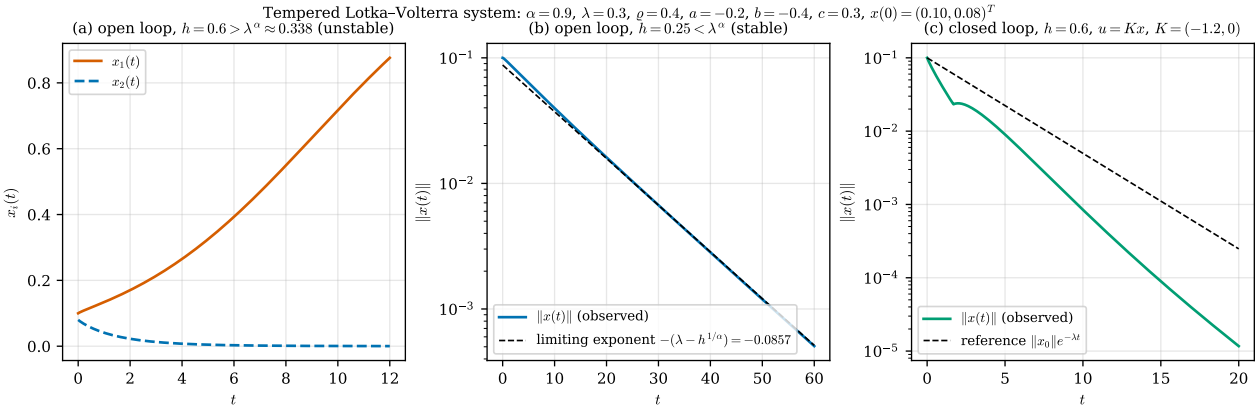


Figure 5. Example 4.4: Tempered Lotka-Volterra system (3.27) with $\alpha = 0.9$, $\lambda = 0.3$, $\varrho = 0.4$, $a = -0.2$, $b = -0.4$, $c = 0.3$, and $x(0) = (0.10, 0.08)^T$: (a) open-loop system with $h = 0.6 > \lambda^\alpha \approx 0.338$ (unstable): the two states; (b) open-loop system with $h = 0.25 < \lambda^\alpha$ (stable): $\|x(t)\|$ on a logarithmic scale, with a line of slope $-(\lambda - h^{1/\alpha}) \approx -0.0857$ (dashed), the limiting exponent of the linear mode; (c) closed-loop system under the feedback $u = Kx$ with $B = (1, 1)^T$ and $K = (-1.2, 0)$, with the reference curve $\|x_0\| e^{-\lambda t}$ (dashed).

Table 3. Subcritical open-loop Lotka-Volterra system of Example 4.4 ($h = 0.25$, $\lambda = 0.3$, $\varrho = 0.4$, $a = -0.2$, $b = -0.4$, $c = 0.3$, $x(0) = (0.10, 0.08)^T$) for several orders α: threshold λ^α, rates guaranteed by Theorem 3.2, limiting exponent $-(\lambda - h^{1/\alpha})$ of the linear mode, observed least-squares slope of $\log \|x(t)\|$ on $[30, 60]$, and slope of the exact linear mode $e^{-\lambda t}E_\alpha(ht^\alpha)$ on the same window.

α	λ^α	Guaranteed rates	$-(\lambda - h^{1/\alpha})$	Observed slope	Slope of linear mode
0.3	0.697	$\beta < 0.2902$	−0.2902	−0.2853	−0.2854
0.5	0.548	$\beta < 0.2375$	−0.2375	−0.2367	−0.2368
0.7	0.431	$\beta < 0.1620$	−0.1620	−0.1620	−0.1620
0.9	0.338	$\beta < 0.0857$	−0.0857	−0.0861	−0.0857

Table 4 summarizes the experiments. In every stable case, the observed decay is at least as fast as the guaranteed envelope, as it must be; the converse inference, from an observed slope to a theoretical rate, is never drawn. The excess over the guaranteed rate is quantitatively accounted for in two cases: in Examples 4.1 and 4.2, it equals the logarithmic derivative $-\alpha/t$ of the algebraic Mittag-Leffler factor on the fitting window, and in the closed-loop case of Example 4.4, it is empirically close to the measured decay of the corresponding untempered system, which is still pre-asymptotic on the fitting window. In the subcritical open-loop case of Example 4.4, the observed slope agrees with the limiting asymptotic

exponent $\lambda - h^{1/\alpha}$ to three decimal places for $\alpha = 0.9$ and $\alpha = 0.7$, and with the slope of the exact linear mode on the same window for every α tested.

Table 4. Summary of the numerical experiments. Observed slopes are least-squares slopes of $\log \|x(t)\|$ (log-log for the algebraic case) on the indicated windows; all runs use the step size 10^{-2} and "guaranteed behavior" refers to the statements of Corollary 2.1, Theorems 3.1 and 3.2, and Proposition 3.2.

Experiment	α	λ	Spectrum	Guaranteed behavior	Window	Observed slope
Ex. 4.1, $\lambda = 0$	0.7	0	$-1 \pm 2i$	algebraic decay $t^{-\alpha}$	$[20, 40]$	-0.694 (log-log)
Ex. 4.1, $\lambda = 0.3$	0.7	0.3	$-1 \pm 2i$	$o(e^{-0.3t})$	$[20, 40]$	-0.324
Ex. 4.1, $\lambda = 0.8$	0.7	0.8	$-1 \pm 2i$	$o(e^{-0.8t})$	$[20, 40]$	-0.824
Ex. 4.2, $\lambda = 0$	0.3–0.9	0	$-1 \pm 2i$	algebraic decay $t^{-\alpha}$	$[20, 40]$	-0.303 to -0.888 (log-log)
Ex. 4.2, $\lambda = 0.3$	0.3–0.9	0.3	$-1 \pm 2i$	$o(e^{-0.3t})$	$[20, 40]$	-0.310 to -0.330
Ex. 4.3	0.7	0.5	$-1 \pm 2i$	$e^{\lambda t} \|x(t)\| \to 0$	—	bounded, $\to 0$
Ex. 4.4, $h = 0.6$	0.9	0.3	$0.6, -0.4$	unstable ($h > \lambda^\alpha$)	—	growth
Ex. 4.4, $h = 0.25$	0.9	0.3	$0.25, -0.4$	$\beta < \lambda - h^{1/\alpha}$ (limit 0.0857)	$[30, 60]$	-0.086
Ex. 4.4, $h = 0.25$	0.3–0.9	0.3	$0.25, -0.4$	$\beta < \lambda - h^{1/\alpha}$ (Table 3)	$[30, 60]$	-0.285 to -0.086
Ex. 4.4, closed loop	0.9	0.3	$-0.6, -0.4$	$o(e^{-0.3t})$	$[10, 20]$	-0.428

5. Conclusions

Two linearization theorems were proved for nonlinear fractional differential equations with the Caputo tempered derivative of order $\alpha \in (0, 1)$ and tempering parameter $\lambda \geq 0$: under the classical sector condition, the trivial solution is asymptotically stable with the explicit envelope $\|x(t)\| \leq Me^{-\lambda t} \|x_0\|$ and the weighted convergence $e^{\lambda t} x(t) \to 0$ (Theorem 3.1), and inner-sector eigenvalues with $\mathrm{Re}(\mu^{1/\alpha}) < \lambda$ are admissible as well (Theorem 3.2), so that the two theorems cover, locally and under the smallness condition (3.2) on the nonlinearity, the full open asymptotic-stability region of the tempered linear part; the geometry of this region and the reasons for excluding its boundary were described in Remark 3.7 and Figure 1. The proofs rest on a tempered Lyapunov-Perron operator on exponentially weighted spaces, in which the tempering factors cancel exactly at the critical weight and remain integrable at subcritical weights (Lemma 3.5); for $\lambda = 0$, the classical theorem of [25] is recovered. The method requires only an eigenvalue computation and complements Lyapunov-function and integral-inequality techniques. For a tempered Lotka–Volterra model, the zero equilibrium is unstable for $h > \lambda^\alpha$, asymptotically stable for $h < \lambda^\alpha$, and stabilizable by linear feedback for every h. The critical case where $h = \lambda^\alpha$ remains open, and the numerical experiments were consistent with the guaranteed rates: the observed subcritical slope agreed with the limiting exponent $\lambda - h^{1/\alpha}$ to three decimal places, and observed slopes were throughout distinguished from guaranteed rates. Experiments for several orders α confirmed that the exponential envelope is independent of α, which affects only the algebraic correction. Future work will address the critical spectral cases, where the linearization is not attractive, as well as delayed, incommensurate, and observer-based tempered systems [16, 19, 23].

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest in this paper.

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A. Proofs of auxiliary results

Proof of Lemma 2.6. Define $g(t) := {}^{C}D^{\alpha,\lambda}_{0+}v(t) - c\,v(t)$ for $t \in (0,T)$. By assumption, g extends to a continuous function on $[0,T)$, and $g \geq 0$ on $(0,T)$, hence on $[0,T)$ by continuity. Moreover, v solves the scalar linear problem

$$ {}^{C}D^{\alpha,\lambda}_{0+}v(t) = c\,v(t) + g(t), \qquad t \in (0,T),$$

whose right-hand side $F(t,w) := c\,w + g(t)$ satisfies the hypotheses of Proposition 2.1 with $T_0 = T$, so v is its unique solution with the initial value $v(0)$. By Lemma 2.3 with $\mu = c$,

$$v(t) = e^{-\lambda t}E_\alpha(c\,t^\alpha)v(0) + \int_0^t (t-w)^{\alpha-1}E_{\alpha,\alpha}(c\,(t-w)^\alpha)e^{-\lambda(t-w)}g(w)\,dw, \qquad t \in [0,T).$$

By Lemma 2.4, $E_{\alpha,\alpha}(x) \geq 0$ for every $x \in \mathbb{R}$, so the integrand is nonnegative. Dropping the integral gives $v(t) \geq e^{-\lambda t}E_\alpha(c\,t^\alpha)v(0)$. The proof is complete. □

Proof of Corollary 2.1. Apply Proposition 2.1 with $F(t,z) := Az$, which is (globally) Lipschitz continuous. By Proposition 2.1(i) and Lemma 2.5, x solves ${}^{C}D^{\alpha,\lambda}_{0+}x = Ax$, $x(0) = x_0$, if and only if $y(t) := e^{\lambda t}x(t)$ solves the classical linear problem ${}^{C}D^{\alpha}_{0+}y = Ay$, $y(0) = x_0$, whose unique global solution is $y(t) = E_\alpha(t^\alpha A)x_0$ [2, 3, 25]. Hence $x(t) = e^{-\lambda t}E_\alpha(t^\alpha A)x_0$. Under the sector condition, the trivial solution of the classical problem is asymptotically stable [3, Theorem 7.20], [25, Theorem 2]; in particular, applying attractivity to the initial values $e_1, \ldots, e_d$ of the standard basis shows that every entry of the continuous matrix function $t \mapsto E_\alpha(t^\alpha A)$ tends to zero as $t \to \infty$. A continuous matrix function converging to zero is bounded, so

$$K := \sup_{t\geq 0} \|E_\alpha(t^\alpha A)\| < \infty,$$

and $K \geq 1$ because $E_\alpha(0) = I_d$. Therefore, for every $x_0 \in \mathbb{R}^d$,

$$\|x(t)\| = e^{-\lambda t}\,\|E_\alpha(t^\alpha A)x_0\| \leq K\,e^{-\lambda t}\,\|x_0\| \quad \text{for all } t \geq 0, \qquad e^{\lambda t}x(t) = E_\alpha(t^\alpha A)x_0 \xrightarrow[t\to\infty]{} 0,$$

which is Definition 2.5(iv) with rate λ, radius ρ arbitrary, and $M = K$. The proof is complete. □

Proof of Lemma 3.1. Recall from Remark 3.2 that f denotes the extension $\tilde{f}$, whose Lipschitz constant satisfies $\ell_{\tilde{f}}(\rho) \le \ell_f(\rho)$, so it suffices to prove the estimate with ℓ_f on the right-hand side. First, $h(0) = 0$ follows from $f(0) = 0$ and linearity of the first summand in (3.5). Let $r > 0$ and $z_1, z_2 \in \bar{B}(0, r) \subset \mathbb{C}^d$. For the linear summand, in the maximum norm, one has $\|N_k w\| = \max_{1\le j\le k-1} \left|w_{j+1}\right| \le \|w\|$ for every $w \in \mathbb{C}^k$, and each block is multiplied by $\delta_i \in \{0, \delta\}$, so

$$\left\|\mathrm{diag}(\delta_1 N_{d_1}, \dots, \delta_n N_{d_n})(z_1 - z_2)\right\| \le \delta \, \|z_1 - z_2\| .$$

For the nonlinear summand, $\left\|TPz_j\right\| \le \|TP\| \, r$ for $j = 1, 2$, hence by the definition of ℓ_f,

$$\left\|(TP)^{-1} f(TPz_1) - (TP)^{-1} f(TPz_2)\right\| \le \left\|(TP)^{-1}\right\| \, \ell_f(\|TP\| \, r) \, \|TP\| \, \|z_1 - z_2\| .$$

Adding the two estimates yields the claimed bound for $\ell_h(r)$; in particular, h is Lipschitz continuous on every ball, hence locally Lipschitz continuous. Finally, (3.2) gives $\ell_f(\|TP\| \, r) \to 0$ as $r \to 0^+$, so $\limsup_{r\to 0^+} \ell_h(r) \le \delta$. The proof is complete. □

Proof of Lemma 3.3. Suppose first that $\xi \in C_\lambda$ with $\xi(0) = x$ solves (3.4) on $[0, \infty)$. Fix $i \in \{1, \dots, n\}$ and define $g_i(t) := h^i(\xi(t))$, which is continuous because h is locally Lipschitz continuous and ξ is continuous. Since $J_i = \mu_i I_{d_i}$, the i-th block of (3.4) reads

$${}^{C}D^{\alpha,\lambda}_{0+} \xi^i(t) = \mu_i \, \xi^i(t) + g_i(t), \qquad \xi^i(0) = x^i.$$

By Lemma 2.3,

$$\xi^i(t) = e^{-\lambda t} E_\alpha(\mu_i t^\alpha) x^i + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\mu_i (t-\tau)^\alpha) e^{-\lambda(t-\tau)} g_i(\tau) \, d\tau = (\mathcal{T}_x \xi)^i(t),$$

so $\mathcal{T}_x \xi = \xi$. Conversely, suppose $\mathcal{T}_x \xi = \xi$ for some $\xi \in C_\lambda$ with $\xi(0) = x$. Fix i and let $g_i(t) := h^i(\xi(t))$ as before. By Lemma 2.3, the right-hand side of (3.8) is the solution of the linear problem ${}^{C}D^{\alpha,\lambda}_{0+} w = \mu_i w + g_i(t)$ with $w(0) = x^i$. Since $\xi^i = (\mathcal{T}_x \xi)^i$, the block ξ^i solves this problem, that is,

$${}^{C}D^{\alpha,\lambda}_{0+} \xi^i(t) = \mu_i \, \xi^i(t) + h^i(\xi(t)), \qquad i = 1, \dots, n,$$

which is exactly (3.4), and $\xi(0) = x$ holds by assumption. The same proof applies verbatim with C_λ replaced by C_β for any $\beta \in [0, \lambda]$, since only the continuity and boundedness of ξ were used. The proof is complete. □



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