AIMS Mathematics, 11(11): 37311–37321.
DOI: 10.3934/math.20261443
Received: 08 July 2026
Revised: 22 October 2026
Accepted: 30 October 2026
Published: 11 November 2026

https://www.aimspress.com/journal/Math

Research article

Certain new weighted Hardy-type integral inequalities and their sharpness analysis

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Abstract: We examine three finite-interval forms of weighted Hardy inequalities: a forward operator with power weights, a forward operator with a logarithmic weight, and a tail integral operator with power weights. Their best constants are respectively $4/p^2$, 4, and $1/\alpha^2$. We give direct proofs, specify the admissible function classes, and construct explicit extremizing sequences, with continuous functions for the logarithmic case. Changes of variables identify the precise correspondence with classical weighted Hardy inequalities. Consequently, the constants coincide with the classical sharp constants for the same operators and weights; they are not strict improvements on those constants. Concrete examples and comparison tables distinguish changes of parameters from improvements for a fixed inequality. The treatment also records non-attainment and the role of the singular endpoint.

Keywords: weighted Hardy inequality; optimal constant; forward integral operator; tail integral operator; logarithmic weight; extremizing sequence
Mathematics Subject Classification: 26D10, 26D15, 47B38

1. Introduction

Hardy's integral inequality is a basic estimate for an integral operator in a Lebesgue space. For $r > 1$, its classical form is

$$\int_0^\infty \left|\frac{1}{x}\int_0^x g(t)dt\right|^r dx \le \left(\frac{r}{r-1}\right)^r \int_0^\infty |g(x)|^r dx.$$

The constant is optimal. Hardy's original integral inequalities [1, 2] and the systematic account of Hardy, Littlewood, and Pólya [3] place this estimate in a broader theory of integral inequalities. For general pairs of weights, Muckenhoupt [4] established a criterion for boundedness and estimates of the best constant. The monograph [5] treats the resulting weighted theory. It is therefore necessary to distinguish an explicit realization of a known sharp estimate from a smaller constant for the same problem.

1.1. Forward operators with power weights

The power-weighted extension of Hardy's inequality [2, 3], with Lebesgue exponent r and weight exponent $a < r - 1$, has the constant $(r/(r-1-a))^r$ in

$$\int_0^1 x^{a-r}\left|\int_0^x g(t)dt\right|^r dx \le \left(\frac{r}{r-1-a}\right)^r \int_0^1 x^a |g(x)|^r dx. \tag{1.1}$$

The modern treatment by Persson et al. [6] includes finite-interval sharpness and a stronger weighted right-hand side. In our first inequality,

$$r = 2, \quad a = 1 - p, \quad \text{and} \quad g(t) = t^{q-1} f(t).$$

Thus, the classical constant becomes $4/p^2$, while q changes the representation of the input. Theorem 3.1 supplies a direct proof and an explicit extremizing family. It does not improve the classical constant.

Gesztesy and Pang [7] derive optimal power-weighted Hardy inequalities on finite intervals in both integral and differential forms and extend them by iteration to higher-order inequalities and annuli. Their observation that the optimal constants in the integral and differential formulations can differ makes the function class and boundary conditions essential to a comparison. This work is particularly relevant to our finite-interval setting, since a common interval alone does not establish equivalence of two estimates, so in Section 4 we compare the complete quadratic forms rather than isolated constants.

1.2. Logarithmic weights on a compact interval

A logarithm in a weight need not define a new operator-norm problem. In the quadratic inequality studied here, put

$$L = \ln b, \quad u = \ln(b/x), \quad \text{and} \quad h(u) = be^{-u} f(be^{-u}).$$

A direct substitution transforms the two quadratic integrals into

$$\int_0^L \left|\int_u^L h(v)dv\right|^2 du, \qquad \int_0^L u^2 |h(u)|^2 du.$$

These are the tail Hardy forms at exponent $\beta = 1/2$ in Lemma 2.1, belonging to the weighted theory described in [3, 5]. Their sharp constant is 4. Theorem 3.2 verifies this correspondence and proves that the same constant is sharp even when f must be nonnegative and continuous on the closed interval. The relevant issue is concentration near the endpoint where the logarithmic weight vanishes, not an improvement over an existing constant.

Gesztesy et al. [8] study Hardy–Rellich inequalities with power weights on multidimensional balls and obtain logarithmic refinements. Their logarithmic terms strengthen differential estimates, including in situations where the uncorrected coefficient degenerates. The accompanying correction [9] supplies the reference to the corrected optimality proof and withdraws an earlier remark. This is related to the present study through logarithmic weights and sharpness, but it concerns a different differential operator. In Theorem 3.2, the logarithmic weight instead arises from an exact coordinate transformation of a first-order tail integral estimate; we do not identify it with a Hardy–Rellich remainder term.

1.3. Tail operators and the scope of the comparison

The tail operator $U \mapsto \int_x^1 U(t)dt$ is the adjoint-direction analogue of the forward Hardy operator; "tail" refers to the integration limits, not to reversing the inequality sign. The weighted framework [4,5] encompasses this direction as well. A direct inversion $y = 1/x$ gives the forward power-weighted inequality on $(1, \infty)$. If

$$k(y) = y^{-2}U(1/y),$$

then the two integrals become

$$\int_1^\infty y^{-2\alpha-1}| \int_1^y k|^2 dy$$

and

$$\int_1^\infty y^{1-2\alpha}|k(y)|^2 dy.$$

Zero extension of k below 1 connects this to the classical power case with $p = 2\alpha$. Theorem 3.3 proves the bound $1/\alpha^2$ directly and justifies sharpness in both the weighted space and the more restrictive integrable class used in the original formulation.

Gogatishvili et al. [10] characterize a four-weight inequality involving Hardy and Copson operators by discretization and antidiscretization, and apply the characterization to embeddings of weighted function spaces. Their work places forward and tail operators in a common weighted framework, whereas our Theorems 3.1 and 3.3 treat individual operators with specified power weights and explicit quadratic constants. García and Salvador [11] characterize the weights for certain weak-type iterated Hardy–Copson inequalities. This provides a complementary recent direction: such weak-type and iterated estimates are not the strong L^2 estimates for a single operator considered here, and their constants cannot be compared directly with ours.

Ruzhansky et al. [12] determine sharp constants for weighted integral Hardy and conjugate Hardy inequalities on homogeneous Lie groups in the range $1 < r \leq s < \infty$. The simultaneous treatment of an operator and its conjugate is relevant to our forward-tail comparison, while the group geometry and potentially unequal exponents distinguish their setting from that of our one-dimensional quadratic forms. Together, these recent studies show that current developments concern exact constants, weight

characterizations, boundary-sensitive formulations, and genuine logarithmic refinements. Our examples address the narrower question of how the classical constants are realized in the three stated finite-interval forms.

Recent sharpness analysis [6] also emphasizes that a stronger inequality can retain the same leading constant while reducing its right-hand-side weight. In particular, in the first setting the known refinement is

$$\int_0^1 x^{-p-1}\left|\int_0^x t^{q-1}f(t)\dot{\mathrm{d}}t\right|^2 \dot{\mathrm{d}}x \le \frac{4}{p^2}\int_0^1 x^{2q-p-1}(1-x^{p/2})|f(x)|^2dx. \tag{1.2}$$

This follows from [6, Theorem 2.3(a)] with its Lebesgue exponent equal to 2 and its weight parameter equal to p. Thus, the uncorrected estimate in Theorem 3.1 is weaker than an available finite-interval refinement. We include this comparison to delimit the scope of the present treatment.

The purpose of this paper is to give a unified, explicit treatment of these three realizations, with admissibility checks and quantitative examples. No claim is made that their constants or underlying operator inequalities are new. Section 2 fixes notation and establishes the endpoint estimate used below. Section 3 gives the three inequalities and their sharpness proofs, followed by parameter comparisons and concrete test families in Subsection 3.4. Section 4 discusses the conclusions and future directions.

2. Foundational framework

2.1. Quadratic forms and optimal constants

All functions are real-valued and measurable unless continuity is specified. Weighted integrals are Lebesgue integrals. For nonnegative quadratic forms A and B, the optimal constant is

$$C_* = \sup_{0<B(f)<\infty}\frac{A(f)}{B(f)}.$$

The factor C_* is the squared operator norm, not the operator norm itself. An extremizing sequence has a quotient tending to C_*; it need not converge to an admissible equality function. Parameters $p, q, and\alpha$ below describe weights or kernels, whereas the Lebesgue exponent is always 2.

2.2. A tail estimate with justified endpoint limits

The following standard weighted Hardy estimate is recorded with a proof that avoids assuming endpoint boundedness of a primitive.

Lemma 2.1. *Let $\beta, L > 0$ and*

$$\int_0^L u^{2\beta+1}|h(u)|^2du < \infty.$$

Then

$$Th(u) = \int_u^L h(v)dv$$

is defined for every $u > 0$, and

$$\int_0^L u^{2\beta-1}|Th(u)|^2du \le \frac{1}{\beta^2}\int_0^L u^{2\beta+1}|h(u)|^2du. \tag{2.1}$$

The constant is sharp, and equality requires $h = 0$ almost everywhere.

Proof. Weighted Cauchy–Schwarz gives integrability on $[s, L]$ for each $s > 0$. Write $Y = Th$,

$$I_s = \int_s^L u^{2\beta-1} Y^2 du \quad \text{and} \quad D_s = \int_s^L u^{2\beta+1} h^2 du.$$

Integration by parts on $[s, L]$, where $Y(L) = 0$, gives

$$I_s = -\frac{s^{2\beta} Y(s)^2}{2\beta} + \frac{1}{\beta} \int_s^L u^{2\beta} Y(u) h(u) du \le \frac{1}{\beta} \sqrt{I_s D_s}.$$

Consequently, $I_s \le D_s/\beta^2$, including when $I_s = 0$. Monotone convergence as $s \downarrow 0$ proves the estimate and finiteness of its left side.

For completeness, the boundary term also tends to zero. Fix $c \in (0, L)$ and split

$$Y(s) = \int_s^c h + \int_c^L h.$$

Then

$$s^\beta \left| \int_s^c h(u) du \right| \le \frac{1}{\sqrt{2\beta}} \left(\int_0^c u^{2\beta+1} |h(u)|^2 du \right)^{1/2}.$$

The remaining term tends to zero for fixed c. Letting $c \downarrow 0$ proves $s^\beta Y(s) \to 0$. Equality in the full Cauchy–Schwarz step would require $h(u) = \beta Y(u)/u$. Since $Y' = -h$ and $Y(L) = 0$, this implies $Y = 0$ and hence $h = 0$.

For $0 < \varepsilon < \beta$ choose

$$h_\varepsilon(u) = u^{-\beta-1+\varepsilon}.$$

Its weighted square integral is $L^{2\varepsilon}/(2\varepsilon)$, and

$$Th_\varepsilon(u) = \frac{u^{-\beta+\varepsilon} - L^{-\beta+\varepsilon}}{\beta - \varepsilon}.$$

Direct integration yields the quotient

$$\frac{1}{(\beta-\varepsilon)^2} \left(1 - \frac{4\varepsilon}{\beta+\varepsilon} + \frac{\varepsilon}{\beta} \right) = \frac{1}{\beta(\beta+\varepsilon)} \longrightarrow \frac{1}{\beta^2}. \tag{2.2}$$

This proves sharpness. □

3. Main results and sharpness analysis

3.1. The forward power-weighted operator

Theorem 3.1. *Let $p, q > 0$ and*

$$D = \int_0^1 x^{2q-p-1} |f(x)|^2 dx < \infty.$$

Then

$$A = \int_0^1 x^{-p-1} \left| \int_0^x t^{q-1} f(t) dt \right|^2 dx \le \frac{4}{p^2} D. \tag{3.1}$$

The constant is sharp, and equality holds only for $f = 0$ almost everywhere.

Proof. The inner integral is absolutely convergent because

$$\int_0^x t^{q-1}|f(t)|dt \le \left(\int_0^x t^{2q-p-1}|f(t)|^2 dt\right)^{1/2} \left(\frac{x^p}{p}\right)^{1/2}.$$

The substitution $t = ux$ and Minkowski's integral inequality give

$$\begin{aligned} A^{1/2} &\le \int_0^1 u^{q-1} \left(\int_0^1 x^{2q-p-1}|f(ux)|^2\, dx\right)^{1/2} du \\ &= \int_0^1 u^{p/2-1} \left(\int_0^u z^{2q-p-1}|f(z)|^2\, dz\right)^{1/2} du \\ &\le \frac{2}{p} D^{1/2}. \end{aligned}$$

These steps can first be applied to $\min(|f|, n)\mathbf{1}_{[1/n,1]}$, followed by monotone convergence and the pointwise bound for signed f. This also avoids presupposing $A < \infty$. For nonzero f, some $a \in (0, 1)$ satisfies

$$\int_a^1 z^{2q-p-1}|f(z)|^2\, dz > 0.$$

The last bound is then strict for every $u < a$, proving nonattainment.

For sharpness, let

$$f_\varepsilon(x) = x^{p/2+\varepsilon-q}, \quad \varepsilon > 0.$$

Direct computation gives

$$D(f_\varepsilon) = \frac{1}{2\varepsilon}, \qquad A(f_\varepsilon) = \frac{1}{2\varepsilon(p/2+\varepsilon)^2}, \qquad \frac{A(f_\varepsilon)}{D(f_\varepsilon)} = \frac{1}{(p/2+\varepsilon)^2} \to \frac{4}{p^2}. \tag{3.2}$$

Every member is admissible; the limiting critical power is not, since its energy contains $\int_0^1 x^{-1}\, dx$. □

3.2. The operator with a logarithmic weight

Theorem 3.2. *Let $b > 1$ and $f \in C[1, b]$ be nonnegative. Then*

$$\int_1^b \frac{1}{x}\left|\int_1^x f(t)dt\right|^2 dx \le 4\int_1^b x(\ln(b/x))^2|f(x)|^2 dx. \tag{3.3}$$

The constant 4 *is sharp for this class, for every fixed $b > 1$. Equality occurs only for $f = 0$.*

Proof. Put $L = \ln b$ and $h(u) = be^{-u}f(be^{-u})$. The substitution $x = be^{-u}$ gives exactly

$$A_b(f) = \int_0^L \left|\int_u^L h(v)dv\right|^2 du, \qquad D_b(f) = \int_0^L u^2|h(u)|^2 du. \tag{3.4}$$

Lemma 2.1 with $\beta = 1/2$ proves the inequality and nonattainment. To establish sharpness within the stated continuous class, take

$$f_\delta(x) = \frac{1}{x(\ln(b/x)+\delta)^{3/2}}, \qquad \delta > 0.$$

These functions are nonnegative and continuous on $[1, b]$. With

$$h_\delta(u) = (u + \delta)^{-3/2},$$

their transformed primitive is

$$\int_u^L h_\delta(v)dv = 2((u + \delta)^{-1/2} - (L + \delta)^{-1/2}).$$

Set $\ell_\delta = \ln((L + \delta)/\delta)$. Elementary integration gives

$$D_b(f_\delta) = \ell_\delta + \frac{2\delta}{L + \delta} - \frac{\delta^2}{2(L + \delta)^2} - \frac{3}{2}, \tag{3.5}$$

$$A_b(f_\delta) = 4\left[\ell_\delta - 4\left(1 - \sqrt{\frac{\delta}{L + \delta}}\right) + \frac{L}{L + \delta}\right]. \tag{3.6}$$

As $\delta \downarrow 0$, both bracketed leading terms grow like ℓ_δ, so $A_b(f_\delta)/D_b(f_\delta) \to 4$. This single regularization parameter proves sharpness without an unqualified exchange of singular limits. □

3.3. The tail power-weighted operator

Theorem 3.3. *Let $\alpha > 0$ and let U be measurable with*

$$D_\alpha(U) = \int_0^1 x^{2\alpha+1}|U(x)|^2dx < \infty.$$

Then

$$A_\alpha(U) = \int_0^1 x^{2\alpha-1}\left|\int_x^1 U(t)dt\right|^2 dx \le \frac{1}{\alpha^2}D_\alpha(U). \tag{3.7}$$

The constant is sharp, even on the subclass $U \in L^1(0, 1)$, and equality holds only for $U = 0$ almost everywhere.

Proof. Apply Lemma 2.1 with $L = 1$, $\beta = \alpha$, and $h = U$. The weighted assumption guarantees that each tail integral exists; integrability at zero is unnecessary. For $0 < \varepsilon < \alpha$, the functions

$$U_\varepsilon(x) = x^{-\alpha-1+\varepsilon}$$

have the quotient

$$\frac{A_\alpha(U_\varepsilon)}{D_\alpha(U_\varepsilon)} = \frac{1}{\alpha(\alpha + \varepsilon)} \longrightarrow \frac{1}{\alpha^2}. \tag{3.8}$$

They need not lie in $L^1(0, 1)$. To retain that additional restriction, use

$$U_{\varepsilon,\delta}(x) = (x + \delta)^{-\alpha-1+\varepsilon}, \ \ \delta > 0.$$

These functions are bounded and integrable on $[0, 1]$. For fixed $0 < \varepsilon < \alpha$, they increase pointwise to U_ε as $\delta \downarrow 0$. Monotone convergence applies separately to the nonnegative energy and primitive-square integrands; both limiting integrals are finite. Hence their quotient tends to the constant in (3.8). Taking $\varepsilon \downarrow 0$ subsequently proves sharpness on the smaller class as well. □

3.4. Parameter comparisons and illustrative examples

3.4.1. Comparisons for the same operator and weights

For a valid comparison, the domain, kernel, weights, and admissible class must agree. In Table 1, r denotes the Lebesgue exponent in the classical forward formulation (1.1), a its weight exponent, and β the tail exponent in (2.1). All comparisons use exponent 2. The transformations are given in Section 1 and (3.4).

Table 1. Exact parameter correspondence; no strict decrease of the classical constant.

Present setting	Classical parameters	Classical C_*	Present C_*
Forward, $p, q > 0$	$r = 2$, $a = 1 - p$; $g = t^{q-1}f$	$4/p^2$	$4/p^2$
Logarithmic, $b > 1$	Tail $\beta = 1/2$, $L = \ln b$	4	4
Tail, $\alpha > 0$	$\beta = \alpha$; forward $p = 2\alpha$ after inversion	$1/\alpha^2$	$1/\alpha^2$

The word "optimal" refers to minimizing C for a fixed inequality. It does not mean choosing p, q, or α to make the displayed number small: those choices change the inequality itself. The constants in the two right-hand columns are equal, not smaller in the present formulation.

3.4.2. Forward case: explicit powers and a known refinement

Example 3.1. Choose $p = 2$ and $q = 1$, so the classical parameters are $r = 2$, $a = -1$, and $g = f$. Theorem 3.1 reads

$$\int_0^1 x^{-3} \left| \int_0^x f(t)dt \right|^2 dx \leq \int_0^1 x^{-1}|f(x)|^2 dx.$$

For $f_\varepsilon(x) = x^\varepsilon$, the exact quotient is $(1 + \varepsilon)^{-2}$. At $\varepsilon = 0.1, 0.01, 0.001$, it equals 0.826446, 0.980296, and 0.998003, respectively. Thus, the best constant is 1, exactly the classical value $4/(1 - a)^2 = 1$. Although $1 < 4$, the number 4 belongs to the different choice $p = 1$ (classical weight $a = 0$), so this numerical comparison is not an improvement for the same inequality.

For a concrete comparison with the stronger known estimate (1.2), take $f(x) = x$. The left-hand side is $1/8$, the uncorrected upper bound is

$$\int_0^1 x dx = 1/2,$$

and the known corrected upper bound is

$$\int_0^1 x(1 - x)dx = 1/6.$$

Hence $1/8 \leq 1/6 < 1/2$. This is a genuine reduction of the upper bound for the same data, already available in the literature, rather than a new smaller leading constant in Theorem 3.1.

3.4.3. Logarithmic case: continuous near-extremizers

Example 3.2. Choose $b = e$, so $L = 1$ and the corresponding tail parameter is $\beta = 1/2$. Use the continuous functions f_δ from Theorem 3.2. Their exact quotients follow from Eqs (3.5) and (3.6);

Table 2 evaluates them for several values of δ. Each quotient is below 4 and approaches it as $\delta \downarrow 0$. The classical constant and the present constant are both 4. Changing b changes L but not the best constant, as is also seen by scaling $u = Ls$ in (3.4). In particular, compactness of $[1, b]$ does not lower the constant because the weight vanishes at $x = b$.

Table 2. Admissible continuous tests in the logarithmic case $b = e$.

δ	$A_e(f_\delta)/D_e(f_\delta)$	Sharp constant
e^{-10}	3.306748	4
e^{-50}	3.876289	4
e^{-200}	3.969773	4

3.4.4. Tail case: power tests and integrable approximations

Example 3.3. Choose $\alpha = 2$, corresponding to tail parameter $\beta = 2$ or forward parameter $p = 4$ after inversion. Theorem 3.3 becomes

$$\int_0^1 x^3 \left| \int_x^1 U(t)dt \right|^2 dx \le \frac{1}{4} \int_0^1 x^5 |U(x)|^2 dx.$$

For

$$U_\varepsilon(x) = x^{-3+\varepsilon},$$

the exact quotient is $1/[2(2+\varepsilon)]$. At $\varepsilon = 0.1, 0.01, 0.001$, it equals 0.238095, 0.248756, and 0.249875, approaching $1/4$. These functions belong to the weighted class of Theorem 3.3. If ordinary integrability is imposed as well, the bounded functions $(x+\delta)^{-3+\varepsilon}$ have quotients tending to the same numbers for fixed ε as $\delta \downarrow 0$. The sequential limit is explicitly justified in the proof.

The classical constant is $4/p^2 = 4/16 = 1/4$, so there is again no decrease. Comparing this number with 1 at $\alpha = 1$ mixes different pairs of weights. For an elementary integrable test, $U = 1$ gives $A_2(U) = 1/60$, $D_2(U) = 1/6$, and the quotient $1/10 < 1/4$. A single test far from equality cannot determine the best constant; the extremizing family provides the required lower bound.

4. Conclusions and future directions

The three inequalities have sharp constants $4/p^2$, 4, and $1/\alpha^2$. Their explicit parameter correspondence shows that these are classical optimal constants for the matching weighted problems. The value of the present treatment lies in the common formulation, complete admissibility and endpoint arguments, continuous regularization, and explicit comparisons. The examples answer the question of whether the constants are smaller: they are equal for corresponding parameters. We have therefore not described these results as three new improvements.

Sharpness persists despite non-attainment because the test families concentrate at a singular or degenerate endpoint. For the forward case, an existing weight correction yields a strictly smaller upper bound for suitable data while retaining the same leading constant. A further research contribution would require additional structure, a new remainder estimate, or a genuinely different operator beyond these classical reformulations. Such developments are outside the scope of the results proved here.

Author contributions

Labeeb Ahmad: led the research and contributed to the conception, methodology, analysis, and original manuscript; Waqas Nazeer: supervised the research and provided critical feedback; Hala H. Taha: assisted with interpretation and manuscript revision; Jongsuk Ro: assisted with interpretation and manuscript revision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2026R899), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia. This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (RS-2026-25471383).

Conflict of interest

The authors have no conflicts of interest.

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