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Research article

Metric dimensions of circulant graphs $C_n(1, 2, \ldots, t)$ for $n \equiv 0, 1 \pmod{2t}$

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Abstract: The metric dimension of a graph is the minimum number of landmarks needed to distinguish its vertices by their distance vectors. We study this invariant for the circulant graph $C_n(1, 2, \ldots, t)$, equivalently the t-th power of a cycle. For fixed t, its metric dimension is known to be eventually periodic in n with period $2t$. Nevertheless, no exact formula was known for general t in either of the classes $n \equiv 0, 1 \pmod{2t}$: The available lower and upper bounds remained separated by quantities linear in t. For $n \equiv 0 \pmod{2t}$, an earlier conjectured lower bound $t + 2$ was subsequently proved, but it still did not meet the available upper bound. We close the gaps in both classes for every $q \geq 2$ and $t \geq 1$ by proving

$$\dim C_{2qt}(1, 2, \ldots, t) = \left\lceil \frac{4t}{3} \right\rceil, \qquad \dim C_{2qt+1}(1, 2, \ldots, t) = \left\lceil \frac{4t + 2}{3} \right\rceil.$$

The proofs use two different structural approaches. For $n = 2qt$, a local counting argument gives the sharp lower bound, while a periodic construction based on antipodal landmark pairs attains it. For $n = 2qt + 1$, an analysis of the gaps between consecutive landmarks yields the lower bound and leads to an explicit construction of matching size. Thus, our results turn the previously nonmatching bounds into exact formulas. Both values are independent of q and have the common leading term $4t/3$, settling these two cases in the eventual periodic classification of this graph family.

Keywords: metric dimension; resolving set; metric basis; circulant graph; power of a cycle
Mathematics Subject Classification: 05C12

1. Introduction

Introduced independently by Slater [1] and by Harary and Melter [2], the metric dimension of a graph is the minimum number of landmarks needed to distinguish all vertices by their distance vectors; see the survey [3] for applications and further background. In this paper we study the circulant graph

$C_n(1, 2, \ldots, t)$, namely the Cayley graph of the cyclic group $\mathbb{Z}_n$ with generating set $\{\pm 1, \pm 2, \ldots, \pm t\}$.

For fixed t, the metric dimension of this graph does not simply grow with n. Chau and Gosselin [4] showed that the family is eventually periodic with period $2t$ and obtained the following bounds in the two congruence classes considered here.

Proposition 1.1. *The following statements hold.*

(i) *For every fixed t, there exists $N = N(t)$ such that, whenever $n \geq N$,*

$$\dim C_n(1, 2, \ldots, t) = \dim C_{n+2t}(1, 2, \ldots, t).$$

(ii) *If $t \geq 4$ and $q \geq 3$, then*

$$t + 2 \leq \dim C_{2qt+1}(1, 2, \ldots, t) \leq 2t - 2.$$

(iii) *If $t \geq 4$ is odd and $q \geq 3$, then*

$$t + 1 \leq \dim C_{2qt}(1, 2, \ldots, t) \leq 2t - 2.$$

Proposition 1.1(i) shows that, for fixed t, determining the metric dimension in each residue class modulo $2t$ is a natural classification problem. Earlier general upper bounds were obtained by Grigorious et al. [5], and Vetrík [6, 7] subsequently improved the estimates in several residue classes. For the residue class 0, Chau and Gosselin further conjectured that the metric dimension should in general be at least $t + 2$; this was later proved by BC and Dueck.

Proposition 1.2 (BC and Dueck [8])**.** *If $t \geq 4$ and $q \geq 2$, then*

$$\dim C_{2qt}(1, 2, \ldots, t) \geq t + 2.$$

Meanwhile, Gao, Xiao, and Zhang [9] proved that, for odd $t \geq 5$ and $k \geq 2$,

$$\dim C_{2kt+t}(1, 2, \ldots, t) = t + 1$$

and obtained new upper bounds in several other congruence classes. For fixed values of t, Grigorious, Kalinowski, Ryan, and Stephen [10] completely determined the metric dimension for $t = 4$, while Knor, Škrekovski, and Vetrík [11] settled the remaining open cases for $t = 5$. Nevertheless, for general t, the residue classes 0 and 1 still had lower and upper bounds separated by a linear gap.

We determine the exact metric dimensions in these two congruence classes.

Theorem 1.3. *Let $q \geq 2$ and $t \geq 1$. Then*

$$\dim C_{2qt}(1, 2, \ldots, t) = \left\lceil \frac{4t}{3} \right\rceil, \qquad \dim C_{2qt+1}(1, 2, \ldots, t) = \left\lceil \frac{4t + 2}{3} \right\rceil.$$

These formulas are not constant-order refinements of the previously known bounds. As t grows, the new lower bounds improve the earlier linear estimates by a linear amount, while the matching upper bounds are attained by explicit constructions. Moreover, both formulas hold for every $q \geq 2$, and the exact values are independent of q. Thus, in the two congruence classes considered here, stabilization

already occurs from the first nontrivial case $q = 2$. Although the two proofs use different structures, both exact values have leading term $4t/3$.

Section 2 gives the basic definitions and the distance formula. Section 3 treats $n = 2qt$: The lower bound follows from local constraints on consecutive columns, while the matching upper bound is achieved by a periodic construction using antipodal landmark pairs. Section 4 treats $n = 2qt + 1$: The lower bound is derived from cyclic-gap constraints on a resolving set, and the upper bound is obtained from a construction suggested by the extremal gap pattern revealed by the lower-bound argument.

2. Preliminaries

Let G be a finite, simple, undirected, connected graph, and let $d_G(u, v)$ denote the distance between vertices $u, v \in V(G)$. For distinct $u, v \in V(G)$ and $w \in V(G)$, we say that w *resolves* the pair $\{u, v\}$ if $d_G(u, w) \neq d_G(v, w)$. For a set $S \subseteq V(G)$, we say that S *resolves* the pair $\{u, v\}$ if there exists a vertex $w \in S$ that resolves $\{u, v\}$. Let $W = (w_1, \ldots, w_k)$ be an ordered set of distinct vertices of G. The metric representation of a vertex v with respect to W is

$$r(v \mid W) := (d_G(v, w_1), \ldots, d_G(v, w_k)).$$

If distinct vertices have distinct metric representations with respect to W, equivalently, if for every $u \neq v$ there exists $w_i \in W$ such that $d_G(w_i, u) \neq d_G(w_i, v)$, then W is called a *resolving set* of G. A resolving set of minimum cardinality is a *metric basis*, and its cardinality is the *metric dimension* of G, denoted by $\dim(G)$.

Let $n \geq 3$ and $1 \leq t \leq \lfloor n/2 \rfloor$, and let $v_0, v_1, \ldots, v_{n-1}$ be distinct vertices. For every integer i, write $v_i = v_r$, where r is the unique integer satisfying $0 \leq r \leq n - 1$ and $i \equiv r \pmod{n}$. The circulant graph $C_n(1, 2, \ldots, t)$ is defined by

$$V(C_n(1, 2, \ldots, t)) = \{v_0, v_1, \ldots, v_{n-1}\},$$

$$E(C_n(1, 2, \ldots, t)) = \{\{v_i, v_{i+s}\} : 0 \leq i \leq n - 1,\ 1 \leq s \leq t\}.$$

The edges $\{v_i, v_{i+1}\}$ form the Hamiltonian cycle $v_0 v_1 \cdots v_{n-1} v_0$. Following the terminology of [4, 8, 9], we call this cycle the *outer cycle*. We write $C_n(t) := C_n(1, 2, \ldots, t)$ for short.

For $0 \leq i, j \leq n - 1$, define

$$\delta_n(i, j) := \min\{|i - j|,\ n - |i - j|\}$$

to be the distance between v_i and v_j on the outer cycle. Then

$$d_{C_n(t)}(v_i, v_j) = \left\lceil \frac{\delta_n(i, j)}{t} \right\rceil. \tag{2.1}$$

Indeed, along a shortest outer-cycle arc of length $\delta_n(i, j)$, each graph edge can traverse at most t outer-cycle positions, while the arc can be partitioned into $\lceil \delta_n(i, j)/t \rceil$ segments of length at most t. This gives both bounds in (2.1). To formulate the local constraints used below, we introduce two families of test pairs. For $\ell \in \{1, 2\}$ and $i \in \mathbb{Z}$, define

$$P_\ell(i) := \{v_i, v_{i+\ell}\}.$$

Thus, $P_1(i)$ is a pair of consecutive vertices on the outer cycle, while, for $n \geq 4$, the two vertices of $P_2(i)$ have outer-cycle distance 2. Every resolving set must resolve all pairs $P_\ell(i)$.

3. The Case $n = 2qt$

3.1. Lower bound

Let $q \geq 2$ and first assume $t \geq 3$; the cases $t = 1, 2$ will be handled separately in the proof of the lower bound. For $a, b \in \mathbb{Z}$, write

$$x_{a,b} := v_{a+bt}.$$

When $0 \leq a < t$ and $0 \leq b < 2q$, this representation is unique; we call a and b the first and second coordinates of $x_{a,b}$, respectively. Moreover, $x_{a+t,b} = x_{a,b+1}, x_{a,b+2q} = x_{a,b}$. For $a \in \mathbb{Z}$, define the ath column by

$$B_a := \{x_{a,b} : 0 \leq b < 2q\}.$$

Thus $B_{a+t} = B_a$, so column indices are understood modulo t. For any resolving set W and $a \in \mathbb{Z}$, define

$$s_a(W) := |W \cap B_a|.$$

Fix an arbitrary resolving set W. Throughout this subsection, all column counts are taken with respect to this same set W; for brevity, write $s_a := s_a(W)$. Then $s_{a+t} = s_a$. To translate the resolving condition into restrictions on the column counts s_a, we first determine which vertices can resolve the test pairs P_1 and P_2.

Lemma 3.1. *For all $a, b, c \in \mathbb{Z}$, the following hold.*

(i) *The pair $P_1(a + bt)$ can be resolved only by vertices in $B_a \cup B_{a+1}$, and*

$$x_{a,c} \text{ resolves } P_1(a + bt) \iff b - c \in \{0, 1, \ldots, q - 1\},$$

$$x_{a+1,c} \text{ resolves } P_1(a + bt) \iff c - b \in \{0, 1, \ldots, q - 1\}.$$

(ii) *The pair $P_2(a + bt)$ can be resolved only by vertices in $B_a \cup B_{a+1} \cup B_{a+2}$, and*

$$\begin{aligned} x_{a,c} \text{ resolves } P_2(a + bt) &\iff b - c \in \{0, 1, \ldots, q - 1\}, \\ x_{a+1,c} \text{ resolves } P_2(a + bt) &\iff b - c \notin \{0, q\}, \\ x_{a+2,c} \text{ resolves } P_2(a + bt) &\iff c - b \in \{0, 1, \ldots, q - 1\}. \end{aligned}$$

In the displayed conditions, $b - c$ and $c - b$ are understood as their least nonnegative residues modulo $2q$.

Proof. We first prove the column restrictions. The endpoints of $P_1(a + bt)$ are consecutive on the outer cycle, so their outer-cycle distances from any fixed vertex differ by 1. If division by t followed by taking ceilings gives different graph distances, then one of these two outer-cycle distances is divisible by t. Since $t \mid n$, the resolving vertex is congruent modulo t to one of the endpoints, and therefore lies in $B_a \cup B_{a+1}$. If a vertex resolves $P_2(a + bt)$, then it must resolve at least one of the two adjacent pairs $P_1(a + bt)$ and $P_1(a + bt + 1)$; otherwise its distances to all three vertices on this two-step arc would be equal. Consequently, every vertex that resolves $P_2(a + bt)$ lies in $B_a \cup B_{a+1} \cup B_{a+2}$.

Translating all vertex indices by $-(a+bt)$ preserves distances, so it remains only to consider the case $a = b = 0$. Let $r \equiv c - b \pmod{2q}$, $0 \le r \le 2q-1$. For convenience, define $h(r) := \min\{r, 2q-r\}$, and

$$A(r) := \begin{cases} 1, & r = 0, \\ r, & 1 \le r \le q, \\ 2q - r + 1, & q+1 \le r \le 2q-1, \end{cases} \qquad B(r) := \begin{cases} 1, & r = 0, \\ r+1, & 1 \le r \le q-1, \\ q, & r = q, \\ 2q - r, & q+1 \le r \le 2q-1. \end{cases}$$

Substitution into (2.1) gives

$$\begin{aligned} d(x_{0,r}, x_{0,0}) &= d(x_{1,r}, x_{1,0}) = d(x_{2,r}, x_{2,0}) = h(r), \\ d(x_{0,r}, x_{1,0}) &= d(x_{0,r}, x_{2,0}) = d(x_{1,r}, x_{2,0}) = A(r), \\ d(x_{1,r}, x_{0,0}) &= d(x_{2,r}, x_{0,0}) = B(r). \end{aligned}$$

We first consider $P_1(0) = \{x_{0,0}, x_{1,0}\}$. The vertex $x_{0,r}$ resolves $P_1(0)$ if and only if $h(r) \ne A(r)$, which holds precisely when

$$r = 0 \quad \text{or} \quad q+1 \le r \le 2q-1.$$

Similarly, $x_{1,r}$ resolves $P_1(0)$ if and only if $B(r) \ne h(r)$, which holds precisely when $0 \le r \le q-1$.

We next consider $P_2(0) = \{x_{0,0}, x_{2,0}\}$. The vertex $x_{0,r}$ resolves $P_2(0)$ if and only if $h(r) \ne A(r)$; hence the condition is again $r = 0$ or $q+1 \le r \le 2q-1$. The vertex $x_{1,r}$ resolves $P_2(0)$ if and only if $B(r) \ne A(r)$, which is equivalent to $r \notin \{0, q\}$. Finally, $x_{2,r}$ resolves $P_2(0)$ if and only if $B(r) \ne h(r)$, which is equivalent to $0 \le r \le q-1$.

These calculations are summarized in the following table:

vertex	pair	values of r for which the vertex resolves the pair
$x_{0,r}$	$P_1(0)$	$r = 0$ or $q+1 \le r \le 2q-1$
$x_{1,r}$	$P_1(0)$	$0 \le r \le q-1$
$x_{0,r}$	$P_2(0)$	$r = 0$ or $q+1 \le r \le 2q-1$
$x_{1,r}$	$P_2(0)$	$r \notin \{0, q\}$
$x_{2,r}$	$P_2(0)$	$0 \le r \le q-1$

Rewriting these conditions in terms of b and c proves the lemma. □

Lemma 3.2. *For every $a \in \mathbb{Z}_t$, $s_a + s_{a+1} \ge 2$.*

Proof. As b ranges over $\mathbb{Z}_{2q}$, the $2q$ pairs $P_1(a+bt)$ can be resolved only by vertices in $B_a \cup B_{a+1}$. By Lemma 3.1(i), each landmark in these two columns resolves exactly q of the $2q$ pairs. Hence fewer than two landmarks cannot resolve all of them. □

The target average of $4/3$ landmarks per column suggests examining three consecutive columns at a time. The next lemma shows that their total can fall below 4 only in one specific pattern.

Lemma 3.3. *If $s_a + s_{a+1} + s_{a+2} \le 3$, then $(s_a, s_{a+1}, s_{a+2}) = (0, 3, 0)$.*

Proof. Recall that, throughout this proof, all quantities $s_j = s_j(W)$ and all intersections $W \cap B_j$ refer to the same fixed resolving set W.

By Lemma 3.2, $s_a + s_{a+1} \geq 2$, $s_{a+1} + s_{a+2} \geq 2$. Together with $s_a + s_{a+1} + s_{a+2} \leq 3$, this leaves, apart from $(0, 3, 0)$, only $(0, 2, 0), (0, 2, 1), (1, 2, 0)$ and $(1, 1, 1)$. Suppose first that $s_a = 0$ and $s_{a+1} = 2$. By Lemma 3.1(i), each landmark in B_{a+1} resolves a cyclic interval of exactly q of the $2q$ pairs $P_1(a + bt)$. Hence the two corresponding q-element parameter sets must be complementary, and the two landmarks are necessarily $x_{a+1,c}, x_{a+1,c+q}$ for some $c \in \mathbb{Z}_{2q}$. By Lemma 3.1(ii), neither landmark resolves either of $P_2(a + ct), P_2(a + (c + q)t)$. This excludes $(0, 2, 0)$. It also excludes $(0, 2, 1)$, since a single landmark in B_{a+2} cannot resolve both of these pairs. The automorphism $v_i \longmapsto v_{2a+2-i}$ exchanges B_a and B_{a+2} while fixing B_{a+1}, so the same argument excludes $(1, 2, 0)$.

Finally, suppose $(s_a, s_{a+1}, s_{a+2}) = (1, 1, 1)$. For this same fixed resolving set W, write

$$W \cap B_a = \{x_{a,\alpha}\}, \qquad W \cap B_{a+1} = \{x_{a+1,\beta}\}, \qquad W \cap B_{a+2} = \{x_{a+2,\gamma}\}.$$

For the first two landmarks to resolve all pairs $P_1(a + bt)$, the two q-element parameter sets in Lemma 3.1(i) must again be complementary. Hence $\beta \equiv \alpha - 1 \pmod{2q}$. Applying the same argument to the pairs $P_1(a + 1 + bt)$ gives $\gamma \equiv \beta - 1 \pmod{2q}$.

Now consider $P_2(a + \beta t)$. In the three conditions of Lemma 3.1(ii), we have

$$\beta - \alpha \equiv 2q - 1, \qquad \beta - \beta = 0, \qquad \gamma - \beta \equiv 2q - 1 \pmod{2q}.$$

The first and third values lie outside $\{0, 1, \ldots, q - 1\}$, while the middle value belongs to the exceptional set $\{0, q\}$. Thus none of the three landmarks resolves $P_2(a + \beta t)$, a contradiction. Therefore the only possible pattern is $(s_a, s_{a+1}, s_{a+2}) = (0, 3, 0)$. □

Proposition 3.4. *If $q \geq 2$ and $t \geq 1$, then*

$$\dim C_{2qt}(t) \geq \left\lceil \frac{4t}{3} \right\rceil.$$

Proof. When $t = 1$, $C_{2q}(1)$ is an even cycle, whose metric dimension is at least $2 = \lceil 4/3 \rceil$. When $t = 2$, [8, Proposition 1.1] gives $\dim C_{4q}(1, 2) = 3 = \left\lceil \frac{8}{3} \right\rceil$. Assume henceforth that $t \geq 3$. For every a, Lemmas 3.2 and 3.3 imply

$$s_a + 2s_{a+1} + 2s_{a+2} + s_{a+3} \geq 8. \tag{3.1}$$

Indeed, if both three-column sums $s_a + s_{a+1} + s_{a+2}, s_{a+1} + s_{a+2} + s_{a+3}$ are at least 4, then (3.1) is immediate. Otherwise one of them is at most 3 and hence equals the pattern $(0, 3, 0)$ by Lemma 3.3; Lemma 3.2 then forces the remaining end column to contribute at least 2, again giving (3.1).

Summing (3.1) over $a \in \mathbb{Z}_t$, each s_a occurs with total coefficient 6. Therefore $6|W| \geq 8t$, and hence $|W| \geq \lceil 4t/3 \rceil$. □

3.2. Upper bound

We retain the coordinates $x_{a,b} = v_{a+bt}$ and the columns B_a. Put $L = qt = n/2$. Assume first that $t \geq 3$ and write

$$t = 3m + r, \qquad r \in \{0, 1, 2\}.$$

Motivated by the three-column counting in the lower-bound proof, we use a three-column periodic construction. For every $0 \le i < m$, choose

$$W_t \cap B_{3i} = \{x_{3i,0}, x_{3i,q}\}, \qquad W_t \cap B_{3i+1} = \{x_{3i+1,1}, x_{3i+1,q+1}\},$$

and choose no landmark in B_{3i+2}. Thus each complete three-column block contains four landmarks: an antipodal pair in each of its first two columns, where the two vertices of each pair are at outer-cycle distance $L = n/2$. The pair in the second column is shifted by one unit in the second coordinate.

If $r = 0$, these blocks cover all columns. If $r = 1$, choose in the remaining column B_{3m} the antipodal pair $x_{3m,0}, x_{3m,q}$. If $r = 2$, choose $x_{3m,0}, x_{3m,q}, x_{3m,1}$ in B_{3m} and no landmark in B_{3m+1}. Equivalently,

$$W_t = \bigcup_{i=0}^{m-1} \{x_{3i,0}, x_{3i,q}, x_{3i+1,1}, x_{3i+1,q+1}\} \cup \begin{cases} \varnothing, & t = 3m, \\ \{x_{3m,0}, x_{3m,q}\}, & t = 3m+1, \\ \{x_{3m,0}, x_{3m,q}, x_{3m,1}\}, & t = 3m+2. \end{cases} \tag{3.2}$$

Hence

$$|W_t| = \begin{cases} 4m, & t = 3m, \\ 4m+2, & t = 3m+1, \\ 4m+3, & t = 3m+2, \end{cases} = \left\lceil \frac{4t}{3} \right\rceil.$$

The three cases differ only in the terminal part of the periodic pattern, as illustrated in Figures 1–3. The figures use the original vertex labels v_j on the outer cycle, rather than the coordinates $x_{a,b}$; recall that $x_{a,b} = v_{a+bt}$. Black dots denote landmarks, crosses mark selected multiples of t, and dashed diameters indicate representative antipodal pairs. In Figure 3, the distinguished landmark at $2t-2$ is the additional landmark $x_{t-2,1}$.

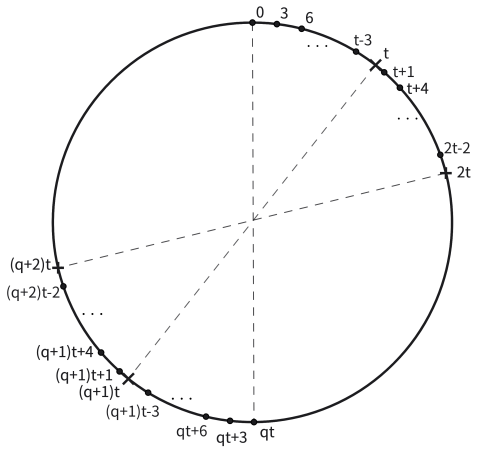


Figure 1. The case $t = 3m$.

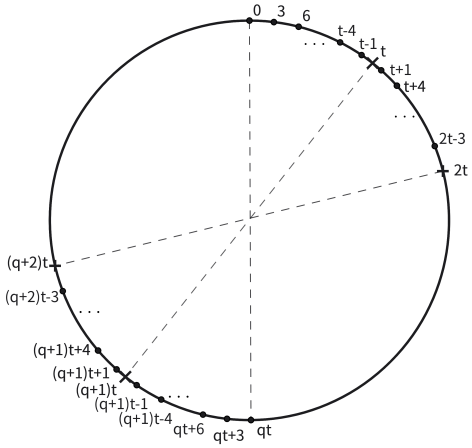


Figure 2. The case $t = 3m + 1$.

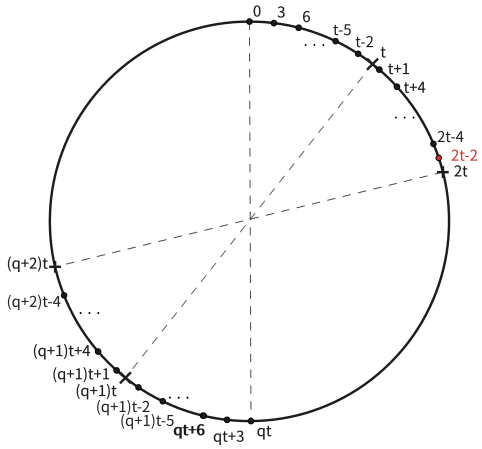


Figure 3. The case $t = 3m + 2$.

We now prove that W_t is a resolving set. The proof has two steps. First, the two antipodal pairs in B_0 and B_1 determine the second coordinate b of a vertex $x_{a,b}$. Once b is fixed, the remaining antipodal pairs determine the first coordinate a. When $t \equiv 2 \pmod 3$, one pair of first-coordinate candidates is not distinguished by the antipodal pairs; the extra landmark $x_{t-2,1}$ distinguishes this remaining pair.

For an antipodal landmark pair $x_{c,\varepsilon}, x_{c,\varepsilon+q}$ with $\varepsilon \in \{0, 1\}$, define

$$\rho_{c,\varepsilon}(x) := \big(d(x, x_{c,\varepsilon}), d(x, x_{c,\varepsilon+q})\big).$$

For any vertex $x_{a,b}$, the two entries of $\rho_{c,\varepsilon}(x_{a,b})$ sum to q if and only if $a = c$, and otherwise sum to $q + 1$. Indeed, if δ is the outer-cycle distance from $x_{a,b}$ to $x_{c,\varepsilon}$, then the outer-cycle distance to its antipode is $L - \delta$. Hence the two graph distances are $\left\lceil \frac{\delta}{t} \right\rceil, \left\lceil \frac{L-\delta}{t} \right\rceil$. Their sum is q exactly when $t \mid \delta$, which, since $0 \le a, c < t$, is equivalent to $a = c$.

Lemma 3.5. *Let $0 \le a, c < t$ and $b, d \in \mathbb{Z}_{2q}$. If $\rho_{0,0}(x_{a,b}) = \rho_{0,0}(x_{c,d})$ and $\rho_{1,1}(x_{a,b}) = \rho_{1,1}(x_{c,d})$, then $b = d$.*

Proof. By the observation above, $\rho_{0,0}(x_{a,b}) = \rho_{0,0}(x_{c,d})$ implies $a = 0$ if and only if $c = 0$, while $\rho_{1,1}(x_{a,b}) = \rho_{1,1}(x_{c,d})$ implies $a = 1$ if and only if $c = 1$. Hence either $a = c = 0$, $a = c = 1$, or $a, c \geq 2$. We show in each case that the second coordinate is uniquely determined by the two distances to $x_{0,0}$ and $x_{1,1}$.

If the first coordinate is 0, then

$$(d(x_{0,b}, x_{0,0}), d(x_{0,b}, x_{1,1})) = \begin{cases} (0, 2), & b = 0, \\ (1, 1), & b = 1, \\ (b, b-1), & 2 \leq b \leq q, \\ (q-1, q), & b = q+1, \\ (2q-b, 2q-b+2), & q+2 \leq b \leq 2q-1. \end{cases}$$

These ordered pairs are distinct for distinct b. If the first coordinate is 1, then

$$(d(x_{1,b}, x_{0,0}), d(x_{1,b}, x_{1,1})) = \begin{cases} (1, 1), & b = 0, \\ (b+1, b-1), & 1 \leq b \leq q-1, \\ (q, q-1), & b = q, \\ (2q-b, 2q-b+1), & q+1 \leq b \leq 2q-1, \end{cases}$$

and these pairs are again distinct. Finally, for every $2 \leq a < t$,

$$(d(x_{a,b}, x_{0,0}), d(x_{a,b}, x_{1,1})) = \begin{cases} (1, 1), & b = 0, \\ (b+1, b), & 1 \leq b \leq q-1, \\ (q, q), & b = q, \\ (2q-b, 2q-b+1), & q+1 \leq b \leq 2q-1. \end{cases}$$

Again the ordered pair determines b. Thus $b = d$ in all three cases. □

Having determined the second coordinate, we now fix it and use the remaining antipodal pairs to recover the first coordinate.

Lemma 3.6. *Let $0 \leq a, c < t$, and fix $b \in \mathbb{Z}_{2q}$. Suppose that $\rho_{u,\varepsilon}(x_{a,b}) = \rho_{u,\varepsilon}(x_{c,b})$ for every antipodal landmark pair $\{x_{u,\varepsilon}, x_{u,\varepsilon+q}\} \subseteq W_t$. Then $a = c$, except possibly when $t \equiv 2 \pmod 3$, $b \in \{0, q\}$, $\{a, c\} = \{t-3, t-1\}$.*

Proof. First suppose $0 \leq b \leq q-1$. We consider separately $b = 0$, $b = 1$, and $2 \leq b \leq q-1$.

(i) $b = 0$. For $0 \leq i < m$, the antipodal pair in B_{3i} gives

$$\rho_{3i,0}(x_{a,0}) = \begin{cases} (0, q), & a = 3i, \\ (1, q), & a \neq 3i. \end{cases}$$

Thus this pair identifies the value $a = 3i$. The antipodal pair in B_{3i+1} gives

$$\rho_{3i+1,1}(x_{a,0}) = \begin{cases} (2, q-1), & a < 3i+1, \\ (1, q-1), & a = 3i+1, \\ (1, q), & a > 3i+1. \end{cases}$$

Hence $\rho_{3i+1,1}(x_{a,0})$ determines whether a lies below, at, or above $3i+1$.

Suppose that $a < c$ and $\rho_{u,\varepsilon}(x_{a,0}) = \rho_{u,\varepsilon}(x_{c,0})$ for every antipodal landmark pair $\{x_{u,\varepsilon}, x_{u,\varepsilon+q}\} \subseteq W_t$. For $0 \le i < m-1$, if both first coordinates lie strictly between $3i+1$ and $3i+4$, the only possibilities are $3i+2$ and $3i+3$. But $3i+3$ is identified by the antipodal pair in B_{3i+3}. Thus each such interval contains at most one possible first coordinate. At the left end, the only value before 1 is 0, which is identified by the pair in B_0.

It remains to inspect the right end. If $t = 3m$, the last column of the form B_{3i+1} is B_{t-2}, leaving only $a = t-1$ to its right. If $t = 3m+1$, the last such column is B_{t-3}; among the two values $t-2, t-1$ to its right, $a = t-1 = 3m$ is identified by the additional antipodal pair in B_{3m}, leaving only $t-2$. If $t = 3m+2$, the last such column is B_{t-4}. Among $t-3, t-2, t-1$, the value $t-2 = 3m$ is identified by the additional antipodal pair in B_{3m}. The remaining first coordinates $t-3$ and $t-1$ satisfy

$$\rho_{u,\varepsilon}(x_{t-3,0}) = \rho_{u,\varepsilon}(x_{t-1,0})$$

for every antipodal landmark pair $\{x_{u,\varepsilon}, x_{u,\varepsilon+q}\} \subseteq W_t$: Both lie above every column B_{3i+1}, and neither is the first coordinate of an antipodal pair in a column B_{3i} or in the additional column $B_{3m} = B_{t-2}$. Therefore the only possible ambiguity for $b = 0$ is $t \equiv 2 \pmod 3, b \in \{0, q\}, \{a, c\} = \{t-3, t-1\}$.

(ii) $b = 1$. For $0 \le i < m$,

$$\rho_{3i,0}(x_{a,1}) = \begin{cases} (1, q), & a < 3i, \\ (1, q-1), & a = 3i, \\ (2, q-1), & a > 3i, \end{cases}$$

so the antipodal pair in B_{3i} determines whether a lies below, at, or above $3i$. On the other hand,

$$\rho_{3i+1,1}(x_{a,1}) = \begin{cases} (0, q), & a = 3i+1, \\ (1, q), & a \ne 3i+1, \end{cases}$$

so the pair in B_{3i+1} identifies the value $a = 3i+1$.

For $0 \le i < m-1$, the only first coordinates strictly between $3i$ and $3i+3$ are $3i+1$ and $3i+2$, and the former is identified by the pair in B_{3i+1}. Hence at most one possible first coordinate remains between two successive columns B_{3i} and B_{3i+3}.

At the right end, if $t = 3m$, the final pair in a column B_{3i} lies in B_{t-3}; of the two larger first coordinates, $t-2$ is identified by the pair in B_{t-2}, leaving only $t-1$. If $t = 3m+1$, the additional antipodal pair in $B_{3m} = B_{t-1}$ completes the same argument at the right end. If $t = 3m+2$, the additional pair in $B_{3m} = B_{t-2}$ distinguishes the possible first coordinate on its left from the only value $t-1$ on its right. Thus $a = c$ when $b = 1$.

(iii) $2 \le b \le q-1$. For $0 \le i < m$, the two types of antipodal pairs give

$$\rho_{3i,0}(x_{a,b}) = \begin{cases} (b, q-b+1), & a < 3i, \\ (b, q-b), & a = 3i, \\ (b+1, q-b), & a > 3i, \end{cases}$$

and

$$\rho_{3i+1,1}(x_{a,b}) = \begin{cases} (b-1, q-b+2), & a < 3i+1, \\ (b-1, q-b+1), & a = 3i+1, \\ (b, q-b+1), & a > 3i+1. \end{cases}$$

Thus $\rho_{3i,0}(x_{a,b})$ determines whether a lies below, at, or above $3i$, while $\rho_{3i+1,1}(x_{a,b})$ determines whether a lies below, at, or above $3i+1$.

In every complete three-column block, antipodal landmark pairs occur in the first two columns. Between consecutive column indices carrying such pairs there is therefore at most one other possible first coordinate. The same remains true at the right end: If $t = 3m$, only $t-1$ follows the last such column B_{t-2}; if $t = 3m+1$, the additional antipodal pair lies in B_{t-1}; and if $t = 3m+2$, it lies in B_{t-2}, with only $t-1$ to its right. Hence, for $2 \le b \le q-1$, if $\rho_{u,\varepsilon}(x_{a,b}) = \rho_{u,\varepsilon}(x_{c,b})$ for every antipodal landmark pair $\{x_{u,\varepsilon}, x_{u,\varepsilon+q}\} \subseteq W_t$, then $a = c$.

It remains to consider $q \le b \le 2q-1$. Write $b = b' + q$, where $0 \le b' \le q-1$. Since $x_{a,b'+q}$ is antipodal to $x_{a,b'}$, translation by $L = qt = n/2$ interchanges the two landmarks in every antipodal pair. Consequently, if $\rho_{u,\varepsilon}(x_{a,b'}) = (r,s)$, then $\rho_{u,\varepsilon}(x_{a,b'+q}) = (s,r)$. Thus, for any a, c,

$$\rho_{u,\varepsilon}(x_{a,b'+q}) = \rho_{u,\varepsilon}(x_{c,b'+q})$$

for every antipodal landmark pair $\{x_{u,\varepsilon}, x_{u,\varepsilon+q}\} \subseteq W_t$ if and only if

$$\rho_{u,\varepsilon}(x_{a,b'}) = \rho_{u,\varepsilon}(x_{c,b'})$$

for every such pair. The conclusions above therefore carry over unchanged. In particular, the only additional possibility is obtained from $b' = 0$, namely $t \equiv 2 \pmod 3$, $b \in \{0,q\}$, $\{a,c\} = \{t-3, t-1\}$. This proves the lemma. □

Proposition 3.7. *If $q \ge 2$ and $t \ge 1$, then*

$$\dim C_{2qt}(t) \le \left\lceil \frac{4t}{3} \right\rceil.$$

Proof. For $t = 1$, the even cycle has metric dimension 2. For $t = 2$, [8, Proposition 1.1] gives $\dim C_{4q}(1,2) = 3$. Assume henceforth that $t \ge 3$, and let W_t be the set defined in (3.2).

Suppose that $x_{a,b}$ and $x_{c,d}$ have the same metric representation with respect to W_t. Since $x_{0,0}, x_{0,q}$, $x_{1,1}, x_{1,q+1} \in W_t$, Lemma 3.5 gives $b = d$. Lemma 3.6 then gives $a = c$, except possibly when $t \equiv 2 \pmod 3$, $b \in \{0,q\}$, $\{a,c\} = \{t-3, t-1\}$.

In this exceptional case, the construction contains the additional landmark $x_{t-2,1} = v_{2t-2}$. Its distances to the two exceptional pairs are

$$\begin{aligned} d(x_{t-2,1}, x_{t-3,0}) &= 2, & d(x_{t-2,1}, x_{t-1,0}) &= 1, \\ d(x_{t-2,1}, x_{t-3,q}) &= q-1, & d(x_{t-2,1}, x_{t-1,q}) &= q. \end{aligned}$$

Thus this landmark distinguishes both exceptional pairs. Hence W_t is a resolving set, and since $|W_t| = \lceil 4t/3 \rceil$, the result follows. □

Combining Propositions 3.4 and 3.7 gives the exact value.

Theorem 3.8. *If $q \ge 2$ and $t \ge 1$, then*

$$\dim C_{2qt}(t) = \left\lceil \frac{4t}{3} \right\rceil.$$

4. The Case $n = 2qt + 1$

4.1. Lower bound

We first assume $q, t \geq 2$. Since $\gcd(n, t) = 1$, the residues $0, t, 2t, \ldots, (n-1)t \pmod{n}$ are all distinct. To make the parameter c in $P_\ell(ct)$ directly serve as a vertex index, we relabel the vertices by

$$x_b := v_{bt} \qquad (b \in \mathbb{Z}).$$

Then $x_{b+n} = x_b$, and $x_0, x_1, \ldots, x_{n-1}$ list all vertices exactly once. For $\ell \in \{1, 2\}$ and $b \in \mathbb{Z}$, define

$$R_\ell(b) := \{c \in \{0, 1, \ldots, n-1\} : x_b \text{ resolves } P_\ell(ct)\}.$$

Because every resolving set must resolve all adjacent pairs $P_1(ct)$, we begin by determining which such pairs can be resolved by a single landmark.

Lemma 4.1. *If $q, t \geq 2$, then*

$$R_1(0) = \{0, 1, \ldots, 2q\} \setminus \{q\}.$$

More generally, for every $b \in \mathbb{Z}$, $R_1(b) = b + R_1(0) = \{b, b+1, \ldots, b+2q\} \setminus \{b+q\}$, with parameters interpreted modulo n.

Proof. Formula (2.1) gives

c	$d(x_0, v_{ct})$	$d(x_0, v_{ct+1})$
$0 \leq c \leq q-1$	c	$c+1$
$c = q$	q	q
$q+1 \leq c \leq 2q$	$2q-c+1$	$2q-c$

For $2q+1 \leq c \leq n-1$, the two distances are equal. Hence x_0 resolves exactly the listed parameters. Translating every vertex index by bt preserves distances and sends x_0 to x_b and $P_1(ct)$ to $P_1((c+b)t)$, proving the general formula. □

Let $W = \{x_{b_1}, x_{b_2}, \ldots, x_{b_k}\}$ be a resolving set, where $0 \leq b_1 < b_2 < \cdots < b_k \leq n-1$. We first use only the necessary condition that W resolve all pairs $P_1(ct)$. Extend the subscripts periodically to the whole integer line by setting $b_{j+k} = b_j + n (j \in \mathbb{Z})$, and define the cyclic gaps of W by

$$g_j := b_{j+1} - b_j.$$

Then $g_{j+k} = g_j, g_j \geq 1, \sum_{j=1}^{k} g_j = n = 2qt + 1$.

All interval arguments involving the b_j below are made on this periodically extended integer line. Since the gap sum is fixed at n, a lower bound on $k = |W|$ will follow from an upper bound on this sum in terms of k.

Lemma 4.2. *If W resolves all pairs $P_1(ct)$, then, for every $j \in \mathbb{Z}$, $g_j \leq 2q+1$ and $\min\{g_{j-1}, g_j\} \leq q$.*

Proof. Suppose first that $g_j \geq 2q+2$, and set $c = b_j + 2q + 1$. By Lemma 4.1, a landmark x_h can resolve $P_1(ct)$ only if

$$h \in \{c-2q, c-2q+1, \ldots, c\} \setminus \{c-q\}.$$

But every landmark index at most $b_j = c - 2q - 1$ lies to the left of this set, while every landmark index at least $b_{j+1} > c$ lies to its right. Hence no landmark resolves $P_1(ct)$, a contradiction.

Next suppose that $g_{j-1} \geq q + 1$ and $g_j \geq q + 1$, and set $c_0 = b_j + q$. By Lemma 4.1, among the parameters $b_j, b_j + 1, \ldots, b_j + 2q$, the single parameter $c_0 = b_j + q$ is the unique one not resolved by x_{b_j}. For every $h \leq j - 1$, $b_h + 2q \leq b_{j-1} + 2q \leq b_j + q - 1 < c_0$, whereas for every $h \geq j + 1$, $b_h \geq b_{j+1} \geq b_j + q + 1 > c_0$. Hence no landmark in W resolves $P_1(c_0 t)$, again a contradiction. □

Lemma 4.2 gives the restrictions arising from the adjacent pairs $P_1(ct)$. We now examine the pairs $P_2(ct)$ to obtain the additional local information needed below.

Lemma 4.3. *If $q, t \geq 2$, then*

$$R_2(0) = \{0, 1, \ldots, 4q\} \setminus \{q, 2q, 3q\}.$$

More generally, $R_2(b) = b + R_2(0)$ for every $b \in \mathbb{Z}$.

Proof. Again, formula (2.1) gives

c	$d(x_0, v_{ct})$	$d(x_0, v_{ct+2})$
$0 \leq c \leq q - 1$	c	$c + 1$
$c = q$	q	q
$q + 1 \leq c \leq 2q - 1$	$2q - c + 1$	$2q - c$
$c = 2q$	1	1
$2q + 1 \leq c \leq 3q - 1$	$c - 2q$	$c - 2q + 1$
$c = 3q$	q	q
$3q + 1 \leq c \leq 4q$	$4q - c + 1$	$4q - c$

Now let $4q + 1 \leq c \leq n - 1$. Write uniquely $c = 2qh + s, h \geq 2, 1 \leq s \leq 2q$. Since $2qt = n - 1$, $x_c = v_{ct} = v_{st-h}, v_{ct+2} = v_{(c-4q)t} = x_{c-4q} = v_{st-(h-2)}$. Substitution into (2.1) yields $d(v_0, v_{ct}) = \min\{s, 2q + 1 - s\} = d(v_0, v_{ct+2})$. Thus x_0 fails exactly for the parameters $q, 2q, 3q$ among $0, \ldots, 4q$ and for every parameter $c \geq 4q + 1$. Translating all indices proves the formula for $R_2(b)$. □

The preceding description is most convenient in the following window form.

Corollary 4.4. *For every $c \in \mathbb{Z}$, the set W resolves $P_2(ct)$ if and only if*

$$\{b_j : j \in \mathbb{Z}\} \cap \Big([c - 4q, c] \setminus \{c - q, c - 2q, c - 3q\}\Big) \neq \varnothing.$$

Proof. By Lemma 4.3, a landmark x_{b_j} resolves $P_2(ct)$ if and only if $c - b_j \in \{0, 1, \ldots, 4q\} \setminus \{q, 2q, 3q\}$, which is equivalent to $b_j \in [c - 4q, c] \setminus \{c - q, c - 2q, c - 3q\}$. The assertion follows. □

We now combine the restrictions supplied by the two families of test pairs. The case $(g_{j-1}, g_j) = (q, 2q + 1)$ is the only one in which both bounds in Lemma 4.2 are attained. Corollary 4.4 imposes an additional restriction on the gaps immediately preceding this pair.

Lemma 4.5. *Suppose that W resolves all pairs $P_1(ct)$ and $P_2(ct)$. If $(g_{j-1}, g_j) = (q, 2q + 1)$, then $g_{j-2} \leq q$. Furthermore, if $g_{j-2} = q$, then $g_{j-3} \leq q$.*

Proof. Suppose first that $g_{j-2} > q$ and set $c_0 = b_j + 2q$. Since $(g_{j-1}, g_j) = (q, 2q+1)$, $b_{j-1} = b_j - q = c_0 - 3q, b_j = c_0 - 2q$. The assumption $g_{j-2} > q$ gives $b_{j-2} < c_0 - 4q$, while $g_j = 2q+1$ gives $b_{j+1} = c_0 + 1$. Hence the only periodically extended landmark indices in $[c_0 - 4q, c_0]$ are $c_0 - 3q, c_0 - 2q$, and both are among the three exceptional positions excluded in Corollary 4.4. Thus $P_2(c_0 t)$ is not resolved, a contradiction. Therefore $g_{j-2} \le q$.

Now assume $g_{j-2} = q$ and, for a contradiction, $g_{j-3} > q$. Set $c_1 = b_j + q$. Since $g_{j-2} = g_{j-1} = q$, $b_{j-2} = c_1 - 3q, b_{j-1} = c_1 - 2q, b_j = c_1 - q$. The assumption $g_{j-3} > q$ gives $b_{j-3} < c_1 - 4q$, while $g_j = 2q+1$ gives $b_{j+1} > c_1$. Hence the landmark indices in $[c_1 - 4q, c_1]$ are exactly $c_1 - 3q, c_1 - 2q, c_1 - q$, again precisely the three exceptional positions in Corollary 4.4. This contradicts the fact that W resolves $P_2(c_1 t)$. Hence $g_{j-3} \le q$. □

With the case $(g_{j-1}, g_j) = (q, 2q+1)$ handled by Lemma 4.5, we can now bound $\sum_{j=1}^{k} g_j$ by grouping consecutive gaps.

Proposition 4.6. *Suppose that $q, t \ge 2$ and W resolves all pairs $P_1(ct)$ and $P_2(ct)$. If $|W| = k$, then $\sum_{j=1}^{k} g_j \le q \left\lfloor \frac{3k}{2} \right\rfloor$.*

Proof. Call a gap $g_j > q$ *long* and a gap $g_j \le q$ *short*. By Lemma 4.2, long gaps are not adjacent. Pair each long gap g_j with the short gap g_{j-1} immediately preceding it; these initial pairs are disjoint.

By Lemma 4.2, every such pair satisfies $g_{j-1} + g_j \le 3q + 1$, and equality can occur only for $(g_{j-1}, g_j) = (q, 2q+1)$. Every other pair therefore has total length at most $3q$.

Consider an exceptional pair $(q, 2q+1)$. By Lemma 4.5, $g_{j-2} \le q$. If $g_{j-2} \le q - 1$, enlarge the pair to the triple g_{j-2}, g_{j-1}, g_j; its total length is at most $(q-1) + q + (2q+1) = 4q$. If $g_{j-2} = q$, then Lemma 4.5 gives $g_{j-3} \le q$. Enlarge instead to the quadruple $g_{j-3}, g_{j-2}, g_{j-1}, g_j$, whose total length is at most $q + q + q + (2q+1) = 5q + 1 \le 6q$. The newly added gaps belong to no initial pair: g_{j-2} is followed by the short gap g_{j-1}, and in the quadruple case g_{j-3} is followed by the short gap g_{j-2}. Hence neither can be the short member of an initial long-gap pair. Moreover, for distinct exceptional long gaps, the added gaps lie in disjoint stretches between consecutive long gaps. Thus the enlarged groups are disjoint. Finally, place each short gap not yet used into a singleton group.

Thus all cyclic gaps are partitioned into disjoint groups satisfying

number of gaps in the group	sum of the gaps in the group
1	$\le q$
2	$\le 3q$
3	$\le 4q$
4	$\le 6q$

Therefore, a group containing $r \in \{1, 2, 3, 4\}$ gaps has total length at most $q \left\lfloor \frac{3r}{2} \right\rfloor$. Let the group sizes be $r_1, \ldots, r_s$. Since $\sum_{\alpha=1}^{s} r_\alpha = k$,

$$\sum_{j=1}^{k} g_j \le q \sum_{\alpha=1}^{s} \left\lfloor \frac{3r_\alpha}{2} \right\rfloor \le q \left\lfloor \frac{3 \sum_{\alpha=1}^{s} r_\alpha}{2} \right\rfloor = q \left\lfloor \frac{3k}{2} \right\rfloor.$$

□

Since the cyclic gaps sum to $n = 2qt + 1$, Proposition 4.6 now gives the desired lower bound on $|W|$.

Proposition 4.7. *If $q \geq 2$ and $t \geq 1$, then*

$$\dim C_{2qt+1}(t) \geq \left\lceil \frac{4t+2}{3} \right\rceil.$$

Proof. Assume first that $q, t \geq 2$, let W be a resolving set, and write $k = |W|$. Proposition 4.6 gives

$$2qt + 1 = \sum_{j=1}^{k} g_j \leq q \left\lfloor \frac{3k}{2} \right\rfloor.$$

Hence $\left\lfloor \frac{3k}{2} \right\rfloor \geq 2t + \frac{1}{q}$. The left-hand side is an integer, so $\left\lfloor \frac{3k}{2} \right\rfloor \geq 2t+1$, and therefore $k \geq \left\lceil \frac{2(2t+1)}{3} \right\rceil = \left\lceil \frac{4t+2}{3} \right\rceil$. If $t = 1$ and $q \geq 2$, then $C_{2q+1}(1)$ is an odd cycle. A single landmark cannot distinguish its two neighbors, so $\dim C_{2q+1}(1) \geq 2 = \lceil (4t+2)/3 \rceil$. □

4.2. Upper bound

The lower-bound argument above reveals two basic restrictions on the cyclic gaps of a resolving set: Every gap is at most $2q + 1$, and a gap larger than q cannot be adjacent to another gap larger than q. To make the landmarks as sparse as possible, it is therefore natural to alternate a long gap of length $2q + 1$ with a compensating short gap of length $q - 1$. We now turn this extremal pattern into an explicit construction and prove that it resolves every pair of vertices, not only the two test families used in the lower bound.

Assume first that $q, t \geq 2$, and continue to use the relabeling $x_b = v_{bt}$. Write

$$n = 3qu + r, \qquad 0 \leq r < 3q.$$

Since $n = 2qt + 1$, the only possibilities are

t	u	r
$3h$	$2h$	1
$3h + 1$	$2h$	$2q + 1$
$3h + 2$	$2h + 1$	$q + 1$

Define

$$B_0 := \{3iq, 3iq + 2q + 1 : 0 \leq i < u\} \cup \{3uq\},$$

and

$$B := \begin{cases} B_0, & r = 1, \\ B_0 \cup \{1\}, & r \in \{q+1, 2q+1\}, \end{cases} \qquad W^\star := \{x_b : b \in B\}. \tag{4.1}$$

Extend B periodically to

$$\widetilde{B} := B + n\mathbb{Z}.$$

Since $x_{c+n} = x_c$, proving that $W^\star$ is a resolving set is equivalent to showing that, for every pair of integer representatives $a < b < a + n$, there is some $c \in \widetilde{B} \cap [a, a + n)$ such that $d(x_c, x_a) \neq d(x_c, x_b)$.

To verify this condition, we first characterize exactly when a landmark x_c fails to distinguish an arbitrary pair x_a, x_b.

Lemma 4.8. *Choose integer representatives $a < b < a + n$ and $c \in (a, a + n) \setminus \{b\}$, and put $\delta = b - a$. Then x_c fails to resolve $\{x_a, x_b\}$ if and only if*

$$\begin{aligned} a < c < b: &\quad \delta \equiv 1 \pmod{2q} \text{ or } 2c \equiv a + b \pmod{2q}, \\ b < c < a + n: &\quad \delta \equiv 0 \pmod{2q} \text{ or } 2c \equiv a + b + 1 \pmod{2q}. \end{aligned}$$

Proof. For $1 \le s, s' < n$, let $\bar{s}, \bar{s}' \in \{1, 2, \dots, 2q\}$ be their respective representatives modulo $2q$. Then, for every $u \in \mathbb{Z}$,

$$d(x_u, x_{u+s}) = \min\{\bar{s}, 2q + 1 - \bar{s}\} =: F(s),$$

and similarly

$$d(x_u, x_{u+s'}) = \min\{\bar{s}', 2q + 1 - \bar{s}'\} =: F(s').$$

It follows that

$$F(s) = F(s') \iff s \equiv s' \pmod{2q} \quad \text{or} \quad s + s' \equiv 1 \pmod{2q}.$$

Suppose first that $a < c < b$. The positive x-index differences from x_c to x_a and x_b are $k_a = a - c + n$ and $k_b = b - c$. Since $n \equiv 1 \pmod{2q}$, $k_a \equiv a - c + 1, k_b \equiv b - c \pmod{2q}$. Thus x_c fails to resolve $\{x_a, x_b\}$ precisely when $F(k_a) = F(k_b)$. By the characterization above, this is equivalent to $\delta \equiv 1 \pmod{2q}$ or $2c \equiv a + b \pmod{2q}$.

Now suppose that $b < c < a + n$. In this case, $k_a = a - c + n$, $k_b = b - c + n$, and hence $k_a \equiv a - c + 1$, $k_b \equiv b - c + 1 \pmod{2q}$. Applying the same characterization gives $\delta \equiv 0 \pmod{2q}$ or $2c \equiv a + b + 1 \pmod{2q}$. □

We now return to the specific construction above. Figure 4 shows the landmark indices of $W^\star$ in one period, after cutting the cyclic order at 0 and unfolding it along the integer line.

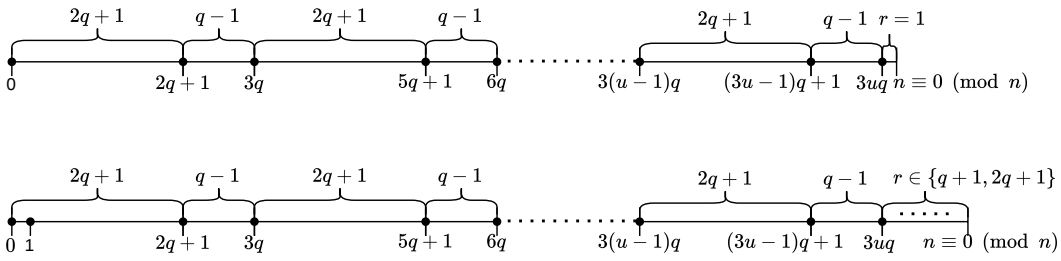


Figure 4. The landmark indices of $W^\star$ in one period. The right endpoint n repeats 0 to indicate the cyclic closure. The upper and lower rows correspond to $r = 1$ and $r \in \{q + 1, 2q + 1\}$, respectively.

Write the elements of $\widetilde{B}$ in increasing order as $\cdots < y_{-1} < y_0 < y_1 < \cdots$ and let $g_j^\star := y_{j+1} - y_j$. To apply Lemma 4.8 to the construction above, we need the following properties of the consecutive gaps in $\widetilde{B}$.

Lemma 4.9. *The construction* (4.1) *satisfies the following properties.*

(i) $g_j^\star \le 2q + 1$ *for every* j;

(ii) *if* $g_j^\star > q$, *then* $g_{j-1}^\star \le q$;

(iii) *every open interval of length at least* $4q$ *contains two elements of* $\widetilde{B}$ *that are incongruent modulo* q.

Proof. Reading the construction in cyclic order, the gap sequence in one period is $((2q+1, q-1)^u, 1)$ when $r = 1$, and $(1, 2q, q-1, (2q+1, q-1)^{u-1}, r)$ when $r \in \{q+1, 2q+1\}$, where the power denotes repetition and the repeated part is empty when $u = 1$. Properties (i) and (ii) are immediate.

We prove (iii). First, every open interval of length at least $4q$ contains at least two elements of $\widetilde{B}$. An interval containing no landmark lies inside a single gap of length at most $2q+1$. If it contains exactly one landmark, its length is at most the sum of the two adjacent gaps. By (i) and (ii), that sum is at most $q + (2q+1) = 3q + 1 < 4q$. Both alternatives are impossible.

Now suppose all landmarks in such an interval were congruent modulo q. Every gap between consecutive landmarks inside the interval would then be divisible by q. In the displayed cyclic gap sequences, the only possible such gap is $2q$, and it is flanked by gaps of lengths 1 and $q-1$. Hence an interval containing landmarks from only one residue class modulo q has length at most $1 + 2q + (q-1) = 3q < 4q$, a contradiction. □

Proposition 4.10. *If* $q \ge 2$ *and* $t \ge 1$, *then*

$$\dim C_{2qt+1}(t) \le \left\lceil \frac{4t+2}{3} \right\rceil.$$

Proof. If $t = 1$, then $C_{2q+1}(1)$ is an odd cycle, whose metric dimension is $2 = \lceil (4t+2)/3 \rceil$. Assume henceforth that $t \ge 2$, and let $W^\star$ be the set defined in (4.1).

Suppose, for a contradiction, that two distinct vertices x_a, x_b are not resolved by $W^\star$. Choose integer representatives $a < b < a + n$, put $\delta = b - a$, and note that neither endpoint belongs to $W^\star$.

Case 1: $\delta \not\equiv 0, 1 \pmod{2q}$. By Lemma 4.8, every landmark index $c \in \widetilde{B} \cap (a, b)$ must satisfy $2c \equiv a + b \pmod{2q}$, whereas every $c \in \widetilde{B} \cap (b, a+n)$ must satisfy $2c \equiv a + b + 1 \pmod{2q}$. The two right-hand sides have opposite parity. Since $2c \equiv M \pmod{2q}$ is solvable only when M is even, one of the two arcs contains no landmark index, while all landmark indices on the other arc are congruent modulo q. Hence all landmark indices in $(a, a+n)$ are congruent modulo q. But $n = 2qt + 1 \ge 4q + 1$, contradicting Lemma 4.9(iii).

Case 2: $\delta \equiv 0$ *or* $1 \pmod{2q}$. If $\delta \equiv 1 \pmod{2q}$, interchange the two endpoints and measure their separation along the complementary arc. The new forward separation is $n - \delta \equiv 0 \pmod{2q}$, since $n \equiv 1 \pmod{2q}$. Thus, after relabeling if necessary, we may assume

$$\delta \equiv 0 \pmod{2q}.$$

Then $b \equiv a \pmod{2q}$. For every landmark index $c \in \widetilde{B} \cap (a, b)$, Lemma 4.8 gives $2c \equiv a + b \pmod{2q}$, and hence $c \equiv a \pmod{q}$. If $\delta \ge 4q$, the interval (a, b) therefore contradicts Lemma 4.9(iii). Since $\delta > 0$ and $\delta \equiv 0 \pmod{2q}$, it remains only to consider

$$\delta = 2q.$$

Choose consecutive landmark indices $y_j < b < y_{j+1}$, and set $d = b - y_j$. By Lemma 4.9(i), $1 \le d \le 2q$. If $d = 2q$, then $y_j = b - 2q = a$, contrary to $x_a \notin W^\star$. Hence $d < 2q$, so $y_j \in (a, b)$. Since every landmark

index in (a, b) is congruent to a modulo q,

$$y_j - a = 2q - d \equiv 0 \pmod{q}.$$

With $1 \le d < 2q$, this forces $d = q$.

Now $y_{j+1} - y_j > q$, because $y_j < b < y_{j+1}$ and $b - y_j = q$. Lemma 4.9(ii) therefore gives $e := y_j - y_{j-1} \le q$. If $e = q$, then $y_{j-1} = a$, again impossible. Thus $e < q$, so $y_{j-1} \in (a, b)$, but

$$y_{j-1} - a = q - e \not\equiv 0 \pmod{q},$$

contradicting the conclusion above.

Thus $W^\star$ resolves every pair of vertices. Finally, writing $t = 3h, 3h + 1$, or $3h + 2$, the construction gives

$$|W^\star| = \begin{cases} 4h + 1, & t = 3h, \\ 4h + 2, & t = 3h + 1, \\ 4h + 4, & t = 3h + 2, \end{cases} = \left\lceil \frac{4t + 2}{3} \right\rceil.$$

Hence the desired upper bound follows. □

Combining Propositions 4.7 and 4.10 gives the exact value.

Theorem 4.11. *If $q \ge 2$ and $t \ge 1$, then*

$$\dim C_{2qt+1}(t) = \left\lceil \frac{4t + 2}{3} \right\rceil.$$

5. Conclusions

This paper determines the exact metric dimension of $C_n(1, 2, \ldots, t)$ for $n \equiv 0, 1 \pmod{2t}$, closing the previously known linear gaps between lower and upper bounds in both classes. The exact values are independent of q and settle these two cases in the eventual periodic classification of this graph family.

Author contributions

Changhong Lu: Conceptualization, Supervision, Funding acquisition, Writing–review and editing; Yuchun Xiong: Conceptualization, Methodology, Formal analysis, Investigation, Validation, Visualization, Writing–original draft, Writing–review and editing. All authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

References

1. P. J. Slater, Leaves of trees, *Congressus Numerantium*, **14** (1975), 549–559.
2. F. Harary, R. A. Melter, On the metric dimension of a graph, *Ars Comb.*, **2** (1976), 191–195.
3. R. C. Tillquist, R. M. Frongillo, M. E. Lladser, Getting the lay of the land in discrete space: a survey of metric dimension and its applications, *SIAM Rev.*, **65** (2023), 919–962. https://doi.org/10.1137/21M1409512
4. K. Chau, S. Gosselin, The metric dimension of circulant graphs and their Cartesian products, *Opuscula Math.*, **37** (2017), 509–534. https://doi.org/10.7494/OpMath.2017.37.4.509
5. C. Grigorious, P. Manuel, M. Miller, B. Rajan, S. Stephen, On the metric dimension of circulant and Harary graphs, *Appl. Math. Comput.*, **248** (2014), 47–54. https://doi.org/10.1016/j.amc.2014.09.045
6. T. Vetrík, The metric dimension of circulant graphs, *Can. Math. Bull.*, **60** (2017), 206–216. https://doi.org/10.4153/CMB-2016-048-1
7. T. Vetrík, On the metric dimension of directed and undirected circulant graphs, *Discuss. Math. Graph Theor.*, **40** (2020), 67–76. https://doi.org/10.7151/dmgt.2110
8. T. BC, S. Dueck, The metric dimension of circulant graphs, *Opuscula Math.*, **45** (2025), 39–51. https://doi.org/10.7494/OpMath.2025.45.1.39
9. R. Gao, Y. Xiao, Z. Zhang, On the metric dimension of circulant graphs, *Can. Math. Bull.*, **67** (2024), 328–337. https://doi.org/10.4153/S0008439523000759
10. C. Grigorious, T. Kalinowski, J. Ryan, S. Stephen, The metric dimension of the circulant graph $C(n, \pm\{1, 2, 3, 4\})$, *Australas. J. Comb.*, **69** (2017), 417–441.
11. M. Knor, R. Škrekovski, T. Vetrík, Metric dimension of circulant graphs with 5 consecutive generators, *Mathematics*, **12** (2024), 1384. https://doi.org/10.3390/math12091384



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