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Research article

Potential-type integral operators and commutators on two-weight Morrey spaces

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Abstract: The strong-type mapping properties of the potential-type integral operators and commutators on two-weight Morrey spaces were established. Analogous results were also obtained for the weak-type mapping properties of these operators on two-weight Morrey spaces.

Keywords: potential-type integral operators; commutator; BMO; two-weight Morrey space
Mathematics Subject Classification: 42B20, 42B25

1. Introduction

For a nonnegative locally integrable function Φ, the potential-type integral operator T_Φ is defined as

$$T_\Phi f(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) dy.$$

Although the basic example is provided by the fractional integral operator I_α, defined by the kernel $\Phi(x) = |x|^{\alpha-n}$, $0 < \alpha < n$, there are other important examples, such as the Bessel potentials. They are denoted by $\mathcal{J}_{\beta,\lambda}$, $\beta, \lambda > 0$, and the kernel $\Phi(x) = \mathcal{K}_{\beta,\lambda}(x)$ is defined by means of its Fourier transform $\hat{\mathcal{K}}_{\beta,\lambda}(\xi) = (\lambda^2 + |\xi|^2)^{-\beta/2}$.

Now, assume that the kernel Φ satisfies the following weak growth condition; there are constants $\delta, C \in (0, \infty)$ and $\varepsilon \in [0, 1)$ such that for any $k \in \mathbb{Z}$,

$$\sup_{2^k < |x| \le 2^{k+1}} \Phi(x) \le \frac{C}{2^{kn}} \int_{\delta(1-\varepsilon)2^k < |y| \le 2\delta(1+\varepsilon)2^k} \Phi(y) dy. \tag{1.1}$$

Associated with any kernel Φ, we denote by $\widetilde{\Phi}$ the positive function defined for $t \geq 0$,

$$\widetilde{\Phi}(t) = \int_{|y|\leq t} \Phi(y)dy.$$

Pérez [18] gave the two-weight inequalities for the potential-type integral operator T_Φ. Ho and Yee [8], Orazbayev and Zaighum [17] obtained the potential-type integral operator T_Φ on Herz spaces. For more results about the weighted inequalities of the potential-type integral operator T_Φ, we refer the readers to [12, 13].

For the kernel Φ satisfying (1.1) and $b \in BMO$, the commutator of T_Φ is defined by

$$T_\Phi^b f(x) = \int_{\mathbb{R}^n} (b(x) - b(y))\Phi(x-y)f(y)dy.$$

Li [11] obtained the two-weight inequalities for T_Φ^b.

Morrey spaces were introduced by Morrey [15] to investigate the local behavior of solutions to second-order elliptic partial differential equations. Komori and Shirai [9] defined the weighted Morrey spaces and obtained the boundedness of the Hardy-Littlewood maximal operator, the Calderón-Zygmund operators and commutators, and the fractional integral operators and commutators on these spaces. Ye and Wang [26], Iida [10], and Tanaka [21] discussed the two-weight norm inequalities for the Hardy-Littlewood maximal operator. Ho [7] gave the mapping properties of the potential-type integral operator T_Φ on the generalized weighted Morrey spaces. Wang [23, 25] obtained the boundedness of the fractional integral operators and commutators on weighted Morrey spaces. For more results about the weighted inequalities of the fractional integral operators, one can see [14,22,24].

For a suitable choice of the parameter, weighted Morrey spaces coincide with weighted Lebesgue spaces. Hence, the conditions required on weighted Morrey spaces are generally stronger; for instance, the weight function in weighted Morrey spaces also needs to satisfy certain doubling conditions or A_∞ conditions.

In this paper, we establish the strong-type and weak-type mapping properties of the potential-type integral operators and commutators on two-weight Morrey spaces.

2. Preliminaries

We will need the following facts about the Young functions and Orlicz spaces; for further information, see [1]. A function $\mathcal{A} : [0, \infty) \to [0, \infty)$ is a Young function if it is convex, increasing, and if $\mathcal{A}(0) = 0$ and $\mathcal{A}(t) \to \infty$ as $t \to \infty$. Given a Young function $\mathcal{A}$, define the mean Luxemburg norm of f on a ball B by

$$\|f\|_{\mathcal{A},B} = \inf\Big\{\lambda > 0 : \frac{1}{|B|}\int_B \mathcal{A}\Big(\frac{|f(y)|}{\lambda}\Big)dy \leq 1\Big\}.$$

Given two Young functions $\mathcal{A}$ and $\mathcal{B}$, if there is a constant $c_1 > 0$ with $\mathcal{A}(t) \leq c_1\mathcal{B}(t)$ for all $t \geq t_0 > 0$, then there exists a constant $c_2 > 0$, depending on $\mathcal{A}$ and $\mathcal{B}$, such that for all balls B and functions f, $\|f\|_{\mathcal{A},B} \leq c_2\|f\|_{\mathcal{B},B}$.

The examples that we are going to use are $\mathcal{A}(t) = t^p \log(e+t)^q$ and $\mathcal{B}(t) = e^t - 1$, where $1 < p < \infty$ and $0 \leq q < \infty$. We denote their mean Luxemburg norms of f on a ball B by $\|f\|_{L^p(\log L)^q,B}$ and $\|f\|_{\exp(L),B}$ respectively.

For a non-decreasing function $\phi : [0,\infty) \to [0,\infty)$ and the Young function $\mathcal{B}(t) = t\log(e+t)^\tau$, $\tau \geq 0$, we define the maximal operator associated with ϕ and $\mathcal{B}$ by

$$M_{\phi,L(\log L)^\tau} f(x) = \sup_{B\ni x} \phi(r(B))\|f\|_{L(\log L)^\tau,B},$$

$r(B)$ is the radius of B. When $\tau = 0$, we write $M_{\phi,L(\log L)^\tau}$ as M_ϕ.

Each Young function $\mathcal{A}$ has an associate complementary Young function $\bar{\mathcal{A}}$ satisfying

$$t \leq \mathcal{A}^{-1}(t)\bar{\mathcal{A}}^{-1}(t) \leq 2t,$$

for all $t > 0$, where $\mathcal{A}^{-1}(t)$ is the inverse function of $\mathcal{A}(t)$.

For $1 < p < \infty$, a Young function $\bar{\mathcal{A}}$ is said to belong to the class B_p, denoted by $\bar{\mathcal{A}} \in B_p$, if there exists a constant $c > 0$ such that

$$\int_c^\infty \frac{\bar{\mathcal{A}}(t)}{t^p}\frac{dt}{t} < \infty.$$

Given a weight w, if there exists a constant $\eta_w > 0$ such that for any ball B, we have $w(2B) \leq 2^{n\eta_w} w(B)$, where $w(B) = \int_B w(x)dx$, we denote $w \in \Delta_2$ for brevity.

When $1 < p < \infty$, a weight $w \in A_p$, if there exists a constant $C > 0$ such that

$$\sup_{B\subset\mathbb{R}^n} \Big(\frac{1}{|B|}\int_B w(y)dy\Big)\Big(\frac{1}{|B|}\int_B w(y)^{-p'/p}dy\Big)^{p/p'} \leq C,$$

where $1/p + 1/p' = 1$. A weight $w \in A_1$, if there exists a constant $C > 0$ for any ball B such that

$$\frac{1}{|B|}\int_B w(y)dy \leq Cw(x), \quad \text{a.e. } x \in B \subset \mathbb{R}^n.$$

We denote $A_\infty = \bigcup\limits_{1\leq p<\infty} A_p$. If $w \in A_\infty$, then we have $w \in \Delta_2$ (see [6]).

A locally integrable function b is said to be in BMO if

$$\|b\|_{BMO} = \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B |b(x) - b_B|dx < \infty,$$

where $b_B = \frac{1}{|B|}\int_B b(y)dy$.

Let $1 \leq p < \infty$ and $0 \leq \kappa < 1$. For two weights v and w, the two-weight Morrey space $L^{p,\kappa}(v,w)$ in [9] is defined by

$$L^{p,\kappa}(v,w) = \{f \in L^p_{loc}(v) : \|f\|_{L^{p,\kappa}(v,w)} < \infty\},$$

where the norm is given by

$$\|f\|_{L^{p,\kappa}(v,w)} = \sup_{B\subset\mathbb{R}^n} \Big(\frac{1}{w(B)^\kappa}\int_B |f(y)|^p v(y)dy\Big)^{1/p}.$$

When $w = 1$, $L^{p,\kappa}(v,w)$ is the weighted Morrey spaces $L^{p,\kappa}(v,dx)$ introduced by Samko in [20]. When $w = v = 1$, $L^{p,\kappa}(v,w)$ is the classical Morrey spaces $L^{p,\kappa}(\mathbb{R}^n)$. The weak two-weight Morrey space $WL^{p,\kappa}(v,w)$ is defined by

$$WL^{p,\kappa}(v,w) = \{f \in L^p_{loc}(v) : \|f\|_{WL^{p,\kappa}(v,w)} < \infty\},$$

where

$$\|f\|_{WL^{p,\kappa}(v,w)} = \sup_B \sup_{t>0} \frac{t}{w(B)^{\kappa/p}} v(\{x \in B : |f(x)| > t\})^{1/p}.$$

When $w = v$, we denote $L^{p,\kappa}(v,w)$ and $WL^{p,\kappa}(v,w)$ as $L^{p,\kappa}(w)$ and $WL^{p,\kappa}(w)$, respectively. When $\kappa = 0$, $L^{p,\kappa}(w)$ is the weighted Lebesgue space $L^p(w)$.

We also need the following lemmas.

Lemma 2.1. *[9] For $w \in \Delta_2$, there are constants $\theta_w > 0$ and $\eta_w > 0$, such that for every ball B,*

$$w(2B) \geq 2^{n\theta_w} w(B), \quad w(2B) \leq 2^{n\eta_w} w(B).$$

Lemma 2.2. *[3] Let $1 < p < \infty$ and $\mathcal{A}$ be a Young function such that $\bar{\mathcal{A}} \in B_p$. Then there exist $C > 0$ and $c_0 > 0$ such that $\mathcal{A}(t) \geq Ct^{p'}$ for every $t \geq c_0 > 0$.*

Lemma 2.3. *[5] Let $w \in A_\infty$, then there are constants $C > 0$ and $r > 1$ such that for any ball B,*

$$\Big(\frac{1}{|B|}\int_B w(y)^r dy\Big)^{1/r} \leq \frac{C}{|B|}\int_B w(y)dy.$$

Lemma 2.4. *[22] Let $0 < q < \infty$, $w \in A_\infty$ and $b \in BMO$, then for every ball B,*

$$\Big(\frac{1}{w(B)}\int_B |b(x) - b_B|^q w(x)dx\Big)^{1/q} \leq C\|b\|_{BMO}.$$

Lemma 2.5. *[19] Let $b \in BMO$, then for every ball B,*

$$\|b - b_B\|_{\exp L,B} \leq C\|b\|_{BMO}.$$

Lemma 2.6. *[16] Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be Young functions such that for all $t > 0$,*

$$\mathcal{A}^{-1}(t)\mathcal{B}^{-1}(t) \leq \mathcal{C}^{-1}(t).$$

Then for all functions f and g and for every ball B,

$$\|fg\|_{\mathcal{C},B} \leq 2\|f\|_{\mathcal{A},B}\|g\|_{\mathcal{B},B}.$$

3. Strong-type boundedness of potential-type integral operators and commutators

In this section, we establish the strong-type boundedness of potential-type integral operators and commutators on two-weight Morrey spaces. Throughout this paper, C always denotes a positive constant independent of the main parameters involved, but it may be different from line to line.

Theorem 3.1. *[18] Let $1 < p \leq q < \infty$, $\bar{\mathcal{A}} \in B_{q'}$, $\bar{\mathcal{B}} \in B_p$ and Φ satisfy condition (1.1). If (w,v) is a pair of weights such that for every ball B,*

$$\widetilde{\Phi}(r(B))|B|^{1/q-1/p}\|w^{1/q}\|_{\mathcal{A},B}\|v^{-1/p}\|_{\mathcal{B},B} \leq C < \infty, \tag{3.1}$$

then T_Φ is bounded from $L^p(v)$ to $L^q(w)$.

Typical examples are given by considering $\mathcal{A}$ and $\mathcal{B}$ such that

$$\mathcal{A}(t) = t^q \log(e+t)^{q-1+\tau} \text{ and } \mathcal{B}(t) = t^{p'} \log(e+t)^{p'-1+\tau},\ \tau > 0.$$

The following is the strong-type boundedness for potential-type integral operators on two-weight Morrey spaces.

Theorem 3.2. *Let $1 < p \le q < \infty$, $\bar{\mathcal{A}} \in B_{q'}$, $\bar{\mathcal{B}} \in B_p$ and Φ satisfy condition (1.1). Suppose that a pair of weights (w, v) satisfies (3.1). If $w \in \Delta_2, u \in \Delta_2$ and $0 < \kappa < \min\{\frac{p}{q}, \frac{p\theta_w}{q\eta_u}\}$, then T_Φ is bounded from $L^{p,\kappa}(v,u)$ to $L^{q,q\kappa/p}(w,u)$.*

Proof. For any $f \in L^{p,\kappa}(v,u)$ and ball $B = B(x,r) \subset \mathbb{R}^n$, write $f_1 = f\chi_{2B}$ and $f_2 = f - f_1$. Then we have

$$\Big(\frac{1}{u(B)^{\kappa q/p}} \int_B |T_\Phi f(z)|^q w(z)dz\Big)^{1/q} \le \sum_{k=1}^{2} \frac{1}{u(B)^{\kappa/p}} \Big(\int_B |T_\Phi f_k(z)|^q w(z)dz\Big)^{1/q}.$$

By Theorem 3.1 and $u \in \Delta_2$, we obtain that

$$\begin{aligned}
&\frac{1}{u(B)^{\kappa/p}} \Big(\int_B |T_\Phi f_1(z)|^q w(z)dz\Big)^{1/q} \\
\le& C \frac{1}{u(B)^{\kappa/p}} \Big(\int_{\mathbb{R}^n} |f_1(z)|^p v(z)dz\Big)^{1/p} \le C \frac{1}{u(B)^{\kappa/p}} \Big(\int_{2B} |f(z)|^p v(z)dz\Big)^{1/p} \\
\le& C \frac{u(2B)^{\kappa/p}}{u(B)^{\kappa/p}} \|f\|_{L^{p,\kappa}(v,u)} \le C\|f\|_{L^{p,\kappa}(v,u)}.
\end{aligned}$$

Let $l \in \mathbb{Z}$ satisfy $2^l < r \le 2^{l+1}$. For $k \ge 1$, $y \in 2^{k+1}B \setminus 2^k B$ and $z \in B$, we have

$$2^{k+l-1} < |y-x| - |z-x| \le |z-y| \le |y-x| + |z-x| \le 2^{k+l+3}.$$

For any $z \in B$, the definitions of T_Φ, $\widetilde{\Phi}$ and condition (1.1) give the following estimate:

$$\begin{aligned}
|T_\Phi f_2(z)| &\le \sum_{k=1}^{\infty} \Big| \int_{2^{k+1}B\setminus 2^k B} \Phi(z-y) f(y)dy\Big| \\
&\le \sum_{k=1}^{\infty} \sup_{2^{k+l-1}<|t|\le 2^{k+l+3}} \Phi(t) \int_{2^{k+1}B\setminus 2^k B} |f(y)|dy \\
&\le C \sum_{k=1}^{\infty} \frac{\widetilde{\Phi}(2\delta(1+\varepsilon)2^{k+l+3})}{2^{(k+l-1)n}} \int_{2^{k+1}B} |f(y)|dy \\
&\le C \sum_{k=1}^{\infty} \frac{\widetilde{\Phi}(2\delta(1+\varepsilon)2^{k+3}r)}{2^{kn}r^n} \int_{2^{k+1}B} |f(y)|dy. \qquad (3.2)
\end{aligned}$$

By Hölder's inequality, condition (3.1), Lemmas 2.1 and 2.2, we obtain that

$$\begin{aligned}
&\frac{1}{u(B)^{\kappa/p}} \Big(\int_B |T_\Phi f_2(z)|^q w(z)dz\Big)^{1/q} \\
\le& C \sum_{k=1}^{\infty} \frac{1}{u(B)^{\kappa/p}} \frac{\widetilde{\Phi}(2\delta(1+\varepsilon)2^{k+3}r)}{2^{kn}r^n} \Big(\int_B w(z)dz\Big)^{1/q} \cdot \Big(\int_{2^{k+1}B} |f(y)|^p v(y)dy\Big)^{1/p} \Big(\int_{2^{k+1}B} v(y)^{-p'/p}dy\Big)^{1/p'}
\end{aligned}$$

$$\leq C\|f\|_{L^{p,\kappa}(v,u)}\sum_{k=1}^{\infty}\frac{\widetilde{\Phi}(2^{k+1}\rho r)}{2^{kn}r^n}w(B)^{1/q}\frac{u(2^{k+1}B)^{\kappa/p}}{u(B)^{\kappa/p}}|2^{k+1}\rho B|^{1/p'}\|v^{-1/p}\|_{\mathcal{B},2^{k+1}\rho B}$$

$$\leq C\|f\|_{L^{p,\kappa}(v,u)}\sum_{k=1}^{\infty}\frac{u(2^{k+1}B)^{\kappa/p}}{u(B)^{\kappa/p}}\frac{w(B)^{1/q}}{|2^{k+1}\rho B|^{1/q}\|w^{1/q}\|_{\mathcal{A},2^{k+1}\rho B}}$$

$$\leq C\|f\|_{L^{p,\kappa}(v,u)}\sum_{k=1}^{\infty}\frac{u(2^{k+1}B)^{\kappa/p}}{u(B)^{\kappa/p}}\frac{w(B)^{1/q}}{w(2^{k+1}B)^{1/q}} \leq C\|f\|_{L^{p,\kappa}(v,u)}\sum_{k=1}^{\infty}2^{n(k+1)(\kappa\eta_u/p-\theta_w/q)} \leq C\|f\|_{L^{p,\kappa}(v,u)},$$

where, and in the sequel, $\rho = \max\{2^3\delta(1+\varepsilon), 1\}$. This completes the proof. □

When $u = w \in \Delta_2$, as a consequence of Theorem 3.2, we obtain the following result for the potential-type integral operators.

Corollary 3.1. *Let $1 < p \leq q < \infty$, $0 < \kappa < p/q$, $\bar{\mathcal{A}} \in B_{q'}$, $\bar{\mathcal{B}} \in B_p$ and Φ satisfy condition (1.1). Suppose that a pair of weights (w, v) satisfy (3.1) and $w \in \Delta_2$, then T_Φ is bounded from $L^{p,\kappa}(v, w)$ to $L^{q,q\kappa/p}(w)$.*

Let $u \equiv 1$ and we obtain the following two-weight inequalities for the potential-type integral operators on weighted Morrey spaces.

Corollary 3.2. *Let $1 < p \leq q < \infty$, $\bar{\mathcal{A}} \in B_{q'}$, $\bar{\mathcal{B}} \in B_p$ and Φ satisfy condition (1.1). Suppose that a pair of weights (w, v) satisfy (3.1). If $w \in \Delta_2$ and $0 < \kappa < \min\{\frac{p}{q}, \frac{p\theta_w}{q}\}$, then T_Φ is bounded from $L^{p,\kappa}(v, dx)$ to $L^{q,q\kappa/p}(w, dx)$.*

Taking $\Phi(t) = |t|^{\alpha-n}, 0 < \alpha < n$, we obtain the following mapping properties for fractional integral operators.

Corollary 3.3. *Let $0 < \alpha < n$, $1 < p \leq q < \infty$, $\bar{\mathcal{A}} \in B_{q'}$ and $\bar{\mathcal{B}} \in B_p$. Suppose that (w, v) is a pair of weights such that for every ball B,*

$$|B|^{\alpha/n+1/q-1/p}\|w^{1/q}\|_{\mathcal{A},B}\|v^{-1/p}\|_{\mathcal{B},B} \leq C < \infty.$$

If $w \in \Delta_2$ and $0 < \kappa < p/q$, then I_α is bounded from $L^{p,\kappa}(v, w)$ to $L^{q,q\kappa/p}(w)$. If $w \in \Delta_2$ and $0 < \kappa < \min\{\frac{p}{q}, \frac{p\theta_w}{q}\}$, then I_α is bounded from $L^{p,\kappa}(v, dx)$ to $L^{q,q\kappa/p}(w, dx)$.

Theorem 3.3. *[11] Let $1 < p \leq q < \infty$, $b \in BMO$ and Φ satisfy condition (1.1). Suppose that (w, v) is a pair of weights such that for some $\tau > 0$ and for every ball B,*

$$\widetilde{\Phi}(r(B))|B|^{1/q-1/p}\|w^{1/q}\|_{L^q(\log L)^{2q-1+\tau},B}\|v^{-1/p}\|_{L^{p'}(\log L)^{2p'-1+\tau},B} \leq C < \infty, \tag{3.3}$$

then T_Φ^b is bounded from $L^p(v)$ to $L^q(w)$.

The following is the strong-type boundedness of the commutators of the potential-type integral operators with *BMO* functions on two-weight Morrey spaces.

Theorem 3.4. *Let $1 < p \leq q < \infty$, $b \in BMO$ and Φ satisfy condition (1.1). Suppose that (w, v) is a pair of weights such that for some $\tau > 0$ and all ball B,*

$$\widetilde{\Phi}(r(B))|B|^{1/q-1/p}\Big(\frac{1}{|B|}\int_B w(x)dx\Big)^{1/q}\|v^{-1/p}\|_{L^{p'}(\log L)^{2p'-1+\tau},B} \leq C < \infty. \tag{3.4}$$

If $w \in A_\infty$, $u \in \Delta_2$ and $0 < \kappa < \min\{\frac{p}{q}, \frac{p\theta_w}{q\eta_u}\}$, then T_Φ^b is bounded from $L^{p,\kappa}(v, u)$ to $L^{q,q\kappa/p}(w, u)$.

Proof. For any $f \in L^{p,\kappa}(v,u)$ and ball $B = B(x,r) \subset \mathbb{R}^n$, write $f_1 = f\chi_{2B}$ and $f_2 = f - f_1$. As $w \in A_\infty$, using condition (3.4), we get that (w,v) satisfies condition (3.3). By Theorem 3.3 and $u \in \Delta_2$, we obtain that

$$\begin{aligned}
&\frac{1}{u(B)^{(q\kappa)/p\cdot 1/q}}\Big(\int_B |T_\Phi^b f_1(z)|^q w(z)dz\Big)^{1/q}\\
\le&\frac{C}{u(B)^{\kappa/p}}\Big(\int_{\mathbb{R}^n} |f_1(z)|^p v(z)dz\Big)^{1/p} \le \frac{C}{u(B)^{\kappa/p}}\Big(\int_{2B} |f(z)|^p v(z)dz\Big)^{1/p}\\
\le&C\frac{u(2B)^{\kappa/p}}{u(B)^{\kappa/p}}\|f\|_{L^{p,\kappa}(v,u)} \le C\|f\|_{L^{p,\kappa}(v,u)}.
\end{aligned}$$

Let $l \in \mathbb{Z}$ satisfy $2^l < r \le 2^{l+1}$. For $k \ge 1$, $y \in 2^{k+1}B \setminus 2^kB$ and $z \in B$, we have

$$2^{k+l-1} < |z-y| \le 2^{k+l+3}.$$

For any $z \in B$, the definitions of T_Φ^b, $\widetilde{\Phi}$ and condition (1.1) give

$$\begin{aligned}
&|T_\Phi^b f_2(z)|\\
\le&\sum_{k=1}^\infty \Big|\int_{2^{k+1}B\setminus 2^kB} (b(z)-b(y))\Phi(z-y)f(y)dy\Big|\\
\le&\sum_{k=1}^\infty \sup_{2^{k+l-1}<|t|\le 2^{k+l+3}}\Phi(t)\int_{2^{k+1}B\setminus 2^kB}|b(z)-b(y)||f(y)|dy\\
\le&C\sum_{k=1}^\infty \frac{\widetilde{\Phi}(2\delta(1+\varepsilon)2^{k+3}r)}{2^{kn}r^n}\Big([|b(z)-b_B|+|b_B-b_{2^{k+1}B}|]\int_{2^{k+1}B}|f(y)|dy+\int_{2^{k+1}B}|b(y)-b_{2^{k+1}B}||f(y)|dy\Big).
\end{aligned}$$

Notice that $|b_B - b_{2^{k+1}B}| \le 2^n(k+1)\|b\|_{BMO}$, by Lemma 2.4, we get

$$\begin{aligned}
&\Big(\frac{1}{u(B)^{\kappa q/p}}\int_B |T_\Phi^b f_2(z)|^q w(z)dz\Big)^{1/q}\\
\le&C\sum_{k=1}^\infty \frac{\widetilde{\Phi}(2\delta(1+\varepsilon)2^{k+3}r)}{2^{kn}r^n u(B)^{\kappa/p}}\Big(\int_B |b(z)-b_B|^q w(z)dz\Big)^{1/q}\int_{2^{k+1}B}|f(y)|dy\\
&+C\|b\|_{BMO}\sum_{k=1}^\infty (k+1)\frac{\widetilde{\Phi}(2\delta(1+\varepsilon)2^{k+3}r)}{2^{kn}r^n u(B)^{\kappa/p}}\Big(\int_B w(z)dz\Big)^{1/q}\int_{2^{k+1}B}|f(y)|dy\\
&+C\sum_{k=1}^\infty \frac{\widetilde{\Phi}(2\delta(1+\varepsilon)2^{k+3}r)}{2^{kn}r^n u(B)^{\kappa/p}}\int_{2^{k+1}B}|b(y)-b_{2^{k+1}B}||f(y)|dy\Big(\int_B w(z)dz\Big)^{1/q}\\
\le&C\|b\|_{BMO}\sum_{k=1}^\infty (k+1)\frac{\widetilde{\Phi}(2\delta(1+\varepsilon)2^{k+3}r)}{2^{kn}r^n u(B)^{\kappa/p}}\Big(\int_B w(z)dz\Big)^{1/q}\int_{2^{k+1}B}|f(y)|dy\\
&+C\sum_{k=1}^\infty \frac{\widetilde{\Phi}(2\delta(1+\varepsilon)2^{k+3}r)}{2^{kn}r^n u(B)^{\kappa/p}}\int_{2^{k+1}B}|b(y)-b_{2^{k+1}B}||f(y)|dy\Big(\int_B w(z)dz\Big)^{1/q}\\
=&\mathrm{I}+\mathrm{II}.
\end{aligned}$$

For I, using Hölder's inequality, condition (3.4), and Lemma 2.1, we obtain

$$\begin{aligned}
\mathrm{I} &\le C\|b\|_{BMO}\sum_{k=1}^{\infty}\frac{(k+1)\widetilde{\Phi}(2^{k+1}\rho r)}{2^{kn}r^n u(B)^{\kappa/p}}w(B)^{1/q}\cdot\Big(\int_{2^{k+1}\rho B}v(y)^{-p'/p}dy\Big)^{1/p'}\Big(\int_{2^{k+1}B}|f(y)|^p v(y)dy\Big)^{1/p}\\
&\le C\|b\|_{BMO}\|f\|_{L^{p,\kappa}(v,u)}\sum_{k=1}^{\infty}\frac{(k+1)\widetilde{\Phi}(2^{k+1}\rho r)}{|B(x,2^{k+1}\rho r)|^{1/p}u(B)^{\kappa/p}}w(B)^{1/q}\cdot\|v^{-1/p}\|_{L^{p'}(\log L)^{2p'-1+\tau},2^{k+1}\rho B}u(2^{k+1}B)^{\kappa/p}\\
&\le C\|b\|_{BMO}\|f\|_{L^{p,\kappa}(v,u)}\sum_{k=1}^{\infty}(k+1)\frac{u(2^{k+1}B)^{\kappa/p}}{u(B)^{\kappa/p}}\frac{w(B)^{1/q}}{w(2^{k+1}B)^{1/q}}\\
&\le C\|b\|_{BMO}\|f\|_{L^{p,\kappa}(v,u)}\sum_{k=1}^{\infty}(k+1)2^{n(k+1)(\kappa\eta_u/p-\theta_w/q)}\\
&\le C\|b\|_{BMO}\|f\|_{L^{p,\kappa}(v,u)}.
\end{aligned}$$

Let $\mathcal{A}(t)\approx t^{p'}\log(e+t)^{p'}$, $\mathcal{B}(t)\approx\exp(t)-1$ and $C(t)=t^{p'}$, we have

$$\mathcal{A}^{-1}(t)\mathcal{B}^{-1}(t)\approx C^{-1}(t).$$

According to Lemmas 2.5 and 2.6, we obtain

$$\begin{aligned}
&\Big(\int_{2^{k+1}B}|b(y)-b_{2^{k+1}B}|^{p'}v(y)^{-p'/p}dy\Big)^{1/p'}\\
=&|2^{k+1}B|^{1/p'}\|(b-b_{2^{k+1}B})v^{-1/p}\|_{C,2^{k+1}B}\\
\le&C|2^{k+1}B|^{1/p'}\|b-b_{2^{k+1}B}\|_{\exp L,2^{k+1}B}\|v^{-1/p}\|_{L^{p'}(\log L)^{p'},2^{k+1}B}\\
\le&C\|b\|_{BMO}|2^{k+1}B|^{1/p'}\|v^{-1/p}\|_{L^{p'}(\log L)^{p'},2^{k+1}B}.
\end{aligned}\tag{3.5}$$

For II, by Hölder's inequality, condition (3.4), and Lemma 2.1, we have

$$\begin{aligned}
\mathrm{II} &\le C\sum_{k=1}^{\infty}\frac{\widetilde{\Phi}(2\delta(1+\varepsilon)2^{k+3}r)}{2^{kn}r^n u(B)^{\kappa/p}}\Big(\int_{2^{k+1}B}|b(y)-b_{2^{k+1}B}|^{p'}v(y)^{-p'/p}dy\Big)^{1/p'}\cdot\Big(\int_{2^{k+1}B}|f(y)|^p v(y)dy\Big)^{1/p}\Big(\int_B w(z)dz\Big)^{1/q}\\
&\le C\|b\|_{BMO}\|f\|_{L^{p,\kappa}(v,u)}\sum_{k=1}^{\infty}\frac{\widetilde{\Phi}(2^{k+1}\rho r)}{2^{kn}r^n u(B)^{\kappa/p}}|2^{k+1}\rho B|^{1/p'}\cdot\|v^{-1/p}\|_{L^{p'}(\log L)^{2p'-1+\tau},2^{k+1}\rho B}w(B)^{1/q}u(2^{k+1}B)^{\kappa/p}\\
&\le C\|b\|_{BMO}\|f\|_{L^{p,\kappa}(v,u)}\sum_{k=1}^{\infty}\frac{u(2^{k+1}B)^{\kappa/p}}{u(B)^{\kappa/p}}\frac{w(B)^{1/q}}{w(2^{k+1}B)^{1/q}}\\
&\le C\|b\|_{BMO}\|f\|_{L^{p,\kappa}(v,u)}\sum_{k=1}^{\infty}2^{n(k+1)(\kappa\eta_u/p-\theta_w/q)}\\
&\le C\|b\|_{BMO}\|f\|_{L^{p,\kappa}(v,u)}.
\end{aligned}$$

This completes the proof. □

By Theorem 3.4, we have the following corollaries.

Corollary 3.4. *Let $1<p\le q<\infty$, $0<\kappa<p/q$, $b\in BMO$, and Φ satisfy condition (1.1). If a pair of weights (w,v) satisfies (3.4) and $w\in A_\infty$, then T_Φ^b is bounded from $L^{p,\kappa}(v,w)$ to $L^{q,q\kappa/p}(w)$.*

Corollary 3.5. *Let $1 < p \le q < \infty$, $b \in BMO$ and Φ satisfy condition (1.1). If a pair of weights (w, v) satisfies condition (3.4), $w \in A_\infty$ and $0 < \kappa < \min\{\frac{p}{q}, \frac{p\theta_w}{q}\}$, then T_Φ^b is bounded from $L^{p,\kappa}(v, dx)$ to $L^{q,q\kappa/p}(w, dx)$.*

Corollary 3.6. *Let $0 < \alpha < n$, $1 < p \le q < \infty$ and $b \in BMO$. Suppose that (w, v) is a pair of weights such that for some $\tau > 0$ and for every ball B,*

$$|B|^{\alpha/n+1/q-1/p}\Big(\frac{1}{|B|}\int_B w(x)dx\Big)^{1/q}\|v^{-1/p}\|_{L^{p'}(\log L)^{2p'-1+\tau},B} \le C < \infty.$$

If $w \in A_\infty$ and $0 < \kappa < p/q$, then I_α^b is bounded from $L^{p,\kappa}(v, w)$ to $L^{q,q\kappa/p}(w)$. If $w \in A_\infty$ and $0 < \kappa < \min\{\frac{p}{q}, \frac{p\theta_w}{q}\}$, then I_α^b is bounded from $L^{p,\kappa}(v, dx)$ to $L^{q,q\kappa/p}(w, dx)$.

4. Weak-type boundedness of potential-type integral operators and commutators

In this section, we establish the weak-type boundedness of the potential-type integral operators and commutators on two-weight Morrey spaces.

Theorem 4.1. *[13] Let $1 < p \le q < \infty$, Φ satisfy conditions (1.1) and $\widetilde{\Phi}(t)/t^n \to 0$ when $t \to \infty$. If (w, v) is a pair of weights for which there exists $\gamma > 1$ and for every ball B,*

$$\widetilde{\Phi}(r(B))|B|^{1/q-1/p}\Big(\frac{1}{|B|}\int_B w(x)^\gamma dx\Big)^{1/\gamma q}\Big(\frac{1}{|B|}\int_B v(x)^{-p'/p}dx\Big)^{1/p'} \le C < \infty, \tag{4.1}$$

then T_Φ is bounded from $L^p(v)$ to $L^{q,\infty}(w)$.

We investigate the boundedness of T_Φ on two-weight Morrey spaces.

Theorem 4.2. *Let $1 < p \le q < \infty$, Φ satisfy condition (1.1) and $\widetilde{\Phi}(t)/t^n \to 0$ when $t \to \infty$. Suppose that the weights (w, v) satisfy (4.1). If $w \in \Delta_2$, $u \in \Delta_2$ and $0 < \kappa < \min\{\frac{p}{q}, \frac{p\theta_w}{q\eta_u}\}$, then T_Φ is bounded from $L^{p,\kappa}(v, u)$ to $WL^{q,q\kappa/p}(w, u)$.*

Proof. For any $f \in L^{p,\kappa}(v, u)$ and ball $B = B(x, r) \subset \mathbb{R}^n$, write $f_1 = f\chi_{2B}$ and $f_2 = f - f_1$. Then for any given $t > 0$, we have

$$\begin{aligned}&\frac{1}{u(B)^{(q\kappa)/p\cdot 1/q}}t[w(\{z \in B : |T_\Phi f(z)| > t\})]^{1/q}\\ \le&\frac{1}{u(B)^{\kappa/p}}t[w(\{z \in B : |T_\Phi f_1(z)| > t/2\})]^{1/q} + \frac{1}{u(B)^{\kappa/p}}t[w(\{z \in B : |T_\Phi f_2(z)| > t/2\})]^{1/q}.\end{aligned}$$

By Theorem 4.1 and $u \in \Delta_2$, we obtain that

$$\begin{aligned}&\frac{1}{u(B)^{\kappa/p}}t[w(\{z \in B : |T_\Phi f_1(z)| > t/2\})]^{1/q}\\ \le&C\frac{1}{u(B)^{\kappa/p}}\Big(\int_{\mathbb{R}^n}|f_1(z)|^p v(z)dz\Big)^{1/p} \le C\frac{u(2B)^{\kappa/p}}{u(B)^{\kappa/p}}\|f\|_{L^{p,\kappa}(v,u)} \le C\|f\|_{L^{p,\kappa}(v,u)}.\end{aligned}$$

By Chebyshev's inequality, inequality (3.2), Hölder's inequality, condition (4.1) and Lemma 2.1, we have

$$\frac{1}{u(B)^{\kappa/p}}t[w(\{z \in B : |T_\Phi f_2(z)| > t/2\})]^{1/q}$$

$$
\begin{aligned}
&\le C\frac{1}{u(B)^{\kappa/p}}\Big(\int_B |T_\Phi f_2(z)|^q w(z)dz\Big)^{1/q}\\
&\le C\frac{w(B)^{1/q}}{u(B)^{\kappa/p}}\sum_{k=1}^{\infty}\frac{\widetilde{\Phi}(2\delta(1+\varepsilon)2^{k+3}r)}{2^{kn}r^n}\int_{2^{k+1}B}|f(y)|dy\\
&\le C\sum_{k=1}^{\infty}\frac{\widetilde{\Phi}(2^{k+1}\rho r)}{2^{kn}r^n u(B)^{\kappa/p}}\Big(\int_{2^{k+1}B}|f(z)|^p v(z)dz\Big)^{1/p}\Big(\int_{2^{k+1}\rho B} v(z)^{-p'/p}dz\Big)^{1/p'} w(B)^{1/q}\\
&\le C\|f\|_{L^{p,\kappa}(v,u)}\sum_{k=1}^{\infty}\frac{u(2^{k+1}B)^{\kappa/p}}{u(B)^{\kappa/p}}\frac{w(B)^{1/q}}{|2^{k+1}\rho B|^{1/q}}\Big(\frac{1}{|2^{k+1}\rho B|}\int_{2^{k+1}\rho B} w(y)^\gamma dy\Big)^{-1/\gamma q}\\
&\le C\|f\|_{L^{p,\kappa}(v,u)}\sum_{k=1}^{\infty}\frac{u(2^{k+1}B)^{\kappa/p}}{u(B)^{\kappa/p}}\frac{w(B)^{1/q}}{|2^{k+1}\rho B|^{1/q}}\Big(\frac{1}{|2^{k+1}\rho B|}\int_{2^{k+1}\rho B} w(y)dy\Big)^{-1/q}\\
&\le C\|f\|_{L^{p,\kappa}(v,u)}\sum_{k=1}^{\infty}\frac{u(2^{k+1}B)^{\kappa/p}}{u(B)^{\kappa/p}}\frac{w(B)^{1/q}}{w(2^{k+1}B)^{1/q}}\\
&\le C\|f\|_{L^{p,\kappa}(v,u)}\sum_{k=1}^{\infty}2^{n(k+1)(\kappa\eta_u/p-\theta_w/q)}\\
&\le C\|f\|_{L^{p,\kappa}(v,u)}.
\end{aligned}
$$

This completes the proof. □

From Theorem 4.2, we obtain the following weighted estimates.

Corollary 4.1. *Let $1 < p \le q < \infty$, $0 < \kappa < p/q$, Φ satisfy condition (1.1) and $\widetilde{\Phi}(t)/t^n \to 0$ when $t \to \infty$. Suppose that the weights (w, v) satisfy condition (4.1) and $w \in \Delta_2$, then T_Φ is bounded from $L^{p,\kappa}(v, w)$ to $WL^{q,q\kappa/p}(w)$.*

Corollary 4.2. *Let $1 < p \le q < \infty$, Φ satisfy condition (1.1) and $\widetilde{\Phi}(t)/t^n \to 0$ when $t \to \infty$. Suppose that the weights (w, v) satisfy condition (4.1). If $w \in \Delta_2$ and $0 < \kappa < \min\{\frac{p}{q}, \frac{p\theta_w}{q}\}$, then T_Φ is bounded from $L^{p,\kappa}(v, dx)$ to $WL^{q,q\kappa/p}(w, dx)$.*

Corollary 4.3. *Let $0 < \alpha < n$ and $1 < p \le q < \infty$. Suppose that (w, v) is a pair of weights for which there exists $\gamma > 1$ and for every ball B,*

$$|B|^{\alpha/n+1/q-1/p}\Big(\frac{1}{|B|}\int_B w(x)^\gamma dx\Big)^{1/\gamma q}\Big(\frac{1}{|B|}\int_B v(x)^{-p'/p}dx\Big)^{1/p'} \le C < \infty.$$

If $w \in \Delta_2$ and $0 < \kappa < p/q$, then I_α is bounded from $L^{p,\kappa}(v, w)$ to $WL^{q,q\kappa/p}(w)$. (We recover the result from [24].) If $w \in \Delta_2$ and $0 < \kappa < \min\{\frac{p}{q}, \frac{p\theta_w}{q}\}$, then I_α is bounded from $L^{p,\kappa}(v, dx)$ to $WL^{q,q\kappa/p}(w, dx)$.

In the following, we give the mapping properties of the potential-type integral operators.

Lemma 4.1. *[4] Let $0 < p < \infty$ and Φ satisfy condition (1.1). Then there is a constant $C > 0$ such that for $w \in A_\infty$ and all f,*

$$\|T_\Phi f\|_{L^{p,\infty}(w)} \le C\|M_{\widetilde{\Phi}} f\|_{L^{p,\infty}(w)}.$$

Lemma 4.2. *Let $1 \le q < \infty$ and Φ satisfy condition (1.1). Suppose that a pair of weights (w, v) satisfies for any ball B,*

$$\widetilde{\Phi}(r(B))|B|^{1/q-1}\Big(\frac{1}{|B|}\int_B w(x)dx\Big)^{1/q}\inf_{x\in B} v(x)^{-1} \le C < \infty. \tag{4.2}$$

Then for every locally integrable function f,

$$w(\{x \in \mathbb{R}^n : M_{\widetilde{\Phi}}f(x) > t\})^{1/q} \le \frac{C}{t}\int_{\mathbb{R}^n} |f(x)|v(x)dx.$$

Proof. By homogeneity, we may assume that $t = 1$. Define the set $\Omega = \{x \in \mathbb{R}^n : M_{\widetilde{\Phi}}f(x) > 1\}$. It is easy to see that Ω is open and we may assume that it is not empty. To estimate the size of Ω, it is enough to estimate the size of every compact set F contained in Ω. We can cover F by a finite family of ball $\{B_j\}$ for which,

$$\frac{\widetilde{\Phi}(r(B_j))}{|B_j|}\int_{B_j} |f(x)|dx > 1.$$

Using Vitali's covering lemma, we can extract a subfamily of disjoint balls $\{B_k\}$ such that $F \subset \bigcup_k 5B_k$.

Hence, using condition (4.2), we have

$$\begin{aligned}
w(F)^{1/q} &\le \sum_k w(5B_k)^{1/q} \\
&\le C\sum_k w(5B_k)^{1/q}\frac{\widetilde{\Phi}(r(B_k))}{|B_k|}\int_{B_k}|f(y)|dy \\
&\le C\sum_k \int_{B_k}|f(y)|\inf_{x\in 5B_k} v(x)dy \\
&\le C\sum_k \int_{B_k}|f(y)|v(y)dy \\
&\le C\int_{\mathbb{R}^n}|f(y)|v(y)dy.
\end{aligned}$$

This ends the proof. □

Theorem 4.3. *Let $1 \le q < \infty$ and Φ satisfy condition (1.1). Suppose that a pair of weights (w, v) satisfies condition (4.2). If $w \in A_\infty, u \in \Delta_2$ and $0 < \kappa < \min\{\frac{1}{q}, \frac{\theta_w}{q\eta_u}\}$, then T_Φ is bounded from $L^{1,\kappa}(v, u)$ to $WL^{q,q\kappa}(w, u)$.*

Proof. For any $f \in L^{1,\kappa}(v, u)$ and ball $B = B(x, r) \subset \mathbb{R}^n$, write $f_1 = f\chi_{2B}$ and $f_2 = f - f_1$. Then for any given $t > 0$, we have

$$\begin{aligned}
&\frac{t}{u(B)^\kappa}w(\{z \in B : |T_\Phi f(z)| > t\})^{1/q} \\
\le&\frac{t}{u(B)^\kappa}w(\{z \in B : |T_\Phi f_1(z)| > t/2\})^{1/q} + \frac{t}{u(B)^\kappa}w(\{z \in B : |T_\Phi f_2(z)| > t/2\})^{1/q}.
\end{aligned}$$

By Lemmas 4.1 and 4.2, and $u \in \Delta_2$, we obtain that

$$\begin{aligned}\frac{t}{u(B)^\kappa} w(\{z \in B : |T_\Phi f_1(z)| > t/2\})^{1/q} &\le \frac{1}{u(B)^\kappa} \sup_{t>0} tw(\{z \in B : |T_\Phi f_1(z)| > t/2\})^{1/q} \\ &\le C\frac{1}{u(B)^\kappa} \sup_{t>0} tw(\{z \in \mathbb{R}^n : M_{\widetilde{\Phi}} f_1(z) > t/2\})^{1/q} \\ &\le C\frac{1}{u(B)^\kappa} \int_{2B} |f(z)|v(z)dz \\ &\le C\|f\|_{L^{1,\kappa}(v,u)}.\end{aligned}$$

By Chebyshev's inequality, inequalities (3.2), condition (4.2) and Lemma 2.1, we have

$$\begin{aligned}&\frac{t}{u(B)^\kappa} w(\{z \in B : |T_\Phi f_2(z)| > t/2\})^{1/q} \\ &\le \frac{C}{u(B)^\kappa}\Big(\int_B |T_\Phi f_2(z)|^q w(z)dz\Big)^{1/q} \\ &\le \frac{C}{u(B)^\kappa} \sum_{k=1}^{\infty} \frac{\widetilde{\Phi}(2\delta(1+\varepsilon)2^{k+3}r)}{2^{kn}r^n}\Big(\int_B w(z)dz\Big)^{1/q}\Big(\int_{2^{k+1}B} |f(y)|v(y)dy\Big) \inf_{y\in 2^{k+1}B} v(y)^{-1} \\ &\le C\|f\|_{L^{1,\kappa}(v,u)} \sum_{k=1}^{\infty} \frac{\widetilde{\Phi}(2^{k+1}\rho r)}{2^{kn}r^n} w(B)^{1/q} \frac{u(2^{k+1}B)^\kappa}{u(B)^\kappa} \inf_{y\in 2^{k+1}\rho B} v(y)^{-1} \\ &\le C\|f\|_{L^{1,\kappa}(v,u)} \sum_{k=1}^{\infty} \frac{u(2^{k+1}B)^\kappa}{u(B)^\kappa} \frac{w(B)^{1/q}}{w(2^{k+1}\rho B)^{1/q}} \\ &\le C\|f\|_{L^{1,\kappa}(v,u)} \sum_{k=1}^{\infty} 2^{n(k+1)(\kappa\eta_u - \theta_w/q)} \\ &\le C\|f\|_{L^{1,\kappa}(v,u)}.\end{aligned}$$

This ends the proof. □

By Theorem 4.3, we have the following corollaries.

Corollary 4.4. *Let $1 \le q < \infty$, $0 < \kappa < 1/q$, and Φ satisfy condition (1.1). If a pair of weights (w, v) satisfies condition (4.2) and $w \in A_\infty$, then T_Φ is bounded from $L^{1,\kappa}(v, w)$ to $WL^{q,q\kappa}(w)$.*

Corollary 4.5. *Let $1 \le q < \infty$ and Φ satisfy condition (1.1). If a pair of weights (w, v) satisfies condition (4.2), $w \in A_\infty$ and $0 < \kappa < \min\{\frac{1}{q}, \frac{\theta_w}{q}\}$, then T_Φ is bounded from $L^{1,\kappa}(v, dx)$ to $WL^{q,q\kappa}(w, dx)$.*

Corollary 4.6. *Let $0 < \alpha < n$ and $1 \le q < \infty$. Suppose that (w, v) is a pair of weights such that for every ball B,*

$$|B|^{\alpha/n+1/q-1}\Big(\frac{1}{|B|}\int_B w(x)dx\Big)^{1/q} \inf_{x\in B} v(x)^{-1} \le C < \infty.$$

If $w \in A_\infty$ and $0 < \kappa < 1/q$, then I_α is bounded from $L^{1,\kappa}(v, w)$ to $WL^{q,q\kappa}(w)$. If $w \in A_\infty$ and $0 < \kappa < \min\{\frac{1}{q}, \frac{\theta_w}{q}\}$, then I_α is bounded from $L^{1,\kappa}(v, dx)$ to $WL^{q,q\kappa}(w, dx)$.

In the following, we give the mapping properties of the commutators of potential-type integral operators on two-weight Morrey spaces.

Lemma 4.3. *Suppose that f is a non-negative bounded function with compact support, Φ satisfies condition (1.1), and*

$$\frac{\widetilde{\Phi}(t)\log(e+t)}{t^n} \to 0, \quad \text{when } t \to \infty. \tag{4.3}$$

Then for each $t > 0$, there exists a disjoint collection of maximal dyadic cubes $\{Q_j\}$ such that

$$\{y \in \mathbb{R}^n : M^d_{\widetilde{\Phi},L\log L} f(y) > t\} = \cup_j Q_j, \ \ \{y \in \mathbb{R}^n : M_{\widetilde{\Phi},L\log L} f(y) > 4^n t\} \subset \cup_j 3Q_j.$$

Proof. We shall omit it since the proof is similar to that of Lemma 5.1 in [18], except using $\|f\|_{L(\log L),Q_j}$. □

Lemma 4.4. *Let $0 < p < \infty$, $b \in BMO$, $w \in A_\infty$ and Φ satisfy condition (1.1), then there is a constant $C > 0$ such that for all f, we have*

$$\|T^b_\Phi f\|_{L^{p,\infty}(w)} \le C\|M_{\widetilde{\Phi},L\log L} f\|_{L^{p,\infty}(w)}.$$

Proof. From [2], we can obtain that

$$\|T^b_\Phi f\|_{L^{1,\infty}(w)} \le C\|M_{\widetilde{\Phi},L\log L} f\|_{L^{1,\infty}(w)},$$

and the desired conclusion follows if we use the extrapolation theorem in [4]. □

Lemma 4.5. *Let $1 < p \le q < \infty$, Φ satisfy conditions (1.1) and (4.3). Suppose that a pair of weights (w, v) satisfies for any cube Q,*

$$\widetilde{\Phi}(l(Q))|Q|^{1/q-1/p}\Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)^{1/q}\|v^{-1/p}\|_{L^{p'}(\log L)^{p'},Q} \le C < \infty, \tag{4.4}$$

where $l(Q)$ is the side-length of Q. Then, for every locally integrable function f,

$$w(\{x \in \mathbb{R}^n : M_{\widetilde{\Phi},L\log L} f(x) > t\})^{1/q} \le \frac{C}{t}\Big(\int_{\mathbb{R}^n} |f(x)|^p v(x)dx\Big)^{1/p}.$$

Proof. Define the set $E_t = \{x \in \mathbb{R}^n : M_{\widetilde{\Phi},L\log L} f(x) > t\}$. If t is such that the set E_t is empty, we have nothing to prove. Otherwise, by Lemma 4.3, there exists $\beta > 0$ and a collection of disjoint dyadic cubes P_j, such that $E_t \subset \bigcup_j 3P_j$ and

$$\widetilde{\Phi}(l(P_j))\|f\|_{L\log L,P_j} > \beta t.$$

By the properties of the Luxemburg norm on Orlicz spaces, let $\mathcal{A}(t) \approx t^{p'}\log(e+t)^{p'}$, $\mathcal{B}(t) \approx t^p$ and $C(t) = t\log(e+t)$, we have

$$\mathcal{A}^{-1}(t)\mathcal{B}^{-1}(t) \approx C^{-1}(t).$$

According to Lemma 2.6, we have

$$\|f\|_{L\log L,P_j} \le 2\|fv^{1/p}\|_{L^p,P_j}\|v^{-1/p}\|_{L^{p'}(\log L)^{p'},P_j}.$$

By condition (4.4), we obtain that

$$\begin{aligned}
tw(E_t)^{1/q} &\le tw(\cup_j 3P_j)^{1/q}\\
&\le C\Big[\sum_j w(3P_j)t^q\Big]^{1/q}\\
&\le C\Big[\sum_j w(3P_j)(\widetilde{\Phi}(l(P_j))\|f\|_{L\log L,P_j})^q\Big]^{1/q}\\
&\le C\Big[\sum_j w(3P_j)(\widetilde{\Phi}(l(3P_j))\|fv^{1/p}\|_{L^p,P_j}\|v^{-1/p}\|_{L^{p'}(\log L)^{p'},3P_j})^q\Big]^{1/q}\\
&\le C\Big[\sum_j |P_j|^{q/p}\|fv^{1/p}\|^q_{L^p,P_j}\Big]^{1/q}\\
&\le C\Big[\sum_j \int_{P_j}|f(x)|^p v(x)dx\Big]^{1/p}\\
&\le C\Big(\int_{\mathbb{R}^n}|f(x)|^p v(x)dx\Big)^{1/p}.
\end{aligned}$$

□

Theorem 4.4. *Let $1 < p \le q < \infty$, $b \in BMO$, Φ satisfy conditions (1.1) and (4.3). Suppose that a pair of weights (w, v) satisfies for any ball B,*

$$\widetilde{\Phi}(r(B))|B|^{1/q-1/p}\Big(\frac{1}{|B|}\int_B w(x)dx\Big)^{1/q}\|v^{-1/p}\|_{L^{p'}(\log L)^{p'},B} \le C < \infty. \tag{4.5}$$

If $w \in A_\infty, u \in \Delta_2$ and $0 < \kappa < \min\{\frac{p}{q}, \frac{p\theta_w}{q\eta_u}\}$, then T^b_Φ is bounded from $L^{p,\kappa}(v,u)$ to $WL^{q,q\kappa/p}(w,u)$.

Proof. For any $f \in L^{p,\kappa}(v,u)$ and ball $B = B(x,r) \subset \mathbb{R}^n$, write $f_1 = f\chi_{2B}$ and $f_2 = f - f_1$. Then for any given $t > 0$, we have

$$\begin{aligned}
&\frac{t}{u(B)^{\kappa/p}}w(\{z \in B : |T^b_\Phi f(z)| > t\})^{1/q}\\
\le&\frac{t}{u(B)^{\kappa/p}}w(\{z \in B : |T^b_\Phi f_1(z)| > t/2\})^{1/q} + \frac{t}{u(B)^{\kappa/p}}w(\{z \in B : |T^b_\Phi f_2(z)| > t/2\})^{1/q}.
\end{aligned}$$

By Lemmas 4.4 and 4.5, and $u \in \Delta_2$, we obtain that

$$\begin{aligned}
&\frac{t}{u(B)^{\kappa/p}}w(\{z \in B : |T^b_\Phi f_1(z)| > t/2\})^{1/q}\\
&\le \frac{1}{u(B)^{\kappa/p}}\sup_{t>0} tw(\{z \in B : |T^b_\Phi f_1(z)| > t/2\})^{1/q}\\
&\le C\frac{1}{u(B)^{\kappa/p}}\sup_{t>0} tw(\{z \in \mathbb{R}^n : M_{\widetilde{\Phi},L\log L}f_1(z) > t/2\})^{1/q}\\
&\le C\frac{1}{u(B)^{\kappa/p}}\Big(\int_{2B}|f(z)|^p v(z)dz\Big)^{1/p}\\
&\le C\|f\|_{L^{p,\kappa}(v,u)}.
\end{aligned}$$

By Chebyshev's inequality, we have

$$\begin{aligned}
&\frac{t}{u(B)^{\kappa/p}} w(\{z \in B : |T_\Phi^b f_2(z)| > t/2\})^{1/q} \\
&\le C\Big(\frac{1}{u(B)^{\kappa q/p}} \int_B |T_\Phi^b f_2(z)|^q w(z)dz\Big)^{1/q} \\
&\le C \sum_{k=1}^{\infty} \frac{\widetilde{\Phi}(2\delta(1+\varepsilon)2^{k+3}r)}{2^{kn} r^n u(B)^{\kappa/p}} \Big(\int_B |b(z) - b_B|^q w(z)dz\Big)^{1/q} \int_{2^{k+1}B} |f(y)|dy \\
&+ C\|b\|_{BMO} \sum_{k=1}^{\infty} (k+1) \frac{\widetilde{\Phi}(2\delta(1+\varepsilon)2^{k+3}r)}{2^{kn} r^n u(B)^{\kappa/p}} \Big(\int_B w(z)dz\Big)^{1/q} \int_{2^{k+1}B} |f(y)|dy \\
&+ C \sum_{k=1}^{\infty} \frac{\widetilde{\Phi}(2\delta(1+\varepsilon)2^{k+3}r)}{2^{kn} r^n u(B)^{\kappa/p}} \int_{2^{k+1}B} |b(y) - b_{2^{k+1}B}||f(y)|dy \Big(\int_B w(z)dz\Big)^{1/q} \\
&\le C\|b\|_{BMO} \sum_{k=1}^{\infty} (k+1) \frac{\widetilde{\Phi}(2\delta(1+\varepsilon)2^{k+3}r)}{2^{kn} r^n u(B)^{\kappa/p}} \Big(\int_B w(z)dz\Big)^{1/q} \int_{2^{k+1}B} |f(y)|dy \\
&+ C \sum_{k=1}^{\infty} \frac{\widetilde{\Phi}(2\delta(1+\varepsilon)2^{k+3}r)}{2^{kn} r^n u(B)^{\kappa/p}} \int_{2^{k+1}B} |b(y) - b_{2^{k+1}B}||f(y)|dy \Big(\int_B w(z)dz\Big)^{1/q} \\
&= \mathrm{I} + \mathrm{II}.
\end{aligned}$$

For I, using Hölder's inequality, condition (4.5), and Lemma 2.1, we obtain

$$\begin{aligned}
\mathrm{I} &\le C\|b\|_{BMO} \sum_{k=1}^{\infty} \frac{(k+1)\widetilde{\Phi}(2^{k+1}\rho r)}{2^{kn} r^n u(B)^{\kappa/p}} w(B)^{1/q} \\
&\cdot \Big(\int_{2^{k+1}\rho B} v(y)^{-p'/p} dy\Big)^{1/p'} \Big(\int_{2^{k+1}B} |f(y)|^p v(y)dy\Big)^{1/p} \\
&\le C\|b\|_{BMO} \|f\|_{L^{p,\kappa}(v,u)} \sum_{k=1}^{\infty} \frac{(k+1)\widetilde{\Phi}(2^{k+1}\rho r)}{|B(x, 2^{k+1}\rho r)|^{1/p} u(B)^{\kappa/p}} w(B)^{1/q} \\
&\cdot \|v^{-1/p}\|_{L^{p'}(\log L)^{p'}, 2^{k+1}\rho B} u(2^{k+1}B)^{\kappa/p} \\
&\le C\|b\|_{BMO} \|f\|_{L^{p,\kappa}(v,u)} \sum_{k=1}^{\infty} (k+1) \frac{u(2^{k+1}B)^{\kappa/p}}{u(B)^{\kappa/p}} \frac{w(B)^{1/q}}{w(2^{k+1}B)^{1/q}} \\
&\le C\|b\|_{BMO} \|f\|_{L^{p,\kappa}(v,u)} \sum_{k=1}^{\infty} (k+1) 2^{n(k+1)(\kappa\eta_u/p - \theta_w/q)} \\
&\le C\|b\|_{BMO} \|f\|_{L^{p,\kappa}(v,u)}.
\end{aligned}$$

For II, by Hölder's inequality, inequality (3.5), condition (4.5), and Lemma 2.1, we have

$$\begin{aligned}
\mathrm{II} &\le C \sum_{k=1}^{\infty} \frac{\widetilde{\Phi}(2\delta(1+\varepsilon)2^{k+3}r)}{2^{kn} r^n u(B)^{\kappa/p}} \Big(\int_{2^{k+1}B} |b(y) - b_{2^{k+1}B}|^{p'} v(y)^{-p'/p} dy\Big)^{1/p'} \\
&\cdot \Big(\int_{2^{k+1}B} |f(y)|^p v(y)dy\Big)^{1/p} \Big(\int_B w(z)dz\Big)^{1/q}
\end{aligned}$$

$$\leq C\|b\|_{BMO}\|f\|_{L^{p,\kappa}(v,u)}\sum_{k=1}^{\infty}\frac{\widetilde{\Phi}(2^{k+1}\rho r)}{2^{kn}r^n u(B)^{\kappa/p}}|2^{k+1}\rho B|^{1/p'}$$
$$\cdot\|v^{-1/p}\|_{L^{p'}(\log L)^{p'},2^{k+1}\rho B}w(B)^{1/q}u(2^{k+1}B)^{\kappa/p}$$
$$\leq C\|b\|_{BMO}\|f\|_{L^{p,\kappa}(v,u)}\sum_{k=1}^{\infty}\frac{u(2^{k+1}B)^{\kappa/p}}{u(B)^{\kappa/p}}\frac{w(B)^{1/q}}{w(2^{k+1}B)^{1/q}}$$
$$\leq C\|b\|_{BMO}\|f\|_{L^{p,\kappa}(v,u)}\sum_{k=1}^{\infty}2^{n(k+1)(\kappa\eta_u/p-\theta_w/q)}$$
$$\leq C\|b\|_{BMO}\|f\|_{L^{p,\kappa}(v,u)}.$$

This completes the proof. □

By Theorem 4.4, we have the following corollaries.

Corollary 4.7. *Let $1 < p \leq q < \infty$, $0 < \kappa < p/q$, $b \in BMO$, Φ satisfy conditions (1.1) and (4.3). Suppose that the weights (w, v) satisfy condition (4.5) and $w \in A_\infty$, then T_Φ^b is bounded from $L^{p,\kappa}(v, w)$ to $WL^{q,q\kappa/p}(w)$.*

Corollary 4.8. *Let $1 < p \leq q < \infty$, $b \in BMO$, Φ satisfy conditions (1.1) and (4.3). Suppose that the weights (w, v) satisfy condition (4.5). If $w \in A_\infty$ and $0 < \kappa < \min\{\frac{p}{q}, \frac{p\theta_w}{q}\}$, then T_Φ^b is bounded from $L^{p,\kappa}(v, dx)$ to $WL^{q,q\kappa/p}(w, dx)$.*

Corollary 4.9. *Let $0 < \alpha < n$, $1 < p \leq q < \infty$ and $b \in BMO$. Suppose that (w, v) is a pair of weights such that for every ball B,*

$$|B|^{\alpha/n+1/q-1/p}\Big(\frac{1}{|B|}\int_B w(x)dx\Big)^{1/q}\|v^{-1/p}\|_{L^{p'}(\log L)^{p'},B} \leq C < \infty.$$

If $w \in A_\infty$ and $0 < \kappa < p/q$, then I_α^b is bounded from $L^{p,\kappa}(v, w)$ to $WL^{q,q\kappa/p}(w)$. (We recover the result from [24].) If $w \in A_\infty$ and $0 < \kappa < \min\{\frac{p}{q}, \frac{p\theta_w}{q}\}$, then I_α^b is bounded from $L^{p,\kappa}(v, dx)$ to $WL^{q,q\kappa/p}(w, dx)$.

5. Conclusions

In this paper, we study the strong-type mapping properties of the potential-type integral operators and commutators on two-weight Morrey spaces in Section 3. In Section 4, we obtain the weak-type mapping properties of the potential-type integral operators and commutators on two-weight Morrey spaces. In the future, we could also study the boundedness of these operators and commutators on generalized two-weight Morrey spaces.

Author contributions

Meichuan Lv: Conceptualization, writing-original draft, methodology, writing-review and editing; Wenming Li: Conceptualization, methodology, writing-review, supervision, language editing and funding acquisition. Both authors have read and agreed to the published version of the manuscript. The manuscript text was jointly written by Lv and Li. All authors reviewed the manuscript.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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