AIMS Mathematics, 11(11): 37099–37132.
DOI: 10.3934/math.20261434
Received: 04 September 2026
Revised: 23 October 2026
Accepted: 04 November 2026
Published: 10 November 2026

https://www.aimspress.com/journal/Math

Research article

Sharp stability boundaries for discrete-time linear delay systems with multiple arithmetic delays

Serbun Ufuk Değer*

Kastamonu Vocational School, Kastamonu University, Kuzeykent Neighborhood, Orgeneral Atilla Ates Pasa Avenue No:15, 37150 Kastamonu, Türkiye

* **Correspondence:** Email: sudeger@kastamonu.edu.tr.

Abstract: In this paper, we investigated the asymptotic stability of a class of linear difference systems with multiple arithmetic delays. The considered system extends earlier two-delay models involving an additional parameter $a \in (0, 1)$ to the case of N arithmetic delays. We analyzed the location of the characteristic roots with respect to the open unit disk. The arithmetic structure of the delays enables the delayed terms to be represented through a finite delay polynomial. Based on this structure, critical parameter values determining the stability boundary were obtained. The results provide a sharp stability threshold and also yield a simple sufficient stability bound. Numerical illustrations were presented to demonstrate the difference between the sharp stability boundary and the sufficient stability region. Furthermore, an application to a nonlinear discrete-time population model with distributed delays and an Allee-type saturating feedback function was presented, showing how the obtained sharp stability threshold can be interpreted as a critical bound for the local stability of a positive population equilibrium.

Keywords: linear delay difference systems; asymptotic stability; sharp stability threshold; Allee-type feedback
Mathematics Subject Classification: 39A30, 39A06, 15A18, 39A21

1. Introduction

Difference equations constitute one of the fundamental mathematical tools used to model discrete-time dynamical processes. In many models arising in population dynamics, economics, biology, control theory, and computational sciences, the current state of a system depends not only on its present value but also on its values at previous time instants. Therefore, delay difference equations play an important role in the investigation of the stability of equilibrium points, the long-term behavior of solutions, and the sensitivity of systems to initial conditions.

In the study of nonlinear discrete-time delay models, the local behavior near an equilibrium point is often governed by the corresponding linearized delay system. Therefore, sharp stability criteria for linear delay systems provide essential tools for understanding local asymptotic stability, bifurcation thresholds, and the onset of oscillatory behavior in nonlinear delayed dynamics. From this viewpoint, the analysis of linear systems with multiple delays is not merely a linear theory problem, but also an important step in the qualitative and numerical investigation of nonlinear discrete-time models.

Discrete-time population models with Allee effects were studied by Elaydi and Sacker [1] through rational nonlinear maps and fitness-function-based formulations. Such models provide a natural motivation for using nonlinear density-dependent feedback functions in discrete population dynamics.

In this study, we consider the linear difference equation system with N delays

$$x_{n+1} - ax_n + B\sum_{j=1}^{N} x_{n-k_j} = 0, \qquad n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}, \tag{1.1}$$

where B is a constant $m \times m$ matrix and k_j, for $(j = 1, 2, \ldots, N)$ are positive integers satisfying $k_1 < k_2 < \cdots < k_N$. Here, we assume that the sequence $\left\{k_j\right\}$ is arithmetic, that is, for some positive integer d,

$$k_{j+1} = k_j + d, \quad j = 1, 2, \ldots, N-1. \tag{1.2}$$

This assumption provides a natural mathematical structure for biological, economic, and computational models in which delayed terms affect the system at regular time intervals. One of the fundamental studies on the asymptotic stability of delay difference equations was presented by Levin and May [2]. They investigated the scalar delay difference equation

$$x_{n+1} - x_n - bx_{n-k} = 0,$$

and obtained necessary and sufficient conditions for the asymptotic stability of the zero solution. Clark [3] subsequently considered the equation

$$x_{n+1} - ax_n - bx_{n-k} = 0,$$

and examined its asymptotic stability under appropriate conditions imposed on the coefficients. Kuruklis [4] derived more detailed necessary and sufficient conditions for scalar equations of the same class with a single delay. These conditions were based on the locations of the characteristic roots within the unit disk.

For delay systems with matrix coefficients, Matsunaga and Hara [5] studied the two dimensional linear delay difference system

$$x_{n+1} - x_n - Ax_{n-k} = 0,$$

and established necessary and sufficient conditions for the asymptotic stability of the zero solution. Matsunaga [6] and Değer and Bolat [7] later investigated matrix coefficient systems with a single delay of the form

$$x_{n+1} - ax_n - Bx_{n-k} = 0.$$

These studies demonstrated the importance of characteristic root analysis and Jordan canonical forms in extending stability results from scalar delay difference equations to systems with matrix

coefficients. The work of Ogita et al. [8] occupies an important place in the study of higher-order scalar difference equations with multiple delays. They considered the scalar difference equation with N delays

$$u_{n+1} - u_n + p \sum_{j=1}^{N} u_{n-k_j} = 0, \qquad n \in \mathbb{Z}_+,$$

and obtained necessary and sufficient asymptotic stability conditions ensuring that all solutions converge to zero. This model is particularly relevant for describing feedback effects involving multiple delays in areas such as mathematical biology and population dynamics. However, the equation has a scalar structure and does not directly cover multidimensional systems with matrix coefficients.

For systems with matrix coefficients and two delays, Nagabuchi [9] investigated the system $x_{n+1} - x_n + A\,(x_{n-k} + x_{n-l}) = 0$, where l, k are positive integers such that $1 < l \leq k$. The locations of the characteristic roots within the unit disk were analyzed through 2×2 Jordan blocks, and necessary and sufficient conditions for asymptotic stability were obtained. Thus, two delay difference systems with matrix coefficients were studied within a more general linear system framework that extended beyond scalar equations. In addition, Kipnis et al. [10], Ivanov et al. [11], and Čermák and Jánský [12] obtained various results concerning stability regions, stability cones, and the behavior of characteristic roots in linear difference systems with two delays. These studies show that stability conditions for delay difference systems depend on the number of delays, the matrix coefficients, and the structure of the characteristic equation.

Stability problems have also been investigated for other classes of dynamical systems using different analytical approaches. He et al. [13] investigated nonlinear impulsive stochastic systems using sliding mode control, while Geng et al. [14] studied asynchronously switched nonlinear systems under convergence rate constraints. Wu et al. [15] developed state-dependent switching conditions for interval stabilization of stochastic positive systems. More recently, Liu et al. [16] investigated predefined-time intralayer and interlayer synchronization of multilayer networks using adaptive control techniques, whereas Ye et al. [17] studied event-triggered exponential synchronization of delayed multi-agent systems with privacy-preserving mechanisms. Although these studies contribute to stability and synchronization theory, their mathematical models and analytical approaches differ from the discrete-time linear systems with multiple arithmetic delays considered in the present paper.

The present study was motivated by the earlier investigations of Nagabuchi [9, 18] and Değer and Bolat [19] on the asymptotic stability of linear delay difference systems with matrix coefficients. Incorporating the parameter a into the two-delay system previously studied by Nagabuchi [9], Değer and Bolat [19] generalized this formulation and investigated the system $x_{n+1} - ax_n + A(x_{n-k} + x_{n-l}) = 0$, where $a \in [-1, 1] \setminus \{0\}$ and $1 \leq l < k$. For the complex conjugate eigenvalue case, they obtained the necessary and sufficient condition for asymptotic stability

$$0 < q < \frac{\sin\left(\dfrac{\pi - 2|\theta|}{k+l}\right)}{2\cos\left(\dfrac{(k-l)(\pi - 2|\theta|)}{2(k+l)}\right)}.$$

Their results also covered real eigenvalues and were extended to higher-dimensional matrix systems. However, the analysis was restricted to two delays.

For systems with matrix coefficients and multiple delays, Nagabuchi [18] investigated the linear difference system

$$x_{n+1} - x_n + A\sum_{j=1}^{N} x_{n-k_j} = 0,$$

where the delays form an arithmetic sequence with common difference d. For the complex conjugate eigenvalue case, the necessary and sufficient condition for asymptotic stability was established as

$$0 < p < \frac{2\sin\left(\frac{\pi/2 - |\theta|}{k_N + k_1 + 1}\right)\sin\left(\frac{d(\pi/2 - |\theta|)}{k_N + k_1 + 1}\right)}{\sin\left(\frac{Nd(\pi/2 - |\theta|)}{k_N + k_1 + 1}\right)}.$$

The results were also extended to higher-dimensional matrix systems, and the asymptotically periodic behavior of solutions in the critical case was examined.

Although these studies established necessary and sufficient conditions for asymptotic stability, they considered different parameter and delay structures. Nagabuchi [18] investigated matrix-coefficient systems with N arithmetic delays in which the coefficient of x_n is fixed at one, whereas Değer and Bolat [19] incorporated the parameter a but restricted their analysis to two delays. Consequently, the existing results do not directly address matrix-coefficient systems combining N arithmetic delays with $a \in (0, 1)$.

The present study addresses this problem by incorporating the parameter a into the arithmetic-delay structure considered by Nagabuchi [18]. Using the characteristic equation, we derive parameter-dependent critical values and establish a sharp necessary and sufficient condition for asymptotic stability. We also obtain an explicitly computable sufficient stability bound that provides a simpler alternative to the exact threshold. Thus, the results extend existing stability criteria to a broader class of discrete-time systems with multiple delays.

In addition, because system (1.1) naturally arises from linearizing nonlinear discrete-time models with delayed feedback around an equilibrium, our stability criteria directly apply to the local stability analysis of such systems.

2. Preliminary

Consider the linear system of difference equations (1.1). Our objective is to investigate the asymptotic stability of this system. Let $x_n = Py_n$, where P is an invertible matrix. Substitution into (1.1) gives

$$\begin{aligned} &Py_{n+1} - aPy_n + B\sum_{j=1}^{N} Py_{n-k_j} = 0, \\ &n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}. \end{aligned} \tag{2.1}$$

Multiplying (2.1) from the left by P^{-1}, we obtain

$$\begin{aligned} &y_{n+1} - ay_n + P^{-1}BP\sum_{j=1}^{N} y_{n-k_j} = 0, \\ &n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}. \end{aligned}$$

Hence, the matrix A is defined by $A = P^{-1}BP$, where A is an $m \times m$ constant matrix. Then, the system in (1.1) can be written as

$$\begin{aligned} &y_{n+1} - ay_n + A\sum_{j=1}^{N} y_{n-k_j} = 0, \\ &n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}. \end{aligned} \tag{2.2}$$

The characteristic equation associated with (2.2) can be decomposed into lower–dimensional problems by reducing A to its real canonical form through an appropriate similarity transformation. In a manner analogous to the canonical reduction employed by Matsunaga [6] for two-dimensional linear delay difference equations, the coefficient matrix in the two-dimensional case can be reduced either to a rotation–dilation block corresponding to a pair of complex conjugate eigenvalues or to an upper triangular block corresponding to real eigenvalues. We therefore begin with a detailed analysis of the 2×2 block associated with a complex conjugate pair of eigenvalues. For higher-dimensional systems, the resulting stability conditions will be interpreted in terms of all Jordan blocks of A, or equivalently, all eigenvalues of A.

Let p, θ, p_1, p_2, and q be real constants, with $0 \le |\theta| < \frac{\pi}{2}$. Assume that A is a nonsingular 2×2 matrix of one of the following forms:

$$\text{(i)} \quad p\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \text{(ii)} \quad \begin{pmatrix} p_1 & q \\ 0 & p_2 \end{pmatrix}.$$

To derive the characteristic equation of (2.2), consider a solution of the form $y_n = v\lambda^n$, where $v \neq 0$. Substituting this expression into (2.2) gives

$$\lambda^{n-k_N}\left[\lambda^{k_N+1}I - a\lambda^{k_N}I + A\left(\lambda^{k_N-k_1} + \lambda^{k_N-k_2} + \cdots + \lambda^{k_N-k_N}\right)\right]v = 0.$$

For $\lambda \neq 0$, the factor λ^{n-k_N} is nonzero. Hence, the existence of a nontrivial characteristic solution $y_n = v\lambda^n$ is equivalent to

$$\Delta(\lambda) = \det\left[\lambda^{k_N+1}I - a\lambda^{k_N}I + A\left(\lambda^{k_N-k_1} + \lambda^{k_N-k_2} + \cdots + \lambda^{k_N-k_N}\right)\right] = 0.$$

Define

$$T(\lambda) := \lambda^{k_N-k_1} + \lambda^{k_N-k_2} + \cdots + \lambda^{k_N-k_N}.$$

Since the delays form an arithmetic sequence (with common difference $d = k_{j+1} - k_j$), this polynomial can be written as

$$T(\lambda) = 1 + \lambda^d + \lambda^{2d} + \cdots + \lambda^{(N-1)d} = \sum_{j=0}^{N-1} \lambda^{jd}.$$

Consequently, the characteristic equation can be written as

$$\Delta(\lambda) = \det\left[\lambda^{k_N+1}I - a\lambda^{k_N}I + T(\lambda)A\right] = 0. \tag{2.3}$$

For the two-dimensional real representation of the complex conjugate pair $pe^{\pm i\theta}$,

$$A = p\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad p > 0, \quad 0 < \theta < \frac{\pi}{2}.$$

Equation (2.3) takes the form

$$\det\begin{pmatrix} L(\lambda)+pT(\lambda)\cos\theta & -pT(\lambda)\sin\theta \\ pT(\lambda)\sin\theta & L(\lambda)+pT(\lambda)\cos\theta \end{pmatrix}=0, \tag{2.4}$$

where $L(\lambda)=\lambda^{k_N+1}-a\lambda^{k_N}$. Therefore,

$$(L(\lambda)+pT(\lambda)\cos\theta)^2+(pT(\lambda)\sin\theta)^2=0.$$

Equivalently,

$$\Big(L(\lambda)+pT(\lambda)e^{i\theta}\Big)\Big(L(\lambda)+pT(\lambda)e^{-i\theta}\Big)=0. \tag{2.5}$$

Define

$$f(\lambda)=\lambda^{k_N+1}-a\lambda^{k_N}+pT(\lambda)e^{i\theta}. \tag{2.6}$$

Since

$$\overline{f(\overline{\lambda})}=\lambda^{k_N+1}-a\lambda^{k_N}+pT(\lambda)e^{-i\theta},$$

the characteristic equation (2.3) can be expressed as $f(\lambda)\,\overline{f(\overline{\lambda})}=0$, where $\overline{\lambda}$ denotes the complex conjugate of λ.

For $-\frac{\pi}{2}<\theta\le 0$, replacing θ by $-\theta$ in $\overline{f(\overline{\lambda})}=0$ yields the equation $f(\lambda)=0$ for $0\le\theta<\frac{\pi}{2}$. Therefore, to investigate the roots of $\overline{f(\overline{\lambda})}=0$, it is sufficient to study the roots of $f(\lambda)=0$ under the condition $0\le\theta<\frac{\pi}{2}$. Hence, the problem of analyzing the roots of the characteristic equation $\Delta(\lambda)=0$ reduces to that of $f(\lambda)=0$. Moreover, when $p=0$, the characteristic equation $f(\lambda)=0$ has the simple root $\lambda=a$ and the root $\lambda=0$ with multiplicity k_N.

3. Some auxiliary results

We first determine the arguments of the complex roots of equation (2.6) lying on the unit circle $\partial D=\{\lambda\in\mathbb{C}:|\lambda|=1\}$ in the case where $p\neq 0$.

Lemma 3.1. *Assume that $0\le\theta<\frac{\pi}{2}$ and $0<a<1$. Then, for each $v\in V$, the arguments $\omega_v\in(-\pi,\pi)$ of the complex roots of $f(\lambda)=0$ on ∂D satisfy the following equations:*

$$\begin{aligned} 2v\pi &= (k_N+k_1)\omega_v+2\arg\Big(e^{i\omega_v}-a\Big), \quad \theta=0,\\ 2\theta+2v\pi &= (k_N+k_1)\omega_v+2\arg\Big(e^{i\omega_v}-a\Big), \quad 0<\theta<\pi/2, \end{aligned} \tag{3.1}$$

where

$$V=\begin{cases} \left\{v\in\mathbb{Z}:-\dfrac{k_N+k_1+2}{2}<v<\dfrac{k_N+k_1+2}{2}\right\}, & \theta=0,\\ \left\{v\in\mathbb{Z}:-\dfrac{k_N+k_1+2}{2}-\dfrac{\theta}{\pi}<v<\dfrac{k_N+k_1+2}{2}-\dfrac{\theta}{\pi}\right\}, & 0<\theta<\dfrac{\pi}{2}. \end{cases} \tag{3.2}$$

Proof. Let $\lambda=e^{i\omega}\in\partial D$ for $\omega_v\in(-\pi,\pi)$. Then λ satisfies (2.6), namely,

$$\lambda^{k_N+1}-a\lambda^{k_N}+pT(\lambda)e^{i\theta}=0,$$

and hence

$$-p = \frac{\lambda^{k_N+1} - a\lambda^{k_N}}{T(\lambda)e^{i\theta}}. \tag{3.3}$$

Since $\overline{\lambda} = 1/\lambda$, we have $T\left(\frac{1}{\lambda}\right) = \frac{T(\lambda)}{\lambda^{(N-1)d}}$. Taking the complex conjugate of both sides of (3.3) yields

$$-p = \frac{e^{i\theta}(1 - a\lambda)}{\lambda^{k_1+1}T(\lambda)}. \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\frac{\lambda^{k_N+1} - a\lambda^{k_N}}{T(\lambda)e^{i\theta}} = \frac{e^{i\theta}(1 - a\lambda)}{\lambda^{k_1+1}T(\lambda)}.$$

Therefore,

$$e^{2i\theta} = \lambda^{k_N+k_1+1}\frac{\lambda - a}{1 - a\lambda}. \tag{3.5}$$

For $\lambda = e^{i\omega}$, Eq (3.5) becomes

$$e^{2i\theta} = e^{i(k_N+k_1+1)\omega}\frac{e^{i\omega} - a}{1 - ae^{i\omega}}.$$

Since $1 - ae^{i\omega} = e^{i\omega}\overline{(e^{i\omega} - a)}$, we obtain

$$\frac{e^{i\omega} - a}{1 - ae^{i\omega}} = e^{-i\omega}\frac{e^{i\omega} - a}{\overline{(e^{i\omega} - a)}}.$$

Then we get

$$\arg\left(\frac{e^{i\omega} - a}{1 - ae^{i\omega}}\right) = \left[-\omega + 2\arg(e^{i\omega} - a)\right] \ (mod(2\pi)).$$

Thus, by (3.5), for some $v \in \mathbb{Z}$, we obtain

$$2\theta + 2v\pi = (k_N + k_1)\omega + 2\arg(e^{i\omega} - a).$$

For $\omega_v \in (-\pi, \pi)$, we define

$$\Phi_a(\omega) = (k_N + k_1)\omega + 2\arg(e^{i\omega} - a).$$

Then, when taking the derivative of $\Phi_a(\omega)$, we obtain

$$\Phi_a'(\omega) = (k_N + k_1) + 2\frac{1 - a\cos\omega}{1 + a^2 - 2a\cos\omega} > 0.$$

Hence Φ_a is strictly increasing on $(-\pi, \pi)$. Furthermore,

$$\lim_{\omega\to-\pi^+}\Phi_a(\omega) = -(k_N + k_1 + 2)\pi,$$
$$\lim_{\omega\to\pi^-}\Phi_a(\omega) = (k_N + k_1 + 2)\pi.$$

Therefore, the equation $\Phi_a(\omega) = 2\theta + 2v\pi$ has a unique solution in $(-\pi, \pi)$ if and only if

$$-(k_N + k_1 + 2)\pi < 2\theta + 2v\pi < (k_N + k_1 + 2)\pi.$$

This yields the index set given in (3.2). It remains to examine the boundary point $\lambda = -1$, which is not represented by the interval $\omega \in (-\pi, \pi)$. Substituting $\lambda = -1$ into (2.6), we obtain

$$(-1)^{k_N}(-1-a) + pe^{i\theta}T(-1) = 0,$$

where

$$T(-1) = \sum_{j=0}^{N-1}(-1)^{jd} = \begin{cases} N, & d \text{ is even,} \\ 1, & d \text{ and } N \text{ are odd,} \\ 0, & d \text{ is odd and } N \text{ is even.} \end{cases}$$

If $T(-1) = 0$, then $f(-1) \neq 0$, since $0 < a < 1$. For $T(-1) > 0$, taking the imaginary part yields

$$pT(-1)\sin\theta = 0.$$

Since $p > 0$ and $0 \leq \theta < \pi/2$, it follows that $\theta = 0$. The characteristic equation therefore reduces to

$$pT(-1) = (-1)^{k_N}(1+a).$$

Consequently, $\lambda = -1$ is a characteristic root for a positive parameter p if and only if k_N is even, $\theta = 0$, and $T(-1) > 0$, in which case

$$\boxed{p = \frac{1+a}{T(-1)}.}$$

Thus, the analysis of the unit-circle roots is completed by including this boundary case in addition to the arguments $\omega_v \in (-\pi, \pi)$ determined by (3.1). The proof is complete. □

Lemma 3.2. *Assume that $0 \leq \theta < \frac{\pi}{2}$ and $0 < a < 1$. Let $\lambda_v = e^{i\omega_v}$, with $\omega_v \in (-\pi, \pi)$ as a root of $f(\lambda) = 0$ on ∂D corresponding to a positive parameter $p = p_v > 0$, for some $v \in V$. Then p_v is given by*

$$p_v = \frac{\sqrt{1+a^2-2a\cos\omega_v}\left|\sin\left(\frac{\omega_v d}{2}\right)\right|}{\left|\sin\left(\frac{N\omega_v d}{2}\right)\right|}, \quad \text{if } \lambda_v^d \neq 1, \tag{3.6}$$

whereas

$$p_v = \frac{\sqrt{1+a^2-2a\cos\omega_v}}{N}, \qquad \text{if } \lambda_v^d = 1, \tag{3.7}$$

where $p = p_v$ and $v \in V$, with ω_v denoting the corresponding solution of the phase equation in Lemma 3.1. Moreover, for each $v \in V$, the corresponding characteristic root $\lambda_v = e^{i\omega_v} \in \partial D$ is simple.

Proof. We begin with the equation

$$p_v = -\frac{e^{-i\theta}\lambda_v^{k_N}(\lambda_v - a)}{T(\lambda_v)}.$$

By Lemma 3.1, p_v is real. In addition, under the standing assumption that $p_v > 0$, this value determines the stability boundary. Then:

$$|p_v| = \frac{|\lambda_v - a|}{|T(\lambda_v)|}.$$

We first calculate

$$|\lambda_v - a|^2 = \left(e^{i\omega_v} - a\right)\left(e^{-i\omega_v} - a\right)$$
$$= 1 - ae^{i\omega_v} - ae^{-i\omega_v} + a^2.$$

Using $e^{i\omega_v} + e^{-i\omega_v} = 2\cos\omega_v$, we obtain

$$|\lambda_v - a| = \sqrt{1 + a^2 - 2a\cos\omega_v},$$

and furthermore, since $T(\lambda_v) = \sum_{j=0}^{N-1} e^{i\omega_v jd}$, for $e^{i\omega_v d} \neq 1$, we get $|T(\lambda_v)| = \left|\frac{\sin(N\omega_v d/2)}{\sin(\omega_v d/2)}\right|$. Consequently, for $p_v > 0$, we obtain

$$p_v = \frac{\sqrt{1 + a^2 - 2a\cos\omega_v}\,|\sin(\omega_v d/2)|}{|\sin(N\omega_v d/2)|}.$$

Now, we show that $T(\lambda_v) \neq 0$. Suppose that $T(\lambda_v) = 0$. Then from $f(\lambda_v) = 0$, we obtain $\lambda_v^{k_N}(\lambda_v - a) = 0$. Since $|\lambda_v| = 1$, it follows that $\lambda_v \neq 0$. Moreover, $0 < a < 1$, which implies $\lambda_v \neq a$, because otherwise, $1 = |\lambda_v| = |a| < 1$, which is impossible. Hence $\lambda_v^{k_N}(\lambda_v - a) \neq 0$,which is a contradiction. Therefore, $T(\lambda_v) \neq 0$.

If $e^{i\omega_v d} \neq 1$, we have

$$\boxed{T(\lambda_v) = e^{i(N-1)d\omega_v/2}\frac{\sin(Nd\omega_v/2)}{\sin(d\omega_v/2)}.}$$

Using $e^{i\omega_v} - a = |e^{i\omega_v} - a|e^{i\arg(e^{i\omega_v} - a)}$, we obtain

$$p_v = -\frac{|e^{i\omega_v} - a|}{\dfrac{\sin(Nd\omega_v/2)}{\sin(d\omega_v/2)}}e^{i\left[\frac{k_N + k_1}{2}\omega_v + \arg(e^{i\omega_v} - a) - \theta\right]}.$$

From (3.1),

$$\frac{k_N + k_1}{2}\omega_v + \arg(e^{i\omega_v} - a) - \theta = v\pi.$$

Therefore,

$$p_v = (-1)^{v+1}\frac{|e^{i\omega_v} - a|\sin(d\omega_v/2)}{\sin(Nd\omega_v/2)}.$$

Consequently, $p_v > 0$ if and only if

$$\boxed{(-1)^{v+1}\frac{\sin(Nd\omega_v/2)}{\sin(d\omega_v/2)} > 0.}$$

If $e^{i\omega_v d} = 1$, then, by the definition of $T(\lambda_v)$, we obtain $T(\lambda_v) = N$. Therefore, in this special case, for $p_v > 0$, we get

$$p_v = \frac{\sqrt{1 + a^2 - 2a\cos\omega_v}}{N}.$$

In this case, from the characteristic equation, we also obtain

$$p_v = -\frac{|e^{i\omega_v} - a|}{N}e^{i[k_N\omega_v + \arg(e^{i\omega_v} - a) - \theta]}.$$

Hence $p_v > 0$ if and only if

$$\boxed{e^{i[k_N\omega_v+\arg(e^{i\omega_v}-a)-\theta]} = -1.}$$

We now show that each λ_v is a simple root. For this purpose, it is sufficient to prove that

$$\left.\frac{\partial f}{\partial \lambda}\right|_{\lambda=\lambda_v} \neq 0.$$

We consider the characteristic Eq (2.6). If we take the derivative of Eq (2.6) at $\lambda = \lambda_v$, we get

$$\left.\frac{\partial f}{\partial \lambda}\right|_{\lambda=\lambda_v} = (k_N + 1)\,\lambda^{k_N} - ak_N\lambda^{k_N-1} + pT'(\lambda)e^{i\theta}.$$

Using $f(\lambda) = 0$ to eliminate the term $pe^{i\theta}$, we have

$$\left.\frac{\partial f}{\partial \lambda}\right|_{\lambda=\lambda_v} = \lambda^{k_N-1}\left[(k_N + 1)\lambda - ak_N - \lambda(\lambda - a)\frac{T'(\lambda)}{T(\lambda)}\right]. \tag{3.8}$$

Expression (3.8) can be written as

$$\frac{\partial f}{\partial \lambda} = \lambda^{k_N-1}\,(\lambda - a)\,G\,(\lambda)\,,$$

where

$$G\,(\lambda) = k_N + \frac{\lambda}{\lambda - a} - \lambda\frac{T'(\lambda)}{T(\lambda)}. \tag{3.9}$$

Since $|a| < 1$ and $|\lambda_v| = 1$, we have $\lambda_v - a \neq 0$. Moreover, $\lambda_v \neq 0$. Consequently, it remains only to show that $G\,(\lambda_v) \neq 0$. Since $\lambda = e^{i\omega}$, we have $\overline{\lambda} = 1/\lambda$. Furthermore, because $\overline{T(\lambda)} = \lambda^{-(N-1)d}T(\lambda)$, we obtain

$$\overline{\frac{T(\lambda)}{T'(\lambda)}} = (N - 1)\,d\lambda - \lambda^2\frac{T'(\lambda)}{T(\lambda)}.$$

Let

$$H(\lambda) = \overline{G(\lambda)}.$$

Using (3.9) together with $|\lambda_v| = 1$, we get

$$H(\lambda) = k_1 + \frac{1}{1 - a\lambda} - \lambda\frac{T'(\lambda)}{T(\lambda)}.$$

It follows that

$$H(\lambda) + G\,(\lambda) = k_N + k_1 + \frac{\lambda}{\lambda - a} + \frac{1}{1 - a\lambda}.$$

For $\lambda = e^{i\omega}$ and $0 < a < 1$, this yields

$$H(\lambda) + G\,(\lambda) = k_N + k_1 + \frac{2\,(1 - a\cos\omega)}{1 + a^2 - 2a\cos\omega}, \tag{3.10}$$

since $H(\lambda) = \overline{G(\lambda)}$, and we have

$$H(\lambda) + G\,(\lambda) = 2\mathrm{Re}G\,(\lambda)\,.$$

Because $0 < a < 1$ and $1 + a^2 - 2a\cos\omega = \left|e^{i\omega_v} - a\right|^2$, the right-hand side of (3.10) is strictly positive. Therefore, Re$G(\lambda) > 0$, and hence $G(\lambda) \neq 0$. Therefore, $\frac{\partial f}{\partial \lambda}|_{\lambda=\lambda_v} \neq 0$, and each λ_v is a simple root.

The following lemma examines the movement of the roots of $f(\lambda) = 0$ near the unit circle as the real parameter p varies in neighborhoods of $p = p_v$ and $p = 0$. In the subsequent proofs, the root branches near $p = p_v$ and $p = 0$ will be denoted, respectively, by $\lambda_v(p_v) = \lambda_v$ and $\lambda_v(0) = a$. □

Lemma 3.3. *Let $p_v > 0$ and let $\lambda_v = e^{i\omega_v}$ be a simple unit-modulus root of $f_\theta(\lambda, p) = 0$ at $p = p_v$. Then there exists a differentiable root branch $\lambda_v(p)$ in a neighborhood of p_v such that $\lambda_v(p_v) = \lambda_v$. Moreover,*

$$\frac{d}{dp}\left|\lambda_v(p)\right|\big|_{p=p_v} > 0. \tag{3.11}$$

Thus, as p increases through p_v, this root crosses the unit circle from inside to outside.

Proof. $f_\theta \in \mathbb{C}^2$, defined by

$$f_\theta(\lambda, p) = \lambda^{k_N+1} - a\lambda^{k_N} + pT(\lambda)e^{i\theta}.$$

Since $\lambda_v(p) = r(p)e^{i\omega(p)}$, we have

$$\frac{d\lambda_v}{dp} = e^{i\omega(p)}\frac{dr}{dp} + ir(p)e^{i\omega(p)}\frac{d\omega}{dp}.$$

Multiplying both sides by $r(p)/\lambda_v(p)$, we obtain

$$\frac{r(p)}{\lambda_v(p)}\frac{d\lambda_v}{dp} = \frac{dr}{dp} + ir(p)\frac{d\omega}{dp}.$$

Taking real parts gives

$$\frac{dr}{dp} = \mathrm{Re}\left\{\frac{r(p)}{\lambda_v(p)}\frac{d\lambda_v}{dp}\right\}. \tag{3.12}$$

It follows from (3.12) that

$$\mathrm{Re}\left\{\frac{r}{\lambda_v}\frac{d\lambda_v}{dp}\right\} = \mathrm{Re}\left[\frac{-\frac{r}{\lambda_v}\frac{\partial f_\theta}{\partial p}}{\frac{\partial f_\theta}{\partial \lambda}}\right] = \frac{\mathrm{Re}\left[-\frac{r}{\lambda_v}\frac{\partial f_\theta}{\partial p}\overline{\left(\frac{\partial f_\theta}{\partial \lambda}\right)}\right]}{\left|\frac{\partial f_\theta}{\partial \lambda}\right|^2}.$$

Since $\frac{\partial f_\theta}{\partial p}|_{p=p_v} = T(\lambda)e^{i\theta}$, it follows from the characteristic equation $f_\theta(\lambda, p) = 0$ that

$$T(\lambda)e^{i\theta} = -\frac{1}{p}\left(\lambda^{k_N+1} - a\lambda^{k_N}\right).$$

Accordingly, we can write

$$\frac{\partial f_\theta}{\partial p}|_{p=p_v} = -\frac{1}{p}\lambda^{k_N}(\lambda - a),$$

and using the function

$$g(\lambda) = k_N + \frac{\lambda}{\lambda - a} - \lambda\frac{T'(\lambda)}{T(\lambda)},$$

defined in Lemma 3.2, we can write

$$\frac{\partial f_\theta}{\partial \lambda} = \lambda^{k_N-1}(\lambda - a)g(\lambda).$$

Consequently, for $\lambda = \lambda_v$ on the unit circle and $r = 1$, the expression $-\frac{r}{\lambda_v}\frac{\partial f_\theta}{\partial p}\overline{\left(\frac{\partial f_\theta}{\partial \lambda}\right)}$ becomes

$$\begin{aligned} -\frac{r}{\lambda_v}\frac{\partial f_\theta}{\partial p}\overline{\left(\frac{\partial f_\theta}{\partial \lambda}\right)} &= -\frac{1}{\lambda}\left(-\frac{1}{p}\lambda^{k_N}(\lambda - a)\right)\overline{(\lambda^{k_N-1}(\lambda - a)g(\lambda))} \\ &= \frac{|\lambda - a|^2}{p}\,\overline{g(\lambda)}. \end{aligned}$$

Taking the real part of this expression yields

$$\mathrm{Re}\left[-\frac{r}{\lambda_v}\frac{\partial f_\theta}{\partial p}\overline{\left(\frac{\partial f_\theta}{\partial \lambda}\right)}\right] = \frac{|\lambda - a|^2}{p}\,\mathrm{Re}(g(\lambda)).$$

Now let $h(\lambda) = \overline{g(\lambda)}$. Then

$$g(\lambda) + h(\lambda) = k_N + k_1 + \frac{\lambda}{\lambda - a} + \frac{1}{1 - a\lambda},$$

for $\lambda = e^{i\omega}$, and we obtain the same result as in Lemma 3.2, that is,

$$2\,\mathrm{Re}\, g(\lambda) = k_N + k_1 + \frac{2(1 - a\cos\omega)}{1 + a^2 - 2a\cos\omega}, \qquad 0 < a < 1.$$

Since $1 - a\cos\omega > 0$ and $1 + a^2 - 2a\cos\omega = \left|e^{i\omega_v} - a\right|^2 > 0$, it follows that $2\,\mathrm{Re}\, g(\lambda) > 0$. Thus, we obtain

$$\frac{dr}{dp}|_{p=p_v} > 0,$$

that is,

$$\frac{d}{dp}|\lambda_v(p)|_{p=p_v} > 0.$$

This shows that $r = |\lambda_v(p)|$ is an increasing function of p in a neighborhood of $p = p_v$. That is, for a sufficiently small neighborhood of $p = p_v$, $r = |\lambda_v(p)|$ increases as p increases. Since the preceding argument applies to every positive critical parameter p_v corresponding to a simple characteristic root λ_v on the unit circle, inequality (3.11) holds at each such parameter value. Consequently, each of these roots crosses the unit circle from inside to outside as p increases through p_v. □

Lemma 3.4. *In a sufficiently small neighborhood of $p = 0$, the following statements hold.*

(i) If $0 < \theta < \frac{\pi}{2}$ and $0 < a < 1$, then for $p \neq 0$, $(Sgn(|\lambda| - a))Sgnp < 0$.

(ii) If $\theta = \pi/2$ and $0 < a < 1$, then

$$\left.\frac{dr}{dp}\right|_{p=0} = 0 \quad \text{and} \quad \left.\frac{d^2r}{dp^2}\right|_{p=0} > 0.$$

Thus, the root $\lambda(0) = a$ moves outward from a at second order. Since $a < 1$, it remains inside the open unit disk for all sufficiently small positive p.

Proof. (i) As in the proof of Lemma 3.3, let $f_\theta \in \mathbb{C}^2$. We consider the characteristic equation

$$f_\theta(p, \lambda) = \lambda^{k_N+1} - a\lambda^{k_N} + pT(\lambda)e^{i\theta}.$$

For $p = 0$, we have $f_\theta(0, \lambda) = \lambda^{k_N}(\lambda - a)$. Hence $\lambda = a$ is a simple root of $f_\theta(0, \lambda) = 0$. Indeed,

$$\frac{\partial f}{\partial \lambda}_{(\lambda,p)} = (k_N + 1)\lambda^{k_N} - ak_N\lambda^{k_N-1} + pT'(\lambda)e^{i\theta},$$

so that

$$\frac{\partial f}{\partial \lambda}_{(\lambda,p)=(a,0)} = (k_N + 1)a^{k_N} - ak_N a^{k_N-1} = a^{k_N} \neq 0.$$

Therefore, by the implicit Function Theorem, there exists in a neighborhood of $p = 0$ a unique function $\lambda(p)$ satisfying $f_\theta(p, \lambda) = 0$ and $\lambda(0) = a$. If we take the derivative of $f_\theta(p, \lambda) = 0$ with respect to p, we have

$$\frac{\partial f}{\partial p} + \frac{\partial f}{\partial \lambda}\frac{d\lambda}{dp} = 0,$$

and consequently

$$\left.\frac{d\lambda}{dp}\right|_{(\lambda,p)=(a,0)} = -\frac{\partial f/\partial p}{\partial f/\partial \lambda} = -\frac{T(a)e^{i\theta}}{a^{k_N}}.$$

Since $T(a) = \sum_{j=0}^{N-1} a^{jd}$, we have $T(a) > 0$ for $0 < a < 1$. Thus, for $r = |\lambda_v(p)|$,

$$\begin{aligned}\left.\frac{dr}{dp}\right|_{p=0} &= \mathrm{Re}\left\{\left.\frac{d\lambda}{dp}\right|_{p=0}\right\} \\ &= \mathrm{Re}\left\{-\frac{T(a)e^{i\theta}}{a^{k_N}}\right\} = -\frac{T(a)\cos\theta}{a^{k_N}} < 0.\end{aligned}$$

Hence $dr/dp < 0$. On the other hand, for $\lambda = \lambda_v(p)$ with $\lambda_v(0) = a$,

$$\frac{dr}{dp} = \lim_{p\to 0}\frac{|\lambda_v(p)| - |\lambda_v(0)|}{p - 0} = \lim_{p\to 0}\frac{|\lambda_v(p)| - a}{p} < 0,$$

so that for $p \neq 0$, $\mathrm{Sgn}(|\lambda| - a)\mathrm{Sgn}p < 0$. This establishes that, at $p = 0$, the root $\lambda = a$ lies inside the unit disk and moves further inward as p increases through positive values.

(ii) When $\theta = \frac{\pi}{2}$, since $\cos\theta = 0$, we have

$$\left.\frac{dr}{dp}\right|_{p=0} = \frac{T(a)\cos\theta}{a^{k_N}} = 0.$$

In this case, it is necessary to examine the second derivative. We also have

$$\frac{d\lambda}{dp}_{p=0} = -\frac{iT(a)}{a^{k_N}},$$

which is purely imaginary. Hence the root does not initially move in the horizontal direction but rather along the vertical direction in the complex plane, so that its modulus does not change at first order. We

now compute the second derivative. If we take the derivative of $f_\theta(p,\lambda) = 0$ twice with respect to p, we have

$$\frac{\partial^2 f_\theta}{\partial \lambda^2}\left(\frac{d\lambda}{dp}\right)^2 + 2\frac{\partial^2 f_\theta}{\partial p\,\partial \lambda}\frac{d\lambda}{dp} + \frac{\partial^2 f_\theta}{\partial p^2} + \frac{\partial f_\theta}{\partial \lambda}\frac{d^2\lambda}{dp^2} = 0.$$

Since the characteristic function $f_\theta(p,\lambda)$ is linear in p, we have $\partial^2 f_\theta/\partial p^2 = 0$. The second derivative therefore reduces to

$$\frac{\partial^2 f_\theta}{\partial \lambda^2}\left(\frac{d\lambda}{dp}\right)^2 + 2\frac{\partial^2 f_\theta}{\partial p\,\partial \lambda}\frac{d\lambda}{dp} + \frac{\partial f_\theta}{\partial \lambda}\frac{d^2\lambda}{dp^2} = 0. \tag{3.13}$$

At $p = 0$ and $\lambda = a$, we have, respectively, $\frac{\partial f_\theta}{\partial \lambda}(0,a) = a^{k_N}$, $\frac{\partial^2 f_\theta}{\partial \lambda^2}(0,a) = 2k_N a^{k_N-1}$, $\frac{\partial^2 f_\theta}{\partial p\,\partial \lambda}(0,a) = iT'(a)$, $\frac{d\lambda}{dp}|_{p=0} = -\frac{iT(a)}{a^{k_N}}$. Substituting these into (3.13), we obtain

$$\frac{d^2\lambda}{dp^2} = \frac{2k_N T(a)^2}{a^{2k_N+1}} - \frac{2T(a)T'(a)}{a^{2k_N}}.$$

for $r = |\lambda_v(p)|$ and

$$\left.\frac{d^2 r}{dp^2}\right|_{p=0} = \mathrm{Re}\left(\frac{d^2\lambda}{dp^2}_{p=0}\right) + \frac{1}{a}\left|\left.\frac{d\lambda}{dp}\right|_{p=0}\right|^2.$$

Therefore, we obtain

$$\left.\frac{d^2 r}{dp^2}\right|_{p=0} = \frac{T(a)}{a^{2k_N+1}}[(2k_N+1)T(a) - 2aT'(a)].$$

We now show that this quantity is positive. We have

$$T(a) = \sum_{j=0}^{N-1} a^{jd}, \qquad aT'(a) = \sum_{j=1}^{N-1} jd\, a^{jd},$$

so that

$$(2k_N+1)T(a) - 2aT'(a) = \sum_{j=0}^{N-1}(2k_N + 1 - 2jd)a^{jd}.$$

Since $k_N = k_1 + (N-1)d$, we have

$$2k_N + 1 - 2jd \geq 2k_1 + 1 > 0, \qquad j = 0, 1, 2, \ldots, N-1.$$

Moreover, $a^{jd} > 0$ for $a \in (0,1)$. It follows that

$$(2k_N+1)T(a) - 2aT'(a) > 0,$$

and consequently

$$\frac{d^2 r}{dp^2}|_{p=0} > 0.$$

Hence, when $\theta = \pi/2$, the root $\lambda(0) = a$ moves outward from a at second order. Since $0 < a < 1$, by the continuity of the root branch $\lambda(p)$, there exists $\delta > 0$ such that $|\lambda(p)| < 1$ for all $0 < p < \delta$.

Therefore, this root remains inside the open unit disk for all sufficiently small positive p. Moreover, suppose that $|\lambda| \geq 1$ for some characteristic root of $f_\theta(\lambda, p) = 0$. From the characteristic equation, we obtain

$$|\lambda|^{k_N}|\lambda - a| = p|T(\lambda)|.$$

Since

$$|\lambda - a| \geq |\lambda| - a \geq 1 - a$$

and

$$|T(\lambda)| \leq \sum_{j=0}^{N-1} |\lambda|^{jd} \leq N|\lambda|^{(N-1)d},$$

it follows that

$$(1 - a)|\lambda|^{k_1} \leq pN.$$

Since $|\lambda| \geq 1$, we have

$$p \geq \frac{1 - a}{N}.$$

Therefore, all characteristic roots lie inside the open unit disk whenever

$$0 < p < \frac{1 - a}{N}.$$

This sufficient condition is independent of θ and remains valid as $\theta \to \pi/2^-$. □

Remark 3.1. *The boundary case $\theta \to \pi/2^-$ does not lead to instability for arbitrarily small positive values of p. Indeed, for $|\lambda| = 1$, we have*

$$|\lambda^{k_N}(\lambda - a)| \geq 1 - a, \qquad |pe^{i\theta}T(\lambda)| \leq pN.$$

Thus, whenever $0 < p < (1 - a)/N$, Rouché's theorem implies that $f_\theta(\lambda, p)$ and $\lambda^{k_N}(\lambda - a)$ have the same number of zeros in the open unit disk. Since all zeros of the latter polynomial lie in this disk, the characteristic roots remain strictly inside the unit circle. This conclusion holds uniformly with respect to θ, including the limiting case $\theta = \pi/2$, and therefore excludes instability caused by second-order effects for arbitrarily small positive p.

We note that the analysis of $\theta = \pi/2$ in Lemma 3.4(ii) concerns the behavior of characteristic roots near $p = 0$ and does not establish a sharp stability threshold for this boundary case. Accordingly, the necessary and sufficient stability conditions in Theorems 4.1 and 4.2 are stated under the restriction $0 \leq \theta < \pi/2$ for the complex eigenvalue case.

We may now discuss the asymptotic stability of the system with the aid of Lemma 3.3 and Lemma 3.4. For $p = 0$, the characteristic equation takes the form $f(\lambda) = \lambda^{k_N}(\lambda - a) = 0$. Hence the characteristic roots consist of $\lambda = 0$ with multiplicity k_N and the simple root $\lambda = a$. Since $|a| < 1$, all characteristic roots lie within the open unit disk D. Because the characteristic roots depend continuously on the parameter p, all characteristic roots remain within D in a sufficiently small neighborhood of $p = 0$. In particular, by Lemma 3.4(i), for $0 \leq \theta < \pi/2$ and $a \in (0, 1)$, the modulus of the root $\lambda(0) = a$ decreases for every sufficiently small $p > 0$; by Lemma 3.4(ii), in the case $\theta = \pi/2$,

although the modulus of this root exceeds a, it nevertheless remains within the open unit disk for sufficiently small values of p. We consider the set

$$\boxed{\wp = \{p_v : p_v > 0,\ v \in V\} \cup \wp_{\mathrm{b}},} \tag{3.14}$$

where p_v denotes a positive critical parameter corresponding to a characteristic root on the unit circle, as determined in Lemma 3.2, and

$$\wp_{\mathrm{b}} = \begin{cases} \left\{\dfrac{1+a}{T(-1)}\right\}, & \text{if } \theta = 0,\ k_N \text{ is even},\ T(-1) > 0, \\ \varnothing, & \text{otherwise.} \end{cases}$$

The set $\wp_{\mathrm{b}}$ accounts for the boundary root $\lambda = -1$ considered in Lemma 3.1. Thus, $\wp$ contains all positive parameter values at which at least one characteristic root lies on the unit circle.

We now show that $\wp \neq \varnothing$. For $p = 0$, the characteristic equation reduces to $f_\theta(\lambda, 0) = \lambda^{k_N}(\lambda - a) = 0$, whose roots lie inside the open unit disk. Let $\lambda_1(p), \ldots, \lambda_{k_N+1}(p)$ denote the characteristic roots, counted with multiplicity. Since the characteristic polynomial is monic and its constant term is $pe^{i\theta}$, Viète's formula [20] gives

$$\prod_{j=1}^{k_N+1} \lambda_j(p) = (-1)^{k_N+1} pe^{i\theta}.$$

Consequently,

$$\prod_{j=1}^{k_N+1} |\lambda_j(p)| = p.$$

Hence, for $p > 1$, at least one characteristic root lies outside the open unit disk. Since the spectral radius $\rho(p) = \max_j |\lambda_j(p)|$ depends continuously on p, there exists $p_0 > 0$ such that $\rho(p_0) = 1$. By the definition of $\wp$, we have $p_0 \in \wp$. Therefore, $\wp \neq \varnothing$.

Since $\wp$ is finite and nonempty, we define

$$\boxed{p^* = \min \wp.} \tag{3.15}$$

By Lemma 3.3, every simple characteristic root on the unit circle crosses from inside to outside as p increases through its corresponding positive critical value. Consequently, p^* is the first critical parameter at which a characteristic root reaches the unit circle.

Lemma 3.5. *Assume that* $0 \leq \theta < \pi/2$ *and* $a \in (0, 1)$ *. Then all roots of the characteristic equation* (2.6) *lie inside the open unit disk if and only if*

$$0 < p < p^*. \tag{3.16}$$

Moreover, we have

$$p^* \geq \frac{1-a}{N}. \tag{3.17}$$

Proof. Sufficiency: Let $0 < p < p^*$. For $p = 0$, the characteristic Eq (2.6) reduces to $f(\lambda) = \lambda^{k_N}(\lambda - a) = 0$. Since $a \in (0, 1)$, all its roots lie inside the open unit disk D. Moreover, by Lemma 3.4, this property

is preserved for sufficiently small positive values of p. Since the coefficients of the characteristic polynomial depend continuously on p, its roots also depend continuously on p.

By Lemmas 3.1 and 3.2, all positive parameter values at which characteristic roots lie on the unit circle, including the boundary case $\lambda = -1$, are contained in $\wp$. Since $p^* = \min \wp$, no element of $\wp$ lies in the interval $0 < p < p^*$, so no characteristic root can reach the unit circle within this interval. As the roots lie in the open unit disk at $p = 0$ and no crossing of the unit circle occurs within this interval, all characteristic roots remain inside the open unit disk for $0 < p < p^*$. The system is therefore asymptotically stable.

Necessity: Suppose now that the system is asymptotically stable. Then $|\lambda| < 1$ must hold for every characteristic root. It therefore suffices to show that $p \geq p^*$ cannot hold.

Case 1: Let $p = p^*$. By definition, $p^* \in \wp$, so at least one characteristic root lies on the unit circle. Since asymptotic stability requires all characteristic roots to lie inside the open unit disk, the system is not asymptotically stable at $p = p^*$.

Case 2: Let $p > p^*$. We know that at $p = p^*$, at least one characteristic root lies on the unit circle. By Lemma 3.2, the corresponding roots with $\omega_v \in (-\pi, \pi)$ are simple. For the boundary case $\lambda = -1$, when it occurs, we have

$$2 \operatorname{Re} G(-1) = k_N + k_1 + \frac{2}{1+a} > 0.$$

Since $T(-1) \neq 0$, Eq (3.8) implies that $f'(-1) \neq 0$. Hence, the boundary root is also simple, and the outward crossing argument of Lemma 3.3 applies to all these roots. In particular,

$$\frac{d}{dp}|\lambda_v(p)|_{p=p_v} > 0, \tag{3.18}$$

and the same outward crossing property also holds for the boundary root $\lambda = -1$, whenever it occurs. Thus, for $p > p^*$ sufficiently close to p^*, at least one characteristic root satisfies $|\lambda(p)| > 1$. Hence, the system loses asymptotic stability immediately after p^*.

Suppose, conversely, that the system becomes asymptotically stable again at some value $p > p^*$. Then the root that exits the unit disk immediately after p^*, or any root lying outside the disk, would have to re-enter the open unit disk. Since the roots depend continuously on p, such a re-entry would require some characteristic root to cross the unit circle from outside to inside at some value $p_0 > p^*$.

However, by Lemmas 3.1 and 3.2, all crossings that can occur on the unit circle occur precisely at the critical values in the set $\wp$, including the boundary case $\lambda = -1$. Furthermore, by Lemma 3.2 and the boundary-case argument above, these roots are simple. According to Lemma 3.3, the outward crossing condition (3.18) holds at each critical value corresponding to $\omega \in (-\pi, \pi)$, and the same property holds for the boundary root. Therefore, as p increases, no characteristic root can cross the unit circle from outside to inside. This contradiction shows that the system cannot become asymptotically stable again for $p > p^*$. Hence, asymptotic stability necessarily requires $0 < p < p^*$.

Furthermore, for every $p \in \wp$, there exists a characteristic root λ with $|\lambda| = 1$. From the characteristic equation, we obtain

$$p = \frac{|\lambda - a|}{|T(\lambda)|}.$$

For $a \in (0, 1)$ and $|\lambda| = 1$, we have

$$|\lambda - a| \geq 1 - a.$$

On the other hand,

$$|T(\lambda)| = \left|\sum_{j=0}^{N-1} \lambda^{jd}\right| \leq \sum_{j=0}^{N-1} |\lambda^{jd}| = N,$$

so that $|T(\lambda)| \leq N$. Consequently, the inequality

$$p = \frac{|\lambda - a|}{|T(\lambda)|} \geq \frac{1-a}{N}$$

holds for every $p \in \wp$, from which we obtain

$$p^* = \min \wp \geq \frac{1-a}{N}.$$

The proof is complete. □

4. Main results

Theorem 4.1. *Assume that $a \in (0,1)$ and that the matrix B satisfies either of the two cases in its Jordan canonical form. Then system* (1.1) *is asymptotically stable if and only if*

$$0 < p < p^*. \tag{4.1}$$

Moreover, since

$$p^* \geq \frac{1-a}{N}, \tag{4.2}$$

we obtain a sufficient condition for the asymptotic stability of system (1.1):

$$0 < p < \frac{1-a}{N}. \tag{4.3}$$

Proof. For the 2×2 Jordan block of the matrix A selected in system (2.2), the characteristic equation reduces to

$$f(\lambda) = \lambda^{k_N+1} - a\lambda^{k_N} + pT(\lambda)e^{i\theta}.$$

In Lemma 3.1, the arguments of the possible roots of this characteristic equation lying on the unit circle were determined, including the boundary case $\lambda = -1$. In Lemma 3.2, the corresponding positive critical parameter values p_v were obtained, and these roots were shown to be simple. The simplicity and outward crossing of the boundary root were also established in Lemma 3.5.

Lemma 3.3 shows that as the parameter p increases, these simple unit-circle roots cross the unit circle from inside to outside. Lemma 3.4 establishes that, in a neighborhood of $p = 0$, the characteristic roots lie inside the open unit disk, and that asymptotic stability is preserved for sufficiently small positive values of p. By Lemma 3.5, the necessary and sufficient condition for all characteristic roots to remain inside the open unit disk, or equivalently for system (2.2) to be asymptotically stable, is $0 < p < p^*$. Furthermore, since Lemma 3.5 establishes that $p^* \geq (1-a)/N$, the condition $0 < p < (1-a)/N$ guarantees the condition $0 < p < p^*$. Therefore, $0 < p < (1-a)/N$ is a sufficient condition for asymptotic stability. The proof is complete. □

Finally, we consider a higher-dimensional linear delay difference system with multiple delays,

$$x_{n+1} - ax_n + B\sum_{j=1}^{N} x_{n-k_j} = 0, \qquad n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}, \tag{4.4}$$

where B is an $m \times m$ constant matrix, $a \in (0, 1)$, and the delays k_j ($j = 1, 2, \ldots, N$) are positive integers forming an arithmetic sequence; that is, $k_{j+1} = k_j + d$, $j = 1, 2, \ldots, N-1$, for some positive integer d. We can then readily verify that the characteristic equation of (4.4) is given by

$$\prod_{j=1}^{m} \left\{\lambda^{k_N+1} - a\lambda^{k_N} + p_j e^{i\theta_j} T(\lambda)\right\} = 0. \tag{4.5}$$

Theorem 4.2. *Let $a \in (0, 1)$ and let B be a nonsingular real $m \times m$ matrix whose eigenvalues are $\mu_j = p_j e^{i\theta_j}$, where $p_j > 0$ and $|\theta_j| < \pi/2$, for $j = 1, \ldots, m$. Suppose that the delays k_j, $j = 1, \ldots, N$, form an arithmetic sequence with common difference $d > 0$. Then system (4.4) is asymptotically stable if and only if*

$$\boxed{0 < p_j < p_j^*, \qquad j = 1, \ldots, m,} \tag{4.6}$$

where p_j^* denotes the sharp stability threshold associated with the eigenvalue argument θ_j, as determined by Lemmas 3.1–3.5. Moreover, since

$$p_j^* \geq \frac{1-a}{N}, \tag{4.7}$$

we obtain a sufficient condition for the asymptotic stability of system (4.4):

$$\boxed{0 < p_j < \frac{1-a}{N}, \qquad j = 1, \ldots, m.} \tag{4.8}$$

Proof. Let $J = P^{-1}BP$ be the Jordan canonical form of B, with Jordan blocks $J_{r_\ell}(\mu_\ell)$, $\ell = 1, \ldots, s$, where the eigenvalues are listed by Jordan blocks and $\sum_{\ell=1}^{s} r_\ell = m$. Since

$$J_{r_\ell}(\mu_\ell) = \mu_\ell I_{r_\ell} + N_{r_\ell},$$

where N_{r_ℓ} is strictly upper triangular, the characteristic determinant of system (4.4) can be written as

$$\begin{aligned}
\Delta(\lambda) &= \det\left[L(\lambda)I_m + T(\lambda)B\right] \\
&= \det\left[L(\lambda)I_m + T(\lambda)J\right] \\
&= \prod_{\ell=1}^{s} \det\left[(L(\lambda) + \mu_\ell T(\lambda))I_{r_\ell} + T(\lambda)N_{r_\ell}\right] \\
&= \prod_{\ell=1}^{s} \left[L(\lambda) + \mu_\ell T(\lambda)\right]^{r_\ell},
\end{aligned}$$

where

$$L(\lambda) = \lambda^{k_N+1} - a\lambda^{k_N}.$$

Thus, the sizes of the Jordan blocks affect only the multiplicities of the characteristic roots, not their locations. Consequently, system (4.4) is asymptotically stable if and only if all roots of the corresponding scalar characteristic equations lie inside the open unit disk. Applying Lemma 3.5 to each scalar factor under the stated assumptions on the eigenvalues yields

$$0 < p_j < p_j^*, \qquad j = 1, \ldots, m.$$

Moreover, since

$$p_j^* \geq \frac{1-a}{N},$$

the inequalities

$$0 < p_j < \frac{1-a}{N}, \qquad j = 1, \ldots, m,$$

provide a sufficient condition for asymptotic stability. The proof is complete. □

5. Numerical illustrations and application

5.1. Sharp stability boundary for the theoretical model

In this section, a numerical example is presented in order to support the asymptotic stability result obtained in the previous section. The figure compares the sharp stability condition $0 < p < p^*$, with the simpler sufficient condition

$$0 < p < \frac{1-a}{N}.$$

Here, p^* defines the exact stability threshold, whereas $(1-a)/N$ provides an easily computable, though more conservative, sufficient bound. For the numerical illustration, we choose $a = 0.4$, $N = 3$, $k_1 = 1$, $d = 2$ in Eqs (2.2), (2.3), and (2.6). In this case, $k_2 = 3$, $k_3 = 5$, and $k_N = 5$. The delay polynomial therefore takes the form

$$T(\lambda) = 1 + \lambda^2 + \lambda^4.$$

When the angle parameter corresponding to the two-dimensional Jordan block associated with the complex conjugate eigenvalue pair is chosen as $\theta = \pi/6$, the corresponding characteristic equation becomes

$$f(\lambda, p) = \lambda^6 - 0.4\lambda^5 + p(1 + \lambda^2 + \lambda^4)e^{i\pi/6}.$$

For this characteristic equation, the spectral radius is defined, as introduced by Elaydi [21], as

$$\rho(p) = \max\{|\lambda| : f(\lambda, p) = 0\}. \tag{5.1}$$

It is a standard fact that all characteristic roots of the system lying within the open unit disk, that is, $\rho(p) < 1$ are equivalent to the asymptotic stability of the system. In the critical case, $\rho(p) = 1$, at least one characteristic root lies on the unit circle. For this example, the first positive critical value obtained by using Lemma 3.1 and Lemma 3.2 is approximately $p^* \approx 0.3851$. On the other hand, since the lower bound given in Lemma 3.5 is $p^* \geq (1-a)/N$, for this example, we obtain

$$\frac{1-a}{N} = \frac{1-0.4}{3} = 0.2.$$

Thus, the interval $0 < p < 0.2$ provides a sufficient region guaranteeing asymptotic stability. However, since the sharp stability interval is $0 < p < 0.3851$, it is seen that the sufficient condition does not cover the entire stability region. Figure 1 shows the variation of $\rho(p)$ with respect to p. In the figure, the horizontal dashed line represents the stability boundary $\rho(p) = 1$. The green vertical line indicates the sufficient stability bound $p = 0.2$, while the red vertical line indicates the sharp stability bound $p^* \approx 0.3851$.

As seen in Figure 1, since $\rho(p) < 1$ holds for $0 < p < (1-a)/N$, the system is asymptotically stable on this interval. However, this interval does not constitute the entire stability region. Indeed, the inequality $\rho(p) < 1$ is also satisfied on the interval $(1-a)/N < p < p^*$. Therefore, the condition $0 < p < (1-a)/N$ is a sufficient condition guaranteeing asymptotic stability, but it is not necessary. In contrast, the condition $0 < p < p^*$ is the sharp stability condition. At $p = p^*$, the spectral radius becomes one, and at least one characteristic root reaches the unit circle. When $p > p^*$, $\rho(p) > 1$ and the system loses asymptotic stability.

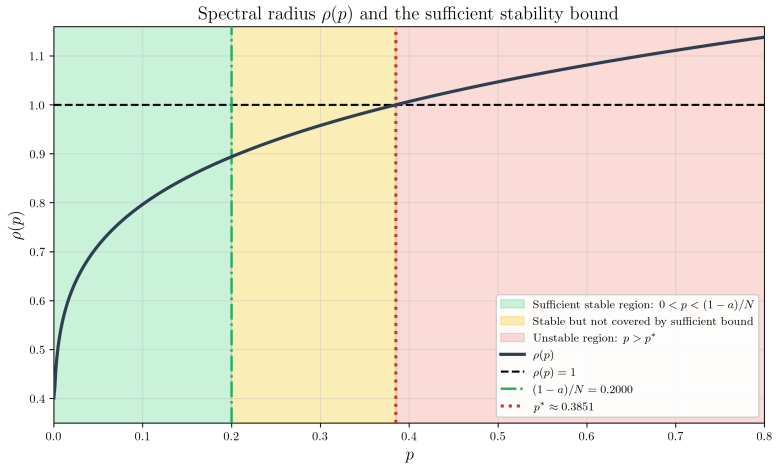


Figure 1. Variation of the spectral radius $\rho(p)$ as a function of p for $a = 0.4$, $N = 3$, $k_1 = 1$, $d = 2$, and $\theta = \pi/6$.

Figure 2 illustrates the variation of the spectral radius as a function of both parameters p and θ. In this case,

$$\rho(p, \theta) = \max\{|\lambda| : f(\lambda, p, \theta) = 0\},$$

where

$$f(\lambda, p, \theta) = \lambda^{k_N+1} - a\lambda^{k_N} + pT(\lambda)e^{i\theta}.$$

The graph is plotted over the interval $0 \leq \theta < \pi/2$, for which the theoretical results are valid. Figure 2 shows that the bound $(1-a)/N$ is independent of θ, whereas the actual critical value $p^*(\theta)$ depends on θ in general. Thus, the condition $0 < p < (1-a)/N$ provides a safe and easily computable sufficient stability region for all values of θ. However, as can be seen from the yellow region, $\rho(p, \theta) < 1$ also holds for certain parameter values lying outside this sufficient bound. This indicates that the bound $(1-a)/N$ is sufficient but not necessary. The sharp boundary, on the other hand, is determined by the curve $p^*(\theta)$ corresponding to the condition $\rho(p, \theta) = 1$.

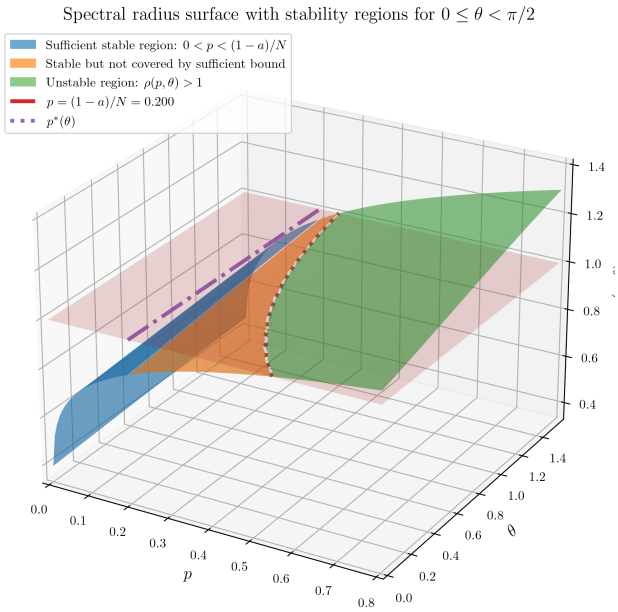


Figure 2. The spectral radius surface $\rho(p, \theta)$ on the interval $0 \leq \theta < \pi/2$.

Finally, in order to illustrate the location of the characteristic roots in the complex plane, let us select three distinct values of p such that

$$p_1 < \frac{1-a}{N} < p_2 < p^* < p_3.$$

Specifically, let $p_1 = 0.10$, $p_2 = 0.30$, and $p_3 = 0.45$. As can be seen, p_1 lies within the sufficient stability region, p_2 lies outside the sufficient bound but below the sharp stability threshold, and p_3 lies above the sharp critical value. Figure 3 shows the location of the characteristic roots in the complex plane for the values $p_1 = 0.10$, $p = (1-a)/N$, $p_2 = 0.30$, $p = p^* \approx 0.3851$, and $p_3 = 0.45$. The unit circle represents the boundary of asymptotic stability. For the values p_1, $p = (1-a)/N$, and p_2, all characteristic roots lie within the open unit disk. In particular, the fact that the roots remain within the unit disk even though $p_2 > (1-a)/N$ demonstrates that the condition $0 < p < (1-a)/N$ is not necessary. By contrast, at $p = p^*$, at least one characteristic root reaches the unit circle, and when $p_3 > p^*$, some roots move outside the unit circle, causing the system to lose its asymptotic stability.

Figure 4 illustrates, in three-dimensional form, the trajectory of the characteristic roots as a function of the parameter p, plotted in the coordinates $(\mathrm{Re}(\lambda), \mathrm{Im}(\lambda), p)$. The gray cylinder represents the stability boundary $|\lambda| = 1$, the green dashed circle represents the sufficient stability boundary $p = (1-a)/N$, and the black dotted circle represents the sharp stability boundary $p = p^*$. To enhance visual clarity, the parameter values $p_1 = 0.10$, $p = (1-a)/N$, $p_2 = 0.30$, $p = p^* \approx 0.3851$, $p_3 = 0.45$, and $p_4 = 0.75$ have been used in this figure. As the figure shows, for the values $p_1 = 0.10$, $p = (1-a)/N$, and $p_2 = 0.30$, the characteristic roots remain inside the unit cylinder. This indicates that the system is asymptotically stable for these parameter values. In particular, the fact that the roots still lie inside the cylinder even though $p_2 = 0.30 > (1-a)/N$ clearly confirms that the condition $0 < p < (1-a)/N$ is not necessary, but only sufficient. By contrast, at the level $p = p^*$, at least one root reaches the unit cylinder, representing the sharp stability boundary. Subsequently, at $p_3 = 0.45$ and $p_4 = 0.75$, some

roots are observed to move outside the cylinder. Consequently, the system is not asymptotically stable for these values. Thus, Figure 4 clearly demonstrates, through the root trajectories, both the distinction between the sufficient stability boundary $(1 - a)/N$ and the sharp boundary p^*, and the transition from the stable region to the unstable region.

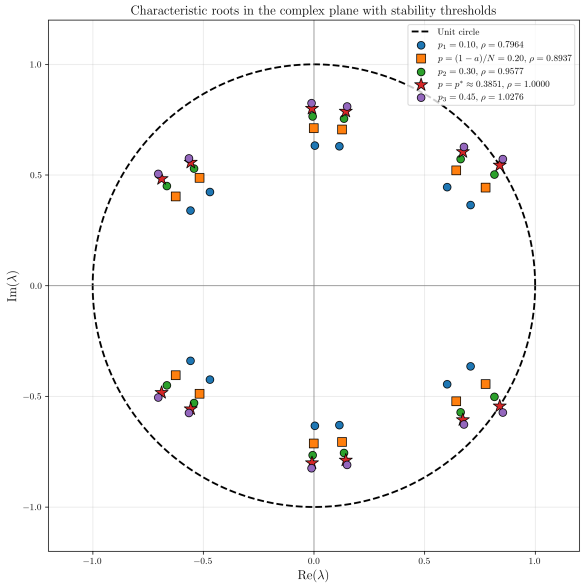


Figure 3. Locations of the characteristic roots in the complex plane for the values $p_1 = 0.10$, $p_2 = 0.30$, and $p_3 = 0.45$.

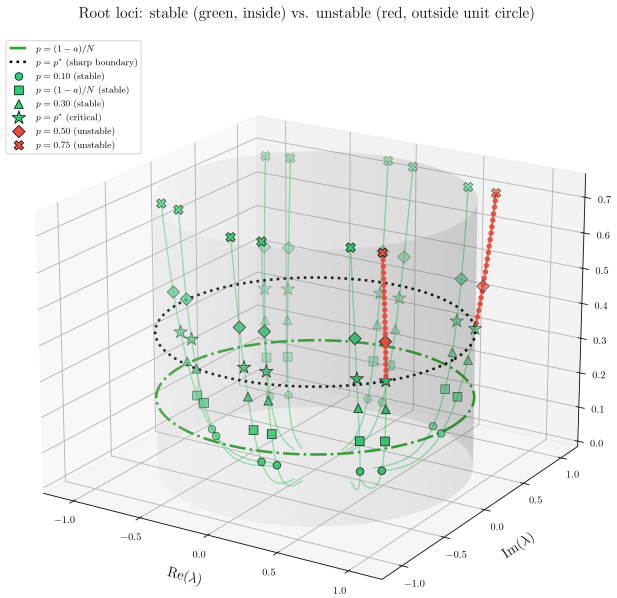


Figure 4. Characteristic root loci with respect to p and stability boundaries.

Figures 1–4 show the same stability transition predicted by the theoretical analysis: the system

remains stable beyond the sufficient bound $(1-a)/N$ and loses stability at $p = p^*$. In particular, stability may persist for $(1-a)/N < p < p^*$, showing that $0 < p < (1-a)/N$ is sufficient but not necessary.

5.2. Higher-dimensional non-diagonalizable matrix example

In this section, a four-dimensional numerical example is presented to illustrate the asymptotic stability conditions established in Theorem 4.2 for higher-dimensional systems with non-diagonalizable coefficient matrices. We retain the parameters $a = 0.4$, $N = 3$, $k_1 = 1$, and $d = 2$ used in Section 5.1, allowing a direct comparison between the two examples. Consider the linear difference system

$$x_{n+1} - ax_n + B(p)\sum_{j=1}^{3} x_{n-k_j} = 0,$$

where $x_n \in \mathbb{R}^4$, and let $a = 0.4, N = 3, k_1 = 1, k_2 = 3, k_3 = 5$. The coefficient matrix is chosen as

$$B(p) = \begin{pmatrix} p & 1 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p\cos(\pi/6) & -p\sin(\pi/6) \\ 0 & 0 & p\sin(\pi/6) & p\cos(\pi/6) \end{pmatrix}, \qquad p > 0.$$

The matrix $B(p)$ contains a Jordan block of order two associated with the eigenvalue p. Since this block is non-diagonalizable, the full coefficient matrix is also non-diagonalizable. Its eigenvalues are

$$\mu_1 = \mu_2 = p, \qquad \mu_3 = pe^{i\pi/6}, \qquad \mu_4 = pe^{-i\pi/6}.$$

Thus, this example involves both a repeated real eigenvalue and a pair of complex conjugate eigenvalues. Since the delays form an arithmetic sequence with common difference $d = 2$, the delay polynomial is

$$T(\lambda) = 1 + \lambda^2 + \lambda^4.$$

Furthermore,

$$L(\lambda) = \lambda^6 - 0.4\lambda^5.$$

By the characteristic determinant decomposition established in Theorem 4.2, we obtain

$$\boxed{\begin{aligned}\Delta(\lambda) = {} & [L(\lambda) + pT(\lambda)]^2 \\ & \times [L(\lambda) + pe^{i\pi/6}T(\lambda)] \\ & \times [L(\lambda) + pe^{-i\pi/6}T(\lambda)].\end{aligned}}$$

The squared factor corresponds to the Jordan block associated with the repeated eigenvalue p. Therefore, the asymptotic stability of the four-dimensional system can be determined by examining the roots of the scalar characteristic equations

$$f_0(\lambda, p) = \lambda^6 - 0.4\lambda^5 + p(1 + \lambda^2 + \lambda^4) = 0$$

and

$$f_{\pi/6}(\lambda, p) = \lambda^6 - 0.4\lambda^5 + pe^{i\pi/6}(1 + \lambda^2 + \lambda^4) = 0.$$

Since the characteristic equation associated with the eigenvalue $pe^{-i\pi/6}$ has conjugate coefficients, its roots are the complex conjugates of those of $f_{\pi/6}(\lambda, p) = 0$. Hence, both equations have the same spectral radius. To determine the critical parameter values, we apply Lemmas 3.1 and 3.2. For the selected parameters, the phase equation becomes

$$2\theta + 2v\pi = 6\omega_v + 2\arg(e^{i\omega_v} - 0.4).$$

The corresponding positive critical parameter values are obtained from

$$p_v = \frac{\sqrt{1.16 - 0.8\cos\omega_v}\,|\sin\omega_v|}{|\sin 3\omega_v|},$$

where $e^{2i\omega_v} \neq 1$, with the special case $e^{2i\omega_v} = 1$ treated according to Lemma 3.2. The critical parameter values were computed by numerically solving the phase equation in Lemma 3.1 and retaining only the positive real parameter values according to Lemma 3.2. The characteristic roots were then calculated from the corresponding scalar polynomial factors, and the spectral radius was obtained as the maximum modulus of these roots. The numerical calculations were performed using Python with NumPy and SciPy. For the real Jordan block, the first positive critical parameter value is $p_0^* \approx 0.546450$, whereas for the complex conjugate pair, we obtain $p_{\pi/6}^* \approx 0.385058$. According to Theorem 4.2, all characteristic roots of the full system lie inside the open unit disk if and only if the stability conditions associated with all eigenvalues are satisfied. Therefore, the sharp stability threshold for this example is

$$\begin{aligned} p^* &= \min\{p_0^*, p_{\pi/6}^*\} \\ &= \min\{0.546450, 0.385058\}. \end{aligned}$$

Consequently,

$$\boxed{p^* \approx 0.385058.}$$

This value agrees with the critical parameter $p^* \approx 0.3851$ obtained in Section 5.1 for $\theta = \pi/6$ and determines the stability boundary of the four-dimensional system. To verify this result, the spectral radii of the scalar characteristic equations are calculated for two values of p. Define

$$\rho_0(p) = \max\{|\lambda| : f_0(\lambda, p) = 0\}$$

and

$$\rho_{\pi/6}(p) = \max\{|\lambda| : f_{\pi/6}(\lambda, p) = 0\}.$$

The numerical results are presented in Table 1.

Table 1. Spectral radii of the characteristic factors for the four-dimensional non-diagonalizable system.

p	$\rho_0(p)$	$\rho_{\pi/6}(p)$	Stability
0.25	0.889604	0.928223	Stable
0.39	0.951633	1.002223	Unstable

For $p = 0.25$, both spectral radii are less than one. Hence, all characteristic roots of the full system lie inside the open unit disk, and the system is asymptotically stable. On the other hand, for $p = 0.39$, we obtain

$$\rho_0(0.39) \approx 0.951633 < 1,$$

whereas

$$\rho_{\pi/6}(0.39) \approx 1.002223 > 1.$$

Thus, although the characteristic roots associated with the real Jordan block remain inside the open unit disk, at least one root associated with the complex conjugate eigenvalue pair lies outside the unit disk. Consequently, the full system is not asymptotically stable. These results are consistent with Theorem 4.2. The stability of the full system is determined by the characteristic roots associated with its eigenvalues, while the Jordan block sizes affect only their algebraic multiplicities.

Figure 5 shows the spectral radii of the scalar characteristic factors associated with the real Jordan block and the complex conjugate pair. The horizontal line $\rho = 1$ represents the stability boundary. The full system remains asymptotically stable as long as both spectral radii are below one. The first loss of asymptotic stability occurs at $p^* \approx 0.385058$, where the complex conjugate branch reaches the unit circle. For $p > p^*$, the corresponding spectral radius exceeds one, and the full system becomes unstable.

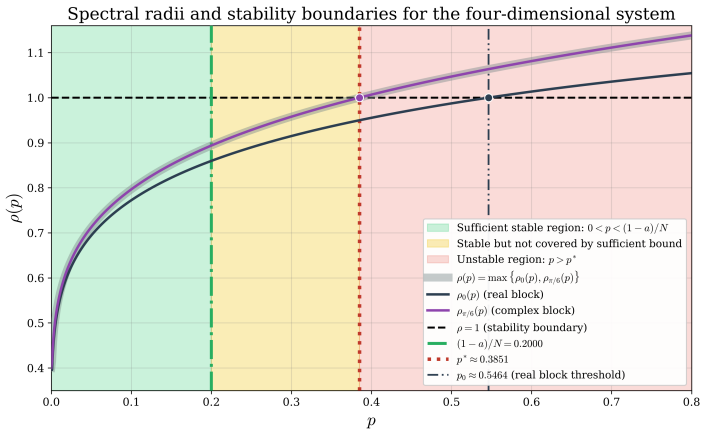


Figure 5. Spectral radii and stability boundaries for the four-dimensional non-diagonalizable system.

5.3. Application to a nonlinear population model with distributed delays

In this subsection, we consider a nonlinear discrete-time population model to illustrate the theoretical results obtained above. The local asymptotic stability of its positive equilibrium is examined through linearization.

For a population observed at discrete time intervals, the current population level may depend not only on its present state but also on population levels recorded at earlier times. Such dependence naturally leads to a model with delayed terms.

Let A_n denote the adult population density at time n, and let $A^* > 0$ be a positive equilibrium density. For $0 < a < 1$ and $S > 0$, consider the following nonlinear discrete-time model:

$$A_{n+1} = A^* + a(A_n - A^*) - \frac{S}{N}\sum_{i=1}^{N}[B(A_{n-i}) - B(A^*)]\,. \tag{5.2}$$

Motivated by the rational fitness-function formulations of discrete-time population models with Allee effects developed by Elaydi and Sacker [1], we take the delayed feedback function in the

following Allee-type saturating form:

$$B(A) = \frac{rA^2}{1 + \left(\frac{A}{K}\right)^2}, \qquad r > 0, \quad K > 0. \tag{5.3}$$

Here, r is a positive coefficient determining the strength of the feedback effect, while K represents the population density scale. Since $B(0) = 0, B'(0) = 0$, the feedback effect is weak at low population densities. Moreover, $B(A)$ is increasing for $A > 0$ and approaches a saturation level as A becomes large. Hence, $B(A)$ represents a density-dependent Allee-type regulatory feedback with saturation. Here, the Allee-type response is incorporated into a distributed-delay feedback term so that the dynamics near the positive equilibrium lead to the multiple-delay linear equation studied in this paper.

We first show that A^* is an equilibrium point of (5.2). If $A_n = A^*$, then all delayed terms also satisfy $A_{n-i} = A^*$ for $i = 1, 2, \ldots, N$. Therefore,

$$B(A_{n-i}) - B(A^*) = 0,$$

and since $A_n - A^* = 0$, the right-hand side of (5.2) becomes

$$A^* + a \cdot 0 - \frac{S}{N} \sum_{i=1}^{N} 0 = A^*.$$

Thus, $A_{n+1} = A^*$, and hence $A^* > 0$ is a positive equilibrium point of the model.

We now analyze the local behavior around this positive equilibrium. Define

$$u_n = A_n - A^*.$$

Here, u_n denotes the perturbation of the population density from the positive equilibrium at time n. Hence,

$$A_n = A^* + u_n.$$

Substituting this relation into (5.2) gives

$$A^* + u_{n+1} = A^* + au_n - \frac{S}{N} \sum_{i=1}^{N} \left[B(A^* + u_{n-i}) - B(A^*)\right].$$

After subtracting A^* from both sides, we obtain

$$u_{n+1} = au_n - \frac{S}{N} \sum_{i=1}^{N} \left[B(A^* + u_{n-i}) - B(A^*)\right]. \tag{5.4}$$

Next, we expand $B(A^* + u_{n-i})$ in a sufficiently small neighborhood of the equilibrium point A^*. Assuming that B is sufficiently smooth near A^*, Taylor's expansion yields

$$B(A^* + u_{n-i}) = B(A^*) + B'(A^*)u_{n-i} + \frac{1}{2}B''(A^*)u_{n-i}^2 + O(u_{n-i}^3).$$

In local stability analysis, only first-order terms are retained, while second- and higher-order terms are neglected. Hence,

$$B(A^* + u_{n-i}) - B(A^*) \approx B'(A^*)u_{n-i}. \tag{5.5}$$

This approximation means that, when the population is sufficiently close to its equilibrium level, the change in the nonlinear feedback function is approximately equal to the product of the perturbation and the derivative of the feedback function at the equilibrium point. Therefore, $B'(A^*)$ measures the local sensitivity of the feedback mechanism to small deviations from equilibrium.

Substituting (5.5) into (5.4), we obtain

$$u_{n+1} = au_n - \frac{S}{N}\sum_{i=1}^{N} B'(A^*)u_{n-i}.$$

Since $B'(A^*)$ is independent of the summation index, it can be taken outside the summation:

$$u_{n+1} = au_n - \frac{S}{N}B'(A^*)\sum_{i=1}^{N} u_{n-i}. \tag{5.6}$$

Thus, the nonlinear perturbation equation is reduced, in a neighborhood of the equilibrium point A^*, to a linear delay difference equation. Define

$$p = \frac{S}{N}B'(A^*),$$

and then (5.6) becomes

$$u_{n+1} = au_n - p\sum_{i=1}^{N} u_{n-i}.$$

Equivalently, we can write

$$u_{n+1} - au_n + p\sum_{i=1}^{N} u_{n-i} = 0. \tag{5.7}$$

Equation (5.7) corresponds to the scalar case of the linear difference system considered in (1.1), with $u_n = x_n$ and $B = p$. In particular, for consecutive delays $k_j = j, j = 1, 2, \ldots, N$, we have $k_1 = 1, d = 1, k_N = N$.

Thus, the characteristic equation of (5.7) is obtained from the main characteristic equation

$$\lambda^{k_N+1} - a\lambda^{k_N} + pT(\lambda)e^{i\theta} = 0, \tag{5.8}$$

by taking $k_1 = 1$, $d = 1$, $k_N = N$, and $\theta = 0$. In this case,

$$T(\lambda) = 1 + \lambda + \lambda^2 + \cdots + \lambda^{N-1},$$

and (5.8) reduces to

$$\lambda^{N+1} - a\lambda^N + p\left(1 + \lambda + \lambda^2 + \cdots + \lambda^{N-1}\right) = 0. \tag{5.9}$$

By the linearization principle for discrete-time dynamical systems [21], if all eigenvalues of the Jacobian matrix at an equilibrium lie strictly inside the open unit disk, then the equilibrium is locally

asymptotically stable. Conversely, if at least one eigenvalue lies outside the closed unit disk, the equilibrium is unstable. Since the delayed system can be represented as a finite-dimensional first-order system by introducing the delayed state variables, this principle applies to the present nonlinear population model. When a characteristic root lies on the unit circle, linearization alone does not determine the local stability of the nonlinear equilibrium [21]. At the critical value $p = p^*$, higher-order nonlinear terms may influence the stability of the equilibrium, and their effects are not determined by the present linear analysis.

Therefore, if all roots of (5.9) lie in the open unit disk, the positive equilibrium A^* of the nonlinear population model is locally asymptotically stable. Equivalently, if all characteristic roots satisfy $|\lambda| < 1$, then $u_n \to 0$. Since $u_n = A_n - A^*$, this implies $A_n \to A^*$, which means that the adult population returns to its positive equilibrium density after sufficiently small environmental or demographic perturbations.

For the population model, p^* gives the limiting value of the delayed feedback coefficient for local asymptotic stability of the positive equilibrium. The delayed feedback coefficient p represents the total regulatory influence of past population densities on the present adult population density. When this effect is sufficiently small, the system returns to equilibrium. However, once the delayed response exceeds a critical level, it may destabilize the equilibrium and generate oscillatory behavior.

Consider an adult population monitored at seasonal intervals. In this setting, the population at the current time may depend not only on the present adult density but also on population levels observed in earlier periods. High densities in previous seasons, for instance, may affect the current population through limited food availability, increased pressure on the habitat, lower juvenile survival, or harvesting. Such delayed effects provide a natural motivation for including multiple delays in the population model.

For this application, let $a = 0.6, N = 3$.

Here, $a = 0.6$ represents the period-to-period persistence effect of the adult population around equilibrium, while $N = 3$ means that the previous three periods influence the current population change. The linearized perturbation equation then becomes

$$u_{n+1} - 0.6u_n + p(u_{n-1} + u_{n-2} + u_{n-3}) = 0.$$

The corresponding characteristic equation is

$$\lambda^4 - 0.6\lambda^3 + p(\lambda^2 + \lambda + 1) = 0.$$

The simple sufficient stability bound is

$$\frac{1-a}{N} = \frac{0.4}{3} = 0.1333333333.$$

The sharp stability threshold, computed as the first positive value of p for which a characteristic root reaches the unit circle, is

$$p^* \approx 0.3177270635.$$

For this example, $(1-a)/N = 0.1333333333$ gives a sufficient stability bound, whereas the computed threshold $p^* \approx 0.3177270635$ gives the actual stability limit of the positive equilibrium.

To illustrate three qualitatively different feedback regimes, choose $p_1 = 0.08, p_2 = 0.22, p_3 = 0.36$.

The value p_1 corresponds to weak delayed feedback and lies in the safe stability region. The value p_2 represents a stronger delayed effect that exceeds the simple sufficient bound but remains below the

true critical threshold p^*. The value p_3 exceeds the critical threshold and therefore corresponds to a destabilizing feedback regime. For the application model, define the spectral radius by

$$\rho(p) = \max\{|\lambda| : \ f(\lambda, p) = 0\},$$

where

$$f(\lambda, p) = \lambda^4 - 0.6\lambda^3 + p(1 + \lambda + \lambda^2).$$

Accordingly,

$$\rho(p) = \max\left\{|\lambda| : \lambda^4 - 0.6\lambda^3 + p(1 + \lambda + \lambda^2) = 0\right\}.$$

Substitution of p_1, p_2, and p_3 gives

$$\begin{aligned}\rho(p_1) &= 0.7068969250,\\ \rho(p_2) &= 0.9082833993,\\ \rho(p_3) &= 1.0343867442.\end{aligned}$$

Hence, for p_1 and p_2, all characteristic roots remain inside the open unit disk, and the positive equilibrium is locally asymptotically stable. In contrast, for p_3, the spectral radius is greater than one; consequently, at least one characteristic root lies outside the unit circle, and the positive equilibrium loses stability.

For $p_2 = 0.22$, the value lies above the sufficient bound $(1 - a)/N$, but the system remains stable since $p_2 < p^*$. Hence, $(1 - a)/N$ provides only a sufficient condition and does not characterize the full stability region. The threshold p^* determines the stability limit of the population equilibrium. This case is shown in Figure 6.

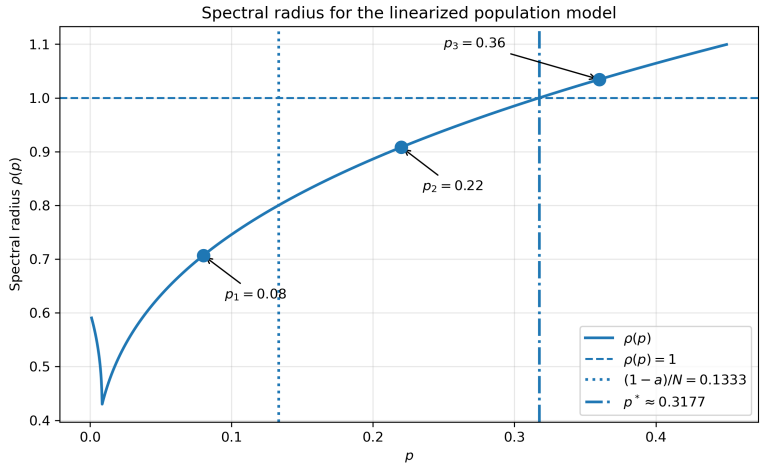


Figure 6. Spectral radius of the linearized population model.

Figure 7 shows the locations of the characteristic roots in the complex plane. The unit circle acts as the stability boundary for local asymptotic stability. For p_1 and p_2, all roots remain inside the unit circle, whereas for p_3, a pair of roots moves outside the unit circle. This indicates that the positive equilibrium loses stability once the delayed feedback exceeds the critical threshold.

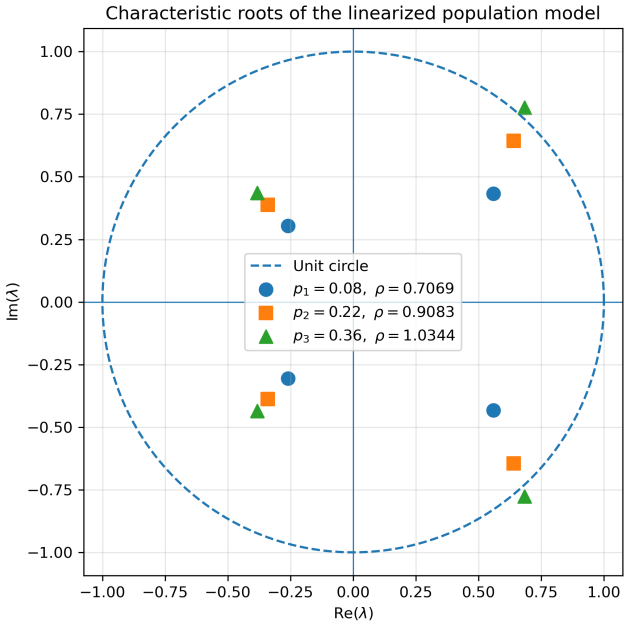


Figure 7. Characteristic roots for selected feedback levels.

Finally, Figure 8 presents the time evolution of the linearized perturbation equation. In the simulation, the initial history is chosen as $u_{-3} = 0.08, u_{-2} = -0.04, u_{-1} = 0.06, u_0 = 0.05$.

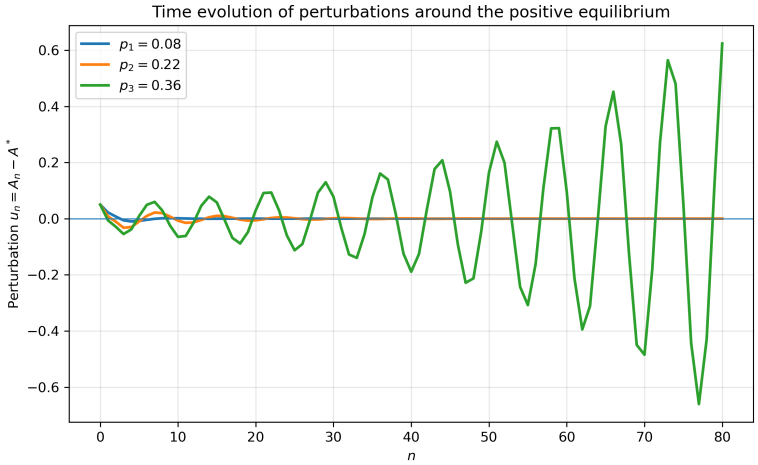


Figure 8. Perturbation dynamics around the positive equilibrium.

For p_1 and p_2, the perturbations tend to zero as time increases. This implies that $A_n \to A^*$, and hence the population returns to its equilibrium level. In contrast, for p_3, the perturbations do not decay, which indicates the instability of the positive equilibrium.

This application gives a direct interpretation of the stability threshold p^* in population terms. The coefficient p measures the effect of past population densities on the current adult population. If $0 < p < p^*$, small perturbations around the equilibrium decay and the population returns to its equilibrium level. When p crosses p^*, the equilibrium loses its local asymptotic stability.

Thus, p^* separates the locally stable regime from the parameter range in which the positive equilibrium is no longer locally asymptotically stable.

6. Conclusions

In this study, the asymptotic stability of a linear system of difference equations with N delays in arithmetic progression is investigated. By reducing the coefficient matrix to Jordan form, the stability problem is expressed in terms of the characteristic equations associated with the corresponding Jordan blocks. The analysis of the characteristic roots on the unit circle then yields the critical parameter values at which stability may be lost.

The system considered here extends the arithmetic multi-delay setting of Nagabuchi [18] by including the parameter $a \in (0, 1)$, which had previously been studied in the two-delay case. The presence of a changes the characteristic equation and, in particular, the determination of the arguments of the unit-circle roots. This leads to a family of critical values from which the first positive crossing value p^* is obtained.

The main stability result is the necessary and sufficient condition:

$$0 < p < p^*.$$

Thus, p^* gives the exact boundary for asymptotic stability. In addition,

$$p^* \geq \frac{1-a}{N},$$

and hence

$$0 < p < \frac{1-a}{N}$$

provides a sufficient condition that can be checked without computing the critical values explicitly. The numerical examples show that this sufficient bound may be strictly smaller than the actual stability interval, since stability can persist for:

$$\frac{1-a}{N} < p < p^*.$$

The nonlinear population model considered in Section 5.3 provides an application of these results. After linearization around the positive equilibrium, the perturbation equation has the same multiple-delay structure studied in the theoretical part of the paper. Accordingly, p^* determines the limiting value of the delayed feedback coefficient for local asymptotic stability of the positive equilibrium.

The arithmetic structure of the delays allows the characteristic equation to be analyzed through a finite delay polynomial, while the extension to higher-dimensional systems is achieved through the eigenvalues of the coefficient matrix. Nevertheless, the analysis is restricted to linear systems with arithmetic delays and $a \in (0, 1)$. Furthermore, determining the exact critical parameter values may require numerical computation. When applied to nonlinear systems, the resulting stability conclusions are local and do not establish global stability. Future research may consider non-arithmetic delay structures, broader parameter ranges, and nonlinear discrete-time systems.

Author contributions

Serbun Ufuk Değer: Conceptualization, Methodology, Formal analysis, Investigation, Software, Validation, Visualization, Writing–original draft, Writing–review and editing. The author has read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares that there is no conflict of interest.

References

1. S. N. Elaydi, R. J. Sacker, Population models with Allee effect: A new model, *J. Biol. Dyn.*, **4** (2010), 397–408. https://doi.org/10.1080/17513750903377434
2. S. A. Levin, R. M. May, A note on difference-delay equations, *Theor. Popul. Biol.*, **9** (1976), 178–187. https://doi.org/10.1016/0040-5809(76)90043-5
3. C. W. Clark, A delayed-recruitment model of population dynamics, with an application to baleen whale populations, *J. Math. Biol.*, **3** (1976), 381–391.
4. S. A. Kuruklis, The asymptotic stability of $x_{n+1} - ax_n + bx_{n-k} = 0$, *J. Math. Anal. Appl.*, **188** (1994), 719–731. https://doi.org/10.1006/jmaa.1994.1457
5. H. Matsunaga, T. Hara, The asymptotic stability of a two-dimensional linear delay difference equation, *Dyn. Contin. Discrete Impuls. Syst.*, **6** (1999), 465–473.
6. H. Matsunaga, Stability regions for a class of delay difference systems, In: *Difference and Differential Equations*, Providence: American Mathematical Society, **42** (2004), 273–283.
7. S. U. Değer, Y. Bolat, Stability conditions for a class of linear delay difference systems, *Cogent Math.*, **4** (2017), 1294445. https://doi.org/10.1080/23311835.2017.1294445
8. R. Ogita, H. Matsunaga, T. Hara, Asymptotic stability condition for a class of linear delay difference equations of higher order, *J. Math. Anal. Appl.*, **248** (2000), 83–96. https://doi.org/10.1006/jmaa.2000.6868
9. Y. Nagabuchi, Asymptotic stability for a linear difference system with two delays, *Nonlinear Anal.-Theory Methods Appl.*, **47** (2001), 4117–4128. https://doi.org/10.1016/S0362-546X(01)00529-6
10. M. M. Kipnis, V. V. Malygina, The stability cone for a matrix delay difference equation, *IJMMS*, **2011** (2011), 860326. https://doi.org/10.1155/2011/860326
11. S. A. Ivanov, M. M. Kipnis, V. V. Malygina, The stability cone for a difference matrix equation with two delays, *ISRN Appl. Math.*, **2011** (2011), 910936. https://doi.org/10.5402/2011/910936

12. J. Čermák, J. Jánský, Stability and periodic investigations of linear planar difference systems, *Math. Methods Appl. Sci.*, **39** (2016), 5343–5354. https://doi.org/10.1002/mma.3919

13. P. He, H. Zhang, S. F. Su, A sliding mode control method with variable convergence rate for nonlinear impulsive stochastic systems, *IEEE Trans. Cybern.*, **55** (2025), 2213–2222. https://doi.org/10.1109/TCYB.2025.3551668

14. H. Geng, H. Zhang, S. F. Su, Asynchronously switched control with variable convergence rate for switched nonlinear systems: A persistent Dwell-time scheme, *IEEE Trans. Fuzzy Syst.*, **32** (2024), 6695–6707. https://doi.org/10.1109/TFUZZ.2024.3459860

15. A. Wu, H. Zhang, J. Xia, Interval stabilization for switched stochastic positive systems: A statedependent switching scheme With convergence rate constraint, *IEEE Trans. Consumer Elect.*, **72** (2026), 3456–3466. https://doi.org/10.1109/TCE.2026.3683018

16. H. Liu, D. Tong, Q. Chen, W. Zhou, Adaptive predefined-time intralayer and interlayer synchronization of multilayer networks, *IEEE Trans. Syst. Man Cybern. Syst.*, **56** (2026), 6467–6478. https://doi.org/10.1109/TSMC.2026.3713217

17. X. Ye, Q. Chen, D. Tong, Event-triggered exponential synchronization for delayed multi-agent systems with privacy-preserving, *Commun. Nonlinear Sci. Numer. Simul.*, **162** (2026), 110444. https://doi.org/10.1016/j.cnsns.2026.110444

18. Y. Nagabuchi, Asymptotic behaviors of a linear difference system with multiple delays, *Funkc. Ekvacioj*, **46** (2003), 89–119. https://doi.org/10.1619/fesi.46.89

19. S. U. Değer, Y. Bolat, Stability conditions for linear difference system with two delays, *Azerbaijan J. Math.*, **11** (2021), 11–27.

20. L. De Carli, A. Echezabal, M. Laporta, Complete homogeneous symmetric polynomials and generalized Vieta's formulas, *Results Math.*, **81** (2026), 81. https://doi.org/10.1007/s00025-026-02632-5

21. S. N. Elaydi, *An Introduction to Difference Equations*, 3 Eds, New York: Springer-Verlag, 2005.



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0)