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Research article

A Riemann–Liouville extension of the Sheffer–Kantorovich framework

Gui-Ying Weng[1], **Edmond Aliaga**[2,*] **and Reşat Aslan**[3]

[1] Department of Mathematics, School of Computer Science and Engineering, Yang-En University, Quanzhou 362014, Fujian, China

[2] Department of Mathematics, University of Prishtina, Prishtina 10000, Kosovo

[3] Department of Mathematics, Van Yüzüncü Yıl University, Van 65080, Türkiye

* **Correspondence:** Email: edmond.aliaga@uni-pr.edu; Tel: +38349110803.

Abstract: We introduce a Riemann–Liouville fractional extension of Kantorovich-type Szász operators generated by multiple Sheffer polynomials. The construction contains the classical Kantorovich form as the particular case $\mu = 1$ and provides a fractional integral modification of the corresponding discrete operators. Algebraic and central moments up to the fourth order are derived directly from the generating structure of the multiple Sheffer polynomials. These identities are subsequently employed to establish local convergence and estimates involving first- and second-order moduli of continuity. The asymptotic behavior of the operators is described by Voronovskaya- and Grüss–Voronovskaya-type formulas. Relations between the discrete, Kantorovich, and fractional constructions are also examined. A numerical illustration complements the theoretical results and demonstrates the influence of the fractional parameter on the approximation process.

Keywords: multiple Sheffer polynomials; Riemann–Liouville fractional integral; Kantorovich-type operators; Voronovskaya-type theorem
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1. Introduction

Positive linear operators constitute an important tool in approximation theory and have been studied through numerous modifications designed to extend their applicability and improve the description of their approximation behavior. A classical example is given by the Szász operators, introduced for the approximation of functions on unbounded intervals [1]. Their subsequent generalizations have been developed in several directions, including constructions based on special polynomial systems. In particular, Jakimovski and Leviatan [2] introduced a generalization involving Appell polynomials, while later investigations extended this approach to Sheffer-type polynomial structures [3, 4]. The

algebraic properties of Sheffer polynomial sequences and related generalizations have also been studied extensively; see, for example, [5–7].

Multiple polynomial systems provide a further extension of this framework. Approximation operators associated with multiple Appell polynomials were considered in [8], whereas Ali and Paris [9] introduced Szász-type operators generated by multiple Sheffer polynomials and investigated their approximation properties. A Kantorovich modification of these operators was subsequently studied by Braha and Mansour [10]. Related Kantorovich-type Szász operators involving Sheffer polynomials have also been considered in [11]. Related convergence and quantitative approximation results for Bernstein–Kantorovich–Stancu operators were investigated in [12]. These developments provide a natural setting for studying integral modifications of multiple Sheffer-based approximation processes. Various extensions and integral modifications have subsequently been introduced in order to enlarge the flexibility of the underlying operators and to study their local and global approximation behavior; see, for instance, [13–21]. Recent developments also include the Baskakov–Schurer–Stancu-type operators studied by Odabaşı et al. [22], who investigated weighted and pointwise approximation properties. Also, Rao et al. [23] presented a new sequence of positive linear operators by Chlodowsky variant of Szász-integral type in terms of general Appell polynomials and proposed some direct approximation results via modulus of continuity of first and second order, Peetre's K-functional, and Lipschitz type class.

Fractional calculus offers a broader integral framework in which classical averaging procedures can be generalized through kernels of noninteger order. Among its fundamental constructions, the Riemann–Liouville fractional integral has played a central role in both the theory and applications of fractional operators; standard references include [24–26]. Motivated by this framework, the present paper introduces a Riemann–Liouville integral extension of the Kantorovich-type multiple Sheffer operators. The normalization is chosen so that the Kantorovich operator is recovered exactly when the fractional order is $\mu = 1$, thereby retaining a direct connection with the previously established construction.

The analysis begins with the derivation of algebraic and central moments required for the fractional setting. These are used to obtain local approximation estimates and asymptotic formulas of Voronovskaya and Grüss–Voronovskaya type. The relations among the discrete, Kantorovich, and Riemann–Liouville constructions are then examined, followed by a numerical comparison illustrating the role of the fractional parameter. The paper is organized accordingly: Section 2 develops the fractional framework and its moment structure; Sections 3 and 4 contain the approximation and asymptotic results, respectively; Section 5 discusses the relations among the three operator families; Section 6 presents the numerical illustration; and Section 7 concludes the paper.

2. Riemann–Liouville integral framework and moment structure

This section develops the integral framework underlying the approximation process considered in the sequel. We begin with the generating structure of the multiple Sheffer polynomial family and briefly recall the associated discrete operator. A normalized Riemann–Liouville weighted mean is then introduced at each sampling interval, giving rise to a fractional integral family that contains the corresponding Kantorovich-type operator as a particular member. The moment structure of the resulting operators is subsequently investigated in a form suitable for the approximation and asymptotic

results developed later.

Let $\{S_{k_1,k_2}(x)\}_{k_1,k_2\geq 0}$ be a family of multiple Sheffer polynomials generated by

$$A(t_1,t_2)\exp(xH(t_1,t_2)) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} S_{k_1,k_2}(x)\frac{t_1^{k_1}t_2^{k_2}}{k_1!k_2!}, \tag{2.1}$$

where A and H are analytic in a neighborhood of $(1,1)$ and satisfy $A(1,1)\neq 0$ and $H_{t_1}(1,1) = H_{t_2}(1,1) = 1$.

In addition, we assume that $S_{k_1,k_2}(x)\geq 0, x\geq 0, k_1,k_2\in\mathbb{N}_0$, so that the operators considered below are positive.

A sufficient condition for this nonnegativity is that both $A(t_1,t_2)$ and $H(t_1,t_2)$ admit power-series expansions with nonnegative coefficients in a common domain of convergence. Indeed, for $x\geq 0$, the coefficients of $A(t_1,t_2)e^{xH(t_1,t_2)}$ are then nonnegative. For instance, the choice $A(t_1,t_2) = 1$ and $H(t_1,t_2) = t_1+t_2$ yields $S_{k_1,k_2}(x) = x^{k_1+k_2}\geq 0$, which is precisely the specialization used in Section 6.

Associated with (2.1), the discrete Szász-type operators are given by

$$\mathcal{G}_n(f;x) = \frac{\exp\left\{-\frac{nx}{2}H(1,1)\right\}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{S_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!}f\left(\frac{k_1+k_2}{n}\right), \qquad x\geq 0. \tag{2.2}$$

For brevity, throughout this section we write

$$k = k_1+k_2.$$

Thus, the sampling point appearing in (2.2) is k/n.

We now replace the point evaluation at k/n by a normalized fractional weighted mean over the adjacent interval

$$\left[\frac{k}{n},\frac{k+1}{n}\right].$$

This leads to the following construction.

Definition 2.1. *Let $\mu>0$, $n\in\mathbb{N}$, and $x\geq 0$. For a function f for which the expression below exists, define*

$$\begin{aligned}\mathcal{R}_{n,\mu}(f;x) = {} & \frac{\exp\left\{-\frac{nx}{2}H(1,1)\right\}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{S_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!}\\ & \times\mu n^{\mu}\int_{\frac{k_1+k_2}{n}}^{\frac{k_1+k_2+1}{n}}\left(\frac{k_1+k_2+1}{n}-t\right)^{\mu-1}f(t)\,dt.\end{aligned} \tag{2.3}$$

The normalization in (2.3) is chosen so that constant functions are preserved. Indeed,

$$\mu n^{\mu}\int_{\frac{k}{n}}^{\frac{k+1}{n}}\left(\frac{k+1}{n}-t\right)^{\mu-1}dt = 1.$$

Consequently,

$$\mathcal{R}_{n,\mu}(e_0; x) = 1, \qquad e_0(t) = 1.$$

The connection with the usual Kantorovich construction follows directly from the definition. In fact, when $\mu = 1$, the fractional kernel becomes constant and

$$\mathcal{R}_{n,1}(f; x) = \frac{n \exp\left\{-\frac{nx}{2}H(1,1)\right\}}{A(1,1)} \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{S_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!} \int_{\frac{k_1+k_2}{n}}^{\frac{k_1+k_2+1}{n}} f(t)\,dt.$$

Hence, $\mathcal{R}_{n,1}$ coincides with the corresponding Kantorovich-type operator associated with the same multiple Sheffer polynomial family.

Remark 2.1. *The normalization of the fractional kernel ensures that $\mathcal{R}_{n,\mu}(e_0; x) = 1$ for every $\mu > 0$, so the proposed operators preserve constant functions. Moreover, when $\mu = 1$, the fractional weighted mean reduces exactly to the ordinary Kantorovich integral, and hence $\mathcal{R}_{n,1} = K_n^{(S)}$. These two properties ensure consistency with the classical Kantorovich construction while allowing the averaging procedure to vary with the fractional parameter μ.*

To analyze the moments of (2.3), it is useful to separate the fractional integral component from the polynomial weights. Making the change of variables

$$t = \frac{k+s}{n}, \qquad 0 \le s \le 1,$$

we obtain

$$\mu n^{\mu} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \left(\frac{k+1}{n} - t\right)^{\mu-1} f(t)\,dt = \mu \int_0^1 (1-s)^{\mu-1} f\left(\frac{k+s}{n}\right) ds. \tag{2.4}$$

For $r \in \mathbb{N}_0$, put

$$c_r(\mu) := \mu \int_0^1 s^r (1-s)^{\mu-1}\,ds.$$

Using the beta integral, one obtains

$$c_r(\mu) = \frac{r!\,\Gamma(\mu+1)}{\Gamma(\mu+r+1)} = \frac{r!}{(\mu+1)(\mu+2)\cdots(\mu+r)}, \qquad r \ge 1,$$

whereas $c_0(\mu) = 1$.

In particular,

$$c_1(\mu) = \frac{1}{\mu+1}, \qquad c_2(\mu) = \frac{2}{(\mu+1)(\mu+2)},$$

and

$$c_3(\mu) = \frac{6}{(\mu+1)(\mu+2)(\mu+3)}, \qquad c_4(\mu) = \frac{24}{(\mu+1)(\mu+2)(\mu+3)(\mu+4)}.$$

For $\mu = 1$, these coefficients reduce to

$$c_r(1) = \frac{1}{r+1}, \qquad r \in \mathbb{N}_0,$$

which is precisely the sequence of coefficients arising from the ordinary uniform Kantorovich mean.

Let $e_j(t) = t^j$, $j \in \mathbb{N}_0$. From (2.4), we have

$$\mu \int_0^1 (1-s)^{\mu-1} \left(\frac{k+s}{n}\right)^j ds = \sum_{r=0}^{j} \binom{j}{r} \frac{k^{j-r}}{n^j} c_r(\mu). \tag{2.5}$$

Consequently, the algebraic moments of the Riemann–Liouville family can be expressed directly through those of the underlying discrete operator.

Lemma 2.1. *Based on the above conditions, we have*

$$\boxed{\mathcal{R}_{n,\mu}(e_j; x) = \sum_{r=0}^{j} \binom{j}{r} \frac{c_r(\mu)}{n^r} \mathcal{G}_n(e_{j-r}; x)} \tag{2.6}$$

for every $x \geq 0$ and $\mu > 0$.

Proof. Substituting (2.5) into (2.3) and interchanging the finite sum with the series, we obtain

$$\mathcal{R}_{n,\mu}(e_j; x) = \sum_{r=0}^{j} \binom{j}{r} c_r(\mu) \frac{1}{n^r} \frac{\exp\left\{-\frac{nx}{2} H(1,1)\right\}}{A(1,1)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{S_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1! k_2!} \left(\frac{k_1+k_2}{n}\right)^{j-r}.$$

The double series in the last expression is exactly $\mathcal{G}_n(e_{j-r}; x)$, which proves (2.6). □

For the first five test functions, relation (2.6) gives

(i) $\mathcal{R}_{n,\mu}(e_0; x) = 1$,

(ii) $\mathcal{R}_{n,\mu}(e_1; x) = \mathcal{G}_n(e_1; x) + \dfrac{1}{(\mu+1)n}$,

(iii) $\mathcal{R}_{n,\mu}(e_2; x) = \mathcal{G}_n(e_2; x) + \dfrac{2}{(\mu+1)n} \mathcal{G}_n(e_1; x) + \dfrac{2}{(\mu+1)(\mu+2)n^2}$,

(iv) $\mathcal{R}_{n,\mu}(e_3; x) = \mathcal{G}_n(e_3; x) + \dfrac{3}{(\mu+1)n} \mathcal{G}_n(e_2; x) + \dfrac{6}{(\mu+1)(\mu+2)n^2} \mathcal{G}_n(e_1; x) + \dfrac{6}{(\mu+1)(\mu+2)(\mu+3)n^3}$,

(v) $\mathcal{R}_{n,\mu}(e_4; x) = \mathcal{G}_n(e_4; x) + \dfrac{4}{(\mu+1)n} \mathcal{G}_n(e_3; x) + \dfrac{12}{(\mu+1)(\mu+2)n^2} \mathcal{G}_n(e_2; x) + \dfrac{24}{(\mu+1)(\mu+2)(\mu+3)n^3} \mathcal{G}_n(e_1; x) + \dfrac{24}{(\mu+1)(\mu+2)(\mu+3)(\mu+4)n^4}$.

Remark 2.2. *For completeness and in order to obtain a self-contained basis for the fractional construction, all moment identities required in the sequel are derived here directly from the generating relation and the integral definition of the operators, rather than being transferred from previously established Kantorovich-type formulas. This independent computation leads, in a few terms, to expressions that differ slightly from some of the corresponding explicit moment formulas reported in [10]. The moment relations obtained here will therefore be used throughout the subsequent approximation and asymptotic analysis.*

For a more compact representation of the moment formulas, introduce the diagonal Euler differential operator

$$\mathcal{D} := t_1 \frac{\partial}{\partial t_1} + t_2 \frac{\partial}{\partial t_2}. \tag{2.7}$$

We shall use the abbreviations

$$a_j := \frac{\mathcal{D}^j A(t_1, t_2)\big|_{(1,1)}}{A(1,1)}, \qquad h_j := \mathcal{D}^j H(t_1, t_2)\big|_{(1,1)}, \qquad j \geq 1. \tag{2.8}$$

In view of $H_{t_1}(1,1) = H_{t_2}(1,1) = 1$, the definition of $\mathcal{D}$ gives $h_1 = (\mathcal{D}H)(1,1) = H_{t_1}(1,1) + H_{t_2}(1,1) = 2$.

The notation in (2.8) also simplifies the higher-order algebraic and central moment formulas, avoiding lengthy expressions involving partial derivatives of the generating functions. The advantage of the notation in (2.8) is that the powers of $k_1 + k_2$ occurring in the moment calculations can be generated directly by successive applications of $\mathcal{D}$ to (2.1). More precisely,

$$\mathcal{D}^j \left[A(t_1, t_2) \exp\left\{ \frac{nx}{2} H(t_1, t_2) \right\} \right] \Big|_{(1,1)} = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} (k_1 + k_2)^j \frac{S_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1! k_2!}. \tag{2.9}$$

Using (2.9), the first algebraic moments of the discrete operators can be written in the following convenient form.

Lemma 2.2. *Let $e_j(t) = t^j$, $j = 0, 1, 2, 3, 4$. Then, the operators $\mathcal{G}_n$ defined by* (2.2) *satisfy*

(i) $$\mathcal{G}_n(e_0; x) = 1, \tag{2.10}$$

(ii) $$\mathcal{G}_n(e_1; x) = x + \frac{a_1}{n}, \tag{2.11}$$

(iii) $$\mathcal{G}_n(e_2; x) = x^2 + \frac{x}{n}\left(2a_1 + \frac{h_2}{2}\right) + \frac{a_2}{n^2}, \tag{2.12}$$

(iv) $$\mathcal{G}_n(e_3; x) = x^3 + \frac{x^2}{n}\left(3a_1 + \frac{3h_2}{2}\right) + \frac{x}{n^2}\left(\frac{3}{2}a_1 h_2 + 3a_2 + \frac{h_3}{2}\right) + \frac{a_3}{n^3}, \tag{2.13}$$

(v) $$\mathcal{G}_n(e_4; x) = x^4 + \frac{x^3}{n}(4a_1 + 3h_2) + \frac{x^2}{n^2}\left(6a_1 h_2 + 6a_2 + 2h_3 + \frac{3}{4}h_2^2\right) + \frac{x}{n^3}\left(2a_1 h_3 + 3a_2 h_2 + 4a_3 + \frac{h_4}{2}\right) + \frac{a_4}{n^4}. \tag{2.14}$$

Proof. The identities follow by successively applying the operator $\mathcal{D}$ in (2.7) to the generating relation (2.1), evaluating at $(t_1, t_2) = (1, 1)$, and then using $h_1 = 2$. For instance,

$$\mathcal{D}\left(Ae^{\frac{nx}{2}H}\right) = \left(\mathcal{D}A + \frac{nx}{2} A\, \mathcal{D}H\right) e^{\frac{nx}{2}H},$$

which, after evaluation at $(1, 1)$ and normalization by $A(1,1)e^{\frac{nx}{2}H(1,1)}$, gives (2.11). Repeating the same procedure up to the fourth order yields (2.12)–(2.14). □

We next turn to the centered moments. For $r \in \mathbb{N}_0$, define

$$\delta_{n,r}(x) := \mathcal{G}_n((t-x)^r; x)$$

and

$$\Delta_{n,r}^{(\mu)}(x) := \mathcal{R}_{n,\mu}((t-x)^r; x).$$

The discrete centered moments corresponding to (2.10)–(2.14) take a particularly simple form.

Lemma 2.3. *For $x \geq 0$, the first four central moments of $\mathcal{G}_n$ are*

$$\textit{(i)} \quad \delta_{n,1}(x) = \frac{a_1}{n}, \tag{2.15}$$

$$\textit{(ii)} \quad \delta_{n,2}(x) = \frac{h_2 x}{2n} + \frac{a_2}{n^2}, \tag{2.16}$$

$$\textit{(iii)} \quad \delta_{n,3}(x) = \frac{x}{2n^2}(h_3 + 3a_1 h_2) + \frac{a_3}{n^3}, \tag{2.17}$$

$$\textit{(iv)} \quad \delta_{n,4}(x) = \frac{3h_2^2 x^2}{4n^2} + \frac{x}{n^3}\left(2a_1 h_3 + 3a_2 h_2 + \frac{h_4}{2}\right) + \frac{a_4}{n^4}. \tag{2.18}$$

Proof. The formulas follow from the identities

$$\delta_{n,r}(x) = \sum_{j=0}^{r} \binom{r}{j} (-x)^{r-j} \mathcal{G}_n(e_j; x),$$

together with (2.10)–(2.14). □

The fractional averaging mechanism produces a simple relation between the centered moments of the discrete and integral families.

Lemma 2.4. *For every $r \in \mathbb{N}_0$,*

$$\boxed{\Delta_{n,r}^{(\mu)}(x) = \sum_{j=0}^{r} \binom{r}{j} \frac{c_j(\mu)}{n^j} \delta_{n,r-j}(x)} \tag{2.19}$$

where $c_j(\mu)$ is defined by (2.5).

Proof. Under the substitution (2.4),

$$t - x = \left(\frac{k}{n} - x\right) + \frac{s}{n}.$$

Therefore,

$$(t-x)^r = \sum_{j=0}^{r} \binom{r}{j} \left(\frac{k}{n} - x\right)^{r-j} \frac{s^j}{n^j}.$$

Integrating with respect to the normalized fractional kernel and using (2.5) gives (2.19). □

In particular,

$$\text{(i)}\quad \Delta_{n,1}^{(\mu)}(x) = \delta_{n,1}(x) + \frac{c_1(\mu)}{n},$$

$$\text{(ii)}\quad \Delta_{n,2}^{(\mu)}(x) = \delta_{n,2}(x) + \frac{2c_1(\mu)}{n}\delta_{n,1}(x) + \frac{c_2(\mu)}{n^2},$$

$$\text{(iii)}\quad \Delta_{n,3}^{(\mu)}(x) = \delta_{n,3}(x) + \frac{3c_1(\mu)}{n}\delta_{n,2}(x) + \frac{3c_2(\mu)}{n^2}\delta_{n,1}(x) + \frac{c_3(\mu)}{n^3},$$

$$\begin{aligned}\text{(iv)}\quad \Delta_{n,4}^{(\mu)}(x) &= \delta_{n,4}(x) + \frac{4c_1(\mu)}{n}\delta_{n,3}(x) + \frac{6c_2(\mu)}{n^2}\delta_{n,2}(x)\\ &\quad + \frac{4c_3(\mu)}{n^3}\delta_{n,1}(x) + \frac{c_4(\mu)}{n^4}.\end{aligned}$$

Consequently,

$$\boxed{\Delta_{n,1}^{(\mu)}(x) = \frac{1}{n}\left(a_1 + \frac{1}{\mu+1}\right)}$$

and

$$\boxed{\Delta_{n,2}^{(\mu)}(x) = \frac{h_2 x}{2n} + \frac{1}{n^2}\left[a_2 + \frac{2a_1}{\mu+1} + \frac{2}{(\mu+1)(\mu+2)}\right].}$$

Furthermore,

$$\begin{aligned}\Delta_{n,3}^{(\mu)}(x) &= \frac{x}{2n^2}\left[h_3 + 3h_2\left(a_1 + \frac{1}{\mu+1}\right)\right] + \frac{1}{n^3}\left[a_3 + \frac{3a_2}{\mu+1} + \frac{6a_1}{(\mu+1)(\mu+2)}\right.\\ &\quad \left. + \frac{6}{(\mu+1)(\mu+2)(\mu+3)}\right].\end{aligned}$$

Finally,

$$\begin{aligned}\Delta_{n,4}^{(\mu)}(x) &= \frac{3h_2^2x^2}{4n^2} + \frac{x}{n^3}\left[2a_1h_3 + 3a_2h_2 + \frac{h_4}{2} + \frac{2h_3 + 6a_1h_2}{\mu+1} + \frac{6h_2}{(\mu+1)(\mu+2)}\right]\\ &\quad + \frac{1}{n^4}\left[a_4 + \frac{4a_3}{\mu+1} + \frac{12a_2}{(\mu+1)(\mu+2)} + \frac{24a_1}{(\mu+1)(\mu+2)(\mu+3)} + \frac{24}{(\mu+1)(\mu+2)(\mu+3)(\mu+4)}\right].\end{aligned}$$

The preceding identities immediately yield the limiting relations required in the subsequent asymptotic analysis.

Lemma 2.5. *For every fixed $x \geq 0$ and $\mu > 0$,*

$$\text{(i)}\quad \lim_{n\to\infty} n\Delta_{n,1}^{(\mu)}(x) = a_1 + \frac{1}{\mu+1}, \tag{2.20}$$

$$\text{(ii)}\quad \lim_{n\to\infty} n\Delta_{n,2}^{(\mu)}(x) = \frac{h_2}{2}x, \tag{2.21}$$

$$\text{(iii)}\quad \lim_{n\to\infty} n^2\Delta_{n,3}^{(\mu)}(x) = \frac{x}{2}\left[h_3 + 3h_2\left(a_1 + \frac{1}{\mu+1}\right)\right], \tag{2.22}$$

$$\text{(iv)}\quad \lim_{n\to\infty} n^2\Delta_{n,4}^{(\mu)}(x) = \frac{3h_2^2}{4}x^2. \tag{2.23}$$

Remark 2.3. *Relations* (2.20)–(2.23) *reveal that the Riemann–Liouville parameter enters the leading asymptotic behavior through the first central moment. The dominant terms of the second and fourth central moments remain unchanged relative to the underlying discrete family. In the special case* $\mu = 1$*, these relations reduce to those associated with the corresponding Kantorovich-type operator.*

3. Approximation properties of the Riemann–Liouville operators

The moment relations obtained in the preceding section provide the basis for studying the approximation behavior of $\mathcal{R}_{n,\mu}$. We first establish local uniform convergence and then derive direct estimates in terms of the first- and second-order moduli of smoothness.

Let

$$C_B(\mathbb{R}_+) = \{f : \mathbb{R}_+ \to \mathbb{R} : f \text{ is continuous and bounded}\},$$

endowed with the norm

$$\|f\|_\infty = \sup_{x\geq 0} |f(x)|.$$

We denote by $UC_B(\mathbb{R}_+)$ the subspace of all bounded and uniformly continuous functions on $\mathbb{R}_+$.

Since $\mathcal{R}_{n,\mu}$ is positive and preserves constants,

$$\left|\mathcal{R}_{n,\mu}(f;x)\right| \leq \mathcal{R}_{n,\mu}(|f|;x) \leq \|f\|_\infty, \qquad f \in C_B(\mathbb{R}_+).$$

For convenience, put

$$\eta_{n,\mu}(x) := \Delta^{(\mu)}_{n,2}(x) = \frac{h_2 x}{2n} + \frac{1}{n^2}\left[a_2 + \frac{2a_1}{\mu+1} + \frac{2}{(\mu+1)(\mu+2)}\right]. \tag{3.1}$$

Theorem 3.1. *Let* $\mu > 0$ *be fixed, and* $f \in C_B(\mathbb{R}_+)$*. Then*

$$\lim_{n\to\infty} \mathcal{R}_{n,\mu}(f;x) = f(x)$$

uniformly with respect to x *on every compact subset of* $\mathbb{R}_+$*.*

Proof. Let $b > 0$ and $x \in [0, b]$. Since f is continuous on compact subsets of $\mathbb{R}_+$ and bounded on $\mathbb{R}_+$, for every $\varepsilon > 0$ there exists a constant $C_{\varepsilon,b} > 0$ such that

$$|f(t) - f(x)| \leq \varepsilon + C_{\varepsilon,b}(t-x)^2, \qquad t \geq 0, \quad x \in [0,b].$$

By positivity and linearity,

$$\left|\mathcal{R}_{n,\mu}(f;x) - f(x)\right| \leq \mathcal{R}_{n,\mu}(|f(t) - f(x)|;x) \leq \varepsilon + C_{\varepsilon,b}\eta_{n,\mu}(x).$$

From (3.1),

$$\sup_{0\leq x\leq b} \eta_{n,\mu}(x) \longrightarrow 0, \qquad n \to \infty.$$

Hence,

$$\limsup_{n\to\infty} \sup_{0\leq x\leq b} \left|\mathcal{R}_{n,\mu}(f;x) - f(x)\right| \leq \varepsilon.$$

Since $\varepsilon > 0$ and $b > 0$ are arbitrary, the assertion follows. □

For $f \in UC_B(\mathbb{R}_+)$, recall the ordinary modulus of continuity

$$\omega(f;\delta) = \sup_{\substack{x,t\geq 0\\ |t-x|\leq\delta}} |f(t)-f(x)|, \qquad \delta > 0,$$

which satisfies

$$|f(t)-f(x)| \leq \left(1+\frac{|t-x|}{\delta}\right)\omega(f;\delta). \tag{3.2}$$

Theorem 3.2. *Let $\mu > 0$ and $f \in UC_B(\mathbb{R}_+)$. Then, for every $x \geq 0$,*

$$\left|\mathcal{R}_{n,\mu}(f;x)-f(x)\right| \leq 2\,\omega\left(f;\sqrt{\eta_{n,\mu}(x)}\right). \tag{3.3}$$

Proof. By positivity, linearity, and (3.2),

$$\left|\mathcal{R}_{n,\mu}(f;x)-f(x)\right| \leq \left[1+\frac{1}{\delta}\mathcal{R}_{n,\mu}(|t-x|;x)\right]\omega(f;\delta).$$

Using the Cauchy–Schwarz inequality,

$$\mathcal{R}_{n,\mu}(|t-x|;x) \leq \sqrt{\mathcal{R}_{n,\mu}((t-x)^2;x)} = \sqrt{\eta_{n,\mu}(x)}.$$

Taking $\delta = \sqrt{\eta_{n,\mu}(x)}$ gives (3.3). The case $\eta_{n,\mu}(x) = 0$ follows immediately. □

We next derive a second-order estimate. Set

$$\beta_{n,\mu} := \mathcal{R}_{n,\mu}(e_1;x)-x = \frac{1}{n}\left(a_1+\frac{1}{\mu+1}\right)$$

and

$$\vartheta_{n,\mu}(x) := \eta_{n,\mu}(x)+\beta_{n,\mu}^2.$$

Let

$$W^2(\mathbb{R}_+) = \{g \in C_B(\mathbb{R}_+) : g', g'' \in C_B(\mathbb{R}_+)\}.$$

For $f \in C_B(\mathbb{R}_+)$, the Peetre K-functional is defined by

$$K_2(f;\delta) = \inf_{g\in W^2(\mathbb{R}_+)} \{\|f-g\|_\infty + \delta\|g''\|_\infty\}.$$

Moreover, there exists an absolute constant $C > 0$ such that

$$K_2(f;\delta) \leq C\,\omega_2(f;\sqrt{\delta}), \tag{3.4}$$

where

$$\omega_2(f;\delta) = \sup_{0<h\leq\delta}\sup_{x\geq 0}|f(x+2h)-2f(x+h)+f(x)|.$$

Theorem 3.3. *Let $\mu > 0$ and $f \in UC_B(\mathbb{R}_+)$. Then, for every $x \geq 0$,*

$$\left|\mathcal{R}_{n,\mu}(f;x)-f(x)\right| \leq C_1\omega_2\left(f;\sqrt{\vartheta_{n,\mu}(x)}\right)+\omega\left(f;|\beta_{n,\mu}|\right), \tag{3.5}$$

where $C_1 > 0$ is an absolute constant.

Proof. Define the auxiliary operator

$$\widehat{\mathcal{R}}_{n,\mu}(f;x) = \mathcal{R}_{n,\mu}(f;x) - f(x+\beta_{n,\mu}) + f(x).$$

Then,

$$\widehat{\mathcal{R}}_{n,\mu}(e_0;x) = 1, \qquad \widehat{\mathcal{R}}_{n,\mu}(e_1;x) = x.$$

For $g \in W^2(\mathbb{R}_+)$, Taylor's formula gives

$$\left|\widehat{\mathcal{R}}_{n,\mu}(g;x) - g(x)\right| \le \frac{1}{2}\vartheta_{n,\mu}(x)\|g''\|_\infty.$$

Consequently,

$$\left|\mathcal{R}_{n,\mu}(f;x) - f(x)\right| \le 2\|f-g\|_\infty + \frac{1}{2}\vartheta_{n,\mu}(x)\|g''\|_\infty + \omega(f;|\beta_{n,\mu}|).$$

Taking the infimum over $g \in W^2(\mathbb{R}_+)$ yields

$$\left|\mathcal{R}_{n,\mu}(f;x) - f(x)\right| \le 2K_2\left(f;\frac{\vartheta_{n,\mu}(x)}{4}\right) + \omega(f;|\beta_{n,\mu}|).$$

Finally, (3.4) gives (3.5). □

4. Asymptotic approximation results

We now investigate the asymptotic behavior of the operators $\mathcal{R}_{n,\mu}$. The central moment limits obtained in the preceding section provide the main ingredients for the Voronovskaya-type and Grüss–Voronovskaya results established below.

Theorem 4.1. *Let $\mu > 0$ be fixed and let $f \in C_B^2(\mathbb{R}_+)$. Then, for every $x > 0$,*

$$\lim_{n\to\infty} n\left[\mathcal{R}_{n,\mu}(f;x) - f(x)\right] = \left(a_1 + \frac{1}{\mu+1}\right)f'(x) + \frac{h_2 x}{4}f''(x). \tag{4.1}$$

Proof. By Taylor's formula at the point x,

$$f(t) = f(x) + f'(x)(t-x) + \frac{f''(x)}{2}(t-x)^2 + r(t,x)(t-x)^2, \tag{4.2}$$

where

$$\lim_{t\to x} r(t,x) = 0.$$

Applying $\mathcal{R}_{n,\mu}$ to (4.2), we obtain

$$\begin{aligned} n\left[\mathcal{R}_{n,\mu}(f;x) - f(x)\right] &= f'(x)n\Delta_{n,1}^{(\mu)}(x) + \frac{f''(x)}{2}n\Delta_{n,2}^{(\mu)}(x) \\ &\quad + n\mathcal{R}_{n,\mu}(r(t,x)(t-x)^2;x). \end{aligned} \tag{4.3}$$

It remains to show that the last term tends to zero. For every $\varepsilon > 0$, choose $\delta > 0$ such that

$$|r(t,x)| < \varepsilon \qquad \text{whenever} \qquad |t-x| < \delta.$$

Since $f \in C_B^2(\mathbb{R}_+)$, the integral form of Taylor's remainder gives

$$r(t,x) = \int_0^1 (1-s)[f''(x+s(t-x)) - f''(x)]\,ds.$$

Consequently,

$$|r(t,x)| \le \|f''\|_\infty =: M_x, \qquad t \ge 0.$$

Thus, the remainder is globally bounded without imposing any additional decay assumption at infinity. Therefore,

$$\begin{aligned} n\left|\mathcal{R}_{n,\mu}(r(t,x)(t-x)^2;x)\right| &\le \varepsilon n\mathcal{R}_{n,\mu}((t-x)^2;x) + \frac{M_x n}{\delta^2}\mathcal{R}_{n,\mu}((t-x)^4;x) \\ &= \varepsilon n\Delta_{n,2}^{(\mu)}(x) + \frac{M_x}{\delta^2} n\Delta_{n,4}^{(\mu)}(x). \end{aligned}$$

By (2.21) and (2.23),

$$n\Delta_{n,2}^{(\mu)}(x) \longrightarrow \frac{h_2 x}{2}, \qquad n\Delta_{n,4}^{(\mu)}(x) \longrightarrow 0.$$

Consequently,

$$\limsup_{n\to\infty} n\left|\mathcal{R}_{n,\mu}(r(t,x)(t-x)^2;x)\right| \le \varepsilon\frac{h_2 x}{2}.$$

Since $\varepsilon > 0$ is arbitrary,

$$\lim_{n\to\infty} n\mathcal{R}_{n,\mu}(r(t,x)(t-x)^2;x) = 0.$$

Finally, using (2.20) and (2.21) in (4.3), we obtain (4.1). □

Remark 4.1. *For $\mu = 1$, the preceding result reduces to*

$$\lim_{n\to\infty} n\left[\mathcal{R}_{n,1}(f;x) - f(x)\right] = \left(a_1 + \frac{1}{2}\right)f'(x) + \frac{h_2 x}{4}f''(x),$$

which gives the corresponding asymptotic formula for the Kantorovich member of the present family.

The fractional parameter μ enters the leading asymptotic error exclusively through the coefficient of $f'(x)$, namely $a_1 + 1/(\mu+1)$, whereas the coefficient of $f''(x)$ remains independent of μ. In particular, $\mu = 1$ yields the Kantorovich coefficient $a_1 + 1/2$, while the fractional contribution to the first-order term vanishes as $\mu \to \infty$. This observation clarifies how the fractional averaging parameter affects the asymptotic behavior of the operators.

Next, we obtain the associated Grüss–Voronovskaya formula.

Theorem 4.2. *Let $\mu > 0$ be fixed, and let $f, g \in C_B^2(\mathbb{R}_+)$. Then, for every $x > 0$,*

$$\lim_{n\to\infty} n\Big[\mathcal{R}_{n,\mu}(fg;x) - \mathcal{R}_{n,\mu}(f;x)\mathcal{R}_{n,\mu}(g;x)\Big] = \frac{h_2 x}{2}f'(x)g'(x). \tag{4.4}$$

Proof. Applying (4.1) to fg, f, and g, respectively, we have

$$\lim_{n\to\infty} n\left[\mathcal{R}_{n,\mu}(fg;x) - f(x)g(x)\right] = \left(a_1 + \frac{1}{\mu+1}\right)(fg)'(x) + \frac{h_2 x}{4}(fg)''(x).$$

Moreover,

$$n\Big[\mathcal{R}_{n,\mu}(fg;x)-\mathcal{R}_{n,\mu}(f;x)\mathcal{R}_{n,\mu}(g;x)\Big]=n\Big[\mathcal{R}_{n,\mu}(fg;x)-f(x)g(x)\Big]\\-g(x)n\Big[\mathcal{R}_{n,\mu}(f;x)-f(x)\Big]-f(x)n\Big[\mathcal{R}_{n,\mu}(g;x)-g(x)\Big]\\-n\Big[\mathcal{R}_{n,\mu}(f;x)-f(x)\Big]\Big[\mathcal{R}_{n,\mu}(g;x)-g(x)\Big]. \tag{4.5}$$

By (4.1),

$$\mathcal{R}_{n,\mu}(f;x)-f(x)=O\left(\frac{1}{n}\right),\qquad \mathcal{R}_{n,\mu}(g;x)-g(x)=O\left(\frac{1}{n}\right),$$

so the last term in (4.5) tends to zero.

Using

$$(fg)'=f'g+fg',\qquad (fg)''=f''g+2f'g'+fg'',$$

the limiting expression simplifies to

$$\begin{aligned}&\left(a_1+\frac{1}{\mu+1}\right)[(fg)'-gf'-fg']+\frac{h_2x}{4}[(fg)''-gf''-fg'']\\=&\frac{h_2x}{4}[f''g+2f'g'+fg''-gf''-fg'']\\=&\frac{h_2x}{2}f'g'.\end{aligned}$$

Hence,

$$\lim_{n\to\infty}n\Big[\mathcal{R}_{n,\mu}(fg;x)-\mathcal{R}_{n,\mu}(f;x)\mathcal{R}_{n,\mu}(g;x)\Big]=\frac{h_2x}{2}f'(x)g'(x),$$

which proves (4.4). □

Remark 4.2. *The limiting expression in* (4.4) *does not depend explicitly on the fractional parameter* μ. *Indeed, the* μ*-dependent first-order displacement appearing in* (4.1) *cancels in the Grüss–Voronovskaya combination.*

5. Comparative analysis and the role of the fractional parameter

Now, we compare the discrete multiple Sheffer operator $\mathcal{G}_n$, its Kantorovich modification $K_n^{(S)}$, and the Riemann–Liouville family $\mathcal{R}_{n,\mu}$. Particular attention is given to the role of the fractional parameter μ, which provides an additional degree of freedom in controlling the leading approximation error.

Recall that

$$\mathcal{R}_{n,1}(f;x)=K_n^{(S)}(f;x),$$

so that the classical Kantorovich modification is contained in the fractional family as the particular case $\mu=1$.

The first central moments of the three operators are

$$\mathcal{G}_n(t-x;x)=\frac{a_1}{n},$$

$$K_n^{(S)}(t-x;x) = \frac{1}{n}\left(a_1 + \frac{1}{2}\right),$$
$$\mathcal{R}_{n,\mu}(t-x;x) = \frac{1}{n}\left(a_1 + \frac{1}{\mu+1}\right).$$

Hence, the fractional parameter modifies the first-order displacement through the quantity $1/(\mu+1)$. In particular,

$$\frac{1}{\mu+1} = \frac{1}{2} \quad \text{for} \quad \mu = 1, \qquad \frac{1}{\mu+1} \longrightarrow 0 \quad \text{as} \quad \mu \to \infty.$$

Thus, the Riemann–Liouville family contains the Kantorovich correction and continuously approaches the discrete first-order displacement as μ increases.

The second central moments are given by

$$\mathcal{G}_n((t-x)^2;x) = \frac{h_2 x}{2n} + \frac{a_2}{n^2},$$
$$K_n^{(S)}((t-x)^2;x) = \frac{h_2 x}{2n} + \frac{1}{n^2}\left(a_2 + a_1 + \frac{1}{3}\right),$$
$$\mathcal{R}_{n,\mu}((t-x)^2;x) = \frac{h_2 x}{2n} + \frac{1}{n^2}\left[a_2 + \frac{2a_1}{\mu+1} + \frac{2}{(\mu+1)(\mu+2)}\right].$$

Consequently, all three operators possess the same leading second-order dispersion,

$$\lim_{n\to\infty} n\, L_n((t-x)^2;x) = \frac{h_2 x}{2},$$

where L_n denotes any of the three operators. The fractional modification therefore changes the lower-order correction without altering the leading dispersion term.

The main advantage of the fractional family becomes more transparent from the corresponding Voronovskaya formulas. For sufficiently smooth f, we have

$$\lim_{n\to\infty} n\left[\mathcal{G}_n(f;x) - f(x)\right] = a_1 f'(x) + \frac{h_2 x}{4} f''(x),$$
$$\lim_{n\to\infty} n\left[K_n^{(S)}(f;x) - f(x)\right] = \left(a_1 + \frac{1}{2}\right) f'(x) + \frac{h_2 x}{4} f''(x),$$
$$\lim_{n\to\infty} n\left[\mathcal{R}_{n,\mu}(f;x) - f(x)\right] = \left(a_1 + \frac{1}{\mu+1}\right) f'(x) + \frac{h_2 x}{4} f''(x). \tag{5.1}$$

Define the leading asymptotic error coefficient of the fractional family by

$$\mathcal{E}_\mu(f;x) := \left(a_1 + \frac{1}{\mu+1}\right) f'(x) + \frac{h_2 x}{4} f''(x). \tag{5.2}$$

The corresponding coefficients for the discrete and Kantorovich operators are

$$\mathcal{E}_D(f;x) = a_1 f'(x) + \frac{h_2 x}{4} f''(x),$$
$$\mathcal{E}_K(f;x) = \left(a_1 + \frac{1}{2}\right) f'(x) + \frac{h_2 x}{4} f''(x).$$

The following observation describes the additional approximation flexibility provided by the fractional parameter.

Proposition 5.1. *Let $f'(x) \neq 0$ and set*

$$q(f;x) = -a_1 - \frac{h_2 x}{4}\frac{f''(x)}{f'(x)}. \tag{5.3}$$

If

$$0 < q(f;x) < 1, \tag{5.4}$$

then the choice

$$\mu^*(f;x) = \frac{1}{q(f;x)} - 1 \tag{5.5}$$

satisfies

$$\mathcal{E}_{\mu^*}(f;x) = 0.$$

Consequently,

$$\lim_{n\to\infty} n\left[\mathcal{R}_{n,\mu^*}(f;x) - f(x)\right] = 0. \tag{5.6}$$

Proof. From (5.5),

$$\frac{1}{\mu^*+1} = q(f;x).$$

Using (5.3), we obtain

$$a_1 + \frac{1}{\mu^*+1} = -\frac{h_2 x}{4}\frac{f''(x)}{f'(x)}.$$

Substitution into (5.2) gives $\mathcal{E}_{\mu^*}(f;x) = 0$, and (5.6) follows from (5.1). □

Remark 5.1. *The preceding proposition does not assert a uniform superiority of the fractional operator for every function and every value of μ. Instead, it identifies a feature unavailable to either fixed endpoint construction: the fractional parameter can be selected to control the leading asymptotic bias. Whenever* (5.4) *holds, an admissible choice of μ eliminates the entire n^{-1} contribution to the approximation error at the point x.*

In particular, if

$$\mathcal{E}_D(f;x) \neq 0 \qquad \text{and} \qquad \mathcal{E}_K(f;x) \neq 0,$$

while (5.4) is satisfied, then

$$\lim_{n\to\infty} n\left[\mathcal{R}_{n,\mu^*}(f;x) - f(x)\right] = 0,$$

whereas

$$\lim_{n\to\infty} n\left[\mathcal{G}_n(f;x) - f(x)\right] = \mathcal{E}_D(f;x) \neq 0,$$
$$\lim_{n\to\infty} n\left[K_n^{(S)}(f;x) - f(x)\right] = \mathcal{E}_K(f;x) \neq 0.$$

Thus, for this class of functions and points, an appropriate fractional choice yields a strictly smaller leading asymptotic error than both the discrete and Kantorovich operators.

Overall, the relations

$$\mathcal{R}_{n,1} = K_n^{(S)}, \qquad \frac{1}{\mu+1} \longrightarrow 0 \quad (\mu \to \infty)$$

show that $\mathcal{R}_{n,\mu}$ provides a parameterized bridge between the Kantorovich and discrete approximation mechanisms. More importantly, the free parameter μ may be used to reduce, and under (5.4) completely eliminate, the leading asymptotic bias. This tunability constitutes the principal approximation advantage of the fractional extension.

6. Numerical illustration and comparative performance

Now, we illustrate numerically the comparative approximation behavior of the discrete multiple Sheffer operator $\mathcal{G}_n$, its Kantorovich modification $K_n^{(S)}$, and the Riemann–Liouville family $\mathcal{R}_{n,\mu}$. The aim is to examine the effect of the fractional parameter μ on the approximation performance and to complement the theoretical comparison established in Section 5.

For the numerical experiment, we consider the particular multiple Sheffer setting

$$A(t_1, t_2) = 1, \qquad H(t_1, t_2) = t_1 + t_2.$$

Under this choice, the multiple Sheffer framework introduced above reduces to the corresponding classical Szász-type setting. The discrete, Kantorovich, and Riemann–Liouville operators are therefore evaluated directly from their definitions given in the preceding sections. In this particular specialization, the multiple Sheffer coefficients reduce to nonnegative Poisson-type weights, so the numerical evaluation does not require direct computation of general high-order multiple Sheffer polynomials or alternating coefficient sums. The adaptive truncation of the resulting series controls the omitted probability mass. Numerical stability for arbitrary multiple Sheffer families, however, is not established by this experiment.

We choose the test function

$$f(x) = \frac{1}{\sqrt{1+x}} + 0.15e^{-0.6x}, \qquad x \in [0, 2].$$

The test function is chosen as a smooth, bounded, strictly decreasing, and convex function combining algebraic and exponential decay. These properties make it suitable for examining the influence of the fractional parameter on the approximation error. Indeed, in the present classical multiple-Sheffer specialization, the leading Voronovskaya term is $f'(x)/(\mu + 1) + xf''(x)/2$, whose two contributions have opposite signs for $x > 0$. Thus, varying μ may produce partial cancellation of the leading error terms. The example is intended to illustrate this parameter-dependent behavior rather than to establish a universal superiority of any particular operator.

The computations were carried out for

$$n \in \{10, 20, 30, 50, 100\}.$$

For the detailed comparison, a uniform grid of 401 points on $[0, 2]$ was employed. The infinite series were truncated adaptively using the Poisson weights $w_0(x) = e^{-nx}$ and $w_{k+1}(x) = w_k(x)\, nx/(k + 1)$. The summation was terminated at the first index K for which

$$\sum_{k=0}^{K} w_k(x) \geq 1 - 10^{-14}, \qquad w_K(x) < 10^{-7},$$

with a maximum truncation index of 5000. The mass defect was evaluated as $\left|1 - \sum_{k=0}^{K} w_k(x)\right|$. The local Kantorovich and Riemann–Liouville integrals were evaluated by a 40-point Gauss–Legendre quadrature rule.

To measure the global approximation performance, we use the mean absolute error

$$\mathrm{MAE} = \frac{1}{N} \sum_{i=1}^{N} |L_n(f; x_i) - f(x_i)|,$$

the root mean square error

$$\mathrm{RMSE} = \left[\frac{1}{N}\sum_{i=1}^{N}\left(L_n(f;x_i) - f(x_i)\right)^2\right]^{1/2},$$

and the maximum absolute error

$$\mathrm{MaxAE} = \max_{1\le i\le N}\left|L_n(f;x_i) - f(x_i)\right|,$$

where L_n denotes the corresponding discrete, Kantorovich, or Riemann–Liouville operator.

The fractional parameter used for the principal comparison is

$$\mu = 2.7.$$

This value is not prescribed in advance. It is selected from the parameter study described below, since it minimizes the MAE of the Riemann–Liouville operator for $n = 20$ over the considered range of fractional parameters.

Table 1 reports the global error measures for the three operators.

Table 1. Global error measures for the discrete, Kantorovich, and Riemann–Liouville operators with $\mu = 2.7$ and $n \in \{10, 20, 30, 50, 100\}$.

n	Discrete			Kantorovich			Riemann–Liouville		
	MAE	RMSE	MaxAE	MAE	RMSE	MaxAE	MAE	RMSE	MaxAE
10	0.007023	0.007207	0.008266	0.006219	0.009384	0.028234	**0.003353**	**0.004470**	0.015502
20	0.003509	0.003598	0.004127	0.003108	0.004725	0.014425	**0.001677**	**0.002238**	0.007860
30	0.002339	0.002397	0.002750	0.002071	0.003158	0.009687	**0.001118**	**0.001493**	0.005265
50	0.001403	0.001438	0.001649	0.001243	0.001898	0.005847	**0.000671**	**0.000896**	0.003171
100	0.000701	0.000719	0.000824	0.000621	0.000950	0.002937	**0.000335**	**0.000448**	0.001590

The results in Table 1 show a consistent reduction of the average approximation error produced by the Riemann–Liouville modification. In particular, for every tested value $n = 10, 20, 30, 50$, and 100, the Riemann–Liouville operator gives the smallest MAE and RMSE among the three approximation procedures.

The additional results for $n = 100$ further illustrate the asymptotic behavior of the operators. In particular, doubling n from 50 to 100 approximately halves the MAE for all three operators consistently with the expected first-order convergence behavior. The Riemann–Liouville operator continues to provide the smallest MAE and RMSE, whereas the discrete operator retains the smallest maximum absolute error.

For instance, when $n = 20$,

$$\mathrm{MAE}_D = 0.003508865, \qquad \mathrm{MAE}_K = 0.003107691,$$

whereas

$$\mathrm{MAE}_{RL} = 0.001677254.$$

Thus, with $\mu = 2.7$, the MAE is reduced by approximately 52.2% relative to the discrete operator and by approximately 46.0% relative to the Kantorovich operator.

A similar improvement is observed for the RMSE:

$$0.002238361 < 0.003598189 < 0.004725239,$$

corresponding respectively to the Riemann–Liouville, discrete, and Kantorovich operators for $n = 20$.

On the other hand, the maximum absolute error behaves differently. For the same value $n = 20$,

$$\mathrm{MaxAE}_D = 0.004126942,$$

$$\mathrm{MaxAE}_{RL} = 0.007859618, \qquad \mathrm{MaxAE}_K = 0.014424604.$$

Consequently, the discrete operator provides the smallest worst-case error, while the fractional operator remains substantially better than the Kantorovich modification. This observation also confirms that the fractional extension should not be interpreted as uniformly superior according to every possible error criterion.

Figure 1 presents the pointwise behavior for $n = 20$ and $\mu = 2.7$.

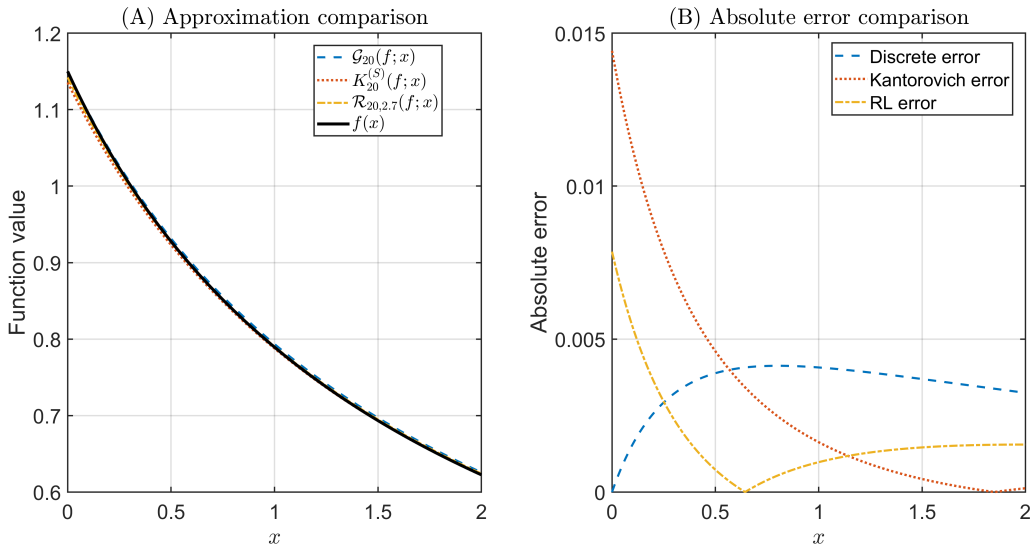


Figure 1. Comparison of the three operators for the function (6.1) with $n = 20$ and $\mu = 2.7$ on $x \in [0, 2]$: (A) approximation curves and (B) corresponding absolute errors.

The left panel of Figure 1 shows that all three operators reproduce the general behavior of the test function with good accuracy. The differences become considerably more visible in the absolute-error representation displayed in the right panel.

At the representative nodes

$$x = 0, 0.2, 0.4, \ldots, 2,$$

the discrete operator gives the smallest error near the left endpoint, the Riemann–Liouville operator performs best over a substantial intermediate portion of the interval, whereas the Kantorovich operator becomes more accurate toward the right-hand part of the interval. For example, at

$$x = 0.4,\ 0.6,\ 0.8,\ 1.0,$$

the Riemann–Liouville operator gives the smallest of the three pointwise errors.

This behavior illustrates an important feature of the fractional construction: the improvement is not uniform over the whole interval, but the additional parameter allows the approximation profile to be shifted relative to both the discrete and the fixed Kantorovich operators.

We next examine the influence of the fractional parameter more systematically. For $n = 20$, the parameter μ was varied over

$$0.5 \leq \mu \leq 15$$

with step size 0.1. Table 2 summarizes the values which optimize three different performance criteria.

Table 2. Selected optimal values of the fractional parameter for $n = 20$ under different error criteria.

Criterion	Optimal μ	Corresponding value
Minimum MAE	2.7	0.001677254
Minimum RMSE	3.9	0.002020392
Maximum pointwise winner rate	1.1	61.845%

An immediate consequence of Table 2 is that there is no single universally optimal fractional parameter. The most favorable choice of μ depends on the criterion used to assess approximation quality. The minimum MAE occurs at

$$\mu = 2.7,$$

whereas the minimum RMSE is obtained at

$$\mu = 3.9.$$

If the objective is instead to maximize the proportion of grid points at which the Riemann–Liouville operator has a strictly smaller error than both competing operators, the best value in the considered parameter range is

$$\mu = 1.1,$$

for which the fractional operator is pointwise superior at approximately 61.85% of the grid points.

The dependence of the three global error measures on the fractional parameter is displayed in Figure 2.

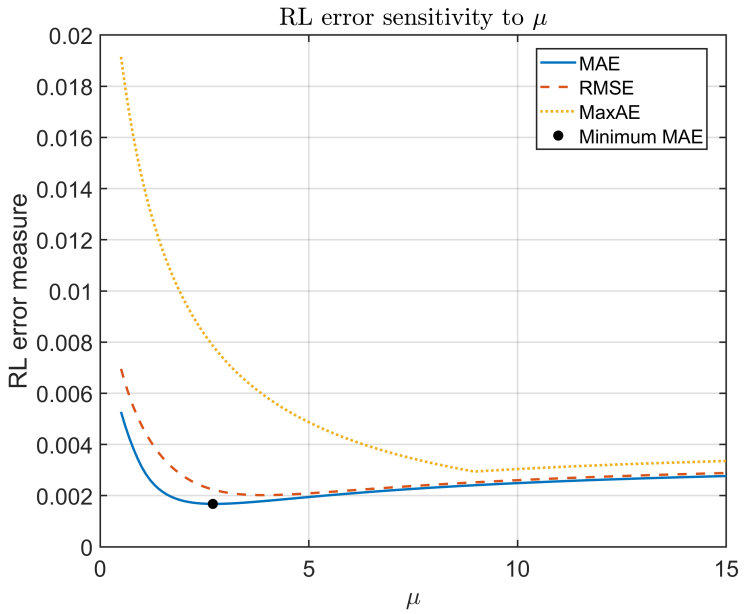


Figure 2. Sensitivity of the Riemann–Liouville approximation error with respect to the fractional parameter μ for $n = 20$ over the interval $0.5 \leq \mu \leq 15$. The marked point corresponds to the minimum MAE at $\mu = 2.7$.

Figure 2 shows that the dependence on μ is nontrivial. In particular, the MAE decreases rapidly for relatively small values of μ, reaches its minimum near $\mu = 2.7$, and then increases slowly. The RMSE exhibits a similar pattern, although its minimum occurs at a somewhat larger value of the fractional parameter. The maximum absolute error behaves differently and decreases over a larger range before reaching its most favorable region.

This numerical behavior is consistent with the theoretical discussion in Section 5. The parameter μ modifies the first-order displacement through

$$\frac{1}{\mu + 1}$$

and consequently provides an additional mechanism for controlling the leading approximation bias. The numerical experiment shows that this freedom can lead to a substantial reduction of averaged approximation errors when μ is selected appropriately.

As a numerical consistency check, we also evaluated the difference between the Kantorovich operator and the Riemann–Liouville operator with $\mu = 1$. For $n = 20$ on the complete grid, the maximum discrepancy was of the order

$$4.4 \times 10^{-16},$$

which is at machine-precision level and provides a numerical consistency check for the exact operator identity

$$\mathcal{R}_{n,1} = K_n^{(S)}.$$

This discrepancy measures the numerical difference between two theoretically identical operators, not the approximation error with respect to the test function. Its magnitude reflects floating-point roundoff and should not be interpreted as evidence of machine-precision approximation accuracy.

Moreover, the maximum mass defect of the truncated probability weights was approximately

$$9.99 \times 10^{-15}$$

for all tested values of n, indicating that the omitted probability mass is small at the prescribed truncation tolerance.

Overall, the experiment indicates that the Riemann–Liouville extension does not produce a universal improvement according to every local or global error criterion. Its principal numerical advantage is instead the additional tunability introduced by the fractional parameter. For the present example, this flexibility yields a pronounced reduction in MAE and RMSE over all tested approximation levels, while different choices of μ can be used to optimize different aspects of the approximation performance.

7. Conclusions and discussion

In this paper, a Riemann–Liouville fractional extension of the Kantorovich-type operators associated with multiple Sheffer polynomials has been introduced and investigated. The proposed construction provides a unified framework connecting the discrete multiple Sheffer operators with their Kantorovich counterparts, the latter being recovered for $\mu = 1$. Explicit algebraic and central moments were derived and used to establish local approximation results, estimates in terms of moduli of continuity, and Voronovskaya- and Grüss–Voronovskaya-type asymptotic formulas.

The comparison of the discrete, Kantorovich, and fractional operators shows that the parameter μ introduces additional flexibility into the approximation process rather than a uniform improvement for every function and every error criterion. This feature was also reflected in the numerical experiment. For the considered test function, a suitable choice of μ produced consistently smaller MAE and RMSE values than both the discrete and Kantorovich operators for all tested values of n, whereas the discrete operator remained preferable with respect to the maximum absolute error. Moreover, different optimal values of μ were obtained for different error criteria, illustrating the tunable character of the fractional extension.

These results indicate that the Riemann–Liouville parameter can be used to adapt the approximation behavior to a prescribed accuracy criterion while preserving the classical Kantorovich operator as a special member of the family. Further investigations may consider related fractional kernels or other polynomial structures within the same general framework.

Possible directions for future research include extensions of the proposed construction to other fractional integral kernels and polynomial families, as well as the investigation of further quantitative approximation properties.

Author contributions

Gui-Ying Weng: Validation, formal analysis, funding acquisition, investigation, writing—original draft preparation, writing—review and editing; Edmond Aliaga: Conceptualization, methodology, validation, formal analysis, investigation, writing—review and editing, visualization; Reşat Aslan: Conceptualization, methodology, validation, formal analysis, investigation, writing—review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

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