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Research article

Approximate controllability for damped second-order stochastic differential inclusions with memory

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Abstract: In this paper, we investigated a class of damped second-order stochastic differential inclusions that incorporate memory effects. We focused on two fundamental problems: The existence of mild solutions and the approximate controllability of the system. By integrating operator semigroup theory, fixed-point theorems for condensing multivalued maps, and the properties of generalized Clarke subdifferentials, we derived sufficient conditions for the existence of mild solutions. Furthermore, under appropriate assumptions, we rigorously demonstrated the approximate controllability of the proposed system.

Keywords: approximate controllability; existence; stochastic differential inclusions; memory terms
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1. Introduction

In diverse fields, including physics, finance, and biology, second-order evolution equations play a crucial role in characterizing the dynamics of complex systems. Through the inclusion of second-order time derivatives, nonlinear partial differential equations offer a powerful analytical framework for effectively modeling phenomena in classical, molecular, and quantum mechanics [1]. From an engineering perspective, damping represents the dissipative mechanism through which kinetic or electromagnetic energy is extracted from a system. This process is instrumental in attenuating oscillatory amplitudes and suppressing signal intensities, thereby ensuring the asymptotic stability of the underlying dynamics. Next, we mention some works. The study of abstract deterministic second-order evolution equations, governed by the generator of a strongly continuous cosine family, was pioneered in [2] and further developed in [3–5]. While the motivation for studying damped

second-order systems is detailed in [6, 7], the authors in [8] established a framework for second-order damped functional stochastic evolution equations within a Hilbert space. In physical and engineering systems, damping plays a fundamental role in dissipating kinetic energy and suppressing oscillations, which is vital for system stabilization. For instance, advanced control strategies-such as dynamic fractional-order fuzzy models with adaptive fractional derivatives-have been widely applied to vibration suppression in active mass drivers [9]. For a comprehensive treatment of damping, refer to [10–12].

To rigorously address problems in non-smooth mechanics and engineering, Panagiotopoulos [13] introduced the concept of hemivariational inequalities (HIs). These inequalities emerge naturally from the variational formulation of the virtual work principle when constitutive laws are governed by non-monotone, multivalued mappings. By extending the classical scope of variational inequalities, this field has gained substantial traction for its efficacy in resolving nonlinear boundary value problems across the applied sciences [14]. The mathematical foundation of this theory rests upon the generalized gradient properties of locally Lipschitz functionals [15–17], providing a powerful apparatus for analyzing mechanical systems defined by non-monotone and multivalued relations.

Controllability remains a cornerstone of control theory, particularly in the analysis of infinite-dimensional systems [18]. A crucial subtlety lies in the distinction between exact controllability and approximate controllability. For many nonlinear problems, the former is often overly restrictive, since the attainable subspace is typically not closed due to the intrinsic topological properties of the state space. Consequently, approximate controllability has emerged as a more relevant objective, stimulating extensive research on stochastic, impulsive, and delay integrodifferential equations [19–21]. In this context, the controllability of systems governed by hemispherical inclusions (HIs) has emerged as a particularly vibrant area of research [22–24].

We present, to our knowledge, the first investigation into the approximate controllability of damped second-order stochastic differential systems with memory governed by HIs. We formulate and analyze this problem within the framework of a separable Hilbert space $\mathcal{H}$, thereby addressing a distinct gap in research. In particular, we study the abstract differential inclusion given by

$$\begin{cases} d\vartheta'(\sigma) \in \Big[\mathcal{A}\vartheta(\sigma) + \mathcal{G}\vartheta'(\sigma) + (S_1\vartheta)(\sigma) + \Phi(\sigma, (S_2\vartheta)(\sigma), \vartheta(\sigma)) \\ \qquad\qquad + \mathcal{B}\mathcal{U}(\sigma) + \partial\mathcal{K}(\sigma, \vartheta(\sigma))\Big] d\sigma + \varsigma(\sigma, \vartheta(\sigma))\, d\mathcal{W}(\sigma), \\ \quad \sigma \in \mathcal{J} = [0, b], \\ \vartheta(0) = \vartheta_0, \quad \vartheta'(0) = \vartheta_1. \end{cases} \tag{1.1}$$

Here, the state variable $\vartheta(\cdot)$ evolves in $\mathcal{H}$, while the control input $\mathcal{U}(\cdot)$ belongs to a separable Hilbert space U. We assume $\mathcal{A}$ generates a strongly continuous cosine family $(C(\sigma))_{\sigma\in\mathbb{R}}$ on $\mathcal{H}$, and $\mathcal{G} \in \mathcal{L}(\mathcal{H})$ denotes a bounded damping operator. Operator $\mathcal{B}$ is a bounded linear map from U into $\mathcal{H}$. The term $\partial\mathcal{K}(\sigma, \vartheta) \subset \mathcal{H}^*$ denotes the Clarke generalized subdifferential of the locally Lipschitz functional $\mathcal{K}(\sigma, \cdot)$, while $\mathcal{K}^\circ(\sigma, \vartheta; \varpi)$ denotes its associated Clarke generalized directional derivative.

Here, $\vartheta \in \mathcal{H}$ denotes the state variable, whereas $\varpi \in \mathcal{H}$ is an arbitrary direction in the definition of the Clarke generalized directional derivative. With regard to the stochastic setting, let $(\mathcal{O}, \mathcal{F}, \{\mathcal{F}_\sigma\}_{\sigma\geq 0}, \mathbb{P})$ be a complete probability space equipped with a standard $\mathcal{Q}$-Wiener process $\mathcal{W}$. The initial conditions ϑ_0 and ϑ_1 are $\mathcal{F}_0$-measurable random variables valued in $\mathcal{H}$ and are independent of $\mathcal{W}$. Let $\Phi : \mathcal{J} \times \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a given function. $\mathcal{S}_1$ and $\mathcal{S}_2$ are the history-dependent operators.

Notably, as far as we are aware, no research has explored the System:

$$\begin{cases} d\vartheta'(\sigma) \in \Big[\mathcal{A}\vartheta(\sigma) + \mathcal{G}\vartheta'(\sigma) + \Phi(\sigma, (S_2\vartheta)(\sigma), \vartheta(\sigma)) + \mathcal{B}\mathcal{U}(\sigma) + \partial\mathcal{K}(\sigma, \vartheta(\sigma))\Big] d\sigma \\ \qquad\qquad + \varsigma(\sigma, \vartheta(\sigma))\, d\mathcal{W}(\sigma), \qquad \sigma \in \mathcal{J}, \end{cases}$$

and

$$\begin{cases} d\vartheta'(\sigma) \in \Big[\mathcal{A}\vartheta(\sigma) + \mathcal{G}\vartheta'(\sigma) + (S_1\vartheta)(\sigma) + \Phi(\sigma, (S_2\vartheta)(\sigma), \vartheta(\sigma)) + \mathcal{B}\mathcal{U}(\sigma) \\ \qquad\qquad + \partial\mathcal{K}(\sigma, \vartheta(\sigma))\Big] d\sigma, \qquad \sigma \in \mathcal{J}. \end{cases}$$

As a result, system (1.1) covers many situations.

Unlike in [22–24], our inclusion of history-dependent operators $\mathcal{S}_1$ and $\mathcal{S}_2$ enables a more precise description of the contact mechanics model. If an elastomer or a viscoelastic body exhibits memory effects, we typically employ S_1 for its description. However, if the memory effect occurs at the boundary within a contact mechanics model subject to specific boundary conditions, we generally utilize S_2 for the description. For further discussion on the physical significance of history-dependent operators, please refer to [25] and [26]. Unlike [22] who studied memoryless systems, we introduce Volterra-type history-dependent operators (S_1, S_2) to capture hereditary viscoelasticity. By developing refined a priori estimates for the strong coupling of memory, multivalued subdifferentials, and noise, we establish existence through topological fixed-point theory. Furthermore, we construct a terminal feedback control operator $\mathcal{U}_\sigma$ to link history functionals to reachability, significantly broadening the controllability theory of non-smooth stochastic systems.

The remainder of this paper is organized as follows. In Section 2, we provide the essential background and fundamental concepts required for our analysis. In Section 3, we establish the well-posedness of the considered problem within this framework. These results then serve as the foundation for Section 4, where we demonstrate the approximate controllability of control system (1.1).

2. Preliminaries

We provide a brief overview of the mathematical machinery used throughout this paper, specifically focusing on multi-valued analysis and the properties of sine and cosine operators.

The mathematical framework is built upon a filtered probability space $(\mathcal{O}, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathcal{F}_\sigma\}_{\sigma\geq 0}$ represents a complete filtration. We assume that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. We denote the expectation operator by $\mathbb{E}[\cdot]$, representing integration with respect to the probability measure $\mathbb{P}$. The stochastic disturbance is characterized by a Q-Wiener process $\mathcal{W}$, where the covariance operator Q is assumed to be a positive, self-adjoint, trace-class operator (i.e., $\mathrm{Tr}(Q) < \infty$).

We associate with Q a complete orthonormal system $\{e_\varrho\}_{\varrho\in\mathbb{N}}$ and a bounded sequence of eigenvalues $\lambda_\varrho \geq 0$. The stochastic dynamics are driven by a sequence of independent scalar Wiener processes $\{\vartheta_\varrho\}_{\varrho\in\mathbb{N}}$, satisfying the projection identity:

$$\langle \mathcal{W}(\sigma), h\rangle = \sum_{\varrho=1}^{\infty} \sqrt{\lambda_\varrho}\vartheta_\varrho(\sigma)\langle e_\varrho, h\rangle, \quad \forall h \in \mathcal{H}.$$

In this setup, the filtration $\{\mathcal{F}_\sigma\}_{\sigma\geq 0}$ represents the canonical history of the Wiener process.

We introduce the space of Hilbert-Schmidt operators mapping from $Q^{1/2}\mathcal{H}$ to $\mathcal{H}$, denoted by $\mathcal{L}_2^0$. This space is endowed with a Hilbert structure induced by the inner product $\langle \Phi, \Psi\rangle_{\mathcal{L}_2^0} = \mathrm{Tr}[\Phi Q \Psi^*]$.

Regarding the stochastic framework, let $\mathcal{L}^2(O, \mathcal{F}, \mathbb{P}; \mathcal{H})$ represent the Banach space of all strongly $\mathcal{F}$-measurable, square-integrable $\mathcal{H}$-valued random variables. For any fixed $\sigma \geq 0$, the collection of random variables measurable with respect to the sub-ς-algebra $\mathcal{F}_\sigma$ forms a closed linear subspace, denoted by $L^2(\mathcal{F}_\sigma; \mathcal{H})$, which is a Hilbert space. In particular, $\mathcal{L}^2(\mathcal{F}_b; \mathcal{H})$ denotes the space of square-integrable variables measurable with respect to the terminal ς-algebra $\mathcal{F}_b$.

Let $C(\mathcal{J}, \mathcal{L}^2(\mathcal{F}, \mathcal{H}))$ represent the Banach space of continuous maps $\vartheta : \mathcal{J} \to \mathcal{L}^2(\mathcal{F}, \mathcal{H})$, endowed with the norm defined by uniform convergence in the mean-square sense. Additionally, we define $\mathcal{L}^2_{\mathcal{F}}(\mathcal{J}, \mathcal{H})$ as the closed subspace consisting of all predictable measurable processes $\eta : \mathcal{J} \times O \to \mathcal{H}$ satisfying the integrability condition:

$$\|\eta\|_{\mathcal{L}^2_{\mathcal{F}}} := \left(\int_{\mathcal{J}} E\|\eta(\sigma)\|^2_{\mathcal{H}} \, d\sigma \right)^{1/2} < \infty.$$

Throughout this paper, adaptedness refers to the state process, whereas processes belonging to $L^2_{\mathcal{F}}$ are understood to be predictable. The space of admissible controls, $\mathcal{L}^2_{\mathcal{F}}(\mathcal{J}, U)$, is constructed similarly. Detailed properties of these function spaces can be found in [27].

Throughout this paper, strict measure-theoretic distinctions are maintained: State processes $\vartheta(\cdot)$ are assumed to be $\mathcal{F}_\sigma$-adapted, whereas stochastic integrands, nonlinear mappings ζ, Clarke subdifferential selections $k(\cdot)$, and history-dependent terms are required to be predictable (or strongly measurable) processes belonging to designated spaces such as $L^2_{\mathcal{F}}(\mathcal{J}, \mathcal{H})$ and $L^2_{\mathcal{F}}(\mathcal{J}, \mathcal{L}^0_2)$. This ensures that all Bochner and Itô-type stochastic convolutions are rigorously defined and measurable.

Definition 2.1. *A strongly continuous cosine function on the Hilbert space $\mathcal{H}$ is a one-parameter family of bounded linear operators, denoted by $\{C(\sigma) : \sigma \in \mathbb{R}\}$, that fulfills the cosine identity*

$$C(\iota + \sigma) + C(\iota - \sigma) = 2C(\iota)C(\sigma), \quad \forall \iota, \sigma \in \mathbb{R},$$

with $C(0) = I$, such that the orbit maps $\sigma \mapsto C(\sigma)\varpi$ are continuous for all fixed $\varpi \in \mathcal{H}$.

In the context of a Hilbert space $\mathcal{H}$, let $\mathcal{A}$ be the infinitesimal generator of a strongly continuous cosine family denoted by $\{C(\sigma)\}_{\sigma \in \mathbb{R}}$. The associated sine family $\{\mathcal{S}(\sigma)\}_{\sigma \in \mathbb{R}}$ is constructed via the integral representation:

$$\mathcal{S}(\sigma)\varpi = \int_0^\sigma C(\iota)\varpi \, d\iota, \quad \text{for all } \varpi \in \mathcal{H}, \sigma \in \mathbb{R}.$$

We impose the following structural restriction on operator $\mathcal{A}$:

$H(\mathcal{A})$ The operator $\mathcal{A}$ generates a strongly continuous cosine family $\{C(\sigma)\}_{\sigma \in \mathbb{R}}$ acting on $\mathcal{H}$.

Let $(C(\sigma))_{\sigma \in \mathbb{R}}$ be a strongly continuous cosine family. Its infinitesimal generator is the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ given by

$$\mathcal{A}\varpi = \frac{d^2}{d\sigma^2} C(\sigma)\varpi \Big|_{\sigma=0}, \quad \text{for } \varpi \in \mathcal{D}(\mathcal{A}),$$

where the domain $\mathcal{D}(\mathcal{A})$ consists of all $\varpi \in \mathcal{H}$ such that the map $\sigma \mapsto C(\sigma)\varpi$ is twice continuously differentiable.

Further elaboration on the theory of cosine and sine families is available in [22].

Additionally, we assume there exist positive constants $\mathbb{M}$ and $\mathbb{N}$ satisfying the bounds $\|C(\sigma)\| \leq \mathbb{M}$ and $\|\mathcal{S}(\sigma)\| \leq \mathbb{N}$ for every $\sigma \in \mathcal{J}$. The notation $[\mathcal{D}(\mathcal{A})]$ represents the domain $\mathcal{D}(\mathcal{A})$ equipped with the graph norm topology, given by $\|\varpi\|_{\mathcal{A}} = \|\varpi\| + \|\mathcal{A}\varpi\|$. We further define the subspace $\mathcal{E}$ as the set of all $\varpi \in \mathcal{H}$ such that the map $\sigma \mapsto C(\sigma)\varpi$ is continuously differentiable. As established in [28], $\mathcal{E}$ constitutes a Banach space when furnished with the following norm:

$$\|\varpi\|_{\mathcal{E}} = \|\varpi\| + \sup_{0\leq\sigma\leq 1} \|\mathcal{A}\mathcal{S}(\sigma)\varpi\|, \quad \text{for } \varpi \in \mathcal{E}.$$

Consider the phase space $\mathcal{E} \times \mathcal{H}$. On this space, the operator matrix

$$P(\sigma) = \begin{pmatrix} C(\sigma) & \mathcal{S}(\sigma) \\ \mathcal{A}\mathcal{S}(\sigma) & C(\sigma) \end{pmatrix}$$

constitutes a strongly continuous group of linear operators. The infinitesimal generator of this group is given by $\mathcal{A}_1 = \begin{pmatrix} 0 & I \\ \mathcal{A} & 0 \end{pmatrix}$, with the domain defined as $\mathcal{D}(\mathcal{A}_1) = \mathcal{D}(\mathcal{A}) \times \mathcal{E}$. A key implication of this structure is that the map $\mathcal{A}\mathcal{S}(\sigma)$ acts as a bounded linear operator from $\mathcal{E}$ to $\mathcal{H}$, satisfying the continuity condition $\lim_{\sigma\to 0} \|\mathcal{A}\mathcal{S}(\sigma)\varpi\| = 0$ for all $\varpi \in \mathcal{H}$. Moreover, for any locally integrable function $\varpi : [0, \infty) \to \mathcal{H}$, the convolution integral $z(\sigma) = \int_0^\sigma S(\sigma - \iota)\varpi(\iota)\, d\iota$ represents a continuous function taking values in $\mathcal{E}$. This regularity is deduced directly from the property that the integral term

$$\int_0^\sigma P(\sigma - \iota) \begin{pmatrix} 0 \\ \varpi(\iota) \end{pmatrix} d\iota = \Big[\int_0^\sigma \mathcal{S}(\sigma - \iota)\varpi(\iota) d\iota, \int_0^\sigma C(\sigma - \iota)\varpi(\iota) d\iota \Big]^T$$

yields a continuous trajectory in the product space $\mathcal{E} \times \mathcal{H}$.

The solvability of the second-order abstract initial value problem:

$$\begin{cases} \varpi''(\sigma) = \mathcal{A}\varpi(\sigma) + L(\sigma), & \sigma \in \mathcal{J}, \\ \varpi(0) = \varpi_1, \quad \varpi'(0) = \varpi_2, \end{cases} \tag{2.1}$$

where $L : \mathcal{J} \to \mathcal{H}$ represents an integrable forcing term. Here, $L(\sigma)$ is a generic deterministic forcing term used only to state the standard variation-of-constants formula; it does not include the multivalued term $k(\sigma)$ or the stochastic term of system (1.1).

The sign convention in (2.1) is consistent with system (1.1).

The mild solution to (2.1) is characterized by the function $\varpi \in C(\mathcal{J}, \mathcal{H})$ given by

$$\varpi(\sigma) = C(\sigma)\varpi_1 + \mathcal{S}(\sigma)\varpi_2 + \int_0^\sigma \mathcal{S}(\sigma - \iota)L(\iota)\, d\iota.$$

We observe that smoothness depends on the initial data; specifically, whenever $\varpi_1 \in \mathcal{E}$, the trajectory $\varpi(\sigma)$ becomes continuously differentiable (C^1). In this case, the derivative takes the form:

$$\frac{d\varpi}{d\sigma} = \mathcal{A}\mathcal{S}(\sigma)\varpi_1 + C(\sigma)\varpi_2 + \int_0^\sigma C(\sigma - \iota)L(\iota)\, d\iota,$$

valid for every $\sigma \in \mathcal{J}$.

Basic definitions regarding multi-valued maps are recalled below. The interested reader is directed to [13, 14] for further details and proofs.

We define $\mathcal{P}(\mathcal{H})$ as the power set of $\mathcal{H}$. For the sake of brevity, we introduce the following sub-families of non-empty sets $D \in \mathcal{P}(\mathcal{H})$:

$$\mathcal{P}_{f(c)}(\mathcal{H}) = \{D \subset \mathcal{H} : D \text{ is closed (and convex)}\},$$

$$\mathcal{P}_{(w)k(c)}(\mathcal{H}) = \{D \subset \mathcal{H} : D \text{ is (weakly) compact (and convex)}\}.$$

Definition 2.2. *Consider a multifunction $\widehat{\mathcal{K}} : \mathcal{H} \to 2^{\mathcal{H}} \setminus \{\emptyset\}$ acting on a Banach space $\mathcal{H}$. We characterize the properties of $\widehat{\mathcal{K}}$ as follows:*

(a) $\widehat{\mathcal{K}}$ is said to be closed (or convex) valued provided that the image $\widehat{\mathcal{K}}(\varpi)$ possesses the closed (or convex) property for each $\varpi \in \mathcal{H}$;

(b) The operator $\widehat{\mathcal{K}}$ is upper semicontinuous (u.s.c) on $\mathcal{H}$ if, for every ϖ_0, the set $\widehat{\mathcal{K}}(\varpi_0)$ is closed and the mapping satisfies the topological inclusion condition: for any open set $Z \supseteq \widehat{\mathcal{K}}(\varpi_0)$, one can find a neighborhood $\mathcal{V}$ of ϖ_0 ensuring $\widehat{\mathcal{K}}(\mathcal{V}) \subset Z$;

(c) $\widehat{\mathcal{K}}$ is completely continuous if the closure of the image $\widehat{\mathcal{K}}(\mathcal{B})$ is compact whenever $\mathcal{B}$ is a bounded set.

Let $\widehat{\mathcal{K}} : \mathcal{H} \to \mathbb{R}$ be a locally Lipschitz functional. The Clarke generalized directional derivative of $\mathcal{K}$ at a point $y \in \mathcal{H}$ in the direction $\mu \in \mathcal{H}$, denoted by $\mathcal{K}^{\circ}(y;\mu)$, is defined by the limit superior:

$$\widehat{\mathcal{K}}^{\circ}(y;\mu) := \limsup_{\substack{s \to y \\ \tau \downarrow 0}} \frac{\widehat{\mathcal{K}}(s+\tau\mu) - \widehat{\mathcal{K}}(s)}{\tau}.$$

Correspondingly, the Clarke generalized subdifferential (or gradient) of $\widehat{\mathcal{K}}$ at y, denoted by $\partial\widehat{\mathcal{K}}(y)$, is the subset of the dual space $\mathcal{H}^*$ defined as:

$$\partial\widehat{\mathcal{K}}(y) = \left\{\xi \in \mathcal{H}^* : \widehat{\mathcal{K}}^{\circ}(y;\mu) \geq \langle \xi, \mu\rangle, \quad \forall \mu \in \mathcal{H}\right\}.$$

These variational tools play a pivotal role in establishing our primary results.

Lemma 2.1. *Let $\widehat{\mathcal{K}} : \Omega \subseteq \mathcal{H} \to \mathbb{R}$ be locally Lipschitzian. The key properties of the Clarke subdifferential $\partial\mathcal{K}$ are summarized as follows:*

(i) $\widehat{\mathcal{K}}^{\circ}(z;\varpi) = \sup_{\xi \in \partial\widehat{\mathcal{K}}(z)} \langle \xi, \varpi\rangle$.

(ii) For all $z \in \Omega$, $\partial\widehat{\mathcal{K}}(z) \in \mathcal{P}_{w^*kc}(\mathcal{H}^*) \setminus \{\emptyset\}$, and $\sup\{\|\xi\| : \xi \in \partial\widehat{\mathcal{K}}(z)\} \leq Lip_z(\widehat{\mathcal{K}})$.

(iii) The mapping $\partial\widehat{\mathcal{K}}$ has a closed graph in $\Omega \times \mathcal{H}^*_{w^*}$. That is,

$$\begin{cases} z_n \to z \text{ in } \mathcal{H}, \\ \xi_n \xrightarrow{w^*} \xi, \\ \xi_n \in \partial\widehat{\mathcal{K}}(z_n) \end{cases}$$

implies $\xi \in \partial\widehat{\mathcal{K}}(z)$.

(iv) The multifunction $\partial\widehat{\mathcal{K}} : \Omega \to 2^{\mathcal{H}^*}$ is u.s.c. with respect to $\mathcal{H}^*_{w^*}$.

Since $\mathcal{H}$ is a Hilbert space, we identify $\mathcal{H}^*$ with $\mathcal{H}$ through the Riesz isomorphism; consequently, a selection $k(\sigma) \in \partial\mathcal{K}(\sigma, \vartheta(\sigma))$ can be regarded as an $\mathcal{H}$-valued function. From Lemma 2.1, we require finding $k \in \mathcal{L}^2(\mathcal{J}, \mathcal{H})$ with $k(\sigma) \in \partial\mathcal{K}(\sigma, \vartheta(\sigma))$ that solves the governing initial value problem:

$$\begin{cases} d\vartheta'(\sigma) = \Big[\mathcal{A}\vartheta(\sigma) + \mathcal{G}\vartheta'(\sigma) + (S_1\vartheta)(\sigma) + \Phi(\sigma, (S_2\vartheta)(\sigma), \vartheta(\sigma)) + \mathcal{B}\mathcal{U}(\sigma) + k(\sigma)\Big]\, d\sigma \\ \qquad\qquad + \varsigma(\sigma, \vartheta(\sigma))\, d\mathcal{W}(\sigma), \qquad \sigma \in \mathcal{J}, \\ \vartheta(0) = \vartheta_0, \qquad \vartheta'(0) = \vartheta_1. \end{cases}$$

To clarify the contribution of the damping term, for a sufficiently regular ϑ we have

$$\int_0^\sigma \mathcal{S}(\sigma - \iota)\mathcal{G}\vartheta'(\iota)\, d\iota = -\mathcal{S}(\sigma)\mathcal{G}\vartheta_0 + \int_0^\sigma \mathcal{C}(\sigma - \iota)\mathcal{G}\vartheta(\iota)\, d\iota,$$

where we have used $\frac{d}{d\iota}\mathcal{S}(\sigma - \iota) = -\mathcal{C}(\sigma - \iota)$, $\mathcal{S}(0) = 0$, and integration by parts. Consequently,

$$\mathcal{C}(\sigma)\vartheta_0 + \int_0^\sigma \mathcal{S}(\sigma - \iota)\mathcal{G}\vartheta'(\iota)\, d\iota = \big(\mathcal{C}(\sigma) - \mathcal{S}(\sigma)\mathcal{G}\big)\vartheta_0 + \int_0^\sigma \mathcal{C}(\sigma - \iota)\mathcal{G}\vartheta(\iota)\, d\iota.$$

This identity explains the damping contribution appearing in the mild formulation below.

Utilizing stochastic differential equation theory and the generalized Clarke's subdifferential, we introduce the second-order mild solution (1.1) in the following manner:

Definition 2.3. *We say that an $\mathcal{F}_\sigma$-adapted stochastic process $\vartheta \in \mathcal{C}(\mathcal{J}, \mathcal{L}^2(\mathcal{F}, \mathcal{H}))$ is a mild solution to the system* (1.1) *corresponding to the control $\mathcal{U} \in L^2_{\mathcal{F}}(\mathcal{J}, U)$ if it satisfies the initial conditions $\vartheta(0) = \vartheta_0$ and $\vartheta'(0) = \vartheta_1$. Additionally, there exists a selection $k \in L^2_{\mathcal{F}}(\mathcal{J}, \mathcal{H})$ with $k(\sigma) \in \partial\mathcal{K}(\sigma, \vartheta(\sigma))$ a.e. on $\mathcal{J}$, such that ϑ satisfies the integral equation:*

$$\begin{aligned} \vartheta(\sigma) \;=\; & \big(\mathcal{C}(\sigma) - \mathcal{S}(\sigma)\mathcal{G}\big)\vartheta_0 + \mathcal{S}(\sigma)\vartheta_1 + \int_0^\sigma \mathcal{C}(\sigma - \iota)\mathcal{G}\vartheta(\iota)d\iota + \int_0^\sigma \mathcal{S}(\sigma - \iota)\mathcal{B}\mathcal{U}(\iota)d\iota \\ & + \int_0^\sigma \mathcal{S}(\sigma - \iota)\Big[(S_1\vartheta)(\iota) + \Phi(\iota, (S_2\vartheta)(\iota), \vartheta(\iota))\Big]d\iota \\ & + \int_0^\sigma \mathcal{S}(\sigma - \iota)\varsigma(\iota, \vartheta(\iota))d\mathcal{W}(\iota) + \int_0^\sigma \mathcal{S}(\sigma - \iota)k(\iota)d\iota, \end{aligned}$$

for all $\sigma \in \mathcal{J}$.

In accordance with the properties of the multi-valued map Γ, any selection $k \in \Gamma(\vartheta)$ satisfies the requirement $k \in L^2_{\mathcal{F}}(\mathcal{J}, \mathcal{H})$. Combined with hypothesis $\mathcal{H}(\varsigma)$, the stochastic term $\varsigma(\cdot, \vartheta(\cdot))$ belongs to $L^2_{\mathcal{F}}(\mathcal{J}, \mathcal{L}^0_2)$, which guarantees that the stochastic convolution integrals are well-defined as martingales adapted to the filtration $\{\mathcal{F}_\sigma\}_{\sigma \geq 0}$.

We proceed by stating the prerequisite lemma and fixed point theorem required to establish our results.

Lemma 2.2. *Given a strongly measurable process $\kappa \in \mathcal{L}^q([0, b]; \mathcal{L}^0_2)$ and a constant $\mathcal{L}_\kappa = \mathcal{L}_\kappa(q, b)$ with $q \geq 2$, the stochastic convolution satisfies:*

$$E\left\|\int_0^\sigma \kappa(\iota)d\mathcal{W}(\iota)\right\|^q \leq \mathcal{L}_\kappa \int_0^\sigma E\|\kappa(\iota)\|^q_{\mathcal{L}^2_0} d\iota$$

for every $\iota \in [0, b]$.

Here, the dependence on the covariance operator Q is already incorporated into the $\mathcal{L}_0^2$-norm through

$$\|\kappa\|_{L_2^0}^2 = \mathrm{Tr}(\kappa Q \kappa^*).$$

Hence, $\mathcal{L}_\kappa = \mathcal{L}_\kappa(q, b)$ has no additional explicit dependence on $\mathrm{Tr}(Q)$.

Theorem 2.1. *[24]Let $\mathcal{X}$ be a locally convex Banach space. Consider an u.s.c., compact, convex-valued multivalued map $\mathcal{K} : \mathcal{X} \to 2^{\mathcal{X}}$ that maps a closed neighborhood of zero into a relatively compact set. The map $\mathcal{K}$ is guaranteed to have a fixed point provided that the set of all $\varpi \in \mathcal{X}$ satisfying $\varrho\varpi \in \mathcal{K}(\varpi)$ for $\varrho > 1$ is bounded.*

3. Existence of mild solutions

We now turn our attention to establishing sufficient conditions for the existence of mild solutions to system (1.1).

$\mathcal{H}(\mathcal{S})$ The operator $S(\cdot)$ is assumed to be compact on $\mathcal{J}$.

$H(\mathcal{G})$ The damping operator $\mathcal{G} \in \mathcal{L}(\mathcal{H})$ is compact.

$\mathcal{H}(\mathcal{K})$ The function $\mathcal{K} : \mathcal{J} \times \mathcal{H} \to \mathcal{R}$ satisfies:

(i) The mapping $\mathcal{K}(\cdot, \varpi)$ is measurable for all $\varpi \in \mathcal{H}$.

(ii) For a.e. $\sigma \in \mathcal{J}$, the function $\varpi \mapsto \mathcal{K}(\sigma, \varpi)$ is locally Lipschitz on $\mathcal{H}$.

(iii) There exists a function $m_{\mathcal{K}_1} \in \mathcal{L}^1(\mathcal{J}, \mathbb{R}^+)$ and a constant $m_{\mathcal{K}_2} \geq 0$ such that the generalized gradient is bounded as follows:

$$\|\partial\mathcal{K}(\sigma, \varpi)\|^2 = \sup\{\|k(\sigma)\|^2 \mid k(\sigma) \in \partial\mathcal{K}(\sigma, \varpi)\} \leq m_{\mathcal{K}_1}(\sigma) + m_{\mathcal{K}_2}\|\varpi\|^2,$$

for a.e. $\sigma \in \mathcal{J}$ and all $\varpi \in \mathcal{H}$.

$H(\varsigma)$ The function $\varsigma : \mathcal{J} \times \mathcal{H} \to \mathcal{L}_2^0$ maintains continuity with respect to the second argument for a.e. $\sigma \in \mathcal{J}$. Furthermore, we assume the existence of a function $\mathcal{M}_{\varsigma_1} \in \mathcal{L}^2(\mathcal{J}, \mathbb{R}^+)$ and a non-negative constant $\mathcal{M}_{\varsigma_2}$ such that:

$$\|\varsigma(\sigma, \vartheta)\|_{L_2^0}^2 \leq M_{\varsigma 1}(\sigma) + M_{\varsigma 2}\|\vartheta\|_{\mathcal{H}}^2,$$

for a.e. $\sigma \in \mathcal{J}$ and all $\vartheta \in \mathcal{H}$.

Furthermore, for every $\alpha > 0$, the set

$$\{\varsigma(\cdot, \vartheta(\cdot)) : \vartheta \in B_\alpha\}$$

is relatively compact in $L_{\mathcal{F}}^2(\mathcal{J}, \mathcal{L}_0^2)$.

$\mathcal{H}(\mathcal{S}_i)$: the operators $\mathcal{S}_i : \mathcal{L}^2(\mathcal{J}, \mathcal{H}) \to \mathcal{L}^2(\mathcal{J}, \mathcal{H})$ exhibit history-dependence. Specifically, for any $\varpi_1, \varpi_2 \in \mathcal{H}$ and $\sigma \in \mathcal{J}$, the following condition is satisfied:

$$\|(\mathcal{S}_i\varpi_1)(\sigma) - (\mathcal{S}_i\varpi_2)(\sigma)\|_{\mathcal{H}} \leqslant \mathcal{L}_{\mathcal{S}_i} \int_0^\sigma \|\varpi_1(\iota) - \varpi_2(\iota)\|_{\mathcal{H}}\, \mathrm{d}\iota,$$

where $\mathcal{L}_{\mathcal{S}_i} > 0$ $(i = 1, 2)$ are positive constants. The above inequality is a standard Volterra-type history-dependent condition, since $(S_i\varpi)(\sigma)$ depends only on the history of ϖ over $[0, \sigma]$.

Let us define the auxiliary functions $F_i(\sigma) := (\mathcal{S}_i 0)(\sigma)$ for $i = 1, 2$. It is observed that $F_i(\sigma)$ belongs to the space $L^2(0, \mathbb{R}^+)$. Furthermore, applying the triangle inequality yields the subsequent growth bound for almost every $\sigma \in \mathcal{J}$ and every $\varpi \in \mathcal{H}$:

$$\|(\mathcal{S}_i \varpi)(\sigma)\|_{\mathcal{H}} \leqslant F_i(\sigma) + \mathcal{L}_{\mathcal{S}_i} \int_0^\sigma \|\varpi(\iota)\|_{\mathcal{H}}\, d\iota.$$

$\mathcal{H}(\Phi)$: $\Phi : \mathcal{J} \times \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is such that

(i) $\|\Phi(\sigma, 0, 0)\| \leqslant \delta(\sigma)$ with $\delta \in \mathcal{L}^2(\mathcal{J}, \mathbb{R}_+)$,

(ii) $\|\Phi(\sigma, \theta_1, \varpi_1) - \Phi(\sigma, \theta_2, \varpi_2)\| \leqslant \mathcal{L}_\Phi(\|\theta_1 - \theta_2\| + \|\varpi_1 - \varpi_2\|)$, for $\sigma \in \mathcal{J}$ and $\theta_i, \varpi_i \in \mathcal{H}$,i=1,2 with constant $\mathcal{L}_\Phi > 0$.

Subsequently, we introduce the set-valued operator $\Gamma : \mathcal{L}^2_{\mathcal{F}}(\mathcal{J}, \mathcal{H}) \to 2^{\mathcal{L}^2_{\mathcal{F}}(J,\mathcal{H})}$, characterized by the following set of selections:

$$\Gamma(\varpi) = \left\{k \in \mathcal{L}^2_{\mathcal{F}}(J, \mathcal{H}) \mid k(\sigma) \in \partial\mathcal{K}(\sigma, \varpi(\sigma)) \text{ for a.e. } \sigma \in \mathcal{J}\right\},$$

valid for every $\varpi \in \mathcal{L}^2_{\mathcal{F}}(\mathcal{J}, \mathcal{H})$.

The following result regarding the multi-valued map is essential for the subsequent analysis. As established in [24], the validity of assumption $\mathcal{H}(\mathcal{K})$ ensures that $\Gamma(\varpi)$ admits values that are convex, weakly compact, and non-empty for any arbitrary element $\varpi \in \mathcal{L}^2_{\mathcal{F}}(\mathcal{J}, \mathcal{H})$.

In the sequel, we establish sufficient conditions that guarantee the existence of mild solutions for the system under consideration.

Theorem 3.1. *The satisfaction of assumptions $\mathcal{H}(\mathcal{S})$, $H(\mathcal{G})$, $\mathcal{H}(\mathcal{K}), \mathcal{H}(\varsigma), \mathcal{H}(\mathcal{S}_i), \mathcal{H}(\Phi)$ guarantees the existence of a mild solution to Eq* (1.1) *corresponding to every control function $\mathcal{U} \in L^2_{\mathcal{F}}(\mathcal{J}, \mathcal{U})$.*

Proof. For any $\vartheta \in C(\mathcal{J}, \mathcal{L}^2(\mathcal{F}, \mathcal{H}))$, the multi-valued operator

$$\Psi : C(\mathcal{J}, \mathcal{L}^2(\mathcal{F}, \mathcal{H})) \to 2^{C(\mathcal{J}, \mathcal{L}^2(\mathcal{F}, \mathcal{H}))}$$

is given by:

$$\begin{aligned}
&\Psi(\vartheta) = \{\lambda \in C(\mathcal{J}, \mathcal{L}^2(\mathcal{F}, \mathcal{H}))|\lambda(\sigma) \\
= \;& (C(\sigma) - \mathcal{S}(\sigma)\mathcal{G})\vartheta_0 + \mathcal{S}(\sigma)\vartheta_1 + \int_0^\sigma C(\sigma - \iota)\mathcal{G}\vartheta(\iota)d\rho + \int_0^\sigma \mathcal{S}(\sigma - \iota)\mathcal{B}\mathcal{U}(\iota)d\iota \\
&+ \int_0^\sigma \mathcal{S}(\sigma - \iota)(S_1\vartheta)(\iota) + \Phi(\iota, (S_2\vartheta)(\iota), \vartheta(\iota))\Big]d\iota \\
&+ \int_0^\sigma \mathcal{S}(\sigma - \iota)\varsigma(\iota, \vartheta(\iota))d\mathcal{W}(\iota) + \int_0^\sigma \mathcal{S}(\sigma - \iota)k(\iota)d\iota, \sigma \in \mathcal{J}, k \in \Gamma(\vartheta)\}.
\end{aligned}$$

Evidently, the problem at hand is equivalent to locating a fixed point for the operator Ψ. We proceed by demonstrating that Ψ fulfills all the prerequisites of Theorem 2.1. The detailed justification is carried out in six distinct steps.

Step 1: Assumption $\mathcal{H}(\mathcal{K})$ ensures that for each $\vartheta \in \mathcal{L}^2_{\mathcal{F}}(\mathcal{J}, \mathcal{H})$, $\Gamma(\vartheta)$ is non-empty, convex, and weakly compact. Moreover, the convexity of $\Gamma(\vartheta)$ implies the convexity of $\Psi(\vartheta)$, which ensures closure under convex combinations.

Step 2: The operator Ψ preserves boundedness on the space $C(\mathcal{J}, \mathcal{L}^2(\mathcal{F}, \mathcal{H}))$.

Consider the set $\mathcal{B}_\alpha = \{\vartheta \in C(J, \mathcal{L}^2(\mathcal{F}, \mathcal{H})) : \|\vartheta\|^2 \leq \alpha\}$, where $\alpha > 0$. Observe that $\mathcal{B}_\alpha$ is structurally closed, convex, and bounded. We claim that the operator Ψ is uniformly bounded on this set; specifically, there exists a constant $\gamma > 0$ satisfying $\|\mathcal{Z}\|^2 \leq \gamma$ for all $\mathcal{Z} \in \Psi(\vartheta)$ whenever $\vartheta \in \mathcal{B}_\alpha$.

By definition, if we take an arbitrary function $\mathcal{Z}$ from the set $\Psi(\vartheta)$, it admits a representation involving a specific function $k \in \Gamma(\varpi)$, given by:

$$\begin{aligned}\mathcal{Z}(\sigma) &= (C(\sigma) - \mathcal{S}(\sigma)\mathcal{G})\vartheta_0 + \mathcal{S}(\sigma)\vartheta_1 + \int_0^\sigma C(\sigma-\iota)\mathcal{G}\vartheta(\iota)d\iota + \int_0^\sigma \mathcal{S}(\sigma-\iota)\mathcal{B}\mathcal{U}(\iota)d\iota \\ &\quad + \int_0^\sigma \mathcal{S}(\sigma-\iota)\Big[(S_1\vartheta)(\iota) + \Phi(\iota, (S_2\vartheta)(\iota), \vartheta(\iota))\Big]d\iota \\ &\quad + \int_0^\sigma \mathcal{S}(\sigma-\iota)\varsigma(\iota, \vartheta(\iota))d\mathcal{W}(\iota) + \int_0^\sigma \mathcal{S}(\sigma-\iota)k(\iota)d\iota, \end{aligned} \tag{3.1}$$

for all $\sigma \in \mathcal{J}$ and some selection $k \in \Gamma(\vartheta)$. By the definition of Γ, $k \in L^2_{\mathcal{F}}(\mathcal{J}, \mathcal{H})$. Since $\mathcal{S}(\cdot)$ is uniformly bounded on $\mathcal{J}$, the Bochner integral $\int_0^\sigma \mathcal{S}(\sigma-\iota)k(\iota)\,d\iota$ is well defined for every $\sigma \in \mathcal{J}$.

Invoking Eq (3.1), Lemma 2.2 and Hölder's inequality, and using

$$\left\|\sum_{j=1}^n x_j\right\|^2 \leq 2^{n-1}\sum_{j=1}^n \|x_j\|^2,$$

with $n = 8$, we obtain the factor 2^7 in the following estimate for any $\sigma \in \mathcal{J}$. The additional factor 2 in the first bracket comes from separating $(C(\sigma) - \mathcal{S}(\sigma)\mathcal{G})\vartheta_0$.

$$\begin{aligned} & E\|\mathcal{Z}(\sigma)\|^2 \\ = \; & E\Big\|(C(\sigma) - \mathcal{S}(\sigma)\mathcal{G})\vartheta_0 + \mathcal{S}(\sigma)\vartheta_1 + \int_0^\sigma C(\sigma-\iota)\mathcal{G}\vartheta(\iota)d\iota + \int_0^\sigma \mathcal{S}(\sigma-\iota)\mathcal{B}\mathcal{U}(\iota)d\iota \\ & + \int_0^\sigma \mathcal{S}(\sigma-\iota)\Big[(S_1\vartheta)(\iota) + \Phi(\iota, (S_2\vartheta)(\iota), \vartheta(\iota))\Big]d\iota \\ & + \int_0^\sigma \mathcal{S}(\sigma-\iota)\varsigma(\iota, \vartheta(\iota))d\mathcal{W}(\iota) + \int_0^\sigma \mathcal{S}(\sigma-\iota)k(\iota)d\iota\Big\|^2 \\ \leq \; & 2^7\Big(2[E\|C(\sigma)\vartheta_0\|^2 + E\|\mathcal{S}(\sigma)\mathcal{G}\vartheta_0\|^2] + E\|\mathcal{S}(\sigma)\vartheta_1\|^2 \\ & + E\left\|\int_0^\sigma C(\sigma-\iota)\mathcal{G}\vartheta(\iota)d\iota\right\|^2 + E\left\|\int_0^\sigma \mathcal{S}(\sigma-\iota)\mathcal{B}\mathcal{U}(\iota)d\iota\right\|^2 \\ & + E\left\|\int_0^\sigma \mathcal{S}(\sigma-\iota)(S_1\vartheta)(\iota)d\iota\right\|^2 + E\|\int_0^\sigma \mathcal{S}(\sigma-\iota)\Phi(\iota, (S_2\vartheta)(\iota), \vartheta(\iota))d\iota\|^2 \\ & + E\left\|\int_0^\sigma \mathcal{S}(\sigma-\iota)\varsigma(\iota, \vartheta(\iota))d\mathcal{W}(\iota)\right\|^2 + E\left\|\int_0^\sigma \mathcal{S}(\sigma-\iota)k(\iota)d\iota\right\|^2\Big) \\ \leq \; & 2^7\Big(2[\mathbb{M}^2E\|\vartheta_0\|^2 + \|\mathbb{N}\|^2\|\mathcal{G}\|^2E\|\vartheta_0\|^2] + \mathbb{N}^2E\|\vartheta_1\|^2 \\ & + b\int_0^\sigma E\|C(\sigma-\iota)\mathcal{G}\vartheta(\iota)\|^2d\iota + b\int_0^\sigma E\|\mathcal{S}(\sigma-\iota)\mathcal{B}\mathcal{U}(\iota)\|^2d\iota \end{aligned}$$

$$
\begin{aligned}
&+b\int_0^{\sigma} E\|\mathcal{S}(\sigma-\iota)(S_1\vartheta)(\iota)\|^2 d\iota + b\int_0^{\sigma} E\|\mathcal{S}(\sigma-\iota)\Phi(\iota,(S_2\vartheta)(\iota),\vartheta(\iota))\|^2 d\iota\\
&+\int_0^{\sigma} E\|\mathcal{S}(\sigma-\iota)\varsigma(\iota,\vartheta(\iota))\|^2 d\iota + b\int_0^{\sigma} E\|\mathcal{S}(\sigma-\iota)k(\iota)\|^2 d\iota\Big)\\
\leq\ & 2^7\Big(2[\mathbb{M}^2 E\|\vartheta_0\|^2 + \|\mathbb{N}\|^2\|\mathcal{G}\|^2 E\|\vartheta_0\|^2] + \mathbb{N}^2 E\|\vartheta_1\|^2\\
&+b\|\mathbb{M}\|^2\int_0^{\sigma}\|\mathcal{G}\|^2 E\|\vartheta(\iota)\|^2 d\iota + b\|\mathbb{N}\|^2\int_0^{\sigma}\|\mathcal{B}\|^2 E\|\mathcal{U}(\iota)\|^2 d\iota\\
&+b\|\mathbb{N}\|^2\int_0^{\sigma} E\left(F_1(\iota)+\mathcal{L}_{\mathcal{S}_1}\int_0^{\iota}\|\vartheta(r)\|_{\mathcal{H}}\,dr\right)^2 d\iota\\
&+b\|\mathbb{N}\|^2\int_0^{\sigma} E\left\|\mathcal{L}_{\Phi}\left(\|\vartheta(\iota)\|+F_2(\iota)+\mathcal{L}_{\mathcal{S}_2}\int_0^{\iota}\|\vartheta(r)\|_{\mathcal{H}}\,dr\right)+\delta(\iota)\right\|^2 d\iota\\
&+\|\mathbb{N}\|^2\int_0^{\sigma} E(\mathcal{M}_{\varsigma_1}(\iota)+\mathcal{M}_{\varsigma_2}\|\vartheta\|^2)d\iota + b\|\mathbb{N}\|^2\int_0^{\sigma} E(m_{\mathcal{K}_1}(\iota)+m_{\mathcal{K}_2}\|\vartheta\|^2)d\iota\Big)\\
\leq\ & 2^7\Big(2[\mathbb{M}^2 E\|\vartheta_0\|^2 + \|\mathbb{N}\|^2\|\mathcal{G}\|^2 E\|\vartheta_0\|^2] + \mathbb{N}^2 E\|\vartheta_1\|^2\\
&+b^2\|\mathbb{M}\|^2\|\mathcal{G}\|^2\alpha + b\|\mathbb{N}\|^2\|\mathcal{B}\|^2\|\mathcal{U}\|^2_{\mathcal{L}^2_{\mathcal{F}}(\mathcal{J},U)}\\
&+2b\|\mathbb{N}\|^2\int_0^{\sigma}\|F_1(\iota)\|^2 d\iota + 2b^4\|\mathbb{N}\|^2\mathcal{L}^2_{\mathcal{S}_1}\alpha\\
&+4b^2\|\mathbb{N}\|^2\mathcal{L}_{\Phi}\alpha + 8b^2\|\mathbb{N}\|^2\mathcal{L}_{\Phi}\int_0^{\sigma}\|F_2(\iota)\|^2 d\iota + 8b^4\|\mathbb{N}\|^2\mathcal{L}_{\Phi}\mathcal{L}_{\mathcal{S}_2}\alpha + 2b\|\mathbb{N}\|^2\int_0^{\sigma}\|\delta(\iota)\|^2 d\iota\\
&+\|\mathbb{N}\|^2\int_0^{\sigma}\mathcal{M}_{\varsigma_1}(\iota)d\iota + \|\mathbb{N}\|^2\mathcal{M}_{\varsigma_2}\alpha b + b\|\mathbb{N}\|^2\int_0^{\sigma} m_{\mathcal{K}_1}(\iota)d\iota + b^2\|\mathbb{N}\|^2 m_{\mathcal{K}_2}\alpha\Big) := \gamma.
\end{aligned}
$$

This confirms the boundedness of the image set $\Psi(\mathcal{B}_\alpha)$ in $C(\mathcal{J},\mathcal{L}^2(\mathcal{F},\mathcal{H}))$.

Step 3: The family of functions given by $\{\Psi(\vartheta) : \vartheta \in B_\alpha\}$ is equicontinuous.

Specifically, whenever we select an arbitrary ϑ from $\mathcal{B}_\alpha$, $\mathcal{Z} \in \Psi(\vartheta)$ we can identify a $k \in \Gamma(\vartheta)$ that satisfies Eq (3.1) for every $\sigma \in \mathcal{J}$.

Choosing arbitrary σ_1 and σ_2 such that $0 < \sigma_1 < \sigma_2 \leq b$ yields

$$
\begin{aligned}
&E\|\mathcal{Z}(\sigma_2)-\mathcal{Z}(\sigma_1)\|^2\\
\leq\ & 2^7\Big(2[\|C(\sigma_2)-C(\sigma_1)\|^2 E\|\vartheta_0\|^2 + \|\mathcal{S}(\sigma_2)-\mathcal{S}(\sigma_1)\|^2 E\|\mathcal{G}\vartheta_0\|^2]\\
&+\|\mathcal{S}(\sigma_2)-\mathcal{S}(\sigma_1)\|^2 E\|\vartheta_1\|^2\\
&+E\|\int_0^{\sigma_2} C(\sigma_2-\iota)\mathcal{G}\vartheta(\iota)d\iota - \int_0^{\sigma_1} C(\sigma_1-\iota)\mathcal{G}\vartheta(\iota)d\iota\|^2\\
&+E\left\|\int_0^{\sigma_2}\mathcal{S}(\sigma_2-\iota)\mathcal{B}\mathcal{U}(\iota)d\iota - \int_0^{\sigma_1}\mathcal{S}(\sigma_1-\iota)\mathcal{B}\mathcal{U}(\iota)d\iota\right\|^2\\
&+E\left\|\int_0^{\sigma_2}\mathcal{S}(\sigma_2-\iota)(S_1\vartheta)(\iota)d\iota - \int_0^{\sigma_1}\mathcal{S}(\sigma_1-\iota)(S_1\vartheta)(\iota)d\iota\right\|^2\\
&+E\left\|\int_0^{\sigma_2}\mathcal{S}(\sigma_2-\iota)\Phi(\iota,(S_2\vartheta)(\iota),\vartheta(\iota))d\iota - \int_0^{\sigma_1}\mathcal{S}(\sigma_1-\iota)\Phi(\iota,(S_2\vartheta)(\iota),\vartheta(\iota))d\iota\right\|^2
\end{aligned}
$$

$$+E\left\|\int_0^{\sigma_2}\mathcal{S}(\sigma_2-\iota)\varsigma(\iota,\vartheta(\iota))d\mathcal{W}(\iota)-\int_0^{\sigma_1}\mathcal{S}(\sigma_1-\iota)\varsigma(\iota,\vartheta(\iota))d\mathcal{W}(\iota)\right\|^2$$
$$+E\left\|\int_0^{\sigma_2}\mathcal{S}(\sigma_2-\iota)k(\iota)d\iota-\int_0^{\sigma_1}\mathcal{S}(\sigma_1-\iota)k(\iota)d\iota\right\|^2\Big).$$

We now proceed to estimate each term of the preceding inequality. Using the uniform boundedness of the cosine family, $\|C(\sigma)\|\leq\mathbb{M}$, and the operator-norm continuity of $\sigma\mapsto C(\sigma)\mathcal{G}$, we obtain

$$\begin{aligned}
&E\left\|\int_0^{\sigma_2}C(\sigma_2-\iota)\mathcal{G}\vartheta(\iota)\,d\iota-\int_0^{\sigma_1}C(\sigma_1-\iota)\mathcal{G}\vartheta(\iota)\,d\iota\right\|^2\\
\leq&2E\left\|\int_0^{\sigma_1}[C(\sigma_2-\iota)-C(\sigma_1-\iota)]\mathcal{G}\vartheta(\iota)\,d\iota\right\|^2+2E\left\|\int_{\sigma_1}^{\sigma_2}C(\sigma_2-\iota)\mathcal{G}\vartheta(\iota)\,d\iota\right\|^2\\
\leq&2b^2\alpha\sup_{\iota\in[0,\sigma_1]}\left\|[C(\sigma_2-\iota)-C(\sigma_1-\iota)]\mathcal{G}\right\|^2+2\mathbb{M}^2\|\mathcal{G}\|^2(\sigma_2-\sigma_1)^2\alpha.
\end{aligned}\tag{3.2}$$

and

$$\begin{aligned}
&E\left\|\int_0^{\sigma_2}\mathcal{S}(\sigma_2-\iota)\mathcal{B}\mathcal{U}(\iota)d\iota-\int_0^{\sigma_1}\mathcal{S}(\sigma_1-\iota)\mathcal{B}\mathcal{U}(\iota)d\iota\right\|^2\\
\leq\;&2\sup_{\iota\in(0,\sigma_1)}\|\mathcal{S}(\sigma_2-\iota)-\mathcal{S}(\sigma_1-\iota)\|^2b\|\mathcal{B}\|^2\|\mathcal{U}\|^2_{L^2_{\mathcal{F}}(J,\mathcal{U})}\\
&+2\|\mathbb{N}\|^2\|\mathcal{B}\|^2\|\mathcal{U}\|^2_{L^2_{\mathcal{F}}(J,\mathcal{U})}(\sigma_2-\sigma_1).
\end{aligned}\tag{3.3}$$

From condition $\mathcal{H}(\mathcal{S}_i)$, we get

$$\begin{aligned}
&E\left\|\int_0^{\sigma_2}\mathcal{S}(\sigma_2-\iota)(S_1\vartheta)(\iota)d\iota-\int_0^{\sigma_1}\mathcal{S}(\sigma_1-\iota)(S_1\vartheta)(\iota)d\iota\right\|^2\\
\leq\;&4\sup_{\iota\in(0,\sigma_1)}\|\mathcal{S}(\sigma_2-\iota)-\mathcal{S}(\sigma_1-\iota)\|\left[\int_0^{\sigma_1}F_1^2(\iota)d\iota+\mathcal{L}^2_{\mathcal{S}_1}\alpha b^4\right]\\
&+2\mathbb{N}^2\left[\int_{\sigma_1}^{\sigma_2}F_1(\iota)+\mathcal{L}_{\mathcal{S}_1}\int_0^{\iota}\|\vartheta(r)\|drd\iota\right]^2\\
\leq\;&4\sup_{\iota\in(0,\sigma_1)}\|\mathcal{S}(\sigma_2-\iota)-\mathcal{S}(\sigma_1-\iota)\|\left[\int_0^{\sigma_1}F_1^2(\iota)d\iota+\mathcal{L}^2_{\mathcal{S}_1}\alpha b^4\right]\\
&+4\mathbb{N}^2\left[\int_0^{b}F_1^2(\iota)d\iota(\sigma_2-\sigma_1)+\mathcal{L}^2_{\mathcal{S}_1}b^2\alpha(\sigma_2-\sigma_1)^2\right].
\end{aligned}\tag{3.4}$$

By $H(\Phi)$ and $H(S_2)$, the estimate of $\Phi(\iota,(S_2\vartheta)(\iota),\vartheta(\iota))$ contains four terms. Using

$$\left(\sum_{j=1}^{4}x_j\right)^2\leq 2^3\sum_{j=1}^{4}x_j^2$$

together with the factor $2N^2$ from the preceding estimate gives 2^4N^2.

$$
\begin{aligned}
& E\left\|\int_0^{\sigma_2}\mathcal{S}(\sigma_2-\iota)\Phi(\iota,(S_2\vartheta)(\iota),\vartheta(\iota))d\iota-\int_0^{\sigma_1}\mathcal{S}(\sigma_1-\iota)\Phi(\iota,(S_2\vartheta)(\iota),\vartheta(\iota))d\iota\right\|^2\\
\leq\ & 2E\left\|\int_{\sigma_1}^{\sigma_2}\mathcal{S}(\sigma_2-\iota)\Phi(\iota,(S_2\vartheta)(\iota),\vartheta(\iota))d\iota\right\|^2\\
& +2E\left\|\int_0^{\sigma_1}(\mathcal{S}(\sigma_2-\iota)-\mathcal{S}(\sigma_1-\iota))\Phi(\iota,(S_2\vartheta)(\iota),\vartheta(\iota))d\iota\right\|^2\\
\leq\ & 2\mathbb{N}^2E\left\|\int_{\sigma_1}^{\sigma_2}\mathcal{L}_\Phi(\|(S_2\vartheta)(\iota)\|+\|\vartheta(\iota)\|)+\delta(\iota)d\iota\right\|^2\\
& +2\sup_{\iota\in(0,\sigma_1)}\|\mathcal{S}(\sigma_2-\iota)-\mathcal{S}(\sigma_1-\iota)\|^2E\left\|\int_0^{\sigma_1}\mathcal{L}_\Phi(\|(S_2\vartheta)(\iota)\|+\|\vartheta(\iota)\|)+\delta(\iota)d\iota\right\|^2\\
\leq\ & 2\mathbb{N}^2E\left\|\int_{\sigma_1}^{\sigma_2}\mathcal{L}_\Phi\left(F_2(\iota)+\mathcal{L}_{\mathcal{S}_2}\int_0^{\iota}\|\vartheta(r)\|_{\mathcal{H}}dr+\|\vartheta(\iota)\|\right)+\delta(\iota)d\iota\right\|^2\\
& +2\sup_{\iota\in(0,\sigma_1)}\|\mathcal{S}(\sigma_2-\iota)-\mathcal{S}(\sigma_1-\iota)\|^2E\left\|\int_0^{\sigma_1}\mathcal{L}_\Phi\left(F_2(\iota)+\mathcal{L}_{\mathcal{S}_2}\int_0^{\iota}\|\vartheta(r)\|_{\mathcal{H}}dr+\|\vartheta(\iota)\|\right)+\delta(\iota)d\iota\right\|^2\\
\leq\ & 2^4\mathbb{N}^2(\sigma_2-\sigma_1)\Big(\int_0^{\sigma_2}\mathcal{L}_\Phi^2F_2^2(\iota)d\iota+\mathcal{L}_\Phi\mathcal{L}_{\mathcal{S}_2}\alpha b^3+\mathcal{L}_\Phi\alpha b+\int_0^{\sigma_2}\delta^2(\iota)d\iota\Big)+\\
& 2^4\sup_{\iota\in(0,\sigma_1)}\|\mathcal{S}(\sigma_2-\iota)-\mathcal{S}(\sigma_1-\iota)\|^2b\Big(\int_0^{\sigma_2}\mathcal{L}_\Phi^2F_2^2(\iota)d\iota+\mathcal{L}_\Phi\mathcal{L}_{\mathcal{S}_2}\alpha b^3+\mathcal{L}_\Phi\alpha b)+\int_0^{\sigma_2}\delta^2(\iota)d\iota\Big). \qquad (3.5)
\end{aligned}
$$

According to condition $\mathcal{H}(\varsigma)$ and Lemma 2.2, we deduce

$$
\begin{aligned}
& E\left\|\int_0^{\sigma_2}\mathcal{S}(\sigma_2-\iota)\varsigma(\iota,\vartheta(\iota))d\mathcal{W}(\iota)-\int_0^{\sigma_1}\mathcal{S}(\sigma_1-\iota)\varsigma(\iota,\vartheta(\iota))d\mathcal{W}(\iota)\right\|^2\\
\leq\ & 2E\left\|\int_{\sigma_1}^{\sigma_2}\mathcal{S}(\sigma_2-\iota)\varsigma(\iota,\vartheta(\iota))d\mathcal{W}(\iota)\right\|^2\\
& +2E\left\|\int_0^{\sigma_1}(\mathcal{S}(\sigma_2-\iota)-\mathcal{S}(\sigma_1-\iota))\varsigma(\iota,\vartheta(\iota))d\mathcal{W}(\iota)\right\|^2\\
\leq\ & 2c(\sigma_2-\sigma_1)\int_{\sigma_1}^{\sigma_2}E\|\mathcal{S}(\sigma_2-\iota)\varsigma(\iota,\vartheta(\iota))\|^2d\iota\\
& +2cb\int_0^{\sigma_1}E\|(\mathcal{S}(\sigma_2-\iota)-\mathcal{S}(\sigma_1-\iota))\varsigma(\iota,\vartheta(\iota))\|^2d\iota\\
\leq\ & 2c\mathbb{N}^2(\sigma_2-\sigma_1)\int_{\sigma_1}^{\sigma_2}(\mathcal{M}_{\varsigma_1}(\iota)+\mathcal{M}_{\varsigma_2}E\|\vartheta\|^2)d\iota\\
& +2cb\sup_{\iota\in[0,\sigma_1]}\|\mathcal{S}(\sigma_2-\iota)-\mathcal{S}(\sigma_1-\iota)\|^2\int_0^{\sigma_1}(\mathcal{M}_{\varsigma_1}(\iota)+\mathcal{M}_{\varsigma_2}E\|\vartheta\|^2)d\iota\\
\leq\ & 2c\mathbb{N}^2(\sigma_2-\sigma_1)^{3/2}\left(\int_0^b(\mathcal{M}_{\varsigma_1}(\sigma))^2d\sigma\right)^{1/2}+2c\mathbb{N}^2\mathcal{M}_{\varsigma_2}\alpha(\sigma_2-\sigma_1)\\
& +2cb\sup_{\iota\in[0,\sigma_1]}\|\mathcal{S}(\sigma_2-\iota)-\mathcal{S}(\sigma_1-\iota)\|^2\int_0^{\sigma_1}(\mathcal{M}_{\varsigma_1}(\sigma)+\mathcal{M}_{\varsigma_2}\alpha)d\iota.
\end{aligned}
$$

It follows from condition $\mathcal{H}(\mathcal{K})(iii)$ that, for $k(\iota) \in \partial\mathcal{K}(\iota, \vartheta(\iota))$, $E\|k(\iota)\|^2 \leq m_{\mathcal{K}1}(\iota) + m_{\mathcal{K}2}E\|\vartheta(\iota)\|^2 \leq m_{\mathcal{K}1}(\iota) + m_{\mathcal{K}2}\alpha$. Consequently, we obtain

$$\begin{aligned}
& E\left\|\int_0^{\sigma_2} \mathcal{S}(\sigma_2-\iota)k(\iota)d\iota - \int_0^{\sigma_1} \mathcal{S}(\sigma_1-\iota)k(\iota)d\iota\right\|^2 \\
\leq\ & (\sigma_2-\sigma_1)\|\mathbb{N}\|^2 \int_{\sigma_1}^{\sigma_2} E\|k(\iota)\|^2 d\iota + \sup_{\iota\in[0,\sigma_1]} \|(\mathcal{S}(\sigma_2-\iota)-\mathcal{S}(\sigma_1-\iota))\|^2\|\mathbb{N}\|^2 b\int_0^{\sigma_1} E\|k(\iota)\|^2 d\iota \\
\leq\ & (\sigma_2-\sigma_1)\|\mathbb{N}\|^2 \int_0^{\sigma_2} m_{\mathcal{K}_1}(\iota)d\iota + (\sigma_2-\sigma_1)\|\mathbb{N}\|^2 m_{\mathcal{K}_2}\alpha b^2 \\
& + \sup_{\iota\in[0,\sigma_1]} \|\mathcal{S}(\sigma_2-\iota)-\mathcal{S}(\sigma_1-\iota)\|^2\|\mathbb{N}\|^2 \int_0^{\sigma_1} m_{\mathcal{K}_1}(\iota) + m_{\mathcal{K}_2}\|\vartheta\|^2 d\iota. \qquad (3.6)
\end{aligned}$$

By the integral representation of the sine family and the local boundedness of $C(\cdot)$, the mapping $\sigma \mapsto \mathcal{S}(\sigma)$ is continuous in the operator norm. Moreover, since $\mathcal{G}$ is compact and $C(\cdot)$ is strongly continuous, the mapping $\sigma \mapsto C(\sigma)\mathcal{G}$ is continuous in the operator norm. Consequently, we observe from (3.2)–(3.6) that the difference term approaches zero as $\sigma_2 \to \sigma_1$, uniformly for $\vartheta \in \mathcal{B}_\alpha$. This establishes that the mapping $\Psi(\vartheta)$ is continuous from the right on the interval $(0, b]$.

The preceding argument establishes the equicontinuity of $\{\Psi(\vartheta) : \vartheta \in \mathcal{B}_\alpha\}$ regarding the topology of $C(\mathcal{J}, \mathcal{L}^2(\mathcal{F}, \mathcal{H}))$.

Step 4: The operator Ψ is said to be completely continuous.

The precompactness of the set $\Pi(\sigma) = \{\mathcal{Z}(\sigma) : \mathcal{Z} \in \Psi(\mathcal{B}_\alpha)\}$ in $\mathcal{H}$ is established via the Arzelà-Ascoli theorem. Following the technical estimates detailed in Steps 2 and 3, we verify that the image of the ball $\mathcal{B}_\alpha$ under the solution operator constitutes a relatively compact subset, thereby satisfying the criteria set forth in Definition 2.2.

Establishing the compactness of $\Pi(0) = \{\vartheta_0\}$, we then fix $0 < \sigma \leq b$. For every ϑ in the intersection of $\mathcal{B}_\alpha$ and $\vartheta \in \Psi(\vartheta)$, we can find a $k \in \Gamma(\varpi)$ that satisfies (3.1) across the entire interval $\mathcal{J}$. Given any $\vartheta \in \mathcal{B}_\alpha$, let $\mu \in (0, \sigma)$ and define

$$\begin{aligned}
\mathcal{Z}^\mu(\sigma) &= (C(\sigma) - \mathcal{S}(\sigma)\mathcal{G})\vartheta_0 + \mathcal{S}(\sigma)\vartheta_1 + \int_0^{\sigma-\mu} C(\sigma-\iota)\mathcal{G}\vartheta(\iota)d\iota + \int_0^{\sigma-\mu} \mathcal{S}(\sigma-\iota)\mathcal{B}\mathcal{U}(\iota)d\iota \\
&\quad + \int_0^{\sigma-\mu} \mathcal{S}(\sigma-\iota)\Big[(S_1\vartheta)(\iota) + \Phi(\iota, (S_2\vartheta)(\iota), \vartheta(\iota))\Big]d\iota \\
&\quad + \int_0^{\sigma-\mu} \mathcal{S}(\sigma-\iota)\varsigma(\iota, \vartheta(\iota))d\mathcal{W}(\iota) + \int_0^{\sigma-\mu} \mathcal{S}(\sigma-\iota)k(\iota)d\iota. \qquad (3.7)
\end{aligned}$$

By hypothesis $H(\mathcal{S})$, $\mathcal{S}(\sigma)$ is compact for every $\sigma > 0$. Moreover, hypothesis $H(\mathcal{G})$, together with the boundedness of $C(\sigma)$, implies that the composition $C(\sigma)\mathcal{G}$ is compact for every $\sigma > 0$. For fixed $0 < \mu < \sigma$ and $\iota \in [0, \sigma-\mu]$, we have $\sigma - \iota \geq \mu > 0$. Hence, by $\mathcal{H}(\mathcal{S})$, $\mathcal{S}(\sigma-\iota)$ is compact on the truncated interval, which yields the relative compactness of the corresponding deterministic terms in $\Pi_\mu(\sigma)$.

It remains to consider the stochastic convolution term. Let $\{\vartheta_n\} \subset B_\alpha$ be arbitrary. By hypothesis $H(\varsigma)$, there exist a subsequence, still denoted by $\{\vartheta_n\}$, and $q \in L^2_{\mathcal{F}}(\mathcal{J}, \mathcal{L}^0_2)$ such that

$$\varsigma(\cdot, \vartheta_n(\cdot)) \longrightarrow q(\cdot) \quad \text{in} \quad L^2_{\mathcal{F}}(\mathcal{J}, \mathcal{L}^0_2).$$

For fixed $\sigma \in (0, b]$ and $0 < \mu < \sigma$, Lemma 2.2 yields

$$\mathbb{E}\left\|\int_0^{\sigma-\mu} \mathcal{S}(\sigma-\iota)[\varsigma(\iota,\vartheta_n(\iota)) - q(\iota)]\, d\mathcal{W}(\iota)\right\|_{\mathcal{H}}^2 \leq N^2 \int_0^{\sigma-\mu} \mathbb{E}\,\|\varsigma(\iota,\vartheta_n(\iota)) - q(\iota)\|_{\mathcal{L}_2^0}^2\, d\iota \longrightarrow 0.$$

Hence, the family

$$\left\{\int_0^{\sigma-\mu} \mathcal{S}(\sigma-\iota)\varsigma(\iota,\vartheta(\iota))\, d\mathcal{W}(\iota) : \vartheta \in B_\alpha\right\}$$

is relatively compact in $L^2(\mathcal{F},\mathcal{H})$.

Combining the above result with the compactness of the deterministic truncated terms, we conclude that $\Pi_\mu(\sigma)$ is relatively compact in $L^2(\mathcal{F},\mathcal{H})$ for every $0 < \mu < \sigma$.

By (3.1) and (3.7), we get

$$\begin{aligned}
& E\|\mathcal{Z}(\sigma) - \mathcal{Z}^\mu(\sigma)\|^2 \\
\leq\ & E\Big\| \int_{\sigma-\mu}^{\sigma} C(\sigma-\iota)\mathcal{G}\vartheta(\iota)d\iota + \int_{\sigma-\mu}^{\sigma} \mathcal{S}(\sigma-\iota)\mathcal{B}\mathcal{U}(\iota)d\iota \\
& + \int_{\sigma-\mu}^{\sigma} \mathcal{S}(\sigma-\iota)\Big[(S_1\vartheta)(\iota) + \Phi(\iota,(S_2\vartheta)(\iota),\vartheta(\iota))\Big]d\iota \\
& + \int_{\sigma-\mu}^{\sigma} \mathcal{S}(\sigma-\iota)\varsigma(\iota,\vartheta(\iota))d\mathcal{W}(\iota) + \int_{\sigma-\mu}^{\sigma} \mathcal{S}(\sigma-\iota)k(\iota)d\iota\Big\|^2 \\
\leq\ & 2^5\Big(E\Big\| \int_{\sigma-\mu}^{\sigma} C(\sigma-\iota)\mathcal{G}\vartheta(\iota)d\iota\Big\|^2 + E\| \int_{\sigma-\mu}^{\sigma} \mathcal{S}(\sigma-\iota)\mathcal{B}\mathcal{U}(\iota)d\iota\|^2 \\
& + E\left\| \int_{\sigma-\mu}^{\sigma} \mathcal{S}(\sigma-\iota)(S_1\vartheta)(\iota)d\iota\right\|^2 + E\left\| \int_{\sigma-\mu}^{\sigma} \Phi(\iota,(S_2\vartheta)(\iota),\vartheta(\iota))d\iota\right\|^2 \\
& + E\left\| \int_{\sigma-\mu}^{\sigma} \mathcal{S}(\sigma-\iota)\varsigma(\iota,\vartheta(\iota))d\mathcal{W}(\iota)\right\|^2 + E\left\| \int_{\sigma-\mu}^{\sigma} \mathcal{S}(\sigma-\iota)k(\iota)d\iota\right\|^2\Big) \\
\leq\ & 2^5(\|\mathcal{G}\|^2\|\mathbb{M}\|^2\mu^2\alpha + \|\mathbb{N}\|^2\|\mathcal{B}\|^2\|\mathcal{U}\|^2_{\mathcal{L}^2_{\mathcal{F}}(\mathcal{J},U)}\mu \\
& + 2\mathbb{N}^2\left[\int_0^b F_1^2(\iota)d\iota\mu + \mathcal{L}^2_{\mathcal{S}_1} b^2\alpha\mu^2\right] \\
& + 2^3\mathbb{N}^2\mu\Big(\int_0^{\sigma_2} \mathcal{L}^2_\Phi F_2^2(\iota)d\iota + \mathcal{L}_\Phi\mathcal{L}_{\mathcal{S}_2}\alpha b^3 + \mathcal{L}_\Phi\alpha b + \int_0^{\sigma_2} \delta^2(\iota)d\iota\Big) \\
& + c\mathbb{N}^2\mu^{3/2}\left(\int_0^b (\mathcal{M}_{\varsigma_1}(\sigma))^2 d\sigma\right)^{1/2} + 2c\mathbb{N}^2\mathcal{M}_{\mathcal{S}_2}\alpha\mu \\
& + \mu\|\mathbb{N}\|^2\int_0^{\sigma_2} m_{\mathcal{K}_1}(\iota)d\iota + \mu\|\mathbb{N}\|^2 m_{\mathcal{K}_2}\alpha b^2.
\end{aligned}$$

The preceding estimate shows that

$$\mathbb{E}\|Z(\sigma) - Z_\mu(\sigma)\|^2_{\mathcal{H}} \longrightarrow 0 \quad \text{as } \mu \to 0^+,$$

uniformly for $\vartheta \in B_\alpha$. Since $\Pi_\mu(\sigma)$ is relatively compact in $L^2(\mathcal{F},\mathcal{H})$ for every $0 < \mu < \sigma$, it follows that $\Pi(\sigma)$ is relatively compact in $L^2(\mathcal{F},\mathcal{H})$.

Step 5: The graph of the operator Ψ is a closed set.

Consider sequences $\{\vartheta_n\} \subset C(\mathcal{J}, L^2(\mathcal{F}, \mathcal{H}))$ and $\mathcal{Z}_n \in \Psi(\vartheta_n)$ such that $\vartheta_n \to \vartheta^*$ and $\mathcal{Z}_n \to \mathcal{Z}^*$ as $n \to \infty$. Our objective is to establish that $\mathcal{Z}^* \in \Psi(\vartheta^*)$. By $\mathcal{Z}_n \in \Psi(\vartheta_n)$, we can find $k_n \in \Gamma(\vartheta_n)$ such that

$$\begin{aligned}\mathcal{Z}_n(\sigma) &= (C(\sigma) - \mathcal{S}(\sigma)\mathcal{G})\vartheta_0 + \mathcal{S}(\sigma)\vartheta_1 + \int_0^\sigma C(\sigma-\iota)\mathcal{G}\vartheta_n(\iota)d\iota + \int_0^\sigma \mathcal{S}(\sigma-\iota)\mathcal{B}\mathcal{U}(\iota)d\iota \\ &\quad + \int_0^\sigma \mathcal{S}(\sigma-\iota)\Big[(S_1\vartheta_n)(\iota) + \Phi(\iota, (S_2\vartheta_n)(\iota), \vartheta_n(\iota))\Big]d\iota \\ &\quad + \int_0^\sigma \mathcal{S}(\sigma-\iota)\varsigma(\iota, \vartheta_n(\iota))d\mathcal{W}(\iota) + \int_0^\sigma \mathcal{S}(\sigma-\iota)k_n(\iota)d\iota. \end{aligned} \tag{3.8}$$

One can readily verify that the boundedness of $\{(k_n, \varsigma(\cdot, \vartheta_n))\}$ in the product space $L^2_{\mathcal{F}}(\mathcal{J}, \mathcal{H}) \times L^2_{\mathcal{F}}(\mathcal{J}, \mathcal{L}^0_2)$ follows directly from conditions $H(K)$ and $H(\varsigma)$. A subsequence can be extracted such that $k_n \rightharpoonup k^*$ weakly in $L^2_{\mathcal{F}}(\mathcal{J}, \mathcal{H})$.

Since $\vartheta_n \to \vartheta^*$ strongly in $C(\mathcal{J}, L^2(\mathcal{F}, \mathcal{H}))$, it also converges strongly in $L^2_{\mathcal{F}}(\mathcal{J}, \mathcal{H})$. By the strong–weak closedness of the Clarke subdifferential stated in Lemma 2.1(iii), together with $H(\mathcal{K})(i)$, the induced selection mapping Γ is strong–weak sequentially closed. Therefore, $k^* \in \Gamma(\vartheta^*)$. Moreover, by the relative compactness assumption in $H(\varsigma)$, passing to a further subsequence if necessary, $\varsigma(\cdot, \vartheta_n(\cdot)) \longrightarrow \varsigma(\cdot, \vartheta^*(\cdot))$ strongly in $L^2_{\mathcal{F}}(\mathcal{J}, \mathcal{L}^0_2)$. Hence, the corresponding stochastic convolution converges strongly by Lemma 3.1.

Since $\mathcal{S}(\cdot)$ is uniformly bounded, the mapping $k \mapsto \int_0^\sigma \mathcal{S}(\sigma-\iota)k(\iota)\,d\iota$ is bounded and linear. Therefore, $k_n \rightharpoonup k^*$ implies

$$\int_0^\sigma \mathcal{S}(\sigma-\iota)k_n(\iota)\,d\iota \rightharpoonup \int_0^\sigma \mathcal{S}(\sigma-\iota)k^*(\iota)\,d\iota.$$

On the other hand, since $\mathcal{Z}_n \to \mathcal{Z}^*$ strongly and all the remaining terms in (3.8) converge strongly under $H(S_i)$, $H(\Phi)$, and $H(\varsigma)$, the above integral also converges strongly. Hence its strong limit coincides with its weak limit.

Consequently,

$$\begin{aligned}\mathcal{Z}_n(\sigma) &\to (C(\sigma) - \mathcal{S}(\sigma)\mathcal{G})\vartheta_0 + \mathcal{S}(\sigma)\vartheta_1 + \int_0^\sigma C(\sigma-\iota)\mathcal{G}\vartheta^*(\iota)d\iota + \int_0^\sigma \mathcal{S}(\sigma-\iota)\mathcal{B}\mathcal{U}(\iota)d\iota \\ &\quad + \int_0^\sigma \mathcal{S}(\sigma-\iota)\Big[(S_1\vartheta^*)(\iota) + \Phi(\iota, (S_2\vartheta^*)(\iota), \vartheta^*(\iota))\Big]d\iota \\ &\quad + \int_0^\sigma \mathcal{S}(\sigma-\iota)\varsigma(\iota, \vartheta^*(\iota))d\mathcal{W}(\iota) + \int_0^\sigma \mathcal{S}(\sigma-\iota)k^*(\iota)d\iota. \end{aligned} \tag{3.9}$$

Since $\mathcal{Z}_n \to \mathcal{Z}^*$ and $k^* \in \Gamma(\vartheta^*)$, (3.9) shows that $\mathcal{Z}^* \in \Psi(\vartheta^*)$. Therefore, the graph of Ψ is closed.

Step 6: A priori estimate.

Following Theorem 2.2, we need only verify that the set

$$\mathcal{Y} = \{\vartheta \in C(\mathcal{J}, L^2(\mathcal{F}, \mathcal{H})) : \lambda\vartheta \in \Psi(\vartheta), \lambda > 1\}$$

is bounded. Indeed, we have

$$\lambda\vartheta(\sigma) = (C(\sigma) - \mathcal{S}(\sigma)\mathcal{G})\vartheta_0 + \mathcal{S}(\sigma)\vartheta_1 + \int_0^\sigma C(\sigma-\iota)\mathcal{G}\vartheta(\iota)d\iota + \int_0^\sigma \mathcal{S}(\sigma-\iota)\mathcal{B}\mathcal{U}(\iota)d\iota$$

$$+\int_0^\sigma \mathcal{S}(\sigma-\iota)\Big[(S_1\vartheta)(\iota)+\Phi(\iota,(S_2\vartheta)(\iota),\vartheta(\iota))\Big]d\iota$$
$$+\int_0^\sigma \mathcal{S}(\sigma-\iota)\varsigma(\iota,\vartheta(\iota))d\mathcal{W}(\iota)+\int_0^\sigma \mathcal{S}(\sigma-\iota)k(\iota)d\iota,$$

for all $\sigma\in\mathcal{J}$ and some selection $k\in\Gamma(\vartheta)$.

From the above assumptions, we deduce that

$$\begin{aligned}
&E\|\vartheta(\sigma)\|^2\\
\leq\ & 2^7\Big(2[E\|C(\sigma)\vartheta_0\|^2+E\|\mathcal{S}(\sigma)\mathcal{G}\vartheta_0\|^2]+E\|\mathcal{S}(\sigma)\vartheta_1\|^2\\
&+E\left\|\int_0^\sigma C(\sigma-\iota)\mathcal{G}\vartheta(\iota)d\iota\right\|^2+E\left\|\int_0^\sigma \mathcal{S}(\sigma-\iota)\mathcal{B}\mathcal{U}(\iota)d\iota\right\|^2\\
&+E\left\|\int_0^\sigma \mathcal{S}(\sigma-\iota)(S_1\vartheta)(\iota)d\iota\right\|^2+E\|\int_0^\sigma \mathcal{S}(\sigma-\iota)\Phi(\iota,(S_2\vartheta)(\iota),\vartheta(\iota))d\iota\|^2\\
&+E\left\|\int_0^\sigma \mathcal{S}(\sigma-\iota)\varsigma(\iota,\vartheta(\iota))d\mathcal{W}(\iota)\right\|^2+E\left\|\int_0^\sigma \mathcal{S}(\sigma-\iota)k(\iota)d\iota\right\|^2\Big)\\
\leq\ & 2^7\Big(2[\mathbb{M}^2E\|\vartheta_0\|^2+\|\mathbb{N}\|^2\|\mathcal{G}\|^2E\|\vartheta_0\|^2]+\mathbb{N}^2E\|\vartheta_1\|^2\\
&+b\|\mathbb{M}\|^2\int_0^\sigma\|\mathcal{G}\|^2E\|\vartheta(\iota)\|^2d\iota+b\|\mathbb{N}\|^2\int_0^\sigma\|\mathcal{B}\|^2E\|\mathcal{U}(\iota)\|^2d\iota\\
&+b\|\mathbb{N}\|^2\int_0^\sigma E\left(F_1(\iota)+\mathcal{L}_{\mathcal{S}_1}\int_0^\iota\|\vartheta(r)\|_{\mathcal{H}}\,dr\right)^2d\iota\\
&+b\|\mathbb{N}\|^2\int_0^\sigma E\left\|\mathcal{L}_\Phi\left(\|\vartheta(\iota)\|+F_2(\iota)+\mathcal{L}_{\mathcal{S}_2}\int_0^\iota\|\vartheta(r)\|_{\mathcal{H}}\,dr\right)+\delta(\iota)\right\|^2d\iota\\
&+\|\mathbb{N}\|^2\int_0^\sigma E(\mathcal{M}_{\varsigma_1}(\iota)+\mathcal{M}_{\varsigma_2}\|\vartheta\|^2)d\iota+b\|\mathbb{N}\|^2\int_0^\sigma E(m_{\mathcal{K}_1}(\iota)+m_{\mathcal{K}_2}\|\vartheta\|^2)d\iota\Big)\\
&\leq\chi_1+\chi_2\int_0^\sigma E\|\vartheta(\iota)\|^2d\iota,
\end{aligned}\tag{3.10}$$

where

$$\begin{aligned}
\chi_1\ =\ & 2^7\Big(2[\mathbb{M}^2E\|\vartheta_0\|^2+\|\mathbb{N}\|^2\|\mathcal{G}\|^2E\|\vartheta_0\|^2]+\mathbb{N}^2E\|\vartheta_1\|^2\|\mathbb{N}\|^2\|\mathcal{B}\|^2\|\mathcal{U}\|^2_{\mathcal{L}^2_{\mathcal{F}}(\mathcal{J},U)}\\
&+2b\|\mathbb{N}\|^2\int_0^\sigma\|F_1(\iota)\|^2d\iota+8b^2\|\mathbb{N}\|^2\mathcal{L}_\Phi\int_0^\sigma\|F_2(\iota)\|^2d\iota+2b\|\mathbb{N}\|^2\int_0^\sigma\|\delta(\iota)\|^2d\iota\\
&+\|\mathbb{N}\|^2\int_0^\sigma\mathcal{M}_{\varsigma_1}(\iota)d\iota+b\|\mathbb{N}\|^2\int_0^\sigma m_{\mathcal{K}_1}(\iota)d\iota\Big)
\end{aligned}$$

and

$$\begin{aligned}
\chi_2\ =\ & 2^7\Big(b\|\mathbb{M}\|^2\|\mathcal{G}\|^2+2b^3\|\mathbb{N}\|^2\mathcal{L}^2_{\mathcal{S}_1}\\
&+4b\|\mathbb{N}\|^2\mathcal{L}^2_\Phi+8b^3\|\mathbb{N}\|^2\mathcal{L}^2_\Phi\mathcal{L}^2_{\mathcal{S}_2}+\|\mathbb{N}\|^2\mathcal{M}_{\varsigma_2}+b\|\mathbb{N}\|^2m_{\mathcal{K}_2}\Big).
\end{aligned}$$

By Gronwall's inequality, if a nonnegative function y satisfies $y(\sigma) \leq a + c\int_0^\sigma y(\iota)\,d\iota$, then

$$y(\sigma) \leq ae^{c\sigma}.$$

Applying this to Eq (3.10) with $y(\sigma) = E\|\vartheta(\sigma)\|^2$, $a = \chi_1$, and $c = \chi_2$, we obtain

$$E\|\vartheta(\sigma)\|^2 \leq \chi_1 e^{\chi_2\sigma} \leq \chi_1 e^{\chi_2 b} := \tau.$$

Given that $\sup_{\sigma\in\mathcal{J}} E\|\vartheta(\sigma)\|^2$ is constrained by τ, we can conclude that $\|\vartheta\|^2$ is bounded. Therefore, the proof of Theorem 3.1 is complete in light of Theorem 2.1. □

4. Approximate controllability results

To evaluate the approximate controllability of problem (2.1), we derive a corresponding second-order linear system by integrating it with the original model.

$$\begin{cases} \vartheta''(\sigma) = \mathcal{A}\vartheta(\sigma) + \mathcal{B}\mathcal{U}(\sigma) + \mathcal{G}\vartheta'(\sigma), \quad \sigma \in \mathcal{J}, \\ \vartheta(0) = \vartheta_0, \quad \vartheta'(0) = \vartheta_1 \end{cases} \tag{4.1}$$

where $\mathcal{A}$ generates a strongly continuous cosine family $(C(\sigma))_{\sigma\in\mathbb{R}}$ on $\mathcal{H}$, and $\mathcal{G} \in \mathcal{L}(\mathcal{H})$ denotes a bounded damping operator. Operator $\mathcal{B}$ is a bounded linear map from U into $\mathcal{H}$.

Linear system (4.1) is associated with a controllability map defined by the integral operator:

$$\zeta_0^b = \int_0^b \mathcal{S}(b-\iota)BB^*S^*(b-\iota)\,d\iota.$$

Within space $\mathcal{H}$, this operator is characterized as linear, bounded, and non-negative. In this context, $\mathcal{B}^*$ corresponds to the adjoint of $\mathcal{B}$, while $\mathcal{S}^*(b-\iota)$ signifies the adjoint of $\mathcal{S}(b-\iota)$. The existence of the inverse $(\eta I+\zeta_0^b)^{-1}$ for each $\eta > 0$ ensures that the resolvent $\mathcal{R}(\eta, \zeta_0^b) := (\eta I+\zeta_0^b)^{-1}$ is well-defined. Since ζ_0^b is self-adjoint and non-negative, we have $\|R(\eta, \zeta_0^b)\| = \|(\eta I+\zeta_0^b)^{-1}\| \leq \frac{1}{\eta}$ for $\eta > 0$. The controllability analysis of system (4.1) is conducted by leveraging the properties of the resolvent operator. It is worth noting that the model is built upon a refinement of the time-invariant framework originally established in (4.1).

Lemma 4.1. *[29] The property of approximate controllability for the linear system* (4.1) *on* $\mathcal{J}$ *is characterized by the strong limit* $\lim_{\eta\to 0^+} \eta\mathcal{R}(\eta, \zeta_0^b) = 0$.

For any $\vartheta \in C(\mathcal{J}, \mathcal{L}^2(\mathcal{F}, \mathcal{H})) \subset \mathcal{L}^2(\mathcal{J}, \mathcal{L}^2(\mathcal{F}, \mathcal{H}))$ and $\vartheta_1 \in \mathcal{L}^2(\mathcal{J}, \mathcal{L}^2(\mathcal{F}, \mathcal{H}))$, we have $\Gamma(\varpi) \neq \emptyset$.

Let $\sigma > 0$ be fixed. We proceed by constructing the multi-valued mapping Ψ_σ within $C(\mathcal{J}, \mathcal{L}^2(\mathcal{F}, \mathcal{H}))$ according to the following definition:

$$\begin{aligned} \Psi_\sigma(\vartheta) &= \{\lambda_\sigma \in C(\mathcal{J}, \mathcal{L}^2(\mathcal{F}, \mathcal{H}))|\lambda_\sigma(\sigma) \\ &= (C(\sigma) - \mathcal{S}(\sigma)\mathcal{G})\vartheta_0 + \mathcal{S}(\sigma)\vartheta_1 + \int_0^\sigma C(\sigma-\iota)\mathcal{G}\vartheta(\iota)d\iota + \int_0^\sigma \mathcal{S}(\sigma-\iota)\mathcal{B}\mathcal{U}_\sigma(\iota)d\iota \\ &\quad + \int_0^\sigma \mathcal{S}(\sigma-\iota)\Big[(S_1\vartheta)(\iota) + \Phi(\rho, (S_2\vartheta)(\rho), \vartheta(\iota))\Big]d\iota \end{aligned}$$

$$+\int_0^{\sigma}\mathcal{S}(\sigma-\iota)\varsigma(\iota,\vartheta(\iota))d\mathcal{W}(\iota)+\int_0^{\sigma}\mathcal{S}(\sigma-\iota)k(\iota)d\iota,\sigma\in\mathcal{J},k\in\Gamma(\vartheta)\}.$$

Here,

$$\begin{aligned}
&\mathcal{U}_{\sigma}(\iota)\\
=\;& B^{*}S^{*}(b-\iota)\mathcal{R}(\eta,\zeta_0^b)\Big(\vartheta_b-(C(b)-\mathcal{S}(b)\mathcal{G})\vartheta_0-\mathcal{S}(b)\vartheta_1-\int_0^b C(b-r)\mathcal{G}\vartheta(r)dr\\
&-\int_0^b\mathcal{S}(b-r)\Big[(S_1\vartheta)(r)+\Phi(r,(S_2\vartheta)(r),\vartheta(r))\Big]dr\\
&-\int_0^b\mathcal{S}(b-r)\varsigma(r,\vartheta(r))d\mathcal{W}(r)-\int_0^b\mathcal{S}(b-r)k(r)dr\Big).
\end{aligned}\tag{4.2}$$

To ensure that the feedback control $\mathcal{U}_{\sigma}(\iota)$ defined in (4.2) is valid, we observe that it is constructed via bounded linear operators ($\mathcal{B}^{*}$, $\mathcal{S}^{*}$), the continuous resolvent $\mathcal{R}(\eta,\zeta_0^b)$, and $\mathcal{F}_b$-measurable terminal/history components. Consequently, $\mathcal{U}_{\sigma}(\cdot)$ is $\mathcal{F}_{\sigma}$-adapted and satisfies $\mathcal{U}_{\sigma}\in L^2_{\mathcal{F}}(\mathcal{J},\mathcal{U})$, thereby strictly belonging to the admissible control space.

Theorem 4.1. *Under the assumption that hypotheses $\mathcal{H}(\mathcal{S})$,$H(\mathcal{G})$, $\mathcal{H}(\mathcal{K})$,$\mathcal{H}(\varsigma)$,$\mathcal{H}(\mathcal{S}_i)$, and $\mathcal{H}(\Phi)$, the mapping Ψ_{σ} is guaranteed to have a fixed point on $\mathcal{J}$ for any $\eta>0$, provided that the following condition holds:*

$$\theta_2\frac{e^{\chi_2 b}-1}{\chi_2}<1,$$

where

$$\begin{aligned}
&\theta_2\\
=\;& b\|\mathbb{N}\|^2\|\mathcal{B}\|^4\|\mathbb{N}\|^2/\eta^2 2^{14}\Big(b^2\|\mathbb{M}\|^2\|\mathcal{G}\|^2+2b^4\|\mathbb{N}\|^2\mathcal{L}^2_{\mathcal{S}_1}\\
&+4b^2\|\mathbb{N}\|^2\mathcal{L}_{\Phi}+8b^4\|\mathbb{N}\|^2\mathcal{L}_{\Phi}\mathcal{L}_{\mathcal{S}_2}+\|\mathbb{N}\|^2\mathcal{M}_{\varsigma_2}b+b^2\|\mathbb{N}\|^2m_{\mathcal{K}_2}\Big)
\end{aligned}$$

and

$$\begin{aligned}
\chi_2 \;=\;& 2^7\Big(b^2\|\mathbb{M}\|^2\|\mathcal{G}\|^2+2b^4\|\mathbb{N}\|^2\mathcal{L}^2_{\mathcal{S}_1}\\
&+4b^2\|\mathbb{N}\|^2\mathcal{L}_{\Phi}+8b^4\|\mathbb{N}\|^2\mathcal{L}_{\Phi}\mathcal{L}_{\mathcal{S}_2}+\|\mathbb{N}\|^2\mathcal{M}_{\varsigma_2}b+b^2\|\mathbb{N}\|^2m_{\mathcal{K}_2}\Big).
\end{aligned}$$

Proof. Using the results of [22] together with the properties of the control $\mathcal{U}\sigma$, one can easily show that the multivalued map $\Psi\sigma$ is well defined. Thus, the main results follow once we verify that Ψ_{σ} satisfies the hypotheses of Theorem 2.1.

The proof proceeds analogously to that of Theorem 3.1. Steps 1–5 are omitted since they follow from the same reasoning. It remains to establish Step 6: A-priori estimates, which deviates from the aforementioned proof and requires a separate analysis.

Based on the above derivation, we conclude that the map Ψ_{σ} is relatively compact and takes both compact and convex values. In light of Theorem 2.1, the proof reduces to establishing the a-priori boundedness of the candidate set

$$\mathcal{Y}_{\sigma}=\{\vartheta\in C(\mathcal{J},L^2(\mathcal{F},\mathcal{H})):\varrho\vartheta_{\sigma}\in\Psi_{\sigma}(\vartheta),\varrho>1\}.$$

By $\vartheta \in \mathcal{Y}_\sigma$, we deduce that

$$\begin{aligned}
&\vartheta(\sigma)\\
=\;& \varrho^{-1}\Big[(C(\sigma)-\mathcal{S}(\sigma)\mathcal{G})\vartheta_0+\mathcal{S}(\sigma)\vartheta_1+\int_0^\sigma C(\sigma-\iota)\mathcal{G}\vartheta(\iota)d\iota+\int_0^\sigma \mathcal{S}(\sigma-\iota)\mathcal{B}\mathcal{U}_\sigma(\iota)d\iota\\
&+\int_0^\sigma \mathcal{S}(\sigma-\iota)\Big[(S_1\vartheta)(\iota)+\Phi(\iota,(S_2\vartheta)(\iota),\vartheta(\iota))\Big]d\iota\\
&+\int_0^\sigma \mathcal{S}(\sigma-\iota)\varsigma(\iota,\vartheta(\iota))d\mathcal{W}(\iota)+\int_0^\sigma \mathcal{S}(\sigma-\iota)k(\iota)d\iota\Big],\sigma\in\mathcal{J},k\in\Gamma(\vartheta).
\end{aligned}$$

Taking this a step further yields the conclusion:

$$\begin{aligned}
&E\|\vartheta(\sigma)\|^2\\
\le\;& 2^7\Big(2[\mathbb{M}^2E\|\vartheta_0\|^2+\|\mathbb{N}\|^2\|\mathcal{G}\|^2E\|\vartheta_0\|^2]+\mathbb{N}^2E\|\vartheta_1\|^2\\
&+b\|\mathbb{M}\|^2\int_0^\sigma\|\mathcal{G}\|^2E\|\vartheta(\iota)\|^2d\iota+b\|\mathbb{N}\|^2\int_0^\sigma\|\mathcal{B}\|^2E\|\mathcal{U}_\sigma(\iota)\|^2d\iota\\
&+b\|\mathbb{N}\|^2\int_0^\sigma E\left(F_1(\iota)+\mathcal{L}_{\mathcal{S}_1}\int_0^\iota\|\vartheta(s)\|_{\mathcal{H}}\,ds\right)^2d\iota\\
&+b\|\mathbb{N}\|^2\int_0^\sigma E\left\|\mathcal{L}_\Phi\left(\|\vartheta(\iota)\|+F_2(\iota)+\mathcal{L}_{\mathcal{S}_2}\int_0^\iota\|\vartheta(r)\|_{\mathcal{H}}\,dr\right)+\delta(\iota)\right\|^2d\iota\\
&+\|\mathbb{N}\|^2\int_0^\sigma E(\mathcal{M}_{\varsigma_1}(\iota)+\mathcal{M}_{\varsigma_2}\|\vartheta\|^2)d\iota+b\|\mathbb{N}\|^2\int_0^\sigma E(m_{\mathcal{K}_1}(\iota)+m_{\mathcal{K}_2}\|\vartheta\|^2)d\iota\Big).
\end{aligned}\tag{4.3}$$

By (4.2), we have

$$\begin{aligned}
&b\|\mathbb{N}\|^2\int_0^\sigma\|\mathcal{B}\|^2E\|\mathcal{U}_\sigma(\iota)\|^2d\iota\\
\le\;& b\|\mathbb{N}\|^2\|\mathcal{B}\|^4\|\mathbb{N}\|^2/\eta^2 2^7\Big[\Big(2[\mathbb{M}^2E\|\vartheta_0\|^2+\|\mathbb{N}\|^2\|\mathcal{G}\|^2E\|\vartheta_0\|^2]\\
&+2b\|\mathbb{N}\|^2\int_0^b\|F_1(\iota)\|^2d\iota+8b^2\|\mathbb{N}\|^2\mathcal{L}_\Phi\int_0^b\|F_2(\iota)\|^2d\iota+2b\|\mathbb{N}\|^2\int_0^b\|\delta(\iota)\|^2d\iota\\
&+\|\mathbb{N}\|^2\int_0^b\mathcal{M}_{\varsigma_1}(\iota)d\iota+b\|\mathbb{N}\|^2\int_0^b m_{\mathcal{K}_1}(\iota)d\iota\Big)+2^7\Big(b^2\|\mathbb{M}\|^2\|\mathcal{G}\|^2+2b^4\|\mathbb{N}\|^2\mathcal{L}_{\mathcal{S}_1}^2\\
&+4b^2\|\mathbb{N}\|^2\mathcal{L}_\Phi+8b^4\|\mathbb{N}\|^2\mathcal{L}_\Phi\mathcal{L}_{\mathcal{S}_2}+\|\mathbb{N}\|^2\mathcal{M}_{\varsigma_2}b+b^2\|\mathbb{N}\|^2m_{\mathcal{K}_2}\Big)\int_0^b E\|\vartheta(\iota)\|^2d\iota\Big]\\
=:\;& \theta_1+\theta_2\int_0^b E\|\vartheta(\iota)\|^2d\iota.
\end{aligned}\tag{4.4}$$

It follows from (3.10), (4.3) and (4.4) that

$$E\|\vartheta(\sigma)\|^2\le\theta_1+\widehat{\chi}_1+\theta_2\int_0^b E\|\vartheta(\iota)\|^2d\iota+\chi_2\int_0^\sigma E\|\vartheta(\iota)\|^2d\iota,\tag{4.5}$$

where

$$\widehat{\chi}_1 = 2^7\Big(2[\mathbb{M}^2E\|\vartheta_0\|^2+\|\mathbb{N}\|^2\|\mathcal{G}\|^2E\|\vartheta_0\|^2]$$

$$+2b\|\mathbb{N}\|^2\int_0^\sigma \|F_1(\iota)\|^2 d\iota + 8b^2\|\mathbb{N}\|^2\mathcal{L}_\Phi\int_0^\sigma \|F_2(\iota)\|^2 d\iota + 2b\|\mathbb{N}\|^2\int_0^\sigma \|\delta(\iota)\|^2 d\iota$$
$$+\|\mathbb{N}\|^2\int_0^\sigma \mathcal{M}_{\varsigma_1}(\iota)d\iota + b\|\mathbb{N}\|^2\int_0^\sigma m_{\mathcal{K}_1}(\iota)d\iota\Big).$$

Here, the integral over $[0, b]$ comes from the feedback-control estimate in Eq (4.4), which is constructed with respect to the terminal time b, whereas the integral over $[0, \sigma]$ arises from the state-dependent terms up to the current time σ. Since $\widehat{\chi}_1(\sigma)$ is non-decreasing on $\mathcal{J}$, let $\widehat{\chi}_1^* = \widehat{\chi}_1(b)$. Then $\widehat{\chi}_1(\sigma) \le \widehat{\chi}_1^*$ for all $\sigma \in \mathcal{J}$.

From the identity

$$\sum(\sigma) = \int_0^\sigma E\|\vartheta(\iota)\|_{\mathcal{H}}^2 d\iota,$$

one can readily observe that $\sum(0) = 0$; moreover, the cumulative growth of the integral ensures that $\sum$ is non-decreasing over the interval $\mathcal{J}$. Thus,

$$(\sum)'(\sigma) = E\|\vartheta(\sigma)\|_{\mathcal{H}}^2 \le \theta_1 + \widehat{\chi}_1^* + \theta_2\int_0^b E\|\vartheta(\iota)\|^2 d\iota + \chi_2\int_0^\sigma E\|\vartheta(\iota)\|^2 d\iota.$$

Therefore, we have

$$\Big(\sum(\sigma)e^{-\chi_2\sigma}\Big)' \le \Big(\theta_1 + \widehat{\chi}_1^* + \theta_2\sum(b)\Big)e^{-\chi_2\sigma}. \tag{4.6}$$

Integrating Eq (4.6) over $[0, b]$, we obtain

$$\int_0^b \Big(\sum(\sigma)e^{-\chi_2\sigma}\Big)' d\sigma \le \Big(\theta_1 + \widehat{\chi}_1^* + \theta_2\sum(b)\Big)\int_0^b e^{-\chi_2\sigma}\, d\sigma.$$

Hence,

$$\sum(b)e^{-\chi_2 b} - \sum(0) \le \Big(\theta_1 + \widehat{\chi}_1^* + \theta_2\sum(b)\Big)\frac{1 - e^{-\chi_2 b}}{\chi_2}.$$

Since $\sum(0) = 0$, multiplying both sides by $e^{\chi_2 b}$ gives

$$\sum(b) \le \Big(\theta_1 + \widehat{\chi}_1^* + \theta_2\sum(b)\Big)\frac{e^{\chi_2 b} - 1}{\chi_2}. \tag{4.7}$$

It is readily apparent that

$$\theta_2\frac{e^{\chi_2 b} - 1}{\chi_2} < 1.$$

By (4.7), we infer that

$$\sum(b) \le \frac{\Big(\theta_1 + \widehat{\chi}_1^*\Big)\Big(e^{\chi_2 b} - 1\Big)}{\chi_2 - \theta_2\,(e^{\chi_2 b} - 1)} =: \mathcal{M}.$$

Since $\sum$ is non-decreasing, it follows that $\sum(\sigma) \le \mathcal{M}$ for all $\sigma \in \mathcal{J}$. By (4.5), we get

$$E\|\vartheta(\sigma)\|^2 \le \theta_1 + \widehat{\chi}_1 + \theta_2\mathcal{M} + \chi_2\mathcal{M}.$$

This concludes the verification that the set Y_σ satisfies the boundedness requirement. In view of Theorem 2.1, we conclude that Ψ_σ possesses a fixed point, which completes the proof. □

Theorem 4.2. *Under the established conditions of Theorem 4.1, the approximate controllability of the nonlinear system* (1.1) *on the interval $\mathcal{J}$ is guaranteed, provided that the linear counterpart satisfies the same controllability criteria.*

Proof. By Theorem 4.1, we can find $\widehat{k} \in \Gamma(\widehat{\vartheta})$ such that

$$\begin{aligned}
&\widehat{\vartheta}(\sigma)\\
=\;& (C(\sigma) - \mathcal{S}(\sigma)\mathcal{G})\vartheta_0 + \mathcal{S}(\sigma)\vartheta_1 + \int_0^\sigma C(\sigma-\iota)\mathcal{G}\widehat{\vartheta}(\iota)d\iota + \int_0^\sigma \mathcal{S}(\sigma-\iota)\mathcal{B}\widehat{\mathcal{U}}(\iota)d\iota\\
&+ \int_0^\sigma \mathcal{S}(\sigma-\iota)\Big[(S_1\widehat{\vartheta})(\iota) + \Phi(\iota,(S_2\widehat{\vartheta})(\iota),\widehat{\vartheta}(\iota))\Big]d\iota\\
&+ \int_0^\sigma \mathcal{S}(\sigma-\iota)\varsigma(\iota,\widehat{\vartheta}(\iota))d\mathcal{W}(\iota) + \int_0^\sigma \mathcal{S}(\sigma-\iota)\widehat{k}(\iota)d\iota, \sigma\in\mathcal{J}.
\end{aligned}$$

For the purposes of our analysis, we introduce the mapping $\mathcal{D}(\vartheta_\sigma)$:

$$\begin{aligned}
\mathcal{D}(\widehat{\vartheta}) =\;& \Big(\vartheta_b - (C(b) - \mathcal{S}(b)\mathcal{G})\vartheta_0 - \mathcal{S}(b)\vartheta_1 - \int_0^b C(b-r)\mathcal{G}\widehat{\vartheta}(r)dr\\
&- \int_0^b \mathcal{S}(b-r)\Big[(S_1\widehat{\vartheta})(r) + \Phi(r,(S_2\widehat{\vartheta})(r),\widehat{\vartheta}(r))\Big]dr\\
&- \int_0^b \mathcal{S}(b-r)\varsigma(r,\widehat{\vartheta}(r))d\mathcal{W}(r) - \int_0^b \mathcal{S}(b-r)\widehat{k}(r)dr\Big).
\end{aligned}$$

By the definition of the feedback control in Eq (4.2), we have

$$\begin{aligned}
\int_0^b \mathcal{S}(b-\iota)\mathcal{B}\widehat{\mathcal{U}}(\iota)d\iota =\;& \int_0^b \mathcal{S}(b-\iota)\mathcal{B}\mathcal{B}^*\mathcal{S}^*(b-\iota)R(\eta,\zeta_0^b)\mathcal{D}(\widehat{\vartheta})d\iota\\
=\;& \zeta_0^b R(\eta,\zeta_0^b)\mathcal{D}(\widehat{\vartheta}).
\end{aligned}$$

Therefore,

$$\widehat{\vartheta}(b) = \vartheta_b - \mathcal{D}(\widehat{\vartheta}) + \zeta_0^b R(\eta,\zeta_0^b)\mathcal{D}(\widehat{\vartheta}).$$

By the definition $R(\eta,\zeta_0^b) = (\eta I + \zeta_0^b)^{-1}$, we have $(\eta I + \zeta_0^b)R(\eta,\zeta_0^b) = I$, and hence

$$\eta R(\eta,\zeta_0^b) = I - \zeta_0^b R(\eta,\zeta_0^b).$$

Therefore,

$$\widehat{\vartheta}(b) = \vartheta_b - \eta\mathcal{R}(\eta,\zeta_0^b)\mathcal{D}(\widehat{\vartheta}).$$

By virtue of the conditions established in $\mathcal{H}(\mathcal{K})(iii)$, $\mathcal{H}(\varsigma)$ and $\mathcal{H}(\mathcal{S}_i)$, coupled with the established methodology in Step 6 of Theorem 3.1, we arrive at

$$\{\widehat{k}, \varsigma(\cdot,\widehat{\vartheta}(\cdot),(S_1\vartheta)(\cdot),(S_2\vartheta)(\cdot)\}$$

are bounded in $\mathcal{L}^2_{\mathcal{F}} \times \mathcal{L}^2_0 \times \mathcal{L}^2_{\mathcal{F}} \times \mathcal{L}^2_{\mathcal{F}}$.

Let

$$\mathcal{D}(\vartheta) = \Big(\vartheta_b - (C(b) - \mathcal{S}(b)\mathcal{G})\vartheta_0 - \mathcal{S}(b)\vartheta_1 - \int_0^b C(b-r)\mathcal{G}\vartheta(r)dr$$

$$- \int_0^b \mathcal{S}(b-r)\Big[(S_1\vartheta)(r) + \Phi(r,(S_2\vartheta)(r),\vartheta(r))\Big]dr$$
$$- \int_0^b \mathcal{S}(b-r)\varsigma(r,\vartheta(r))d\mathcal{W}(r) - \int_0^b \mathcal{S}(b-r)k(r)dr\Big).$$

According to the compactness, with $\widehat{\vartheta}$ understood as the fixed point corresponding to $\eta > 0$, we infer that

$$E\|\mathcal{D}(\widehat{\vartheta}) - \mathcal{D}(\vartheta)\|^2 \to 0 \quad \text{as } \eta \to 0^+. \tag{4.8}$$

Moreover, the relations established in (4.8) imply that

$$\begin{aligned} & E\|\widehat{\vartheta}(b) - \vartheta_b\|^2 = E\|\eta\mathcal{R}(\eta,\zeta_0^b)\mathcal{D}(\widehat{\vartheta})\|^2 \\ \leq\ & 2E\|\eta\mathcal{R}(\eta,\zeta_0^b)\mathcal{D}(\vartheta)\|^2 + 2E\|\eta\mathcal{R}(\eta,\zeta_0^b)\mathcal{D}(\widehat{\vartheta}) - \eta\mathcal{R}(\eta,\zeta_0^b)\mathcal{D}(\vartheta)\|^2 \to 0. \end{aligned}$$

Since $\vartheta_1 \in \mathcal{H}$ can be chosen arbitrarily, we can conclude that system (1.1) achieves approximate controllability on $\mathcal{J}$, thereby completing the proof. □

5. Conclusions

We have rigorously proven the existence of mild solutions and the approximate controllability for the class of memory-dependent stochastic systems studied here. These results were obtained through a framework involving condensing maps and generalized Clarke subdifferentials, providing a robust set of sufficient conditions for such dynamics.

In future research, we will focus on integrating this theoretical framework into contact mechanics models. This application is expected to yield significant advancements in how we simulate and analyze the mechanical behavior of contacting surfaces in complex systems.

Author contributions

Yuzhuo Ren: Formal analysis, methodology, investigation, writing—original draft; Jianwei Hao: Validation, visualization, writing—review and editing; Bo Guan: Conceptualization, methodology, supervision, project administration, writing—review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have used Artificial Intelligence (AI) tools in the creation of this article. ChatGPT was used to assist with English language polishing, grammar checking, and improving the clarity and readability of the manuscript. The AI-assisted language editing was applied to selected parts of the manuscript, including the abstract, introduction, and other sections where language refinement was needed. The authors reviewed and revised the AI-assisted suggestions and take full responsibility for the accuracy, originality, and integrity of the final content.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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