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Research article

A matrix approach to hyperfibonacci quaternions

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Abstract: In this paper, we introduced hyperfibonacci quaternions by taking consecutive hyperfibonacci numbers as the coefficients of a quaternion. We established fundamental recurrence relations and several algebraic identities that extended the properties of Fibonacci quaternions. Moreover, we derived an explicit representation of hyperfibonacci quaternions in terms of Fibonacci quaternions and binomial coefficients. We developed a matrix approach based on two linear operators acting on the sequence index and the generation parameter. This gave rise to a two-dimensional operator model for hyperfibonacci quaternions and lead to a closed matrix formula for vectors of consecutive hyperfibonacci numbers. Finally, we obtained generating functions for the sequence of hyperfibonacci quaternions of arbitrary generation and an explicit identity relating quaternionic value at a fixed generation.

Keywords: Fibonacci quaternions; hyperfibonacci numbers; quaternion algebra; matrix representation; generating functions
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1. Introduction

Fibonacci-type sequences and quaternion algebras have each been extensively studied [1–3]. An overview of the fundamental properties of Fibonacci quaternions can be found in [4–6]. Motivated by the numerous generalizations of Fibonacci numbers, various extensions of Fibonacci quaternions have also been introduced and investigated [7–9]. In addition, quaternion sequences associated with other classical and generalized number sequences have attracted considerable interest [10–12]. Fibonacci polynomials and quaternionic structures have also found applications in areas such as cryptography and image encryption [13–15]. Quaternions themselves play an important role in representing rotations in three-dimensional space, with applications in computer graphics, robotics, and aerospace navigation [16].

Since Fibonacci quaternions arise naturally by using consecutive Fibonacci numbers as quaternion coefficients, it is natural to ask whether analogous constructions can be developed for generalized Fibonacci sequences. In this paper, we consider hyperfibonacci numbers as quaternion coefficients and introduce the resulting family of hyperfibonacci quaternions, with particular emphasis on their algebraic properties, matrix representation, and generating functions.

The main results and organization of the paper are summarized as follows:

- In Section 2, we recall the necessary preliminaries on quaternions, Fibonacci quaternions, and hyperfibonacci numbers.
- In Section 3, we introduce the n-th hyperfibonacci quaternion of the r-th generation. We establish fundamental recurrence relations and algebraic identities, and express hyperfibonacci quaternions directly in terms of Fibonacci quaternions and binomial coefficients (Theorem 1).
- In Section 4, we develop a two-dimensional matrix-operator framework. By constructing two 4×4 shift matrices M and N, acting on the sequence index n and the generation parameter r, respectively, we derive a closed-form matrix representation for vectors of consecutive hyperfibonacci numbers (Theorem 2).
- In Section 5, we derive generating functions for hyperfibonacci quaternions of arbitrary generation and establish an explicit identity relating quaternionic values at a fixed generation (Proposition 5).

2. Preliminaries

Quaternions, introduced by William Rowan Hamilton in 1843 [17, 18], form a natural four-dimensional extension of the complex numbers. The set of quaternions, denoted $\mathbb{H}$, consists of elements of the form

$$q = q_0 + iq_1 + jq_2 + kq_3, \qquad q_0, q_1, q_2, q_3 \in \mathbb{R},$$

where the imaginary units i, j, k satisfy the fundamental relations

$$i^2 = j^2 = k^2 = ijk = -1, \tag{2.1}$$

from which the noncommutative multiplication rules follow:

$$ij = k = -ji, \qquad jk = i = -kj, \qquad ki = j = -ik. \tag{2.2}$$

The algebra $\mathbb{H}$ is a four-dimensional associative but noncommutative division algebra over $\mathbb{R}$, and may be viewed as a classic example of a Clifford algebra.

For a quaternion $q = q_0 + iq_1 + jq_2 + kq_3$, the conjugate is defined as

$$\bar{q} = q_0 - iq_1 - jq_2 - kq_3,$$

the norm as

$$N(q) = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2},$$

and, for $q \neq 0$, the inverse as

$$q^{-1} = \bar{q}/N^2(q),$$

which equips $\mathbb{H}$ with the structure of a normed division algebra.

Building on the algebra of quaternions introduced above, Horadam [4] was the first to combine the quaternionic and Fibonacci settings by introducing the nth Fibonacci quaternion:

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3},\ n \geq 0, \tag{2.3}$$

where F_n denotes the nth Fibonacci number. Since each quaternionic component of Q_n is itself a Fibonacci number, the sequence $(Q_n)_{n\geq 0}$ inherits the defining recurrence of the Fibonacci sequence:

$$Q_{n+2} = Q_{n+1} + Q_n.$$

Fibonacci quaternions have subsequently been studied from several perspectives, including algebraic identities, generating functions, matrix representations, and extensions associated with other Fibonacci-type sequences (e.g., [6, 19, 20]). These developments suggest that many properties of the underlying scalar sequence admit natural quaternionic analogues.

Accordingly, the conjugate of Q_n is given by

$$\bar{Q}_n = F_n - iF_{n+1} - jF_{n+2} - kF_{n+3}.$$

Horadam [4] established a few relations for the Fibonacci quaternions, using standard (noncommutative) quaternion multiplication:

$$Q_n\bar{Q}_n = \sum_{m=0}^{3} F_{n+m}^2 = 3F_{2n+3}, \tag{2.4}$$

$$Q_n^2 = 2F_nQ_n - Q_n\bar{Q}_n, \tag{2.5}$$

and, for $Q_n \neq 0$,

$$2F_n = Q_n + \bar{Q}_n, \tag{2.6}$$

which extracts the scalar part of the quaternion.

These identities show that Fibonacci quaternions retain much of the recursive and algebraic richness of the Fibonacci sequence, while embedding it within the noncommutative algebra $\mathbb{H}$, and they will serve as the natural starting point for the hyperfibonacci generalization introduced in the next section. This raises the question of whether analogous identities can be established when the Fibonacci coefficients are replaced by hyperfibonacci numbers, whose recursive structure involves higher-order summation processes.

Hyperfibonacci numbers were introduced by Dil and Mezö in [21] and have since been the subject of further investigation (e.g., [22, 23]).

For nonnegative integers n and r, the hyperfibonacci numbers of the rth generation [21], denoted by $F_n^{(r)}$, are given by the recurrence relation

$$F_n^{(r)} = \sum_{k=0}^{n} F_k^{(r-1)}, \text{ for } r > 0, \tag{2.7}$$

with initial conditions $F_n^{(0)} = F_n$, for $n \geq 0$, and $F_0^{(r)} = 0$, $F_1^{(r)} = 1$, for $r \geq 0$.

Hyperfibonacci numbers generalize the Fibonacci sequence through repeated summation. The parameter r, called the generation, measures the number of times the summation process is applied. Directly from the definition, one can find the relation

$$F_n^{(r)} = F_{n-1}^{(r)} + F_n^{(r-1)}, \text{ for } n, r > 0. \tag{2.8}$$

For example, for $n = r = 1$, the relation (2.8) gives

$$F_1^{(1)} = F_0^{(1)} + F_1^{(0)} = 0 + F_1 = 1.$$

Using the relation (2.7), we obtain Hyperfibonacci numbers $F_n^{(r)}$, for $n = 0, 1, \ldots, 9$ and $r = 0, 1, \ldots, 5$. The results are listed in Table 1.

Table 1. The Hyperfibonacci numbers.

$r\backslash n$	0	1	2	3	4	5	6	7	8	9	$\cdots$
0	0	1	1	2	3	5	8	13	21	34	$\cdots$
1	0	1	2	4	7	12	20	33	54	88	$\cdots$
2	0	1	3	7	14	26	46	79	133	221	$\cdots$
3	0	1	4	11	25	51	97	176	309	530	$\cdots$
4	0	1	5	16	41	92	189	365	674	1204	$\cdots$
5	0	1	6	22	63	155	344	709	1383	2587	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Cristea et al. in [23] proved the identity

$$F_n^{(r)} = F_{n+2r} - \sum_{k=0}^{r-1} \binom{n+r+k}{r-k-1},$$

from which the following identity can be established by using Vandermonde's convolution

$$F_n^{(r)} = F_{n+2r} - \sum_{k=0}^{r-1} \binom{n+r-1}{k} F_{r-k+1}. \tag{2.9}$$

Namely,

$$\begin{aligned}
\sum_{k=0}^{r-1} \binom{n+r-1}{k} F_{r-k+1} &= \sum_{k=0}^{r-1} \binom{n+r-1}{k} \sum_{i=0}^{r-k} \binom{r-k-i}{i} \\
&= \sum_{k=0}^{r-1} \sum_{j=k}^{r} \binom{n+r-1}{k} \binom{r-j}{j-k} \\
&= \sum_{j=0}^{r} \sum_{k=0}^{\min(j,r-1)} \binom{n+r-1}{k} \binom{r-j}{j-k} \\
&= \sum_{j=0}^{r-1} \sum_{k=0}^{j} \binom{n+r-1}{k} \binom{r-j}{j-k}
\end{aligned}$$

$$= \sum_{j=0}^{r-1} \binom{n+2r-j-1}{j}$$
$$= \sum_{j=0}^{r-1} \binom{n+r+j}{r-j-1}.$$

Formula (2.9) expresses hyperfibonacci numbers directly in terms of Fibonacci numbers and will play a key role throughout the paper, allowing quaternionic hyperfibonacci identities to be reduced to identities involving Fibonacci quaternions.

Motivated by the theory of Fibonacci quaternions and recent developments concerning hyperfibonacci numbers, we introduce hyperfibonacci quaternions and establish their fundamental properties. Unlike Fibonacci numbers, hyperfibonacci numbers incorporate both the Fibonacci recurrence and an additional summation hierarchy determined by the generation parameter r. Casting these sequences into the quaternion algebra provides a natural framework for studying the interaction between recursive combinatorial structures and noncommutative algebra.

3. Hyperfibonacci Quaternions

The following definition introduces the hyperfibonacci quaternion.

For nonnegative integers n and r, the nth hyperfibonacci quaternion of the rth generation is defined as

$$Q_n^{(r)} := F_n^{(r)} + iF_{n+1}^{(r)} + jF_{n+2}^{(r)} + kF_{n+3}^{(r)}, \tag{3.1}$$

where $F_n^{(r)}$ denotes the hyperfibonacci number of the rth generation.

For $r = 0$, Definition (3.1) reduces to the Fibonacci quaternion (2.3). The hyperfibonacci quaternions inherit the fundamental recurrence (2.8) of the underlying hyperfibonacci numbers. The conjugate of $Q_n^{(r)}$ is defined by

$$\bar{Q}_n^{(r)} := F_n^{(r)} - iF_{n+1}^{(r)} - jF_{n+2}^{(r)} - kF_{n+3}^{(r)}. \tag{3.2}$$

Proposition 1. *For positive integers n and r, we have*

$$Q_n^{(r)} = Q_{n-1}^{(r)} + Q_n^{(r-1)}.$$

Proof. Adding $Q_{n-1}^{(r)}$ and $Q_n^{(r-1)}$ componentwise and applying (2.8) to each coordinate yields

$$\begin{aligned} Q_{n-1}^{(r)} + Q_n^{(r-1)} &= (F_{n-1}^{(r)} + iF_n^{(r)} + jF_{n+1}^{(r)} + kF_{n+2}^{(r)}) + (F_n^{(r-1)} + iF_{n+1}^{(r-1)} + jF_{n+2}^{(r-1)} + kF_{n+3}^{(r-1)}) \\ &= (F_{n-1}^{(r)} + F_n^{(r-1)}) + i(F_n^{(r)} + F_{n+1}^{(r-1)}) + j(F_{n+1}^{(r)} + F_{n+2}^{(r-1)}) + k(F_{n+2}^{(r)} + F_{n+3}^{(r-1)}) \\ &= F_n^{(r)} + iF_{n+1}^{(r)} + jF_{n+2}^{(r)} + kF_{n+3}^{(r)} \\ &= Q_n^{(r)}. \end{aligned}$$

□

The next result gives the squared norm of a hyperfibonacci quaternion.

Proposition 2. *For nonnegative integers n and r, hyperfibonacci quaternions satisfy*

$$Q_n^{(r)}\bar{Q}_n^{(r)} = \left(F_n^{(r)}\right)^2 + \left(F_{n+1}^{(r)}\right)^2 + \left(F_{n+2}^{(r)}\right)^2 + \left(F_{n+3}^{(r)}\right)^2. \tag{3.3}$$

Proof. The result follows from direct quaternion multiplication, the definition of quaternion conjugation (3.2), and by applying Eqs (2.1) and (2.2). □

Quaternion multiplication is not commutative in general, but $Q_n^{(r)}\bar{Q}_n^{(r)} = \bar{Q}_n^{(r)}Q_n^{(r)}$ holds.

Example 1. *For $n = 4$ and $r = 3$, hyperfibonacci quaternion $Q_4^{(3)}$ is*

$$Q_4^{(3)} = 25 + 51i + 97j + 176k,$$

and its conjugate is

$$\bar{Q}_4^{(3)} = 25 - 51i - 97j - 176k.$$

By the relation (3.3), we have

$$Q_4^{(3)}\bar{Q}_4^{(3)} = 25^2 + 51^2 + 97^2 + 176^2 = 43611,$$

so it follows that the norm of $Q_4^{(3)}$ is

$$N(Q_4^{(3)}) = \sqrt{43611}.$$

Proposition 3. *For all nonnegative integers n and r, hyperfibonacci quaternions satisfy*

$$\left(Q_n^{(r)}\right)^2 = 2F_n^{(r)}Q_n^{(r)} - Q_n^{(r)}\bar{Q}_n^{(r)}. \tag{3.4}$$

Proof. The result follows from direct computation using the properties of quaternion multiplication (2.1) and (2.2) and the relation (3.3).

$$\begin{aligned}
\left(Q_n^{(r)}\right)^2 &= \left(F_n^{(r)} + iF_{n+1}^{(r)} + jF_{n+2}^{(r)} + kF_{n+3}^{(r)}\right)^2 \\
&= \left(F_n^{(r)}\right)^2 + iF_n^{(r)}F_{n+1}^{(r)} + jF_n^{(r)}F_{n+2}^{(r)} + kF_n^{(r)}F_{n+3}^{(r)} + iF_n^{(r)}F_{n+1}^{(r)} \\
&+ i^2\left(F_{n+1}^{(r)}\right)^2 + ijF_{n+1}^{(r)}F_{n+2}^{(r)} + ikF_{n+1}^{(r)}F_{n+3}^{(r)} + jF_n^{(r)}F_{n+2}^{(r)} + jiF_{n+1}^{(r)}F_{n+2}^{(r)} \\
&+ j^2\left(F_{n+2}^{(r)}\right)^2 + jkF_{n+2}^{(r)}F_{n+3}^{(r)} + kF_n^{(r)}F_{n+3}^{(r)} + kiF_{n+1}^{(r)}F_{n+3}^{(r)} + kjF_{n+2}^{(r)}F_{n+3}^{(r)} + k^2\left(F_{n+3}^{(r)}\right)^2 \\
&= 2F_n^{(r)}\left(F_n^{(r)} - iF_{n+1}^{(r)} - jF_{n+2}^{(r)} - kF_{n+3}^{(r)}\right) - Q_n^{(r)}\bar{Q}_n^{(r)} \\
&= 2F_n^{(r)}Q_n^{(r)} - Q_n^{(r)}\bar{Q}_n^{(r)}.
\end{aligned}$$

□

Setting $r = 0$ in identities (3.3) and (3.4) yields (2.4) and (2.5), respectively.

Also, as an analogue of (2.6), for all nonnegative integers n and r, summing the Definitions (3.1) and (3.2) gives

$$2F_n^{(r)} = Q_n^{(r)} + \bar{Q}_n^{(r)}.$$

Formula (2.9) enables us to express hyperfibonacci quaternions directly in terms of Fibonacci quaternions. Applying (2.9) to each of the four scalar components of $Q_n^{(r)}$ in turn – with n replaced by n, $n + 1$, $n + 2$, $n + 3$, we obtain the following theorem.

Theorem 1. *Hyperfibonacci quaternions satisfy*

$$Q_n^{(r)} = Q_{n+2r} - \sum_{m=0}^{r-1} C_{n,m}^{(r)},$$

where quaternions $C_{n,m}^{(r)}$ are given by

$$C_{n,m}^{(r)} := \binom{n+r-1}{m} F_{r-m+1} + i\binom{n+r}{m} F_{r-m+1} + j\binom{n+r+1}{m} F_{r-m+1} + k\binom{n+r+2}{m} F_{r-m+1}, \tag{3.5}$$

for nonnegative integers n, m, and a positive integer r with $m < r$.

Proof. Using (2.9), for each $s \in \{0, 1, 2, 3\}$, we obtain

$$F_{n+s}^{(r)} = F_{n+s+2r} - \sum_{m=0}^{r-1} \binom{n+s+r-1}{m} F_{r-m+1}.$$

Multiplying these identities by 1, i, j, and k, respectively, and adding them together, yields

$$\begin{aligned} Q_n^{(r)} &= (F_{n+2r} + iF_{n+2r+1} + jF_{n+2r+2} + kF_{n+2r+3}) \\ &\quad - \sum_{m=0}^{r-1} \left(\binom{n+r-1}{m} F_{r-m+1} + i\binom{n+r}{m} F_{r-m+1} + j\binom{n+r+1}{m} F_{r-m+1} + k\binom{n+r+2}{m} F_{r-m+1} \right). \end{aligned}$$

By Definitions (2.3) and (3.5), the first term is precisely Q_{n+2r}, while the expression inside the summation equals $C_{n,m}^{(r)}$. Hence,

$$Q_n^{(r)} = Q_{n+2r} - \sum_{m=0}^{r-1} C_{n,m}^{(r)},$$

which proves the theorem. □

4. A matrix approach

In this section, we develop a matrix framework for hyperfibonacci quaternions. By encoding four consecutive hyperfibonacci numbers as vectors in $\mathbb{R}^4$, we obtain matrix representations that describe the evolution of the sequence with respect to both the index n and the generation parameter r.

For Fibonacci numbers, we have

$$v_0 := \begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} \qquad \text{and} \qquad v_n := \begin{bmatrix} F_n \\ F_{n+1} \\ F_{n+2} \\ F_{n+3} \end{bmatrix}.$$

Let M denote the matrix

$$M := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Now we have

$$Mv_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} = v_1.$$

Furthermore, a straightforward computation shows that

$$v_{n+1} = Mv_n.$$

Consequently, for a nonnegative integer n,

$$v_n = M^n v_0. \tag{4.1}$$

Extending this construction to the hyperfibonacci setting, we introduce the analogous generation vector $v_n^{(r)}$, tracking four consecutive hyperfibonacci numbers of the rth generation:

$$v_0^{(r)} := \begin{bmatrix} F_0^{(r)} \\ F_1^{(r)} \\ F_2^{(r)} \\ F_3^{(r)} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ r+1 \\ \frac{r^2+3r+4}{2} \end{bmatrix}, \qquad v_n^{(r)} := \begin{bmatrix} F_n^{(r)} \\ F_{n+1}^{(r)} \\ F_{n+2}^{(r)} \\ F_{n+3}^{(r)} \end{bmatrix}.$$

By a brief calculation, using the relation (2.8) component-wise,

$$\begin{bmatrix} F_n^{(r)} \\ F_{n+1}^{(r)} \\ F_{n+2}^{(r)} \\ F_{n+3}^{(r)} \end{bmatrix} = \begin{bmatrix} F_{n-1}^{(r)} \\ F_n^{(r)} \\ F_{n+1}^{(r)} \\ F_{n+2}^{(r)} \end{bmatrix} + \begin{bmatrix} F_n^{(r-1)} \\ F_{n+1}^{(r-1)} \\ F_{n+2}^{(r-1)} \\ F_{n+3}^{(r-1)} \end{bmatrix},$$

yields

$$v_n^{(r)} = v_{n-1}^{(r)} + v_n^{(r-1)}, \text{ for } n, r > 0. \tag{4.2}$$

The role of matrix M in the n-direction is mirrored, in the r-direction, by the matrix N defined as

$$N := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

For $n = 0$, direct matrix multiplication yields the following result.

Proposition 4. *For a nonnegative integer r, we have*

$$v_0^{(r)} = N^r v_0. \tag{4.3}$$

For example,

$$Nv_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \end{bmatrix} = v_0^{(1)}.$$

The two relations (4.1) and (4.3), together with the component-wise recurrence, indicate that $v_n^{(r)}$ admits a closed expression as a linear combination of powers of M and N applied to v_0. This is presented in the following theorem.

Theorem 2. *For positive integers n and r, we have*

$$v_n^{(r)} = \left(\sum_{s=1}^{r} \binom{n+r-s-1}{r-s} N^s + \sum_{t=1}^{n} \binom{n+r-t-1}{r-1} M^t \right) v_0.$$

Proof. We proceed by induction on $m = n + r$. For the base case $n = r = 1$, using (4.2) gives

$$(N + M)v_0 = v_0^{(1)} + v_1^{(0)} = v_1^{(1)}.$$

It is also convenient to record the following special cases. For $r = 1$ and $n \geq 1$, using relations (4.2) and (4.3), we obtain

$$\left(N + \sum_{t=1}^{n} M^t \right) v_0 = v_0^{(1)} + v_1^{(0)} + v_2^{(0)} + \cdots + v_n^{(0)} = v_n^{(1)}.$$

Similarly, for $n = 1$ and $r \geq 1$,

$$\left(\sum_{s=1}^{r} N^s + M \right) v_0 = v_0^{(1)} + v_0^{(2)} + \cdots + v_0^{(r)} + v_1^{(0)} = v_1^{(r)}.$$

Using relation (4.2) again, the induction step proceeds as follows

$$\begin{aligned}
v_n^{(r)} &= v_{n-1}^{(r)} + v_n^{(r-1)} \\
&= \left(\sum_{s=1}^{r} \binom{n+r-s-2}{r-s} N^s + \sum_{t=1}^{n-1} \binom{n+r-t-2}{r-1} M^t \right) v_0 \\
&\quad + \left(\sum_{s=1}^{r-1} \binom{n+r-s-2}{r-s-1} N^s + \sum_{t=1}^{n} \binom{n+r-t-2}{r-2} M^t \right) v_0 \\
&= \left[\sum_{s=1}^{r-1} \left(\binom{n+r-s-2}{r-s} + \binom{n+r-s-2}{r-s-1} \right) N^s + \binom{n-2}{0} N^r \right] v_0 \\
&\quad + \left[\sum_{t=1}^{n-1} \left(\binom{n+r-t-2}{r-1} + \binom{n+r-t-2}{r-2} \right) M^t + \binom{r-2}{r-2} M^n \right] v_0 \\
&= \left(\sum_{s=1}^{r-1} \binom{n+r-s-1}{r-s} N^s + \binom{n+r-r-1}{r-r} N^r \right) v_0 \\
&\quad + \left(\sum_{t=1}^{n-1} \binom{n+r-t-1}{r-1} M^t + \binom{n+r-n-1}{r-1} M^n \right) v_0 \\
&= \left(\sum_{s=1}^{r} \binom{n+r-s-1}{r-s} N^s + \sum_{t=1}^{n} \binom{n+r-t-1}{r-1} M^t \right) v_0.
\end{aligned}$$

□

The matrices M and N admit a natural interpretation in terms of the two parameters indexing the vectors $v_n^{(r)}$. The matrix M acts as a shift operator in the index n, moving from $v_n^{(r)}$ to $v_{n+1}^{(r)}$, while the matrix N acts in the direction of the generation parameter r, generating higher-order vectors.

Remark 1. *Since the components of the vector $v_n^{(r)}$ are precisely the scalar coefficients of the hyperfibonacci quaternion $Q_n^{(r)}$, the vector $v_n^{(r)}$ may be regarded as the coordinate representation of $Q_n^{(r)}$ with respect to the basis $\{1, i, j, k\}$ of $\mathbb{H}$. Therefore, all matrix identities derived for $v_n^{(r)}$ translate directly into corresponding identities for hyperfibonacci quaternions.*

Example 2. *To illustrate the application of Theorem 2, let us consider the case $n = 3$ and $r = 2$. Our goal is to determine the vector $v_3^{(2)}$, which tracks four consecutive hyperfibonacci numbers of the second generation starting from index 3,*

$$v_3^{(2)} = \begin{bmatrix} F_3^{(2)} \\ F_4^{(2)} \\ F_5^{(2)} \\ F_6^{(2)} \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \\ 26 \\ 46 \end{bmatrix}.$$

By applying the explicit formula from the theorem, we can express $v_3^{(2)}$ as a linear combination of the matrix powers acting on the initial vector v_0,

$$v_3^{(2)} = \left(\sum_{s=1}^{2} \binom{4-s}{2-s} N^s + \sum_{t=1}^{3} \binom{4-t}{1} M^t \right) v_0.$$

Evaluating the binomial coefficients for each term yields:

$$v_3^{(2)} = \left(3N + N^2 + 3M + 2M^2 + M^3\right) v_0.$$

Using the shifting properties of matrices M and N, i.e., $M^t v_0 = v_t$ and $N^s v_0 = v_0^{(s)}$, the expression simplifies to the following combination of base vectors:

$$v_3^{(2)} = 3v_0^{(1)} + v_0^{(2)} + 3v_1 + 2v_2 + v_3.$$

Substituting the respective vector values:

$$v_3^{(2)} = 3\begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 3 \\ 7 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \\ 26 \\ 46 \end{bmatrix}.$$

This agrees exactly with the values obtained directly from the recurrence relation.

5. Generating functions

In this section, we derive generating functions for hyperfibonacci quaternions of arbitrary generation, first recalling the scalar case and then extending it to the quaternionic setting.

Recall that the generating function for the sequence of Fibonacci numbers $(F_n)_{n\geq 0}$ is

$$F(z) = \sum_{n\geq 0} F_n z^n = \frac{z}{1-z-z^2}. \tag{5.1}$$

Using (5.1), we determine the generating function for the sequence of the hyperfibonacci numbers of the first generation $(F_n^{(1)})_{n\geq 0}$.

$$\begin{aligned}
F^{(1)}(z) &= \sum_{n\geq 0} F_n^{(1)} z^n \\
&= \sum_{n\geq 0} \left(\sum_{m=0}^{n} F_m \right) z^n \\
&= F_0 \sum_{n\geq 0} z^n + F_1 \sum_{n\geq 1} z^n + F_2 \sum_{n\geq 2} z^n + \cdots + F_m \sum_{n\geq m} z^n + \cdots \\
&= F_0 \frac{1}{1-z} + F_1 \frac{z}{1-z} + F_2 \frac{z^2}{1-z} + \cdots + F_m \frac{z^m}{1-z} + \cdots \\
&= \frac{1}{1-z} \left(F_0 + F_1 z + \cdots + F_m z^m + \cdots \right) \\
&= \frac{F(z)}{1-z} \\
&= \frac{z}{(1-z)(1-z-z^2)}.
\end{aligned}$$

We proceed to determine the generating function for the sequence of the hyperfibonacci numbers of arbitrary generation $(F_n)_{n\geq 0}^{(r)}$. By mathematical induction, we obtain that the generating function is given by

$$F^{(r)}(z) = \sum_{n\geq 0} F_n^{(r)} z^n = \frac{z}{(1-z)^r (1-z-z^2)}.$$

Furthermore, by shifting, we find the following generating functions

$$\begin{aligned}
\sum_{n\geq 0} F_{n+1}^{(r)} z^n &= \frac{F^{(r)}(z)}{z}, \\
\sum_{n\geq 0} F_{n+2}^{(r)} z^n &= \frac{F^{(r)}(z) - z}{z^2}, \\
\sum_{n\geq 0} F_{n+3}^{(r)} z^n &= \frac{F^{(r)}(z) - z - (r+1)z^2}{z^3}.
\end{aligned}$$

Inserting these, we have

$$\begin{aligned}
Q^{(r)}(z) &= \sum_{n\geq 0} Q_n^{(r)} z^n \\
&= \sum_{n\geq 0} \left(F_n^{(r)} + i F_{n+1}^{(r)} + j F_{n+2}^{(r)} + k F_{n+3}^{(r)} \right) z^n \\
&= F^{(r)}(z) + i \frac{F^{(r)}(z)}{z} + j \frac{F^{(r)}(z) - z}{z^2} + k \frac{F^{(r)}(z) - z - (r+1)z^2}{z^3}.
\end{aligned}$$

Thus, the generating function for the sequence of the hyperfibonacci quaternions has a compact form

$$Q^{(r)}(z) = \frac{z}{D_r(z)} + i\frac{1}{D_r(z)} + j\frac{1 - D_r(z)}{zD_r(z)} + k\frac{1 - D_r(z) - (r+1)zD_r(z)}{z^2 D_r(z)},$$

where $D_r(z)$ is

$$D_r(z) = (1-z)^r(1-z-z^2).$$

For $r = 0$, this reduces to the known generating function for the sequence of Fibonacci quaternions $(Q_n)_{n\geq 0}$,

$$Q(z) = \frac{z + i + j(z+1) + k(z+2)}{1 - z - z^2}.$$

Next, we transform the above generating function using a notation in which the imaginary units i, j, and k are not explicitly displayed.

An alternative form of the generating function for the sequence $(Q_n)_{n\geq 0}$ is

$$Q(z) = \frac{Q_0 + (Q_1 - Q_0)z}{1 - z - z^2}, \tag{5.2}$$

where $Q_0 = i + j + 2k$ and $Q_1 = 1 + i + 2j + 3k$. Namely, by summing

$$Q(z) = \sum_{n\geq 0} Q_n z^n,$$

$$-zQ(z) = \sum_{n\geq 0} (-Q_n) z^{n+1},$$

and

$$-z^2 Q(z) = \sum_{n\geq 0} (-Q_n) z^{n+2}$$

we get the generating function (5.2).

We obtain the generating function for the sequence of the hyperfibonacci quaternions of the first generation as follows.

$$\begin{aligned} Q^{(1)}(z) &= \sum_{n\geq 0} Q_n^{(1)} z^n \\ &= Q_0^{(1)} + \sum_{n\geq 1} \left(Q_{n-1}^{(1)} + Q_n^{(0)}\right) z^n \\ &= Q_0^{(1)} + \sum_{n\geq 0} Q_n^{(1)} z^{n+1} + Q(z) - Q_0^{(0)} \\ &= Q_0^{(1)} - Q_0 + zQ^{(1)}(z) + Q(z). \end{aligned}$$

This leads to

$$Q^{(1)}(z) = \frac{Q_0^{(1)} - Q_0}{1-z} + \frac{Q_0 + (Q_1 - Q_0)z}{(1-z)(1-z-z^2)} = \frac{z + i + j\,(2 - z^2) + k\,(4 - z - 2z^2)}{(1-z)(1-z-z^2)}.$$

Moreover, by an inductive argument, the generating function of the hyperfibonacci quaternions of arbitrary generation $(Q_n^{(r)})_{n\geq 0}$ is

$$\begin{aligned} Q^{(r)}(z) &= \sum_{n\geq 0} Q_n^{(r)} z^n \\ &= \sum_{m=1}^{r} \frac{Q_0^{(m)} - Q_0^{(m-1)}}{(1-z)^{r-m+1}} + \frac{Q_0 + (Q_1 - Q_0)z}{(1-z)^r(1-z-z^2)}. \end{aligned} \tag{5.3}$$

This notation is convenient for deriving the identity involving these quaternions.

Proposition 5. *For a positive integer $r \geq 2$, we have*

$$Q_r^{(r)} = \sum_{m=1}^{r}(-1)^{m+1}\binom{r+1}{m}Q_{r-m}^{(r)} + \sum_{m=2}^{r}(-1)^m\binom{r}{m-2}Q_{r-m}^{(r)} + (-1)^r\left((2-r)Q_0^{(r)} + (r-3)Q_0^{(r-1)} + Q_0^{(r-2)}\right).$$

Proof. Multiplying both sides of the relation (5.3) by $(1-z)^r(1-z-z^2)$ gives

$$(1-z)^r(1-z-z^2)Q^{(r)}(z) = (1-z-z^2)\sum_{m=1}^{r}(1-z)^{m-1}(Q_0^{(m)} - Q_0^{(m-1)}) + Q_0 + (Q_1 - Q_0)z.$$

Then, by applying the binomial theorem, it follows

$$\begin{aligned} &(1-z-z^2)\sum_{m=0}^{r}\binom{r}{m}(-1)^m z^m \sum_{n\geq 0} Q_n^{(r)} z^n \\ &= (1-z-z^2)\sum_{m=1}^{r}(Q_0^{(m)} - Q_0^{(m-1)})\sum_{l=0}^{m-1}\binom{m-1}{l}(-1)^l z^l + Q_0 + (Q_1 - Q_0)z, \end{aligned}$$

i.e.,

$$\begin{aligned} &(1-z-z^2)\sum_{n\geq 0}\left(\sum_{m=0}^{\min(n,r)}(-1)^m\binom{r}{m}Q_{n-m}^{(r)} z^n\right) \\ &= (1-z-z^2)\sum_{m=1}^{r}(Q_0^{(m)} - Q_0^{(m-1)})\sum_{l=0}^{m-1}\binom{m-1}{l}(-1)^l z^l + Q_0 + (Q_1 - Q_0)z. \end{aligned} \tag{5.4}$$

Finally, we compare the coefficients of z^r for $r \geq 2$ on both sides of the Equation (5.4). For the left-hand side, we obtain

$$\begin{aligned} &Q_r^{(r)} + \sum_{m=1}^{r}(-1)^m\binom{r}{m}Q_{r-m}^{(r)} + \sum_{m=1}^{r}(-1)^m\binom{r}{m-1}Q_{r-m}^{(r)} + \sum_{m=2}^{r}(-1)^{m+1}\binom{r}{m-2}Q_{r-m}^{(r)} \\ &= Q_r^{(r)} + \sum_{m=1}^{r}(-1)^m\binom{r+1}{m}Q_{r-m}^{(r)} - \sum_{m=2}^{r}(-1)^m\binom{r}{m-2}Q_{r-m}^{(r)}, \end{aligned}$$

and for the right-hand side we have

$$\begin{aligned} &(-1)^r(Q_0^{(r)} - Q_0^{(r-1)})\binom{r-1}{r-1} + (-1)^{r-1}(Q_0^{(r)} - Q_0^{(r-1)})\binom{r-1}{r-2} + (-1)^{r-1}(Q_0^{(r-1)} - Q_0^{(r-2)})\binom{r-2}{r-2} \\ &= (-1)^r((2-r)Q_0^{(r)} + (r-3)Q_0^{(r-1)} + Q_0^{(r-2)}). \end{aligned}$$

This completes the proof. □

Example 3. *From Proposition 5, for r = 2, we have*

$$\begin{aligned} Q_2^{(2)} &= \binom{3}{1} Q_1^{(2)} - \binom{3}{2} Q_0^{(2)} + \binom{2}{0} Q_0^{(2)} - Q_0^{(1)} + Q_0^{(0)} \\ &= 3\,(1 + 3i + 7j + 14k) - 3\,(i + 3j + 7k) + i + 3j + 7k - i - 2j - 4k + i + j + 2k \\ &= 3 + 7i + 14j + 26k. \end{aligned}$$

6. Conclusions

In this paper, we introduced hyperfibonacci quaternions and established their fundamental algebraic and matrix properties. Besides deriving recurrence relations, explicit representations, and generating functions, we developed a two-dimensional matrix-operator framework that provides a unified description of their evolution with respect to both the sequence index and the generation parameter.

In addition to providing a unified theoretical representation, the matrix-operator framework offers a significant computational advantage. While direct evaluation of the state vector via the two-dimensional recurrence requires $O(nr)$ operations, evaluating the explicit matrix sums in Theorem 2 requires only $O(n + r)$ operations, making the proposed approach highly efficient for evaluating hyperfibonacci quaternions with large indices n and high generations r.

The proposed framework opens several directions for further research, including extensions to generalized Fibonacci and hyper-Lucas quaternions and their polynomial generalizations, algorithmic implementations motivated by cryptographic applications, and generalizations to higher-dimensional hypercomplex algebras.

Author contributions

All authors contributed equally to this work. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

All authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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