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Research article

A Markov-switching logGARCH stochastic volatility model: Probabilistic properties and GMM estimation

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Abstract: In this paper, we introduced a new Markov-switching extension of the recently proposed logarithmic generalized autoregressive conditional heteroskedasticity (logGARCH) stochastic volatility model by allowing all model coefficients to evolve according to an unobserved finite-state Markov chain. The proposed specification extended the classical logGARCH stochastic volatility framework by incorporating regime-dependent volatility dynamics while preserving its logarithmic structure and latent stochastic volatility formulation. The probabilistic properties of the proposed model were investigated in detail. Sufficient conditions were established for the existence of a unique causal, strictly stationary, and ergodic solution. Additional spectral-radius conditions were derived to guarantee the existence of finite second-order and higher-order moments, and explicit representations of the stationary solution and its unconditional moments were obtained. To estimate the model parameters, a Generalized Method of Moments procedure was developed based on the theoretical moment conditions implied by the stationary distribution of the process. Finally, an extensive Monte Carlo simulation study was conducted under different parameter configurations, innovation distributions, and sample sizes.

Keywords: Markov-switching stochastic volatility; logGARCH; periodic model; stationarity; ergodicity; higher-order moments; Generalized Method of Moments
Mathematics Subject Classification: Primary 62M10; Secondary 62M05

1. Introduction

Modeling volatility has long been a central topic in financial econometrics and time series analysis because of its essential role in risk management, portfolio allocation, asset pricing, and economic forecasting. The autoregressive conditional heteroskedastic (ARCH) model introduced by Engle [1], followed by the generalized ARCH (GARCH) model proposed by Bollerslev [2], laid the foundation

for modern volatility modeling by allowing the conditional variance to evolve dynamically according to past observations and previous levels of volatility. Owing to their analytical simplicity and strong empirical performance, these approaches have become standard tools for describing volatility persistence and clustering across a broad range of financial and economic applications. As financial systems have become increasingly interconnected, research has gradually progressed from univariate volatility specifications to multivariate formulations capable of capturing the dynamic dependence among several financial variables simultaneously. This transition has significantly expanded the scope of volatility analysis and enhanced its applicability to problems involving interconnected markets and high-dimensional financial data. To better represent more complex dynamic behaviors, subsequent studies have introduced a variety of nonlinear extensions. Among them, threshold bilinear stochastic processes provide a flexible framework for modeling nonlinear regime-dependent dynamics, while asymmetric multivariate stochastic difference volatility models account for both asymmetric effects and cross-sectional dependence among multiple financial variables (see, e.g., [3, 4]). These developments have substantially broadened the volatility modeling literature, providing increasingly flexible frameworks for analyzing the dynamic behavior of financial and economic time series.

Despite the remarkable success of ARCH- and GARCH-type models in describing volatility clustering and persistence, these models assume that the conditional variance is a deterministic function of past observations and past conditional variances. As a result, once the historical information is available, future volatility is entirely determined by the recursive structure of the model. However, empirical evidence indicates that financial market volatility is influenced by numerous latent factors that cannot be fully explained by past returns alone, suggesting that volatility itself evolves as an unobserved stochastic process. These observations motivated the development of stochastic volatility (SV) models, in which the latent volatility follows its own stochastic dynamics, providing a more flexible and realistic representation of financial time series. The pioneering work of Taylor [5] introduced one of the earliest stochastic volatility formulations and laid the foundation for a broad class of latent volatility models. Since then, substantial theoretical and empirical advances have demonstrated the ability of SV models to reproduce several important stylized facts of financial returns, including heavy tails, excess kurtosis, long-memory behavior, and complex dependence structures, while offering greater flexibility than traditional conditional heteroskedastic models. Several influential studies have further advanced this literature, including the investigation of persistence and kurtosis by Carnero et al. [6], the analysis of volatility dynamics and market uncertainty by Black [7], the Bayesian estimation framework developed by Jacquier et al. [8], the long-memory studies of Ding et al. [9] and Harvey [10], and the extension of stochastic volatility models to the multivariate setting by Harvey, Ruiz, and Shephard [11]. Together, these contributions have established stochastic volatility models as one of the principal frameworks for modeling time-varying uncertainty in financial econometrics and have inspired a wide range of subsequent methodological developments.

Numerous extensions of the GARCH framework have been developed to overcome some of the limitations of the classical specification and to provide a more flexible description of financial volatility. Among the most influential is the Exponential GARCH (EGARCH) model, which introduced a logarithmic representation of the conditional variance and enabled the modeling of asymmetric volatility responses to positive and negative shocks. Building on this idea, the logGARCH model emerged as an attractive alternative by modeling the logarithm of the conditional variance directly while preserving a relatively simple dynamic structure. Its logarithmic formulation eliminates the

need for positivity constraints on the volatility coefficients and can naturally accommodate asymmetric effects while effectively capturing the persistence of both large and small volatility movements. These features, together with its convenient statistical properties and autoregressive moving average (ARMA) representation, have made the logGARCH model an appealing framework for modeling highly persistent financial time series and have stimulated extensive research on its estimation, inference, and practical implementation. More recent studies have further strengthened the theoretical foundations of the logGARCH framework. These contributions include estimation methods that account for the presence of zero returns, efficient quasi-maximum likelihood estimation procedures based on ARMA representations, a comprehensive ARMA characterization of the model, and extensions to periodic time-varying asymmetric logGARCH processes (see, [12–15]). As a result, the logGARCH model has become a well-established specification within the GARCH family and an effective alternative for modeling complex volatility dynamics in financial and economic time series.

Although classical volatility models and their subsequent extensions have substantially advanced the analysis of financial time series, they generally share a common underlying assumption: The model parameters remain constant over time. This assumption underlies the standard ARCH and GARCH models, stochastic volatility specifications, and logGARCH models, all of which are governed by a single set of fixed coefficients throughout the observation period. While this assumption greatly facilitates theoretical analysis and statistical inference, it may not adequately reflect the behavior of many real-world financial and economic systems. In practice, financial markets and many economic variables—including macroeconomic indicators, energy prices, and exchange rates—are continuously influenced by policy changes, financial crises, technological innovations, geopolitical events, and unexpected external shocks. As a result, these systems often evolve through different regimes, each characterized by distinct volatility patterns and dynamic behaviors. Under such circumstances, assuming a single set of constant parameters over the entire sample period may be overly restrictive and may fail to capture structural changes, abrupt transitions, and regime-dependent dynamics that are frequently observed in empirical data. These limitations have motivated the development of more flexible modeling frameworks in which model parameters are allowed to vary over time according to specified mechanisms. By accommodating temporal changes in the underlying dynamics, such frameworks provide a more realistic and adaptable representation of complex financial and economic time series.

A natural extension of constant-parameter volatility models is to allow the model coefficients to vary systematically over time. Rather than assuming that the parameters remain fixed throughout the entire observation period, periodic models introduce coefficients that evolve according to a predetermined periodic structure, thereby capturing regular seasonal or cyclical variations frequently observed in financial and economic time series. This idea gained considerable attention following the pioneering periodic autoregressive conditional heteroskedastic (PARCH) model proposed by Bollerslev and Ghysels [16], which demonstrated that periodic conditional heteroskedasticity provides a flexible framework for describing recurring patterns in volatility. Since then, periodic volatility modeling has been extended in several directions. Ghezal et al. [17] established quasi-maximum likelihood estimation for periodic bilinear models and periodic autoregressive moving average (PARMA) models with periodic bilinear innovations, and subsequently [18] developed the theoretical framework for periodic time-varying bilinear GARCH models. Further advances include the asymptotic inference for periodic time-varying bilinear processes developed by Bibi and Ghezal [19], the investigation

of probabilistic properties and estimation methods for periodic threshold autoregressive stochastic volatility models by Alzeley [20], and the study of the consistency and asymptotic normality of M-estimators for periodic threshold GARCH models by Zemmouri [21]. Taken together, these studies have substantially enriched the theory of periodically varying time series and demonstrated the effectiveness of periodic coefficients in capturing regular temporal variations across a broad range of linear and nonlinear stochastic models. Despite these advances, periodic models still rely on deterministic periodic mechanisms to describe the evolution of model parameters. In many practical applications, however, changes in financial markets, macroeconomic indicators, commodity prices, and exchange rates are rarely governed by perfectly periodic patterns. Instead, they are often characterized by irregular structural changes, abrupt transitions, and unpredictable shifts between different regimes. Consequently, periodic coefficients alone may not provide sufficient flexibility to represent the complex dynamics observed in real-world data, thereby motivating the development of more general modeling frameworks capable of accommodating random regime changes.

To overcome the limitations of deterministic parameter variations, a more general and flexible modeling framework has emerged through the incorporation of Markov-switching mechanisms. Instead of assuming that model parameters remain constant or evolve according to a predetermined periodic structure, Markov-switching models allow them to change probabilistically according to an unobserved finite-state Markov chain. This stochastic mechanism enables the model to adapt to different regimes, providing a realistic representation of financial and economic systems subject to structural breaks, abrupt transitions, market crises, policy interventions, and other unexpected events. As a result, Markov-switching models have become a widely used framework for capturing regime-dependent dynamics, nonlinear behaviors, and heterogeneous volatility patterns that cannot be adequately represented by fixed or periodically varying parameters. The flexibility of the Markov-switching framework has encouraged its integration with a broad range of linear and nonlinear time series models. For example, Bibi and Ghezal [22] investigated the probabilistic structure, stationarity, higher-order moments, and mixing properties of Markov-switching bilinear processes, and subsequently [23] established the strong consistency of the quasi-maximum likelihood estimator for these models. Ghezal et al. [24–26] further developed estimation procedures based on minimum distance estimation, derived the spectral representation of Markov-switching bilinear processes, and proposed a doubly Markov-switching autoregressive model together with its probabilistic properties and consistency results. The framework was subsequently extended to Markov-switching BILINEAR-GARCH models by Bibi and Ghezal [27], providing a unified approach that combines nonlinear dynamics with conditional heteroskedasticity. More recently, Cavicchioli and Zemmouri [28] developed a comprehensive spectral and bispectral analysis of Markov-switching bilinear time series, further strengthening the theoretical understanding of these models.

The success of the Markov-switching paradigm has also led to its application in several advanced volatility models. Zemmouri [29] investigated Markov-switching asymmetric logGARCH models by establishing their stationarity conditions and estimation theory, and later [30] introduced Markov-switching autoregressive stochastic volatility processes. Balegh and Zemmouri [31] proposed Markov-switching threshold stochastic volatility models capable of describing volatility regime changes, while Alraddadi [32] developed the Markov-switching threshold bilinear generalized autoregressive conditional heteroskedasticity (BLGARCH) model and, more recently [33], introduced the logarithmic threshold GARCH stochastic volatility (logTG-SV) model, a threshold-based logarithmic stochastic

volatility framework for modeling exchange-rate dynamics. Taken together, these contributions demonstrate that integrating Markov-switching mechanisms with increasingly sophisticated volatility models substantially enhances modeling flexibility, improves the representation of structural changes and regime-dependent dynamics, and provides a more realistic framework for analyzing complex financial and economic time series. These advances naturally motivate the development of Markov-switching extensions of the logGARCH stochastic volatility framework.

Despite the substantial progress achieved in regime-switching volatility modeling, an important research gap remains. In a recent study, Guerbyenne et al. [34] introduced the logGARCH stochastic volatility model, which combines the logarithmic GARCH specification with a latent stochastic volatility process, thereby providing a flexible framework for modeling persistent and asymmetric volatility dynamics. Nevertheless, as in most stochastic volatility models, the proposed specification assumes that the model parameters remain constant throughout the entire observation period. Such an assumption may be restrictive in many financial applications, where market conditions evolve through successive expansion and contraction phases, financial crises, policy interventions, and other structural changes that give rise to distinct volatility regimes. To the best of our knowledge, the incorporation of Markov-switching dynamics into the logGARCH stochastic volatility framework has not yet been investigated. This gap in the existing literature provides the primary motivation for the present study, which aims to extend the logGARCH stochastic volatility model by allowing its parameters to evolve according to an unobserved finite-state Markov chain, thereby offering a more flexible framework for capturing regime-dependent volatility dynamics.

Motivated by this gap in the existing literature, the present paper proposes a Markov-switching extension of the recently introduced logGARCH stochastic volatility model of Guerbyenne et al [34]. Specifically, a hidden Markov chain is incorporated into the model to allow the parameters governing the volatility process to evolve across different latent regimes. This extension provides a more flexible framework for capturing structural changes, regime-dependent dynamics, and heterogeneous volatility patterns commonly observed in financial and economic time series. From a theoretical perspective, we derive sufficient conditions for the strict stationarity of the proposed model and establish the existence of higher-order moments. From a computational perspective, the model parameters are estimated using a Generalized Method of Moments (GMM) approach. In addition, an extensive simulation study is carried out to assess the finite-sample performance and statistical efficiency of the proposed estimation procedure. Together, these contributions extend the logGARCH stochastic volatility framework to accommodate regime-switching dynamics and provide a flexible methodology for modeling complex volatility behavior in financial and economic time series.

The remainder of this paper is organized as follows. Section 2 presents the complete mathematical formulation of the proposed model. Section 3 establishes several probabilistic properties of the model, with particular emphasis on strict stationarity, as well as the existence of higher-order moments. In Section 4, the model parameters are estimated using the GMM method. Finally, Section 5 provides a numerical simulation study to assess the performance of the proposed estimation procedure and to examine its statistical efficiency under various scenarios.

2. The Markov-switching logGARCH stochastic volatility model

The growing complexity of financial and economic markets has driven the development of more sophisticated volatility models capable of capturing the nonlinear and highly persistent behavior commonly observed in empirical time series. Among these developments, stochastic volatility models have emerged as an important alternative to traditional conditional heteroskedastic specifications by treating volatility as an unobserved stochastic process rather than as a deterministic function of past information. This modeling framework provides a more realistic representation of financial uncertainty and has proven effective in describing latent volatility dynamics, heavy-tailed distributions, and long-range dependence in financial returns. Among the recent contributions in this direction, Guerbyenne et al. [34] proposed the logGARCH stochastic volatility model, which combines the logarithmic GARCH formulation with a latent stochastic volatility process. The resulting specification retains the principal advantages of the logGARCH framework, including the absence of positivity constraints on the model coefficients, the ability to accommodate asymmetric volatility responses, and the ability to capture persistent volatility dynamics through its logarithmic formulation. More specifically, the standard logGARCH stochastic volatility model is defined as follows:

$$\begin{cases} \xi_t = & \eta_t^{(1)} \exp\left(\psi_t/2\right), \\ \psi_t = & \beta_1 + \beta_2 \log \xi_{t-1}^2 + \beta_3 \psi_{t-1} + \beta_4 \eta_t^{(2)}, \end{cases} \tag{1}$$

where the latent process (ψ_t) evolves according to an autoregressive stochastic process driven by past observations and random innovations, while $(\eta_t^{(1)})$ and $(\eta_t^{(2)})$ are two mutually independent sequences of independent and identically distributed random variables with zero mean and unit variance. Although this formulation provides a flexible and mathematically tractable framework for modeling stochastic volatility, it relies on the assumption that a single set of parameters governs the entire data-generating process. Consequently, the relationship between current volatility, historical observations, and previous volatility is assumed to remain unchanged throughout the observation period. In practice, however, financial markets rarely evolve under homogeneous conditions. Instead, they experience successive periods of expansion and recession, tranquil and turbulent episodes, policy interventions, speculative bubbles, financial crises, and other unexpected events, each of which may give rise to distinct volatility dynamics. Under these circumstances, the assumption of constant parameters may be overly restrictive and may fail to adequately capture the heterogeneous behavior commonly observed in empirical financial time series.

To overcome this limitation, we propose a Markov-switching extension of the logGARCH stochastic volatility model in which the parameters governing the latent volatility process are allowed to vary across latent regimes. Specifically, the parameter vector is assumed to depend on a finite-state Markov chain whose evolution is governed by fixed transition probabilities, allowing the model to switch probabilistically between distinct volatility regimes over time. This extension substantially enhances the flexibility of the original specification by allowing volatility persistence, the impact of past returns, innovation effects, and the unconditional volatility level to differ across regimes.

Let (λ_t) denote a homogeneous Markov chain taking values in the finite state space $\Lambda = \{1, ..., r\}$, with transition probability matrix $(\mathbb{P} = (p_{lk}))$ and stationary distribution $\underline{\Pi} = (\Pi_1, ..., \Pi_r))$. The chain is assumed to be irreducible, aperiodic, and stationary, and to be independent of the innovation sequences $(\eta_t^{(1)}, \eta_t^{(2)})$. At each time point, the state of the Markov chain determines the latent volatility

regime governing the dynamics of the model. The proposed Markov-switching logGARCH stochastic volatility (MSlogGARCH-SV) model is then defined as

$$\begin{cases} \xi_t = & \eta_t^{(1)} \exp(\psi_t/2), \\ \psi_t = & \beta_1(\lambda_t) + \beta_2(\lambda_t) \log \xi_{t-1}^2 + \beta_3(\lambda_t)\psi_{t-1} + \beta_4(\lambda_t)\eta_t^{(2)}, \end{cases} \tag{2}$$

where $(\eta_t^{(1)})$ and $(\eta_t^{(2)})$ are two mutually independent sequences of independent and identically distributed random variables with zero mean and unit variance, which are also independent of the Markov chain. For each state (λ_t), the coefficients $(\beta_i(\lambda_t))$, $i = 1, ..., 4$, define a distinct volatility regime, allowing the latent volatility process to adjust to changing market conditions over time. The proposed specification preserves the fundamental stochastic volatility structure of the original logGARCH stochastic volatility model while introducing additional flexibility through regime-dependent dynamics. By allowing the volatility mechanism itself to evolve according to an unobserved Markov chain, the model is able to capture structural changes, regime-dependent persistence, asymmetric market behavior, and abrupt volatility transitions within a unified probabilistic framework. Consequently, it represents a natural and theoretically consistent extension of the recently proposed logGARCH stochastic volatility model, providing a richer and more realistic representation of the complex dynamics frequently observed in financial and economic time series.

3. Probabilistic properties of the MSlogGARCH-SV model

This section develops the probabilistic theory underlying the proposed MSlogGARCH-SV model. Owing to the simultaneous presence of latent stochastic volatility and Markov-switching dynamics, the resulting process belongs to the class of nonlinear stochastic recurrence equations with regime-dependent random coefficients. Establishing its probabilistic properties is therefore essential for ensuring that the model is mathematically well defined and statistically suitable for inference. The main objectives of this section are to derive sufficient conditions for strict stationarity, ergodicity, and the existence of moments of arbitrary order, while also characterizing the structural behavior of the latent volatility process. These theoretical results provide the probabilistic foundation required for the estimation methodology developed later and justify the applicability of the proposed model to financial time series exhibiting regime changes. To facilitate the subsequent analysis, we first derive an equivalent Markov-switching autoregressive representation of the latent log-volatility process. This representation transforms the original nonlinear stochastic volatility specification into a first-order stochastic recurrence equation whose coefficients evolve according to the hidden Markov chain. Such a formulation constitutes the key analytical tool throughout the remainder of the paper, since it enables the probabilistic properties of the model to be investigated using the theory of Markov-driven stochastic recurrence equations.

Using the first equation of model (2), we obtain $\log \xi_t^2 = \log\left(\eta_t^{(1)}\right)^2 + \psi_t$. Substituting this identity into the second equation of model (2) yields

$$\psi_t = \beta_1(\lambda_t) + (\beta_2(\lambda_t) + \beta_3(\lambda_t))\psi_{t-1} + \left(\beta_4(\lambda_t)\eta_t^{(2)} + \beta_2(\lambda_t)\log\left(\eta_{t-1}^{(1)}\right)^2\right), \tag{3}$$

which is referred to as the Markov-switching autoregressive representation of the latent log-volatility process. This formulation shows that the latent volatility evolves as a first-order autoregressive

process whose intercept, persistence coefficient, and innovation structure all depend on the current hidden regime. Consequently, unlike the standard logGARCH-SV model, where a single parameter vector governs the entire sample, the proposed specification allows the persistence mechanism and the response to shocks to adjust dynamically as the Markov chain switches between regimes. This representation serves as the cornerstone of the theoretical developments that follow, providing a unified framework for establishing strict stationarity, moment existence, ergodicity, and the asymptotic properties of the proposed estimation procedure.

Remark 3.1. *The latent process associated with the proposed MSlogGARCH-SV model admits a natural Markovian representation. More precisely, the augmented process* (ψ_t, λ_t) *consisting of the latent log-volatility and the hidden regime variable forms a time-homogeneous Markov chain on* $\mathbb{R} \times \Lambda$. *This follows immediately from the recursive definition of the latent volatility equation, since the future evolution of* (ψ_t, λ_t) *depends only on its current state and the contemporaneous innovations, independently of the previous history of the process. This Markovian representation provides the foundation for the probabilistic analysis of the proposed model. In particular, it offers a natural framework for establishing strict stationarity, ergodicity, higher-order moment conditions, and the asymptotic properties of statistical estimators. Furthermore, it enables the regime-switching dynamics and the latent stochastic volatility process to be analyzed jointly within the theory of Markov chains, thereby facilitating the study of the long-run behavior of the model.*

Before investigating the probabilistic properties of the proposed MSlogGARCH-SV model, it is first necessary to establish whether the stochastic recurrence defining the latent volatility process admits a well-defined stationary solution. The existence of such a solution is fundamental to both the theoretical analysis and the practical applicability of the model, as it ensures that the volatility dynamics remain stable over time and do not depend on arbitrary initial conditions. Moreover, strict stationarity provides the foundation for deriving further probabilistic properties, including the existence of higher-order moments, ergodicity, and the asymptotic behavior of statistical estimators. Owing to the presence of regime-dependent coefficients, the proposed model belongs to the class of stochastic recurrence equations with random coefficients driven by a hidden Markov chain. Consequently, the classical stationarity conditions established for fixed-parameter logGARCH stochastic volatility models are no longer applicable and must be replaced by conditions that explicitly account for the stochastic evolution of the latent regimes. The following theorem establishes a sufficient condition under which the proposed MSlogGARCH-SV process admits a unique strictly stationary and causal solution, together with an explicit infinite-series representation.

Theorem 3.1. *Let* (ξ_t) *be the MSlogGARCH-SV process introduced in (2). Assume that*

$$\prod_{j=1}^{r} |\beta_2(j) + \beta_3(j)|^{\Pi_j} < 1. \tag{4}$$

Moreover, assume that

$$\mathbb{E}\left[\log^+ |D_0|\right] < \infty,$$

where

$$D_t = \beta_1(\lambda_t) + \beta_4(\lambda_t)\eta_t^{(2)} + \beta_2(\lambda_t) \log\left((\eta_{t-1}^{(1)})^2\right),$$

and $\log^+ x = \max\{\log x, 0\}$ *for* $x > 0$, *with* $\log^+ 0 = 0$. *Then the stochastic recurrence equation associated with the model admits a unique ergodic solution which is both strictly stationary and causal. Moreover, this solution possesses the explicit infinite-product representation*

$$\xi_t = \eta_t^{(1)} \prod_{l\geq 0} \exp\left(\frac{1}{2}\left\{\prod_{k=0}^{l-1}\left(\beta_2\left(\lambda_{t-k}\right)+\beta_3\left(\lambda_{t-k}\right)\right)\right\}\left(\beta_1\left(\lambda_{t-l}\right)+\beta_4\left(\lambda_{t-l}\right)\eta_{t-l}^{(2)}+\beta_2\left(\lambda_{t-l}\right)\log\left(\eta_{t-l-1}^{(1)}\right)^2\right)\right). \quad (5)$$

Furthermore, the above infinite representation converges almost surely.

Proof. The proof relies on the stochastic recurrence representation of the latent log-volatility process. Indeed, from the autoregressive formulation given in (3), the latent process satisfies

$$\psi_t = C\left(\lambda_t\right)\psi_{t-1} + D_t\left(\lambda_t\right),$$

where $C\left(\lambda_t\right) = \beta_2\left(\lambda_t\right) + \beta_3\left(\lambda_t\right)$. Iterating the above recursion, for every integer $m \geq 1$, we obtain

$$\psi_t = \left\{\prod_{k=0}^{m-1} C\left(\lambda_{t-k}\right)\right\}\psi_{t-m} + \sum_{l=0}^{m-1}\left\{\prod_{k=0}^{l-1} C\left(\lambda_{t-k}\right)\right\} D_{t-l}\left(\lambda_{t-l}\right).$$

We now show how condition (4) guarantees the convergence of the latter series. Since (λ_t) is a stationary, irreducible, and aperiodic finite-state Markov chain, it is ergodic. Hence, by the ergodic theorem,

$$(1/m)\log\left|\prod_{k=0}^{m-1} C\left(\lambda_{t-k}\right)\right| = (1/m)\sum_{k=0}^{m-1}\log\left|C\left(\lambda_{t-k}\right)\right| \longrightarrow \sum_{j=1}^{r}\Pi_j \log|C(j)| \text{ a.s.}$$

Condition (4) is equivalent to

$$\gamma_C := \sum_{j=1}^{r}\Pi_j\log|C(j)| = \log\left(\prod_{j=1}^{r}|C(j)|^{\Pi_j}\right) < 0.$$

Therefore,

$$\limsup_{m\to\infty}\left|\prod_{k=0}^{m-1} C\left(\lambda_{t-k}\right)\right|^{1/m} = e^{\gamma_C} < 1 \text{ a.s.}$$

To justify the application of the Borel–Cantelli lemma, recall the additional assumption in the theorem that

$$\mathbb{E}\left[\log^+|D_0|\right] < \infty.$$

Since (D_t) is stationary, for every $\varepsilon > 0$ we have

$$\begin{aligned}\sum_{m=1}^{\infty}\mathbb{P}\left(|D_{t-m}| > e^{\varepsilon m}\right) &= \sum_{m=1}^{\infty}\mathbb{P}\left(|D_0| > e^{\varepsilon m}\right)\\ &\leq \frac{1}{\varepsilon}\mathbb{E}\left[\log^+|D_0|\right] < \infty.\end{aligned}$$

Consequently, by the first Borel–Cantelli lemma,

$$\mathbb{P}\left(|D_{t-m}| > e^{\varepsilon m} \text{ infinitely often}\right) = 0.$$

It follows that

$$\limsup_{m\to\infty} |D_{t-m}|^{1/m} \leq e^{\varepsilon} \quad \text{a.s.}$$

Taking a countable sequence of positive values of ε converging to zero yields

$$\limsup_{m\to\infty} |D_{t-m}|^{1/m} \leq 1 \quad \text{a.s.}$$

Combining this result with the preceding ergodic limit, we obtain

$$\limsup_{m\to\infty} \left| \left\{ \prod_{k=0}^{m-1} C(\lambda_{t-k}) \right\} D_{t-m}(\lambda_{t-m}) \right|^{1/m} \leq e^{\gamma_C} < 1 \quad \text{a.s.}$$

Thus, by the Cauchy root test, $\sum_{l\geq 0} \left\{ \prod_{k=0}^{l-1} C(\lambda_{t-k}) \right\} D_{t-l}(\lambda_{t-l})$ converges absolutely almost surely. Noting that $\xi_t = \eta_t^{(1)} \exp(\psi_t/2)$, substituting the previous expression for (ψ_t) yields directly the infinite-product representation given in (5). Since the solution is a measurable functional of the stationary ergodic driving sequence $(\lambda_t, \eta_t^{(1)}, \eta_t^{(2)})$, it follows that (ξ_t) is strictly stationary, causal, and ergodic. Furthermore, the almost sure convergence of the stochastic series implies the almost sure convergence of the corresponding infinite product, thereby establishing the desired result. A detailed treatment of analogous convergence arguments for related Markov-switching stochastic volatility models can be found in Ghezal and Zemmouri [30]. □

Remark 3.2. *An important implication of condition (4) is that the stationarity of the proposed MSlogGARCH-SV model is governed by the global contraction mechanism induced by the underlying Markov chain rather than by the stability of each individual regime. Indeed, the condition depends only on the weighted long-run average of the regime-specific persistence coefficients through the stationary distribution of the Markov chain. Consequently, local instability in one or more regimes does not necessarily prevent the existence of a globally stationary solution. More precisely, the presence of highly persistent or even locally unstable volatility regimes does not preclude the existence of a unique strictly stationary, causal, and ergodic solution, provided that the average contraction induced by the Markov chain satisfies condition (4). In other words, transient high-volatility regimes may occur, but their long-run effect is offset by sufficiently stable regimes that are visited with adequate frequency. This highlights one of the principal advantages of Markov-switching specifications over fixed-parameter volatility models, namely, their ability to accommodate heterogeneous volatility dynamics while preserving global probabilistic stability. Furthermore, if the stochastic volatility innovation term is removed from the latent volatility equation, the proposed MSlogGARCH-SV model reduces to a Markov-switching logGARCH-type process with regime-dependent coefficients. In this setting, condition (4) continues to hold, indicating that the additional stochastic volatility component enriches the volatility dynamics without altering the fundamental Markov-switching contraction mechanism responsible for strict stationarity and ergodicity.*

Example 3.1. *To illustrate the stationarity condition (4) established in Theorem 3.1, consider the proposed MSlogGARCH-SV model with two regimes $r = 2$. Let $\varphi_1 = \beta_2(1) + \beta_3(1) := (2a) + (b)$ and*

$\varphi_2 = \beta_2(2)+\beta_3(2) := (a)+(b+1)$, and assume that the stationary distribution of the underlying Markov chain is given by $\Pi_1 = 1/3$. In this case, the stationarity condition derived from Theorem (4) becomes $|\varphi_1|^{\Pi_1}|\varphi_2|^{\Pi_2} < 1$. This condition demonstrates that the existence of a strictly stationary solution depends on the joint contribution of the two regimes through their stationary probabilities, rather than on a single constant autoregressive coefficient. By contrast, for the standard logGARCH stochastic volatility model, the corresponding stationarity condition reduces to $|\varphi_1| < 1$. The stationarity regions associated with the proposed Markov-switching model and the standard logGARCH stochastic volatility model are illustrated in the Figure 1.

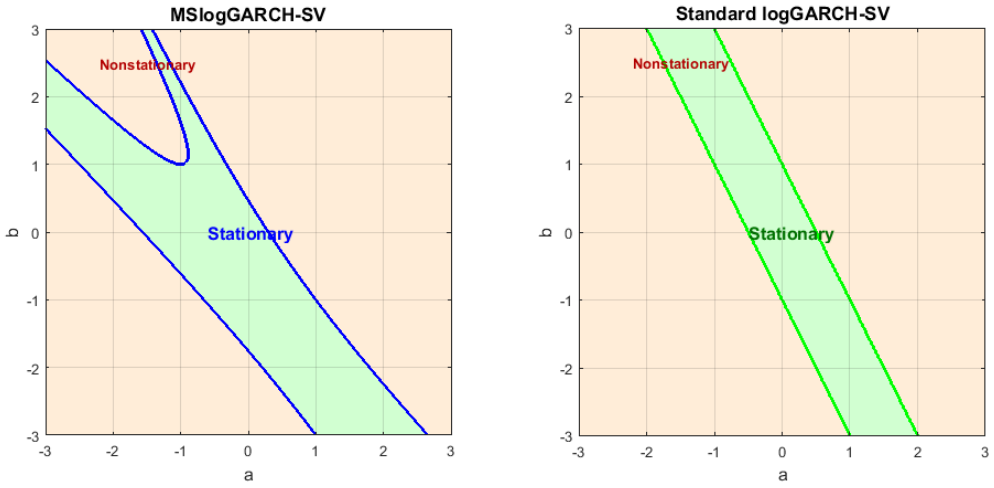


Figure 1. Comparison of the strict stationarity regions for the proposed MSlogGARCH-SV model and the standard logGARCH-SV model.

Figure 1 provides a graphical comparison of the strict stationarity regions associated with the proposed MSlogGARCH-SV model and the classical logGARCH-SV specification. Several important observations emerge from this comparison. Unlike the standard logGARCH-SV model, whose stationarity region is characterized by a simple linear strip, the admissible parameter region of the proposed model exhibits a nonlinear geometry induced by the interaction between the regime-specific persistence coefficients and the stationary distribution of the underlying hidden Markov chain. Consequently, strict stationarity is governed by the long-run average contraction generated by the Markov-switching mechanism rather than by the magnitude of a single autoregressive coefficient. The figure also reveals that the stationarity region of the proposed MSlogGARCH-SV model extends beyond that of the standard logGARCH-SV specification. This enlargement is a direct consequence of the regime-switching dynamics. In particular, the process may remain globally strictly stationary even when one regime is locally highly persistent, provided that transitions toward more stable regimes occur with sufficient frequency. Hence, the overall stability of the model is determined by the combined effect of the regime-specific persistence coefficients and the transition probabilities of the hidden Markov chain, rather than by the behavior of any individual regime in isolation. This graphical comparison provides clear numerical support for the theoretical result established in Theorem 3.1, demonstrating that local instability within a particular regime does not necessarily preclude the

existence of a unique globally strictly stationary and ergodic solution.

Following the establishment of strict stationarity and ergodicity, the next fundamental question concerns the existence of finite second-order moments for the proposed MSlogGARCH-SV process. This property is essential for both the probabilistic analysis and the statistical inference of the model, since finite second-order moments ensure a well-defined unconditional volatility level and provide the foundation for studying autocovariances, dependence structures, and the asymptotic behavior of estimation procedures. Unlike the strict stationarity condition, which is governed by the long-run contraction of the underlying stochastic recurrence equation, the existence of second-order moments requires stronger conditions on the regime-dependent dynamics. The following theorem provides a sufficient condition under which the proposed process admits a unique second-order stationary solution while maintaining the strict stationarity and ergodicity established above.

Theorem 3.2. *Let (ξ_t) denote the MSlogGARCH-SV process defined in (2) and suppose that the stationarity condition (4) holds, so that the process admits a unique strictly stationary and causal solution given by (5). Assume further that the second-order Markov operator associated with the stochastic recurrence satisfies*

$$\rho\left(\mathbb{P}\left(\left(\underline{\beta}_2+\underline{\beta}_3\right)^{(2)}\right)\right)<1, \tag{6}$$

where $\underline{\beta}_j'=\left(\beta_j(1),...,\beta_j(r)\right)$, $j=2,3$, $\rho(.)$ denotes the spectral radius and

$$\mathbb{P}\left(\left(\underline{\beta}_2+\underline{\beta}_3\right)^{(2)}\right)=\left(p_{ij}\left(\beta_2(j)+\beta_3(j)\right)^2\right)_{1\le i,j\le r}.$$

Suppose, in addition, that the innovation term D_t in the stochastic recurrence representation of ψ_t satisfies $\mathbb{E}[D_t^2]<\infty$. Then the latent log*-volatility process (ψ_t) has a unique causal strictly stationary solution with finite second moment. In particular, $E\left\{\psi_t^2\right\}<\infty$, and the infinite series representation of ψ_t converges both almost surely and in $\mathbb{L}^2$. Furthermore, suppose that $E\{\exp(\psi_t)\}<\infty$. Then the observed process (ξ_t) has a finite second moment, that is, $E\left\{\xi_t^2\right\}<\infty$. Consequently, (ξ_t) is second-order stationary.*

Proof. By Theorem 3.1, under condition (4), the latent log-volatility process admits the causal stationary representation

$$\psi_t=\sum_{l=0}^{\infty}\left\{\prod_{k=0}^{l-1}C(\lambda_{t-k})\right\}D_{t-l}.$$

The series converges almost surely by Theorem 3.1. We now establish its convergence in $\mathbb{L}^2$ under condition (6). For $l\ge 0$, define

$$R_{t,l}:=\left\{\prod_{k=0}^{l-1}C(\lambda_{t-k})\right\}D_{t-l},$$

with the empty product equal to 1. Set

$$c_j=\beta_2(j)+\beta_3(j),\qquad d_j=\mathbb{E}[D_t^2\mid\lambda_t=j],$$

and let

$$M=\mathbb{P}\left(\left(\underline{\beta}_2+\underline{\beta}_3\right)^{(2)}\right)=(p_{ij}c_j^2)_{1\le i,j\le r}.$$

Since the Markov chain has a finite state space, its stationary probabilities are positive, and $\mathbb{E}[D_0^2] < \infty$, each d_j is finite. Let $\boldsymbol{\pi} = (\Pi_1, \ldots, \Pi_r)$ denote the stationary distribution and let $\mathbf{d} = (d_1, \ldots, d_r)'$. Using the independence of the driving innovations and the Markov chain, together with the Markov transition probabilities, gives

$$\begin{aligned}\mathbb{E}[R_{t,l}^2] &= \mathbb{E}\left[D_{t-l}^2 \prod_{k=0}^{l-1} c_{\lambda_{t-k}}^2\right] \\ &= \boldsymbol{\pi}\ \mathrm{diag}(\mathbf{d})\, M^l \mathbf{1},\end{aligned}$$

where $\mathbf{1}$ is the column vector of ones. Indeed, each transition from state i to state j contributes the factor $p_{ij}c_j^2$, which is precisely the (i, j) entry of M.

By condition (6), $\rho(M) < 1$. Hence, for any number q satisfying $\rho(M) < q < 1$, there exists a finite constant $K > 0$ such that

$$\|M^l\| \le K q^l, \qquad l \ge 0.$$

Consequently, there exists a finite constant $K_1 > 0$ such that

$$\mathbb{E}[R_{t,l}^2] \le K_1 q^l, \qquad l \ge 0.$$

It follows that

$$\|R_{t,l}\|_2 = \left(\mathbb{E}[R_{t,l}^2]\right)^{1/2} \le \sqrt{K_1}\, q^{l/2}.$$

Therefore,

$$\sum_{l=0}^{\infty} \|R_{t,l}\|_2 \le \sqrt{K_1} \sum_{l=0}^{\infty} q^{l/2} < \infty.$$

By Minkowski's inequality, the partial sums of the series form a Cauchy sequence in $\mathbb{L}^2$. Thus, the series converges in $\mathbb{L}^2$. Its almost-sure convergence follows from Theorem 3.1. In particular, its limit satisfies

$$\mathbb{E}[\psi_t^2] < \infty.$$

The stationary and causal properties, as well as uniqueness, follow from Theorem 3.1. Hence, the latent process is second-order stationary.

Furthermore, suppose that $\mathbb{E}[\exp(\psi_t)] < \infty$. Since

$$\xi_t^2 = (\eta_t^{(1)})^2 \exp(\psi_t),$$

and $\eta_t^{(1)}$ is independent of ψ_t with unit variance, we obtain

$$\mathbb{E}[\xi_t^2] = \mathbb{E}[(\eta_t^{(1)})^2]\, \mathbb{E}[\exp(\psi_t)] = \mathbb{E}[\exp(\psi_t)] < \infty.$$

Thus, the observed process has a finite second moment. Since it is strictly stationary by Theorem 3.1, the existence of its second moment implies that it is second-order stationary. The distinction between strict stationarity and the additional conditions required for finite moments in the logGARCH-SV setting is also consistent with the discussion in Guerbyenne et al. [34]. This completes the proof. □

Remark 3.3. *An immediate consequence of Theorem 3.2 is obtained by considering the special case $m = 2$, which guarantees the existence of finite second-order moments for the proposed MSlogGARCH-SV process whenever condition (6) is satisfied. Consequently, the unconditional variance of the process*

is well defined and admits an explicit representation in terms of the stationary distribution of the underlying Markov chain and the infinite stochastic expansion given in (5). This result highlights an important structural feature of the proposed model. Unlike the classical logGARCH-SV specification, where the unconditional variance is governed by a single persistence mechanism, the second-order behavior of the MSlogGARCH-SV process depends on the sequence of latent regimes generated by the hidden Markov chain. Each regime is characterized by its own volatility persistence, innovation sensitivity, and long-run volatility level, allowing the unconditional variance to adapt naturally as the process switches between different market environments.

Example 3.2. *To illustrate the second-order stationarity condition (6) established in Theorem 3.2, we consider the proposed MSlogGARCH-SV model with two regimes* $r = 2$. *The regime-specific persistence coefficients are defined by* $\varphi_1 = \beta_2(1) + \beta_3(1) := (2a) + (b)$ *and* $\varphi_2 = \beta_2(2) + \beta_3(2) := (a) + (b+1)$. *The underlying Markov chain is assumed to have transition probabilities:* $p_{12} = 1/5$ *and* $p_{21} = 3/5$. *For this specification, the second-order stationarity region determined by condition (6) is computed numerically and compared with the corresponding region obtained for the standard logGARCH-SV model, whose second-order stationarity condition reduces to* $\varphi_1^2 < 1$. *The resulting stability regions are displayed in Figure 2.*

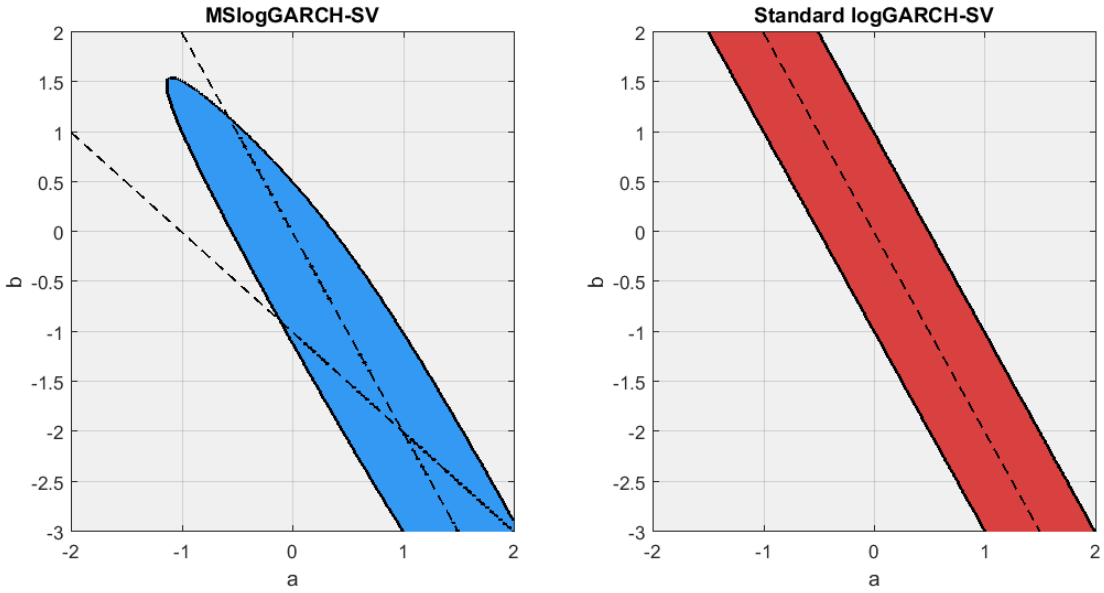


Figure 2. Second-order stationarity regions of the proposed MSlogGARCH-SV model and the standard logGARCH-SV model.

Figure 2 provides a graphical comparison of the second-order stationarity regions associated with the proposed MSlogGARCH-SV model and the classical logGARCH-SV specification. Several important observations emerge from this comparison. In contrast to the standard logGARCH-SV model, whose admissible region is governed solely by the condition ($\varphi_1^2 < 1$), *the proposed MSlogGARCH-SV model satisfies a spectral-radius condition involving all latent regimes simultaneously. Consequently, the boundary separating stable and unstable parameter values becomes nonlinear and depends explicitly on the transition probability matrix of the hidden Markov chain. This*

demonstrates that the existence of finite second-order moments is determined by the global dynamics of the Markov-switching mechanism rather than by the persistence of any individual regime. Another important observation is that the second-order stationarity region is noticeably smaller than the strict stationarity region. This behavior is fully consistent with the theoretical results established in Section 3. Since the existence of finite second-order moments requires a stronger contraction condition than strict stationarity, certain parameter combinations that generate a unique strictly stationary solution fail to satisfy the more restrictive spectral-radius condition ensuring finite unconditional variances. Overall, the graphical comparison provides clear numerical support for the theoretical hierarchy established in Section 3, demonstrating that second-order stationarity constitutes a stronger requirement than strict stationarity for the proposed MSlogGARCH-SV model.

Having established the existence of finite second-order moments, we now investigate the existence of moments of arbitrary order for the proposed MSlogGARCH-SV process. Higher-order moments are fundamental for understanding the tail behavior and distributional properties of financial returns, and they provide important information about the probabilistic characteristics of the underlying volatility process. Their existence is also essential for deriving stronger probabilistic results and for establishing the asymptotic properties of statistical estimation procedures. Since the proposed model is driven by regime-dependent random coefficients, the existence of higher-order moments depends on an appropriate contraction condition for the corresponding higher-order Markov operator. The following theorem establishes a sufficient condition for the existence of moments of arbitrary order and, when applicable, provides an explicit expression for the corresponding unconditional moments.

Theorem 3.3. *Let (ξ_t) be the MSlogGARCH-SV process defined by (2). Let $m \geq 1$ be an integer and assume that $E\left\{\left|\eta_t^{(j)}\right|^m\right\} < \infty$, $j = 1, 2$. Let*

$$C(j) = \beta_2(j) + \beta_3(j), \qquad j = 1, \ldots, r,$$

and define the matrix

$$M_m = \left(p_{ij}|C(j)|^m\right)_{1\leq i,j\leq r}.$$

Assume that

$$\rho(M_m) < 1, \tag{7}$$

and that the innovation term D_t in the stochastic recurrence representation $\psi_t = C(\lambda_t)\psi_{t-1} + D_t$ satisfies $E[|D_t|^m] < \infty$. Furthermore, suppose that the exponential moment required at order m is finite, namely, $E\left\{\exp(\frac{m}{2}\psi_t)\right\} < \infty$. Then the proposed MSlogGARCH-SV process admits a unique causal, strictly stationary, and ergodic solution possessing finite moments up to order m.

Proof. The assumed exponential-moment condition at order m,

$$E\left[\exp\left(\frac{m}{2}\psi_t\right)\right] < \infty,$$

provides the exponential integrability required to establish the m-th absolute moment of the observed process. To establish the corresponding polynomial moments of the latent log-volatility, we use the stochastic recurrence representation

$$\psi_t = C(\lambda_t)\psi_{t-1} + D_t, \qquad C(j) = \beta_2(j) + \beta_3(j),$$

where D_t denotes the innovation term in this recurrence.

In addition to the spectral-radius condition (7), assume that $E[|D_t|^m] < \infty$. Define the nonnegative matrix

$$M_m = \left(p_{ij}|C(j)|^m\right)_{1\le i,j\le r}.$$

Since $\rho(M_m) < 1$, the powers of M_m decay geometrically. Under the Markov-switching recurrence assumptions, this contraction property, together with the finite m-th absolute moment of D_t, yields convergence of the series representation of ψ_t in $\mathbb{L}^m$. Consequently,

$$E[|\psi_t|^m] < \infty.$$

This matrix-operator argument is standard in the analysis of higher-order moments for Markov-switching stochastic recurrence processes; see Bibi and Ghezal [22].

Since $E[|\eta_t^{(1)}|^m] < \infty$ and $E[\exp(m\psi_t/2)] < \infty$, and since $\eta_t^{(1)}$ is independent of ψ_t, we obtain

$$\begin{aligned} E[|\xi_t|^m] &= E[|\eta_t^{(1)}|^m]E\left[\exp\left(\frac{m}{2}\psi_t\right)\right] \\ &< \infty. \end{aligned}$$

Moreover, for every q satisfying $0 < q \le m$, we have

$$\exp\left(\frac{q}{2}x\right) \le 1 + \exp\left(\frac{m}{2}x\right), \qquad x \in \mathbb{R}.$$

It follows that $E[\exp(q\psi_t/2)] < \infty$. Furthermore, $E[|\eta_t^{(1)}|^q] < \infty$, because $q \le m$. By independence, we therefore obtain

$$E[|\xi_t|^q] = E[|\eta_t^{(1)}|^q]E\left[\exp\left(\frac{q}{2}\psi_t\right)\right] < \infty.$$

Hence, the observed process possesses finite absolute moments of all orders q with $0 < q \le m$.

Finally, the uniqueness, causality, strict stationarity, and ergodicity of the solution follow from Theorem 3.1, under its stated assumptions. Thus, the spectral-radius condition controls the contraction of the Markov-switching recurrence, whereas the additional exponential-moment condition ensures the finiteness of the corresponding moments of the observed process. This distinction between stationarity and the additional requirements for higher-order moments is consistent with the analysis of the logGARCH-SV model in Guerbyenne et al. [34]. This completes the proof. □

Remark 3.4. *The special case $m = 4$ in Theorem 3.3 guarantees the existence of finite fourth-order moments for the proposed MSlogGARCH-SV process whenever condition (7) is satisfied. Consequently, the kurtosis coefficient is well defined, providing a rigorous basis for studying the tail behavior and the propensity of the process to generate extreme observations. The existence of finite fourth-order moments reveals an important feature of the proposed model. Although each latent regime is characterized by its own persistence and innovation sensitivity, the overall tail behavior is determined by the cumulative effect of the regime-switching dynamics rather than by any individual regime alone. Recurrent transitions between volatility regimes may substantially amplify the variability of the latent volatility process, thereby increasing the likelihood of large observations while preserving finite fourth-order moments. This property is particularly relevant in financial applications, where abrupt changes between tranquil and turbulent market conditions are frequently associated with heavy-tailed return distributions and pronounced volatility clustering. The proposed MSlogGARCH-SV model is therefore able to accommodate these empirical features within a mathematically well-defined probabilistic framework under the imposed spectral-radius condition.*

4. Parameter estimation via the GMM

The estimation of Markov-switching stochastic volatility models remains a challenging problem because both the latent volatility process and the regime sequence are unobservable. In the proposed MSlogGARCH-SV model, the observed series is generated by a latent log-volatility process whose dynamics depend on an underlying hidden Markov chain. Consequently, the likelihood function requires integration over the latent volatility process while simultaneously summing over all possible realizations of the hidden regimes. Since the number of admissible regime paths grows exponentially with the sample size, direct likelihood-based estimation quickly becomes computationally demanding. To overcome these computational challenges, moment-based estimation procedures have emerged as attractive alternatives for hidden nonlinear time-series models. Among them, the GMM has become one of the most flexible and robust estimation techniques. Rather than relying on the full specification of the likelihood function, GMM exploits theoretical moment conditions implied by the probabilistic structure of the model. This approach is particularly well suited to the proposed MSlogGARCH-SV process, since Section 3 established sufficient conditions for strict stationarity together with the existence of second-order and higher-order moments. These theoretical results naturally provide the moment conditions required for GMM estimation.

Let $\underline{\theta} = \{\beta_i(j), p_{lk;} i = 1, 2, 3, 4, l, k \in \Lambda, l \neq k\}$ denote the unknown parameter vector, where the coefficients governing the latent volatility dynamics depend on the hidden Markov state and the transition probability matrix $\mathbb{P}$. Let $(\xi_1, ..., \xi_m)$ be an observed realization of the proposed MSlogGARCH-SV process generated under the true parameter vector $\underline{\theta}_0$. Motivated by the moment representations established in Theorems 3.2 and 3.3, we construct a vector of theoretical moment conditions based on the unconditional moments representations established in Theorems 3.2 and 3.3. Let $g(\underline{\theta}, \xi_t)$ denote the corresponding vector of moment functions satisfying the orthogonality condition $E_{\underline{\theta}_0}\left\{g(\underline{\theta}_0, \xi_t)\right\} = 0$. Specifically, for a suitable positive integer q, we define the components of $g(\underline{\theta}, \xi_t)$ as $\xi_t^{2j} - E_{\underline{\theta}}\left\{\xi_t^{2j}\right\}$, $j = 1, ..., q$. Thus, the moment conditions are constructed from the selected even-order unconditional moments of the observed process. The value of q is chosen so that all moments entering the GMM procedure are finite under the innovation distributions considered. The sample counterpart is defined by $G_m(\underline{\theta}) = m^{-1}\sum_{t=1}^{m} g(\underline{\theta}, \xi_t)$. The GMM estimator is then obtained as the minimizer of the quadratic criterion $Q_m(\underline{\theta}) = G_m'(\underline{\theta}) W_m G_m(\underline{\theta})$, where W_m is a sequence of positive definite weighting matrices converging to a symmetric positive definite matrix W. Accordingly, $\widehat{\underline{\theta}}_m = \arg_{\underline{\theta}\in\Phi} \min Q_m(\underline{\theta})$, where Φ denotes the admissible parameter space satisfying the stationarity and moment conditions established in Section 3. Under the contraction assumptions of Theorems 3.1–3.3, together with the usual identifiability, continuity, and regularity assumptions required in the GMM framework, the sequence of empirical moments converges almost surely to its theoretical counterpart by the ergodic theorem. Consequently, $G_m(\underline{\theta}) \xrightarrow[m]{a.s.} E_{\underline{\theta}_0}\left\{g(\underline{\theta}, \xi_t)\right\}$, which implies the consistency of the GMM estimator. Furthermore, under standard differentiability assumptions and the nonsingularity of the Jacobian matrix associated with the moment conditions, the estimator is asymptotically normal (see, for example, Harris and Mátyás, [35]), namely, $\sqrt{m}(\widehat{\underline{\theta}}_m - \underline{\theta}_0) \rightsquigarrow \mathcal{N}(0, \Omega)$, where Ω denotes the classical GMM asymptotic covariance matrix (see Harris and Mátyás, [35]). The practical implementation of the proposed GMM estimator, together with the numerical optimization procedure adopted throughout

this paper, is described in the following simulation study.

Remark 4.1. *The proposed MSlogGARCH-SV model belongs to the class of latent regime-switching stochastic volatility models, for which likelihood-based inference is generally computationally demanding because both the latent volatility process and the underlying Markov states are unobservable. The proposed GMM procedure avoids repeated evaluation of the full likelihood function and thereby circumvents the exponential growth in the number of latent regime paths that typically arises in maximum-likelihood estimation. Instead, parameter estimation is based on moment conditions derived directly from the probabilistic properties established in Section 3. Consequently, the proposed methodology provides a computationally attractive alternative while retaining the consistency and asymptotic normality of the resulting estimators under the standard regularity conditions. This moment-based framework is particularly well suited to models combining latent stochastic volatility with Markov-switching dynamics, where exact likelihood evaluation may become computationally prohibitive.*

4.1. Numerical implementation of the GMM estimator

The numerical estimation proceeds through the following steps.

4.1.1. Step 1: Specification of the parameter space

For a given sample of observations, the parameter vector is initialized at an admissible value $\underline{\theta}_0$ belonging to the parameter space Φ. The parameter space is restricted to values satisfying the transition-probability constraints and the stationarity and moment conditions established in Section 3. In particular, the transition probabilities are constrained to lie in $(0, 1)$, with each row of the transition matrix summing to one.

4.1.2. Step 2: Construction of the theoretical moment vector

For each candidate parameter vector $\underline{\theta} \in \Phi$, the theoretical unconditional moments implied by Theorems 3.2 and 3.3 are evaluated. These moments are used to construct the corresponding vector of moment conditions $g(\underline{\theta}, \xi_t)$. Specifically, the selected even-order moments of the observed process are centered around their theoretical values under $\underline{\theta}$.

4.1.3. Step 3: Computation of the sample moments

For each candidate value $\underline{\theta}$, the sample moment vector is computed as $G_m(\underline{\theta})$. Thus, the empirical moments measure the discrepancy between the sample moments of the observed series and the corresponding theoretical moments implied by the candidate parameter vector.

4.1.4. Step 4: Construction of the GMM objective function

The sample moment vector is used to construct the GMM criterion $Q_m(\underline{\theta})$. In the Monte Carlo experiments, the identity weighting matrix $W_m = I_q$ is used for all sample sizes, parameter configurations, and innovation distributions. Hence, the numerical procedure corresponds to a one-step GMM estimator.

4.1.5. Step 5: Numerical minimization

The GMM estimator is obtained by numerically minimizing $Q_m(\underline{\theta})$ over the admissible parameter space Φ: $\widehat{\underline{\theta}}_m$. The optimization is performed subject to the parameter restrictions defining the transition probabilities and the admissibility conditions required by the theoretical results in Section 3.

4.1.6. Step 6: Repetition over Monte Carlo samples

For each combination of parameter configuration, innovation distribution, and sample size, the preceding estimation procedure is repeated independently for 1000 simulated samples. For each replication, the estimated parameter vector is stored.

4.1.7. Step 7: Monte Carlo summaries

For each parameter, the reported estimate is the empirical mean of the 1000 estimates, and the value in parentheses is the corresponding empirical standard deviation. The same estimation procedure and weighting matrix are used throughout all simulation experiments, allowing the finite-sample behavior of the estimator to be compared across sample sizes and innovation distributions.

This procedure provides a direct numerical implementation of the GMM estimator based on the theoretical unconditional moments established in Section 3.

5. Simulation study

To assess the finite-sample performance of the proposed GMM estimation procedure, we conduct a Monte Carlo simulation study for the MSlogGARCH-SV model. The primary objective is to evaluate the ability of the estimator to recover the true model parameters under different sample sizes and innovation distributions, with particular attention to the estimation accuracy and dispersion of the regime-specific coefficients and the transition probabilities of the hidden Markov chain. Two parameter configurations are considered to examine the behavior of the estimator under different persistence structures. In the simulation study, the model is simulated using Gaussian innovations and standardized Student-t innovations with 10 degrees of freedom. The latter are scaled to have zero mean and unit variance, thereby allowing us to examine the effect of heavier-tailed innovations on the finite-sample behavior of the estimator. Two sample sizes, namely, $m = 1500$ and $m = 4000$, are considered to investigate the effect of increasing the sample size on the estimation results. For each experimental setting, 1000 independent Monte Carlo replications are generated. The reported estimates are the empirical means of the estimated parameters across the 1000 replications, while the values in parentheses denote the corresponding empirical standard deviations. In all Monte Carlo experiments, the parameters are estimated using the one-step GMM procedure described in Section 4 with the identity weighting matrix, $W_m = I_q$, where q is the number of moment conditions used in the estimation. Thus, the same weighting matrix is employed for both innovation distributions, both sample sizes, and all parameter configurations. The estimation is therefore based directly on the theoretical unconditional moment conditions derived in Section 3.

Finally, to illustrate the advantages of incorporating regime-switching dynamics, the results obtained for the proposed MSlogGARCH-SV model are systematically compared with those of the classical logGARCH-SV specification.

Tables 1 and 2 summarize the Monte Carlo results obtained using the proposed GMM estimation procedure for both the MSlogGARCH-SV model and the classical logGARCH-SV model under Gaussian innovations and standardized Student-t innovations with 10 degrees of freedom. The reported values correspond to the empirical means computed over repeated simulations, while the quantities in parentheses represent the associated empirical standard deviations. Overall, the proposed estimation procedure successfully recovers the true parameter values across all simulation scenarios, providing convincing numerical evidence of the effectiveness of the GMM approach for estimating Markov-switching stochastic volatility models. A comparison between the two model specifications shows that, under the regime-switching data-generating mechanism considered in the present simulation study, the proposed MSlogGARCH-SV model generally provides more accurate recovery of the regime-specific parameters than the standard logGARCH-SV model. Under Gaussian innovations and standardized Student-t innovations with 10 degrees of freedom, the estimates of the regime-dependent parameters remain consistently close to their true values, whereas the constant-parameter specification exhibits larger estimation errors. This finding is consistent with the data-generating mechanism adopted in the simulation study, where the observations are generated from a regime-switching process with distinct regime-specific coefficients. Since the classical logGARCH-SV model imposes a common parameter vector across regimes, it does not explicitly account for such regime heterogeneity. Thus, the observed difference in estimation accuracy should be interpreted in the context of the present simulation design rather than as a general claim of superiority of the MSlogGARCH-SV model. The results indicate that accounting for regime heterogeneity can be important when the underlying data-generating process exhibits switching volatility dynamics. The results also demonstrate the robustness of the proposed estimation procedure with respect to the innovation distribution. As expected, replacing Gaussian innovations with heavier-tailed standardized Student-t innovations with 10 degrees of freedom leads to a moderate increase in the empirical standard deviations of the parameter estimates. Nevertheless, the estimated coefficients remain close to their true values under both distributions, indicating that the proposed GMM estimator maintains a satisfactory level of accuracy even when the Gaussian assumption is relaxed. The influence of the sample size is also evident from the simulation results. Under comparable innovation distributions, increasing the sample size from $m = 1500$ to $m = 4000$ generally leads to smaller empirical standard deviations for the estimated parameters. This reduction in estimation variability is observed for most of the regime-specific coefficients and transition probabilities. The results therefore indicate the expected improvement in estimation precision as the sample size increases, in accordance with the asymptotic properties of the GMM estimator discussed in Section 4.

Table 1. GMM estimation results for the standard and MSlogGARCH-SV models under Gaussian and standardized Student-t_{10} innovations for $m = 1500$.

Parameters	Tv	MSlogGARCH-SV		Standard logGARCH-SV	
		Gaussian	Standardized Student-t_{10}	Gaussian	Standardized Student-t_{10}
$\beta_1(1)$	1.500	1.4945(0.0117)	1.4901(0.0136)	1.4887(0.0179)	1.4865(0.0183)
$\beta_1(2)$	2.000	2.0049(0.0161)	2.0092(0.0188)	–	–
$\beta_2(1)$	−0.750	−0.7456(0.0095)	−0.7419(0.0113)	−0.7330(0.0172)	−0.7296(0.0224)
$\beta_2(2)$	−0.250	−0.2461(0.0087)	−0.2442(0.0101)	–	–
$\beta_3(1)$	0.015	0.0161(0.0063)	0.0172(0.0082)	0.0175(0.0091)	0.0186(0.0121)
$\beta_3(2)$	0.000	0.0027(0.0084)	0.0036(0.0094)	–	–
$\beta_4(1)$	0.050	0.0538(0.0096)	0.0552(0.0138)	0.0569(0.0167)	0.0594(0.0224)
$\beta_4(2)$	0.150	0.1467(0.0058)	0.1546(0.0085)	–	–
p_{12}	0.250	0.2473(0.0032)	0.2468(0.0064)	–	–
p_{21}	0.750	0.7544(0.0050)	0.7587(0.0092)	–	–

Table 2. GMM estimation results for the standard and MSlogGARCH-SV models under Gaussian and standardized Student-t_{10} innovations for $m = 4000$.

Parameters	Tv	MSlogGARCH-SV		Standard logGARCH-SV	
		Gaussian	Standardized Student-t_{10}	Gaussian	Standardized Student-t_{10}
$\beta_1(1)$	1.500	1.4997(0.0024)	1.4932(0.0033)	1.4956(0.0044)	1.4890(0.0059)
$\beta_1(2)$	2.000	2.0004(0.0019)	2.013(0.0026)	–	–
$\beta_2(1)$	−0.750	−0.7478(0.0022)	−0.7443(0.0029)	−0.7367(0.0038)	−0.7318(0.0046)
$\beta_2(2)$	−0.250	−0.2488(0.0023)	−0.2451(0.0042)	–	–
$\beta_3(1)$	0.015	0.0156(0.0014)	0.0160(0.0025)	0.0171(0.0027)	0.0178(0.0035)
$\beta_3(2)$	0.000	0.0011(0.0007)	0.0024(0.0015)	–	–
$\beta_4(1)$	0.050	0.0506(0.0008)	0.0522(0.0024)	0.0542(0.0094)	0.0567(0.0126)
$\beta_4(2)$	0.150	0.1494(0.0010)	0.1518(0.0021)	–	–
p_{12}	0.250	0.2498(0.0005)	0.2476(0.0019)	–	–
p_{21}	0.750	0.7521(0.0022)	0.7543(0.0036)	–	–

6. Conclusions

In this paper, we introduced a novel Markov-switching extension of the recently proposed logGARCH stochastic volatility model by allowing the parameters governing the latent volatility dynamics to evolve according to a hidden finite-state Markov chain. This extension substantially enhances the flexibility of the original specification and enables the model to capture structural changes, heterogeneous persistence, and regime-dependent volatility behavior that frequently arise in financial and economic time series. From a theoretical perspective, we established a comprehensive probabilistic framework for the proposed model. By rewriting the latent volatility equation as a Markov-switching stochastic recurrence equation, we derived sufficient conditions for the existence of a unique causal, strictly stationary, and ergodic solution. Additional spectral-radius conditions were obtained to guarantee the existence of finite second-order and higher-order moments, together with explicit representations of the stationary solution and its unconditional moments. These results provide a rigorous mathematical foundation for the proposed model and demonstrate that incorporating regime-switching dynamics preserves a well-defined probabilistic structure while considerably enriching the

underlying volatility dynamics. We also developed a GMM estimation procedure specifically designed for the proposed MSlogGARCH-SV model. The estimator exploits the moment conditions implied by the theoretical properties of the process and provides a computationally attractive alternative to likelihood-based estimation. Under the stated regularity conditions, it possesses the standard consistency and asymptotic normality properties associated with the GMM framework. The Monte Carlo simulation study further confirmed the practical effectiveness of the proposed methodology. Across different sample sizes, persistence structures, and innovation distributions, the GMM estimator successfully recovered the true model parameters while exhibiting satisfactory finite-sample accuracy and precision. The numerical results also indicate that, under the regime-switching data-generating mechanisms considered in the Monte Carlo study, allowing for regime-dependent dynamics can improve parameter recovery relative to the classical logGARCH stochastic volatility specification, which imposes common parameters across regimes.

Overall, the proposed MSlogGARCH-SV model provides a mathematically rigorous and computationally tractable framework for modeling stochastic volatility in the presence of structural changes and regime-dependent dynamics. By combining the flexibility of hidden Markov switching with the advantages of the logGARCH stochastic volatility formulation, the proposed model offers a realistic and versatile framework for analyzing financial time series characterized by volatility clustering, persistence, and abrupt changes in market conditions. Although the present study focuses on the theoretical properties and Monte Carlo performance of the proposed MSlogGARCH-SV model, an empirical investigation using real financial time series would provide an important complementary assessment of its practical relevance. In particular, such an analysis would allow the regime-dependent dynamics of the model to be examined in actual financial markets and would provide a natural empirical comparison with the classical logGARCH-SV specification. We therefore consider the application of the proposed framework to real financial and macroeconomic datasets, together with a systematic evaluation of its empirical and forecasting performance, an important direction for future research. Several directions for future research naturally arise from the present work. These include extending the proposed framework to multivariate settings, incorporating exogenous covariates or leverage effects, developing Bayesian and Sequential Monte Carlo estimation procedures, investigating forecasting performance, and applying the model to real financial and macroeconomic datasets.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflicts of interest in this paper.

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