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Research article

Rough statistical data dependence of fixed points under persistent and sparse operator perturbations

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Abstract: We formulate localized fixed-point data dependence using the statistical upper radius of a nonnegative error sequence. The scalar transfer rules are elementary consequences of density-zero exceptional sets and are explicitly treated as tools rather than new fixed-point existence principles. For contractive and Reich-type perturbations, these tools quantify persistent errors using the defect at a reference fixed point, even when a global operator discrepancy is infinite. A diagonal Picard estimate separates computational error from perturbation error. A residual criterion also treats families with contraction factors tending to one, provided the normalized residual tends statistically to zero; no stability follows from an unnormalized vanishing defect alone. A genuinely nonlinear Fredholm example computes forcing and kernel defects explicitly and admits finite statistical bounds despite sparse unbounded forcing errors. Exact optimal amplification is established only in the scalar affine class. The framework describes typical asymptotic errors and leaves exceptional-index errors uncontrolled.

Keywords: fixed point; data dependence; rough statistical convergence; statistical limsup; persistent perturbation; sparse perturbation; nonlinear operator; Reich contraction; Fredholm integral equation
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1. Introduction

The stability of fixed points under perturbations of the underlying operator is a fundamental problem in nonlinear analysis. Let (X, d) be a metric space and let

$$T : X \longrightarrow X$$

be a nonlinear operator possessing a fixed point x^*, that is,

$$Tx^* = x^*.$$

Consider a sequence of perturbed operators

$$T_n : X \longrightarrow X, \qquad n \in \mathbb{N},$$

with arbitrarily selected corresponding fixed points

$$T_n x_n = x_n.$$

Classical data-dependence theory commonly investigates estimates of the form

$$d(x_n, x^*) \leq C\Delta_n,$$

where Δ_n measures the perturbation of T_n relative to T. When

$$\Delta_n \longrightarrow 0,$$

such an estimate typically yields

$$d(x_n, x^*) \longrightarrow 0.$$

However, perturbations arising in nonlinear models need not vanish uniformly. A persistent background error may remain present, while large disturbances may occur at infinitely many indices. If those exceptional indices have natural density zero, ordinary convergence may provide an unnecessarily restrictive description of the long-term behavior.

Statistical convergence, introduced independently by Fast [1] and Steinhaus [2], provides a natural mechanism for ignoring exceptional behavior occurring on sets of natural density zero. The fundamental properties of statistical convergence are subsequently investigated by Fridy [3]. Rough statistical convergence is introduced by Aytar [4] to permit a nonzero tolerance radius around the limiting state.

For the underlying upper-bound terminology, Fridy and Orhan [5] develop statistical limit superior and inferior in relation to statistical convergence and cluster points. Phu [6] studies rough convergence and rough limit sets in normed spaces before the rough statistical formulation. These are foundations of the present radius-based description, rather than concepts introduced by this paper.

The purpose of the present paper is not to introduce statistical or rough statistical convergence itself. Instead, these concepts are used to develop a quantitative data-dependence framework for fixed points subject to persistent, sparse, and statistically exceptional operator perturbations.

The first basic mechanism considered in this work is

$$\text{st}-\limsup d(x_n, x^*) \leq C \ \text{st}-\limsup \Delta_n.$$

Thus a deterministic data-dependence coefficient becomes an amplification factor for statistically persistent perturbations.

The framework is then extended to the nonuniform estimate

$$d(x_n, x^*) \leq C_n e_n,$$

where both the perturbation magnitude e_n and its sensitivity coefficient C_n may vary with n. Rather than requiring

$$\sup_n C_n < \infty,$$

we use the weaker statistical requirement

$$\mathrm{st}-\limsup C_n < \infty.$$

A further feature of the theory is localization. Instead of requiring the global discrepancy

$$\sup_{x\in X} d(T_n x, Tx),$$

we show that, for important classes of contractive operators, the local defect

$$e_n = d(T_n x^*, Tx^*)$$

is sufficient for quantitative fixed-point stability.

This distinction is especially useful on unbounded spaces or under nonuniform perturbations, where a global operator discrepancy may be infinite while the behavior near the reference equilibrium remains statistically controlled.

Related work and scope of the increment. Fixed-point data dependence predates statistical convergence applications. Rus, Petruşel, and Sîntămărian [7] treat multivalued weakly Picard operators, common fixed points, and a Fredholm inclusion. Rus [8] develops retraction-displacement conditions and admissible weakly Picard perturbations. Our single-valued selected-point estimates do not replace these set-valued or retraction-based results. Gürsoy [9] uses collage estimates for quasi-contractive operators; Doğan et al. [10] analyze iterative computations and data dependence. Data-dependence problems have also been studied in controlled metric spaces [11] and for contractive mappings defined through auxiliary functions [12]. The diagonal Picard bound here combines a classical geometric tail with a statistical defect radius; it is not a new iteration scheme.

Further data-dependence precedents include Chifu and Petruşel [13], who treat fixed points and strict fixed points of contractive-type multivalued operators, and Gürsoy, Karakaya, and Rhoades [14], who establish approximation and data-dependence results for multi-step and S-iterations of contractive-like operators. More recently, Kumar et al. [15] analyze stability and data dependency of an implicit Jungck scheme with nonlinear integral-equation applications. These algorithmic and multivalued settings differ from independent Picard solves indexed by the perturbation family, but they are relevant antecedents when interpreting our computational estimates. We cite the Kumar et al. article together with its published figure correction [16]; no quantitative superiority over those iteration schemes is claimed here.

The summability literature likewise already contains much broader convergence notions than the scalar natural-density setting used here. Mursaleen and Edely [17] study statistical double sequences. Kişi and Dündar [18] combine roughness, ideals, and lacunary double sequences. Weighted rough limits are investigated by Das et al. [19], Ghosal and Ghosh [20], and Bayram et al. [21]. Difference sequences are treated by Demir and Gümüş [22]; Khatun and Banerjee [23] study statistical and rough statistical convergence in an S-metric space. Ghosal and Mandal [24] further study rough weighted ideal $\alpha\beta$-statistical convergence in locally solid Riesz spaces. These works concern convergence structures and limit sets, whereas our working quantity is the radius of a selected fixed-point error about a specified reference point. No priority is claimed for the underlying radius concept or for finite-union transfer arguments.

Two additional operator-theoretic comparisons have different aims. Ramaswamy et al. [25] study a corrected distance $d - \psi$ in a perturbed controlled metric space and prove fixed-point existence results for that geometry. Here the metric is fixed, and the operators T_n vary; the perturbation is measured by $d(T_n x^*, T x^*)$ and summarized through natural-density tails. Gnanaprakasam et al. [26] establish fixed-point results for orthogonal F-Khan contractions and apply them to an integral equation. Our Fredholm estimates concern dependence of already uniquely determined solutions on forcing and kernel changes. The shared use of integral equations therefore does not identify the underlying questions. Neither comparison is presented as a direct statistical-data-dependence predecessor or as a theorem subsumed by our results.

Theorem-level comparison with classical contraction data dependence. For a common contraction factor $0 \le \lambda < 1$, the standard Banach perturbation argument [27] gives $d(x_n, x^*) \le e_n/(1-\lambda)$ whenever T_n and T have the stipulated fixed points and T_n is λ-contractive. Taking a statistical upper radius therefore gives $\mathrm{st{-}lim\,sup} d(x_n, x^*) \le (1-\lambda)^{-1}\,\mathrm{st{-}lim\,sup} e_n$; this implication alone is not new. The distinct hypotheses in Proposition 6.5 allow $\lambda_n \uparrow 1$ with an unbounded inverse gap, but require direct control of $h_n = e_n/(1-\lambda_n)$ and the computed Picard tail b_n. In that regime, the separate-radius estimate based on a finite statistical upper bound for $(1-\lambda_n)^{-1}$ cannot be invoked. This is a precise difference in applicability, not a claim that the classical pointwise estimate is false or that the earlier multivalued and iterative theorems are subsumed. In particular, $T_n(x) = (1-n^{-2})x + n^{-3}$ has inverse gap n^2 and $h_n = 1/n \to 0$, while replacing n^{-3} with $1/(2n^2)$ produces $e_n \to 0$ but $x_n = 1/2$. The latter is a counterexample to stability from unnormalized residuals alone.

Direct comparison with a published single-valued data-dependence theorem. Theorem 5 of [28] assumes two contractions $T, \widetilde{T} : X \to X$ sharing a constant $q \in (0,1)$ and a uniform operator discrepancy $\sup_{x \in X} \|Tx - \widetilde{T}x\| \le \varepsilon$. Its fixed-point conclusion is $\|x^* - \widetilde{x}^*\| \le \varepsilon/(1-q)$; it also establishes a bound for a particular accelerated iterative scheme. The present work does not improve this constant, and its Picard-diagonal bounds do not imply the published accelerated-iteration estimate. In contrast, our varying-sensitivity formulation controls the reference-point residual $e_n = d(T_n x^*, T x^*)$ for each independently defined operator, without requiring a uniform-in-x discrepancy or a single contraction constant across n. The relevant conclusion concerns $\mathrm{st{-}lim\,sup}_n d(x_n, x^*)$ under density-based assumptions on $e_n/(1-\lambda_n)$; it is not a uniform bound for one pair of operators.

For a separating example on $X = \mathbb{R}$, take $T(x) = 0$ and $T_n(x) = (1-n^{-2})x + n^{-3}$ for $n \ge 2$. Then $x^* = 0$, $x_n = 1/n$, $e_n = n^{-3}$, and $\lambda_n = 1 - n^{-2}$. Hence $e_n/(1-\lambda_n) = 1/n \to 0$, whereas $\sup_x |T_n x - Tx| = \infty$ and there is no common $q < 1$ valid for all T_n. Thus the cited theorem's hypotheses cannot be applied uniformly to this family, while the normalized-residual estimate yields convergence. This is a difference in hypotheses and asymptotic scope, not a contradiction or a claim of a stronger fixed-n inequality.

Comparison with recent iterative data-dependence work. In particular, Theorem 3 of Doğan et al. [10] considers a contraction $\mathbb{S}$ and an approximating mapping $\widetilde{\mathbb{S}}$ in the setting of the iteration methods referred to there as DF and AR (named after their originators, Filali et al. and Alam and Rohen); the perturbed mapping need not itself be contractive. Their data-dependence conclusions are obtained from a uniform approximation condition and hypotheses on the iterative schemes. By contrast, Proposition 6.5 assumes independently solved contraction problems T_n, with potentially varying factors

λ_n, and controls the statistical upper radius of the reference-point normalized residual together with the diagonal Picard tail. Thus our assumptions are not uniformly weaker: Our result requires contraction of each T_n off the exceptional set, whereas their analysis allows a noncontractive approximating operator under its stated hypotheses. Conversely, the common uniform-error formulation does not directly supply a family-level statistical conclusion when no uniform error or common contraction factor exists. We claim neither a generalization of their Theorem 3 nor improved DF/AR error constants.

The concrete distinction is the simultaneous presentation of reference-point residuals, statistically varying conditioning, independently computed Picard diagonals, and a nonlinear kernel calculation. The scalar lemmas are technical tools, affine sharpness is restricted to its stated class, and neither local residual estimates nor Picard tail bounds are claimed as independently novel. The normalized-residual criterion retains information when separate sensitivity and defect radii would yield an unusable product. The scope of this paper is quantitative statistical data dependence and independently computed solutions, rather than new convergence structures or a generalized fixed-point existence theory.

Table 1 compares the objects and questions, rather than ranking the papers. In particular, a perturbation of the metric is different from a perturbation of the operator, and a theorem about rough limit sets is different from an estimate for solution sensitivity. The comparison explains the scope of the present combination without treating an omitted topic as a deficiency of earlier work.

The scalar transfer lemmas below apply to arbitrary nonnegative sequences. Their proofs do not use fixed-point structure, and we do not claim a new general existence or uniqueness theory. Our contributions are organized into three components:

(1) Localized statistical estimates for already existing contractive and Reich-type fixed points, with all sensitivity and selection assumptions explicit.

(2) A diagonal Picard error decomposition and a normalized-residual criterion that remains applicable without statistically bounded inverse contraction gaps.

(3) A nonlinear Fredholm example with explicit residual components, together with affine sharpness and separation of ordinary, statistical, and rough statistical stability.

A conceptual summary of these estimates is the class-dependent statistical perturbation amplification factor $\kappa_{\mathrm{st}}(T;\mathfrak{P})$ of Section 10. It is the supremum of the ratio of the selected fixed-point error radius to a positive finite local-defect radius over an explicitly specified admissible class. It summarizes statistical conditioning about the chosen reference equilibrium; it is not an invariant of T alone. In the scalar affine class, it equals the classical factor $(1-\lambda)^{-1}$. The remainder of the paper is organized as follows. Section 2 recalls statistical and rough statistical convergence and introduces the statistical upper radius. Section 3 establishes the basic transfer principle. Section 4 develops the variable-sensitivity framework. Section 5 gives the nonlinear transfer principle. Section 6 treats localized contractive perturbations. Section 7 addresses Reich-type operators. Section 8 gives a nonlinear scalar example. Section 9 studies affine sharpness and sparse perturbations. Section 10 specifies the amplification class. Section 11 presents the integral-equation application. Section 12 compares stability notions; Sections 13–15 discuss scope, limitations, and conclusions.

Table 1. Comparison of research questions and scope. The entries describe the cited results; they do not assert that the present paper subsumes them.

Study	Framework and object	Main question or result	Relation to this paper
Aytar (2008) [4]	Rough statistical convergence of sequences	Rough limit sets and statistical cluster points	Supplies the convergence foundation; no new radius concept is claimed here.
Rus et al. (2003) [7]	Multivalued weakly Picard operators	Fixed-point-set data dependence, common fixed points, and integral inclusion	Our bounds concern selected single-valued fixed points about one reference equilibrium.
Filali et al. (2026), Thm. 5 [28]	Two contractions with a common factor $q < 1$ and uniform operator error ε	Fixed-point distance $\leq \varepsilon/(1-q)$ and an accelerated-iteration bound	Our family estimate uses local reference-point defects and varying λ_n; neither theorem subsumes the other.
Doğan et al. (2025), Thm. 3 [10]	DF/AR iteration and uniform approximation, allowing a noncontractive approximant	Iterative data-dependence estimate under algorithmic hypotheses	Our Picard diagonals require contractions off an exceptional set; no improvement of their DF/AR bounds is asserted.
Kişi–Dündar (2018) [18]	Rough ideal/lacunary statistical double sequences	Statistical limit sets in normed spaces	Our natural-density radius is applied to an operator-error sequence, not a new ideal or double-sequence theory.
Das et al. (2018) [19]	Rough weighted statistical convergence	Characterization and properties of weighted rough limit sets	We use unweighted natural density and quantify local-defect amplification.
Ramaswamy et al. (2026) [25]	Perturbed controlled metric geometry $d - \psi$	Fixed-point results and an integral-equation application	We keep the metric fixed and vary T_n; metric perturbations and operator perturbations differ.
Gnanaprakasam et al. (2023) [26]	Orthogonal F-Khan contractions	Existence/uniqueness and integral-equation solutions	We estimate changes in already determined solutions, rather than establish an F-Khan existence theorem.
Present paper	Local defects of operator families in a fixed metric	Statistical error radii, varying sensitivity, Picard diagonals, and nonlinear kernel residuals	Combines classical pointwise estimates with statistical aggregation; exceptional errors remain uncontrolled.

2. Preliminaries

Let

$$\mathbb{N} = \{1, 2, \ldots\}.$$

Definition 2.1. Let $E \subseteq \mathbb{N}$. The natural density of E, whenever it exists, is defined by

$$\delta(E) = \lim_{N\to\infty} \frac{1}{N} |E \cap \{1, 2, \ldots, N\}| .$$

Definition 2.2. A sequence (x_n) in a metric space (X, d) is statistically convergent to $x \in X$ if, for every $\varepsilon > 0$,

$$\delta(\{n \in \mathbb{N} : d(x_n, x) \geq \varepsilon\}) = 0.$$

We write

$$\text{st}-\lim_{n\to\infty} x_n = x.$$

Definition 2.3. Let $r \geq 0$. A sequence $(x_n) \subseteq X$ is rough statistically convergent to x with roughness degree r if, for every $\varepsilon > 0$,

$$\delta(\{n \in \mathbb{N} : d(x_n, x) \geq r + \varepsilon\}) = 0.$$

We now introduce the statistical upper quantity used throughout the paper.

Definition 2.4. Let (a_n) be a nonnegative sequence. Define

$$\mathcal{R}_{\text{st}}(a) = \inf \{r \geq 0 : \delta(\{n \in \mathbb{N} : a_n > r + \varepsilon\}) = 0 \text{ for every } \varepsilon > 0\} .$$

We use $\inf \varnothing = +\infty$. For all nonnegative sequences, we write

$$\mathcal{R}_{\text{st}}(a) = \text{st}-\limsup_{n\to\infty} a_n.$$

The quantity $\mathcal{R}_{\text{st}}(a)$ is an asymptotic upper bound modulo sets of natural density zero: For every $\varepsilon > 0$, values above $\mathcal{R}_{\text{st}}(a) + \varepsilon$ occur only on a density-zero set when the radius is finite. This resembles an essential upper bound, with negligibility measured by natural density rather than a measure on $\mathbb{N}$, and is consistent with the classical statistical-limsup viewpoint of Fridy and Orhan [5]. It does not require one fixed exceptional set to work for every tolerance. We use st$-\limsup$ as the main notation below; $\mathcal{R}_{\text{st}}$ denotes the same statistical upper radius.

Proposition 2.5. *Let $a_n = d(x_n, x)$ and $r_x = \text{st}-\limsup a_n$. If $r_x < \infty$, it is itself an admissible roughness degree and is the smallest such degree. If $r_x = +\infty$, no finite roughness degree is admissible.*

Proof. Suppose $r_x < \infty$. For $\varepsilon > 0$, the defining infimum supplies an admissible $r < r_x + \varepsilon/2$. Then

$$\{n : a_n \geq r_x + \varepsilon\} \subseteq \{n : a_n > r + \varepsilon/2\},$$

whose density is zero. Thus r_x is admissible, including the non-strict threshold in rough statistical convergence. Every admissible roughness degree belongs to the defining threshold class, so it is at least r_x. If a finite degree were admissible when $r_x = +\infty$, that same observation would contradict the definition. □

Proposition 2.6. *Let (a_n) and (b_n) be nonnegative sequences with finite statistical upper radii, and let $C \geq 0$. Then*

$$\mathcal{R}_{\text{st}}(Ca) = C\mathcal{R}_{\text{st}}(a)$$

and

$$\mathcal{R}_{\text{st}}(a + b) \leq \mathcal{R}_{\text{st}}(a) + \mathcal{R}_{\text{st}}(b).$$

Proof. The first assertion is immediate for $C = 0$. For $C > 0$, it follows by rescaling the threshold in Definition 2.4.

For the second assertion, let

$$r = \mathcal{R}_{\mathrm{st}}(a), \qquad s = \mathcal{R}_{\mathrm{st}}(b).$$

For every $\varepsilon > 0$,

$$\{n : a_n + b_n > r + s + \varepsilon\}$$

is contained in

$$\{n : a_n > r + \varepsilon/2\} \cup \{n : b_n > s + \varepsilon/2\}.$$

Both sets on the right have natural density zero. Hence their union has natural density zero, which proves the result. □

3. A general statistical data-dependence principle

The following result provides the basic transfer mechanism.

Theorem 3.1 (Statistical-limsup transfer principle)**.** *Let* (a_n) *and* (b_n) *be nonnegative sequences. Suppose that there exist* $C > 0$ *and* $E \subseteq \mathbb{N}$ *satisfying*

$$\delta(E) = 0$$

such that

$$a_n \le C b_n, \qquad n \notin E.$$

If

$$\rho = \mathrm{st}\text{–}\limsup_{n\to\infty} b_n < \infty,$$

then

$$\mathrm{st}\text{–}\limsup_{n\to\infty} a_n \le C\rho.$$

Proof. Fix $\eta > 0$. The set

$$F_\eta = \{n : b_n > \rho + \eta\}$$

has natural density zero.

For

$$n \notin E \cup F_\eta,$$

we have

$$a_n \le C b_n \le C(\rho + \eta).$$

Hence

$$\{n : a_n > C\rho + C\eta\} \subseteq E \cup F_\eta.$$

The right-hand side has natural density zero. Letting $\eta \to 0^+$ gives

$$\mathrm{st}\text{–}\limsup a_n \le C\rho.$$

□

Corollary 3.2. *Let $T : X \to X$ possess a fixed point x^*, and let $T_n : X \to X$ possess fixed points x_n. Suppose that*

$$d(x_n, x^*) \leq C\Delta_n$$

outside a set of natural density zero.

Then

$$\text{st}-\limsup d(x_n, x^*) \leq C \,\text{st}-\limsup \Delta_n.$$

4. Data dependence with statistically varying sensitivity

The preceding result assumes a fixed data-dependence coefficient. We now allow the sensitivity coefficient itself to vary with the perturbation index.

Theorem 4.1 (Variable-sensitivity transfer principle)**.** *Let (a_n), (b_n), and (C_n) be nonnegative sequences. Suppose that there exists $E \subseteq \mathbb{N}$ satisfying*

$$\delta(E) = 0$$

such that

$$a_n \leq C_n b_n, \qquad n \notin E.$$

Assume that

$$C_* = \text{st}-\limsup_{n\to\infty} C_n < \infty$$

and

$$\rho = \text{st}-\limsup_{n\to\infty} b_n < \infty.$$

Then

$$\text{st}-\limsup_{n\to\infty} a_n \leq C_*\rho.$$

Proof. Fix $\eta > 0$ and define

$$E_C(\eta) = \{n \in \mathbb{N} : C_n > C_* + \eta\}$$

and

$$E_b(\eta) = \{n \in \mathbb{N} : b_n > \rho + \eta\}.$$

By the definitions of C_* and ρ,

$$\delta(E_C(\eta)) = 0$$

and

$$\delta(E_b(\eta)) = 0.$$

For every

$$n \notin E \cup E_C(\eta) \cup E_b(\eta),$$

we obtain

$$a_n \leq C_n b_n \leq (C_* + \eta)(\rho + \eta).$$

Therefore,

$$\{n : a_n > (C_* + \eta)(\rho + \eta)\}$$

is contained in a set of natural density zero.

It follows that

$$\text{st}-\limsup_{n\to\infty} a_n \le (C_* + \eta)(\rho + \eta).$$

Letting $\eta \to 0^+$, we conclude that

$$\text{st}-\limsup_{n\to\infty} a_n \le C_*\rho.$$

□

Corollary 4.2. *Suppose that*

$$d(x_n, x^*) \le C_n e_n$$

outside a set of natural density zero.

If

$$C_* = \text{st}-\limsup_{n\to\infty} C_n < \infty$$

and

$$\rho_e = \text{st}-\limsup_{n\to\infty} e_n < \infty,$$

then

$$\text{st}-\limsup_{n\to\infty} d(x_n, x^*) \le C_*\rho_e.$$

Remark 4.3. Theorem 4.1 does not require

$$\sup_n C_n < \infty.$$

The sensitivity coefficients may become arbitrarily large on an infinite set of indices, provided that such exceptional behavior is statistically negligible and

$$\text{st}-\limsup C_n < \infty.$$

The hypothesis of Theorem 4.1 is strictly weaker than uniform boundedness. For example, set $C_n = n$ at perfect-square indices and $C_n = 2$ elsewhere. Then $\sup_n C_n = +\infty$, but $\text{st}-\limsup C_n = 2$, because the number of squares not exceeding N is $\lfloor \sqrt{N} \rfloor$. This relaxation permits arbitrarily large sensitivity on sparse indices but does not permit persistent ill-conditioning on a set of positive density. Moreover, the product of separate radii may be conservative; the normalized-residual analysis in Section 6 retains their indexwise coupling.

5. Nonlinear statistical transfer functions

The statistical transfer mechanism can also be formulated using a nonlinear control function.

Theorem 5.1 (Nonlinear statistical transfer principle)**.** *Let* (a_n) *and* (b_n) *be nonnegative sequences and let*

$$\Psi : [0, \infty) \longrightarrow [0, \infty)$$

be continuous and nondecreasing.

Suppose that there exists $E \subseteq \mathbb{N}$ satisfying

$$\delta(E) = 0$$

such that

$$a_n \leq \Psi(b_n), \qquad n \notin E.$$

If

$$\rho = \text{st}-\limsup_{n\to\infty} b_n < \infty,$$

then

$$\text{st}-\limsup_{n\to\infty} a_n \leq \Psi(\rho).$$

Proof. Fix $\varepsilon > 0$.

By continuity of Ψ at ρ, there exists $\eta > 0$ such that

$$\Psi(\rho + \eta) < \Psi(\rho) + \varepsilon.$$

The set

$$F_\eta = \{n : b_n > \rho + \eta\}$$

has natural density zero.

For

$$n \notin E \cup F_\eta,$$

we obtain

$$a_n \leq \Psi(b_n) \leq \Psi(\rho + \eta) < \Psi(\rho) + \varepsilon.$$

Consequently,

$$\{n : a_n \geq \Psi(\rho) + \varepsilon\} \subseteq E \cup F_\eta.$$

The right-hand side has natural density zero. Therefore,

$$\text{st}-\limsup a_n \leq \Psi(\rho).$$

□

Remark 5.2. The linear transfer principle is recovered by choosing

$$\Psi(t) = Ct.$$

Thus Theorem 3.1 is a special case of Theorem 5.1.

6. Localized perturbations of contractive operators

Let $T : X \to X$ possess a fixed point x^*, and let $T_n : X \to X$ possess a fixed point x_n for each n. Define the local perturbation defect

$$e_n = d(T_n x^*, T x^*).$$

Since

$$T x^* = x^*,$$

we may equivalently write

$$e_n = d(T_n x^*, x^*).$$

We first allow the contraction coefficient itself to vary with n.

Theorem 6.1 (Statistically varying contractive sensitivity)**.** *Suppose that there exists $E \subseteq \mathbb{N}$ satisfying*

$$\delta(E) = 0$$

such that, for every $n \notin E$,

$$d(T_n x, T_n y) \le \lambda_n d(x, y), \qquad 0 \le \lambda_n < 1.$$

Define $\lambda_n = 0$ on E for bookkeeping. Define

$$\kappa_n = \frac{1}{1 - \lambda_n}.$$

Assume

$$\kappa_* = \text{st}-\limsup_{n\to\infty} \kappa_n < \infty$$

and

$$\rho_e = \text{st}-\limsup_{n\to\infty} e_n < \infty.$$

Then

$$\text{st}-\limsup_{n\to\infty} d(x_n, x^*) \le \kappa_* \rho_e.$$

Proof. For $n \notin E$,

$$\begin{aligned} d(x_n, x^*) &= d(T_n x_n, T x^*) \\ &\le d(T_n x_n, T_n x^*) + d(T_n x^*, T x^*) \\ &\le \lambda_n d(x_n, x^*) + e_n. \end{aligned}$$

Therefore,

$$(1 - \lambda_n) d(x_n, x^*) \le e_n,$$

and hence

$$d(x_n, x^*) \le \frac{e_n}{1 - \lambda_n} = \kappa_n e_n.$$

The conclusion follows from Theorem 4.1. □

A convenient uniform version follows immediately.

Corollary 6.2 (Statistically uniform contractivity)**.** *Assume the fixed-point and nonexceptional contractivity hypotheses of Theorem 6.1, with finite* $\mathrm{st}-\limsup e_n$*. Suppose that there exist* $\tau \in [0,1)$ *and* $E \subseteq \mathbb{N}$*, with*

$$\delta(E) = 0,$$

such that

$$\lambda_n \le \tau \qquad (n \notin E).$$

Then

$$\mathrm{st}-\limsup_{n\to\infty} d(x_n, x^*) \le \frac{1}{1-\tau}\,\mathrm{st}-\limsup_{n\to\infty} e_n.$$

Proof. For $n \notin E$,

$$\kappa_n = \frac{1}{1-\lambda_n} \le \frac{1}{1-\tau}.$$

The result follows from Theorem 6.1. □

Corollary 6.3. *Under all hypotheses of Theorem 6.1, especially finite* κ_**, if*

$$\mathrm{st}-\lim_{n\to\infty} e_n = 0,$$

then

$$\mathrm{st}-\lim_{n\to\infty} x_n = x^*.$$

Remark 6.4. The preceding results use only the local quantity

$$e_n = d(T_n x^*, T x^*).$$

No global estimate of the form

$$\sup_{x\in X} d(T_n x, T x) < \infty$$

is required.

Picard computation and normalized residuals as follows.

Proposition 6.5 (Diagonal Picard error)**.** *Let* X *be complete. Off a density-zero set* E*, let* T_n *be a contraction with factor* $\lambda_n < 1$*. Let* $Tx^* = x^*$*,* $e_n = d(T_n x^*, x^*)$*,* $z_{n,0} \in X$*, and* $z_{n,j+1} = T_n z_{n,j}$*. For arbitrary integers* $m_n \ge 0$*, put*

$$h_n = \frac{e_n}{1-\lambda_n}, \qquad b_n = \frac{\lambda_n^{m_n}}{1-\lambda_n} d(z_{n,1}, z_{n,0}).$$

Use $0^0 = 1$ *and set* $h_n = b_n = 0$ *on* E*. Then, for every* $n \notin E$*, the unique fixed point* x_n *of* T_n *satisfies*

$$d(z_{n,m_n}, x_n) \le b_n, \qquad d(z_{n,m_n}, x^*) \le b_n + h_n.$$

Consequently, if $\mathrm{st}-\limsup h_n < \infty$ *and* $\mathrm{st}-\lim b_n = 0$*, then*

$$\mathrm{st}-\limsup d(z_{n,m_n}, x^*) \le \mathrm{st}-\limsup h_n.$$

No bound on $\mathrm{st}-\limsup(1-\lambda_n)^{-1}$ *is needed for this conclusion. In particular, if both* $\mathrm{st}-\lim h_n = 0$ *and* $\mathrm{st}-\lim b_n = 0$*, then the computed diagonal converges statistically to* x^**.*

Proof. Banach's theorem supplies x_n at nonexceptional indices. The consecutive Picard increments are at most $\lambda_n^j d(z_{n,1}, z_{n,0})$. Summing their geometric tail gives the first inequality. The localized fixed-point estimate gives $d(x_n, x^*) \le h_n$, and the triangle inequality gives the second. At indices in E, the displayed geometric estimates are not asserted: Define $h_n = b_n = 0$ there as above and choose any finite values for the computed errors. The inequalities and statistical-radius conclusion are applied on $\mathbb{N} \setminus E$; the union of E with any further density-zero exceptional set remains density zero. Subadditivity and $\mathrm{st-lim\,sup}\, b_n = 0$ prove the assertion. □

Example 6.6 (Vanishing contraction gap)**.** On $\mathbb{R}$, take $T(x) = 0$ and $T_n(x) = (1 - n^{-2})x + n^{-3}$. Then $e_n = n^{-3}$, $h_n = n^{-1}$, and $x_n = n^{-1}$, although the inverse gap is n^2. Starting at zero with $m_n = n^3$, one has $b_n = (1 - n^{-2})^{n^3}/n \to 0$. As a counterexample to the assertion that an unnormalized vanishing defect forces fixed-point convergence, replacing n^{-3} by $1/(2n^2)$ gives $e_n \to 0$ but $x_n = 1/2$ for every n. Thus unnormalized defect convergence is insufficient.

For comparison with iteration-based data dependence, see [14, 15]; our estimate concerns a diagonal of independently solved perturbation problems and does not analyze their multi-step or implicit Jungck algorithms. The proposition combines classical geometric Picard estimates with statistical radii. It is an explicit computational extension, not a claim of a new Picard convergence principle.

7. Rough statistical data dependence for Reich-type operators

Throughout this section, the existence of the fixed points x^* and x_n is assumed. The Reich-type contractive condition [29] is used specifically to derive quantitative data-dependence estimates.

Suppose that

$$T : X \longrightarrow X$$

possesses a fixed point x^*, and that

$$T_n : X \longrightarrow X$$

possesses a fixed point x_n.

Assume that there exist constants

$$\alpha, \beta, \gamma \ge 0$$

satisfying

$$\alpha + \beta + \gamma < 1$$

such that, outside a set $E \subseteq \mathbb{N}$ with

$$\delta(E) = 0,$$

we have

$$d(T_n x, T_n y) \le \alpha d(x, y) + \beta d(x, T_n x) + \gamma d(y, T_n y)$$

for all $x, y \in X$.

Again define

$$e_n = d(T_n x^*, T x^*).$$

Theorem 7.1 (Reich-type statistical data dependence)**.** *Under the preceding assumptions, for every $n \notin E$,*

$$d(x_n, x^*) \le \frac{1 + \min\{\beta, \gamma\}}{1 - \alpha}\, e_n.$$

Consequently,

$$\operatorname{st-lim\,sup}_{n\to\infty} d(x_n, x^*) \le \frac{1 + \min\{\beta, \gamma\}}{1 - \alpha} \operatorname{st-lim\,sup}_{n\to\infty} e_n.$$

Proof. Fix $n \notin E$ and put $D = d(x_n, x^*)$. Since $T_n x_n = x_n$ and $Tx^* = x^*$, the triangle inequality and Reich condition with $(x, y) = (x_n, x^*)$ give

$$D \le d(T_n x_n, T_n x^*) + e_n \le \alpha D + (1 + \gamma) e_n.$$

Applying the same all-pairs Reich condition with the ordered pair $(x, y) = (x^*, x_n)$ gives independently

$$D \le d(T_n x^*, T_n x_n) + e_n \le \alpha D + (1 + \beta) e_n.$$

Combining these inequalities proves the pointwise estimate. The statistical bound follows by the nonnegative-sequence transfer rule outside E. □

We next allow the Reich coefficients to depend on n.

Suppose that, outside $E \subseteq \mathbb{N}$ with $\delta(E) = 0$,

$$d(T_n x, T_n y) \le \alpha_n d(x, y) + \beta_n d(x, T_n x) + \gamma_n d(y, T_n y),$$

where

$$\alpha_n, \beta_n, \gamma_n \ge 0.$$

For every $n \notin E$, require $\alpha_n, \beta_n, \gamma_n \ge 0$ and $\alpha_n + \beta_n + \gamma_n < 1$, and require the displayed inequality for all $x, y \in X$. Set these three coefficients to zero on E, so that $\widehat{C}_n$ is defined for every index. Fixed points are assumed to exist at every index. Outside E they are unique: Two fixed points z, w satisfy $d(z, w) \le \alpha_n d(z, w)$. Choices on E do not change the statistical radius.

Define, for all indices (using the zero-coefficient extension on E),

$$\widehat{C}_n = \frac{1 + \min\{\beta_n, \gamma_n\}}{1 - \alpha_n}.$$

Theorem 7.2 (Variable Reich sensitivity)**.** *Under the complete index-dependent Reich assumptions just stated, suppose*

$$\widehat{C}_* = \operatorname{st-lim\,sup}_{n\to\infty} \widehat{C}_n < \infty, \qquad \rho_e = \operatorname{st-lim\,sup}_{n\to\infty} e_n < \infty.$$

Then

$$\operatorname{st-lim\,sup}_{n\to\infty} d(x_n, x^*) \le \widehat{C}_* \rho_e.$$

Proof. For each $n \notin E$, apply the all-pairs Reich inequality first to (x_n, x^*) and then to (x^*, x_n), exactly as in Theorem 7.1. This yields

$$(1 - \alpha_n) d(x_n, x^*) \le (1 + \min\{\beta_n, \gamma_n\}) e_n,$$

so that $d(x_n, x^*) \le \widehat{C}_n e_n$ for $n \notin E$. The nonnegative-sequence estimate of Theorem 4.1 then gives the assertion. No pointwise inequality is claimed on E. □

Proposition 7.3 (Sharpness of the two-sided Reich sensitivity constant)**.** *Let $\alpha, \beta, \gamma \geq 0$ with $\alpha+\beta+\gamma < 1$, and put*

$$m = \min\{\beta, \gamma\}, \qquad q = \frac{\alpha + m}{1 + m}.$$

For any sequence $e_n \geq 0$, on the complete metric space $X = \mathbb{R}$ consider $T(x) = 0$ and $T_n(x) = qx + e_n$. Then every T_n satisfies the Reich inequality with the prescribed coefficients α, β, γ for all ordered pairs $x, y \in \mathbb{R}$. Its unique fixed point p_n satisfies

$$d(p_n, 0) = \frac{1 + m}{1 - \alpha} e_n.$$

Consequently the coefficient in Theorem 7.1 is sharp throughout the admissible parameter region, even for affine maps.

Proof. Since $q < 1$, $p_n = e_n/(1 - q)$ is the unique fixed point and $|x - T_n x| = (1 - q)|x - p_n|$. The triangle inequality gives $|x - p_n| + |y - p_n| \geq |x - y|$. Thus

$$\begin{aligned} \alpha|x - y| + \beta|x - T_n x| + \gamma|y - T_n y| &\geq \alpha|x - y| + m(1 - q)(|x - p_n| + |y - p_n|) \\ &\geq [\alpha + m(1 - q)]|x - y| = q|x - y| = |T_n x - T_n y|, \end{aligned}$$

where the penultimate equality follows from the definition of q. Finally, $1 - q = (1 - \alpha)/(1 + m)$, proving the exact equality. □

Remark 7.4. No smaller coefficient depending only on α, β, γ can replace the displayed constant uniformly over the stated operator class.

8. A nonlinear illustrative example

Consider

$$X = [0, \infty)$$

with the usual metric and define

$$T(x) = \frac{x}{4 + x}.$$

Since

$$T(0) = 0,$$

the reference fixed point is

$$x^* = 0.$$

Moreover,

$$T'(x) = \frac{4}{(4 + x)^2},$$

so

$$0 < T'(x) \leq \frac{1}{4} \qquad (x \geq 0).$$

Thus T is a contraction on X.

For a nonnegative perturbation sequence (a_n), define

$$T_n(x) = \frac{x}{4+x} + a_n.$$

Each T_n has the same Lipschitz coefficient at most $1/4$. Its fixed-point equation is

$$x_n = \frac{x_n}{4+x_n} + a_n.$$

Multiplication by $4 + x_n$ gives

$$x_n(4+x_n) = x_n + a_n(4+x_n),$$

and hence

$$x_n^2 + (3-a_n)x_n - 4a_n = 0.$$

The nonnegative fixed point is

$$x_n = \frac{a_n - 3 + \sqrt{a_n^2 + 10a_n + 9}}{2}.$$

Define

$$\Psi(a) = \frac{a - 3 + \sqrt{a^2 + 10a + 9}}{2}.$$

Then

$$x_n = \Psi(a_n).$$

A direct differentiation gives

$$\Psi'(a) = \frac{1}{2} + \frac{a+5}{2\sqrt{a^2+10a+9}} > 0, \qquad a \geq 0.$$

Hence Ψ is continuous and strictly increasing on $[0, \infty)$.

The local defect is

$$e_n = |T_n(0) - T(0)| = a_n.$$

Therefore, if

$$\rho = \text{st}-\limsup a_n < \infty,$$

Theorem 5.1 gives

$$\text{st}-\limsup x_n \leq \Psi(\rho).$$

If, in addition, $a_n \to \rho$ outside a set of natural density zero, then

$$\text{st}-\limsup x_n = \Psi(\rho).$$

The nonlinear control gives a sharper, magnitude-dependent bound than the linear estimate $x_n \leq (4/3)a_n$. Indeed, for $a > 0$ the fixed point is positive and

$$\frac{\Psi(a)}{a} = \frac{4+\Psi(a)}{3+\Psi(a)} < \frac{4}{3}, \qquad \lim_{a\to\infty} \frac{\Psi(a)}{a} = 1.$$

Thus $\Psi(\rho) < (4/3)\rho$ for $\rho > 0$. For instance, $\rho = 1$ gives $\Psi(1) = \sqrt{5} - 1 \approx 1.2361$ versus the linear bound $4/3 \approx 1.3333$. The gain comes from the exact nonlinear fixed-point response, not from a stronger statistical transfer mechanism.

9. Sharpness and separation examples

We now show that the classical contractive amplification factor is optimal and use explicit perturbation sequences to separate ordinary, statistical, and rough statistical stability.

9.1. Sharpness of the contractive amplification factor

Theorem 9.1 (Sharpness)**.** *Let*

$$0 \le \lambda < 1$$

and let (a_n) *be a nonnegative sequence with finite statistical upper radius. Define*

$$T(x) = \lambda x$$

and

$$T_n(x) = \lambda x + a_n$$

on $\mathbb{R}$.

Then

$$x^* = 0, \qquad x_n = \frac{a_n}{1-\lambda},$$

and the local perturbation defect satisfies

$$e_n = a_n.$$

Consequently,

$$|x_n - x^*| = \frac{e_n}{1-\lambda},$$

and

$$\operatorname{st-lim\,sup}_{n\to\infty} |x_n - x^*| = \frac{1}{1-\lambda} \operatorname{st-lim\,sup}_{n\to\infty} e_n.$$

Proof. The reference fixed-point equation

$$x = \lambda x$$

has the unique solution

$$x^* = 0.$$

For T_n, the fixed-point equation is

$$x_n = \lambda x_n + a_n.$$

Therefore,

$$(1-\lambda)x_n = a_n,$$

and hence

$$x_n = \frac{a_n}{1-\lambda}.$$

Moreover,

$$e_n = |T_n x^* - T x^*| = |T_n(0) - T(0)| = a_n.$$

Thus

$$|x_n - x^*| = \frac{e_n}{1-\lambda}.$$

Applying the homogeneity property of the statistical upper radius gives

$$\text{st}-\limsup |x_n - x^*| = \frac{1}{1-\lambda} \text{st}-\limsup e_n.$$

□

Remark 9.2. Theorem 9.1 shows that the coefficient

$$\frac{1}{1-\lambda}$$

in the contractive data-dependence estimate cannot be replaced by a smaller universal constant.

9.2. Ordinary versus statistical stability

Let

$$E = \{m^2 : m \in \mathbb{N}\}.$$

Since

$$|E \cap \{1, \dots, N\}| = \lfloor \sqrt{N} \rfloor,$$

we have

$$\delta(E) = \lim_{N\to\infty} \frac{\lfloor \sqrt{N} \rfloor}{N} = 0.$$

Define

$$a_n = \begin{cases} 1, & n \in E, \\ 0, & n \notin E. \end{cases}$$

Then

$$a_n \nrightarrow 0$$

in the ordinary sense because

$$a_{m^2} = 1$$

for every $m \in \mathbb{N}$.

Nevertheless,

$$\text{st}-\lim_{n\to\infty} a_n = 0.$$

For

$$T_n(x) = \lambda x + a_n,$$

we have

$$x_n = \frac{a_n}{1-\lambda}.$$

Therefore,

$$x_n \nrightarrow 0$$

ordinarily, whereas

$$\text{st}-\lim_{n\to\infty} x_n = 0.$$

This example shows that statistical fixed-point stability is strictly weaker than ordinary asymptotic stability.

9.3. Sparse unbounded perturbations

The following single example simultaneously exhibits a persistent background and sparse unbounded perturbations, subsuming the bounded-exceptional illustration.

Let

$$a_n = \begin{cases} n, & n = m^2, \\ \rho + \dfrac{1}{n}, & n \neq m^2, \end{cases} \qquad \rho > 0.$$

Then

$$\sup_{n \geq 1} a_n = \infty.$$

Indeed,

$$a_{m^2} = m^2 \longrightarrow \infty.$$

Moreover,

$$a_n \nrightarrow 0.$$

Nevertheless, outside the density-zero set of square indices,

$$a_n \longrightarrow \rho.$$

Hence

$$\text{st}-\limsup_{n \to \infty} a_n = \rho.$$

For

$$T_n(x) = \lambda x + a_n,$$

we obtain

$$x_n = \frac{a_n}{1 - \lambda}.$$

Thus

$$\sup_n |x_n| = \infty,$$

while

$$\text{st}-\limsup_{n \to \infty} |x_n| = \frac{\rho}{1 - \lambda} < \infty.$$

Therefore a finite rough statistical fixed-point radius may coexist with unbounded fixed-point errors on an infinite set of indices.

Remark 9.3. A finite rough statistical radius does not imply uniform boundedness of the errors. In the preceding example,

$$\sup_n |x_n| = \infty,$$

although

$$\text{st}-\limsup |x_n| < \infty.$$

Thus rough statistical robustness describes typical asymptotic behavior outside a statistically negligible set. It should not be interpreted as worst-case robustness.

9.4. Sparse sensitivity blow-up

The variable-sensitivity principle also permits the sensitivity coefficient itself to become unbounded on a sparse set.

Let

$$C_n = \begin{cases} n, & n = m^2, \\ C + \dfrac{1}{n}, & n \neq m^2, \end{cases} \qquad C > 0,$$

and let

$$b_n = \begin{cases} 1, & n = m^2, \\ \rho + \dfrac{1}{n}, & n \neq m^2, \end{cases} \qquad \rho > 0.$$

Then

$$\sup_n C_n = \infty,$$

while

$$\text{st}-\limsup C_n = C.$$

Similarly,

$$\text{st}-\limsup b_n = \rho.$$

If

$$a_n \leq C_n b_n$$

outside a density-zero exceptional set, then Theorem 4.1 yields

$$\text{st}-\limsup a_n \leq C\rho.$$

This example shows that neither the perturbation sensitivity nor the perturbation magnitude needs to possess a useful global supremum for the statistical transfer principle to remain informative.

10. Statistical perturbation amplification

For a fixed reference pair (T, x^*), $Tx^* = x^*$, let $\mathfrak{P}(T)$ be a specified, nonempty class of perturbation families with selected fixed points $T_n x_n = x_n$. Define $e_n = d(T_n x^*, x^*)$, $r_e = \text{st}-\limsup e_n$, and $r_{\text{FP}} = \text{st}-\limsup d(x_n, x^*)$. Selection matters if fixed points are nonunique. Only families with $0 < r_e < \infty$ enter the following ratio; $r_{\text{FP}} = \infty$ is permitted and $\sup \varnothing = 0$.

Definition 10.1. The statistical amplification factor of the specified class is

$$\kappa_{\text{st}}(T; \mathfrak{P}) = \sup_{((T_n),(x_n)) \in \mathfrak{P}(T),\, 0 < r_e < \infty} \frac{r_{\text{FP}}}{r_e}.$$

If every admissible family obeys $d(x_n, x^*) \leq Ce_n$ outside a density-zero set with the same C, Theorem 3.1 immediately gives $\kappa_{\text{st}} \leq C$. The bound is attained in the nonnegative affine class $T(x) = \lambda x$, $T_n(x) = \lambda x + a_n$, $0 \leq \lambda < 1$, $a_n \geq 0$, and $0 < \text{st}-\limsup a_n < \infty$, with $x_n = a_n/(1 - \lambda)$: here $r_{\text{FP}} = r_e/(1 - \lambda)$, so $\kappa_{\text{st}} = (1 - \lambda)^{-1}$. This is an interpretation of the classical sensitivity constant, not a separate fixed-point theorem.

11. Application to a nonlinear Fredholm integral equation

Let $X = C([0,1],\mathbb{R})$ with $\|u\|_\infty = \sup_{t\in[0,1]} |u(t)|$, and consider

$$(Tu)(t) = g(t) + \int_0^1 K(t,s,u(s))\,ds.$$

Assume $g \in C([0,1])$, K is jointly continuous on $[0,1]^2 \times \mathbb{R}$, and

$$|K(t,s,\xi) - K(t,s,\eta)| \le L(t,s)|\xi-\eta|, \qquad q := \sup_t \int_0^1 L(t,s)\,ds < 1,$$

where $L \ge 0$ has integrable sections. Joint continuity makes T a self-map of the complete space X, and the displayed bound makes it a contraction. Consequently T has a unique fixed point u^*.

11.1. Perturbed integral equations

For each $n \in \mathbb{N}$, consider

$$u_n(t) = g_n(t) + \int_0^1 K_n(t,s,u_n(s))\,ds.$$

Define

$$(T_n u)(t) = g_n(t) + \int_0^1 K_n(t,s,u(s))\,ds.$$

For every n, assume g_n is continuous, K_n is jointly continuous on $[0,1]^2 \times \mathbb{R}$, and $|K_n(t,s,\xi) - K_n(t,s,\eta)| \le L_n(t,s)|\xi-\eta|$, where $L_n \ge 0$ has integrable sections and $q_n = \sup_t \int_0^1 L_n(t,s)\,ds < 1$. Thus T_n is a contraction self-map of X and has a unique fixed point u_n. The estimates remain valid when these assumptions hold only outside a density-zero set E, provided fixed points are selected on E and q_n is assigned a value in $[0,1)$ there.

Define

$$\kappa_n = \frac{1}{1-q_n}$$

and the local residual

$$R_n = \|T_n u^* - T u^*\|_\infty.$$

Theorem 11.1 (Variable-sensitivity stability of Fredholm solutions)**.** *Under all the continuity, integrability, self-map, fixed-point, and nonexceptional contractivity hypotheses stated above, suppose*

$$\kappa_* = \text{st}-\limsup_{n\to\infty} \kappa_n < \infty$$

and

$$\rho_R = \text{st}-\limsup_{n\to\infty} R_n < \infty.$$

Then

$$\text{st}-\limsup_{n\to\infty} \|u_n - u^*\|_\infty \le \kappa_* \rho_R.$$

Proof. For $n \notin E$, the fixed-point equations and contractivity give $\|u_n - u^*\|_\infty \le q_n\|u_n - u^*\|_\infty + R_n$, hence $\|u_n - u^*\|_\infty \le R_n/(1-q_n)$. Apply Theorem 4.1. In particular, if $q_n \le q_0 < 1$ off a density-zero set, the bound becomes $(1-q_0)^{-1}\,\text{st}-\limsup R_n$. □

Forcing and kernel residual decomposition. Set

$$A_n = \|g_n - g\|_\infty, \qquad B_n = \sup_{t\in[0,1]} \int_0^1 |K_n(t,s,u^*(s)) - K(t,s,u^*(s))|\,ds.$$

The triangle inequality gives $R_n \le A_n + B_n$, and therefore

$$\text{st}-\limsup \|u_n - u^*\|_\infty \le \kappa_*(\text{st}-\limsup A_n + \text{st}-\limsup B_n),$$

provided the radii on the right are finite. This is the forcing–kernel decomposition of the same fixed-point estimate, not an additional independent theorem.

11.2. A time- and integration-dependent nonlinear example

A time- and integration-dependent nonlinear kernel is obtained by setting

$$w(t,s) = 1 + \tfrac{1}{4}ts, \qquad K(t,s,\xi) = \tfrac{1}{4}w(t,s)\tanh\xi, \qquad g(t) = 1 - \tfrac{1}{4}\tanh(1)\int_0^1 w(t,s)\,ds.$$

Then $u^*(t) \equiv 1$ solves the reference equation. Define $S = \{k^2 : k \in \mathbb{N}\}$ and, for $n \ge 1$, set $a_n = n$ and $c_n = 1/16$ when $n \in S$, while $a_n = 1/10$ and $c_n = 0$ when $n \notin S$. Set $g_n = g + a_n$ and $K_n = (\frac{1}{4} + c_n)w(t,s)\tanh\xi$. Since $\sup_{t\in[0,1]} \int_0^1 w(t,s)\,ds = 9/8$ and $|\tanh\xi - \tanh\eta| \le |\xi - \eta|$, the integral operator has a Lipschitz constant at most $q_n = (\frac{1}{4} + c_n)9/8 \le 45/128 < 1$, and thus has a unique fixed point in $C[0,1]$. At u^* the forcing defect is $A_n = |a_n|$, and the kernel defect satisfies

$$B_n = \sup_t \left| c_n \tanh(1) \int_0^1 w(t,s)\,ds \right| = \tfrac{9}{8}c_n \tanh(1).$$

Both perturbation terms are nonnegative at the reference solution, hence $R_n = A_n + B_n$. Because the square indices have density zero, $\text{st}-\limsup R_n = 1/10$, whereas $\text{st}-\limsup(1 - q_n)^{-1} = 32/23$. Therefore the fixed-point stability theorem yields the finite bound

$$\text{st}-\limsup \|u_n - u^*\|_\infty \le \frac{16}{115}.$$

This establishes a nonlinear nonconstant-kernel application without claiming a sharp bound. The density-zero exceptional square indices may have unbounded forcing defects, while the statistical radius remains finite. In fact, the uniform bound obtained from the same pointwise residual estimate is infinite: $R_{k^2} \ge a_{k^2} = k^2$, so $\sup_n R_n/(1 - q_n) = +\infty$. Moreover, the actual solution errors are unbounded: Since $g_n = g + k^2$ and the nonlinear integral term is bounded uniformly by $(5/16)(9/8)$ in absolute value, $\|u_{k^2} - u^*\|_\infty \ge k^2 - C$ for a fixed finite C. Thus $\sup_n \|u_n - u^*\|_\infty = +\infty$, in contrast to the finite statistical bound $16/115$.

12. Hierarchy of stability notions

The preceding examples naturally lead to three levels of fixed-point stability.

Definition 12.1. The sequence of perturbed fixed points (x_n) is said to be ordinarily asymptotically stable with respect to x^* if

$$d(x_n, x^*) \longrightarrow 0.$$

Definition 12.2. The sequence (x_n) is said to be statistically stable with respect to x^* if

$$\text{st}-\lim_{n\to\infty} x_n = x^*.$$

Definition 12.3. The sequence (x_n) is said to be rough statistically stable with respect to x^* if

$$r_{\text{FP}} = \text{st}-\limsup_{n\to\infty} d(x_n, x^*) < \infty.$$

Proposition 12.4. *The following implications hold:*

$$\textit{ordinary stability} \Longrightarrow \textit{statistical stability} \Longrightarrow \textit{rough statistical stability}.$$

Neither converse holds in general.

Proof. If

$$d(x_n, x^*) \to 0,$$

then, for every $\varepsilon > 0$, the set

$$\{n : d(x_n, x^*) \geq \varepsilon\}$$

is finite and hence has natural density zero. Therefore ordinary stability implies statistical stability.

If

$$\text{st}-\lim x_n = x^*,$$

then

$$\text{st}-\limsup d(x_n, x^*) = 0,$$

so the rough statistical radius is finite.

The converse implications fail by the examples of Section 9. The perfect-square perturbation with zero background level is statistically but not ordinarily stable, whereas a positive persistent background level produces rough statistical stability without statistical convergence. □

13. Discussion

The results developed above highlight several distinctions between classical perturbation theory and rough statistical data dependence.

A concrete interpretation is a sequence of independently solved discretized problems with rare corrupted input batches. Density-zero corruption is acceptable only if a typical asymptotic guarantee is the intended criterion. This assumption does not automatically follow from random sensor failures, and these independent-solve estimates do not control propagation of an exceptional error through one nonautonomous iteration. Such propagation requires separate dynamical hypotheses.

First, uniform perturbation theory is sensitive to every perturbation index. A single unbounded subsequence therefore forces a global worst-case quantity such as

$$\sup_n e_n$$

to become infinite.

The statistical framework instead distinguishes the typical asymptotic perturbation level from exceptional disturbances occurring on sets of natural density zero. Thus it is possible to have

$$\sup_n e_n = \infty$$

while simultaneously

$$\text{st}-\limsup e_n < \infty.$$

This does not imply that the exceptional perturbations are dynamically insignificant. At exceptional indices, the corresponding fixed-point errors may also become arbitrarily large. The theory therefore describes typical asymptotic robustness rather than worst-case robustness.

Second, the local perturbation defect

$$e_n = d(T_n x^*, T x^*)$$

may provide substantially more information than the global operator discrepancy

$$\sup_{x \in X} d(T_n x, T x).$$

The latter may be infinite on an unbounded domain even when the perturbation at the reference equilibrium remains controlled.

Third, the variable-sensitivity theorem shows that uniform control of the data-dependence coefficients is not necessary. A sequence (C_n) may satisfy

$$\sup_n C_n = \infty$$

while

$$\text{st}-\limsup C_n < \infty.$$

Consequently, statistically exceptional losses of conditioning can be incorporated into the same framework.

Fourth, the nonlinear transfer theorem demonstrates that the method is not restricted to Lipschitz-type data-dependence estimates. Whenever one has

$$d(x_n, x^*) \leq \Psi(e_n)$$

with Ψ continuous and nondecreasing, one obtains

$$\text{st}-\limsup d(x_n, x^*) \leq \Psi(\text{st}-\limsup e_n).$$

The nonlinear example of Section 8 illustrates this mechanism explicitly.

14. Limitations and further directions

The proposed framework has several limitations that indicate directions for further research.

First, natural density is used to identify negligible perturbation indices. Some perturbation patterns may not possess a natural density. Such cases motivate extensions based on ideals, weighted densities, lacunary densities, or matrix summability methods. Second, the statistical transfer principles require an underlying deterministic data-dependence inequality. Consequently, rough statistical stability cannot compensate for an intrinsically unstable or severely ill-conditioned fixed-point problem. Third, a finite statistical upper radius does not provide control over the magnitude of errors on the exceptional density-zero set. Thus the present framework complements rather than replaces uniform robustness analysis. Fourth, the current application is deterministic. Random operator perturbations would require probabilistic or almost-sure analogues of the statistical radii considered here.

Possible future developments include rough statistical data dependence for multivalued mappings, common and coupled fixed points, generalized contractive mappings, stochastic perturbations, fractional integral operators, and nonlinear equations in generalized metric structures.

15. Conclusions

This paper developed a quantitative framework for the data dependence of fixed points under persistent, sparse, and statistically exceptional operator perturbations.

The first main result established a statistical transfer principle: Whenever

$$a_n \le C b_n$$

outside a set of natural density zero,

$$\text{st}-\limsup a_n \le C \ \text{st}-\limsup b_n.$$

The framework was subsequently strengthened by allowing the sensitivity coefficient itself to vary with the perturbation index. If

$$a_n \le C_n b_n,$$

then, under finite statistical upper radii,

$$\text{st}-\limsup a_n \le (\text{st}-\limsup C_n)(\text{st}-\limsup b_n).$$

This result permits the sensitivity coefficients to be unbounded on statistically negligible sets.

A nonlinear transfer principle was also obtained. For continuous nondecreasing Ψ,

$$a_n \le \Psi(b_n)$$

statistically almost everywhere implies

$$\text{st}-\limsup a_n \le \Psi(\text{st}-\limsup b_n).$$

For sequences of contractive operators, the global operator discrepancy was replaced by the localized defect

$$e_n = d(T_n x^*, T x^*).$$

With

$$\kappa_n = \frac{1}{1-\lambda_n},$$

the resulting fixed-point estimate becomes

$$d(x_n, x^*) \leq \kappa_n e_n.$$

Consequently,

$$\text{st}-\limsup d(x_n, x^*) \leq (\text{st}-\limsup \kappa_n)(\text{st}-\limsup e_n).$$

The framework was extended to Reich-type operators with fixed and index-dependent parameters. Applying the all-pairs Reich condition in both orders yields the two-sided sensitivity coefficient

$$\widehat{C}_n = \frac{1 + \min\{\beta_n, \gamma_n\}}{1-\alpha_n}.$$

The affine construction in Proposition 7.3 attains the corresponding constant-parameter coefficient exactly, showing that this bound is sharp over the stated Reich class. This sharpness assertion does not, by itself, establish priority over existing fixed-point data-dependence results.

The examples demonstrated three distinct regimes: ordinary stability, statistical stability, and rough statistical stability. In particular, a finite rough statistical fixed-point radius may persist even when both the perturbation sequence and the corresponding fixed-point errors are unbounded on an infinite density-zero set. Finally, the theory was applied to nonlinear Fredholm integral equations. The resulting estimates show how persistent perturbations of the forcing term and integral kernel propagate quantitatively to the associated solutions.

The factor $\kappa_{\text{st}}(T; \mathfrak{P})$ summarizes these bounds as a class-dependent measure of statistical conditioning. Its finite upper bound requires a common finite sensitivity estimate; its exact value is established here for the scalar affine class. The nonlinear kernel example and the diagonal Picard estimate show how local data errors and computational tolerances can be combined. In the diagonal Picard conclusion, both the normalized fixed-point defect h_n and the computational tail b_n must tend statistically to zero to obtain statistical convergence to x^*. The scalar transfer arguments are elementary consequences of natural-density tails.

Future work: A natural next step is to identify genuinely operator-level stability conditions for nonlinear and Reich-type mappings that are not reducible to scalar statistical transfer estimates. The stability of nonautonomous Picard iterations under persistent perturbations also merits separate investigation. These questions are beyond the scope of the present work.

Use of Generative-AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest in this paper.

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