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Research article

Spectral sampling and stable reconstruction for fractional-order Fourier transforms

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Abstract: We develop a spectral sampling framework for reconstructing fractional Fourier transform orbits from finitely many samples of the transform order. Using the Hermite spectral representation, the orbit $\alpha \mapsto \mathcal{F}_\alpha f$ is viewed as an $L^2(\mathbb{R})$-valued periodic Fourier series whose frequencies in the order variable coincide with the Hermite indices of the signal. For finite Hermite expansions, we derive an exact interpolation formula for recovering the complete orbit from equispaced order samples. For general $L^2(\mathbb{R})$ signals, an explicit modulo-M aliasing representation is obtained, showing that the reconstruction error is determined by the unresolved Hermite coefficients and their folding modulo M. This yields a uniform error bound and an exact root-mean-square (RMS) error identity. We further establish stability under perturbed transform samples: worst-case deterministic uniform errors are controlled by the associated trigonometric Lebesgue constant, which grows logarithmically with the number of samples, whereas the discrete-to-continuous RMS reconstruction is exactly isometric. Numerical tests with a finite Hermite expansion and a chirped Gaussian illustrate the interpolation and aliasing identities. For the specified Gaussian and transform orders, an integral-kernel evaluation provides an independent comparison with the Hermite-spectral fractional Fourier transform (FrFT) computation. Finite-dimensional Gaussian-noise simulations are also compared with the expected mean-square reconstruction-error formula. A synthetic, discontinuous, windowed chirp and an observational atmospheric CO_2 time series, embedded as a piecewise-linear L^2 signal, additionally illustrate the Hermite-tail bounds. These tests do not establish numerical accuracy for arbitrary signals or noise models.

Keywords: fractional-order Fourier transform; spectral sampling; Hermite functions; stable reconstruction; aliasing
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1. Introduction

The fractional Fourier transform (FrFT) is a parameterized extension of the classical Fourier transform in which the transform order provides a continuous transition between intermediate time–frequency representations. Since the early operator formulation of Namias [1] and its subsequent mathematical development [2], the FrFT has become a standard tool in harmonic analysis, optics, and signal processing. Its time–frequency interpretation was clarified by Almeida [3], and efficient digital realizations and discrete formulations were developed in [4, 5]. A systematic account of the transform, its Hermite representation, and its computational properties can be found in [6].

The mathematical theory of the FrFT continues to develop. Chen et al. [7] established an L^p theory involving inversion, approximation, multipliers, and Littlewood–Paley estimates. More recent work has connected fractional Fourier analysis with fractional Riesz potentials and imaging problems [8]. At the same time, the transform-order parameter has become increasingly important in applications involving nonstationary signals. For example, FrFT-based time–frequency representations have been used for instantaneous-frequency identification [9], and parameterized fractional domains have been incorporated into time-series prediction [10] and trainable signal-processing architectures [11]. These developments emphasize the transform order is not merely a formal parameter, but an effective degree of freedom in analysis and computation.

Weighted fractional Fourier transforms (WFRFTs) constitute a related but distinct family of parameterized transform-domain constructions. Dong et al. [12] studied multicomponent signal-domain secure hybrid precoding for millimeter-wave systems, including a WFRFT-based realization, and Dong et al. [13] combined spatial and WFRFT domains for physical-layer security in rate-splitting multiple-access systems. These examples suggest possible communication-oriented applications of transform-order representations, but they do not establish that our classical FrFT sampling identities apply to WFRFT. Such an extension would require a separate spectral analysis.

Related developments in learning-based image analysis further illustrate how transformed representations and cross-modal information can support robust feature extraction. Liu et al. [14] developed robust feature representations for accurate human parsing, whereas Wang et al. [15] developed mask-guided Mamba fusion for visible–infrared vehicle detection. These studies concern distinct computer-vision tasks and do not establish results for FrFT order-domain sampling; they provide broader motivation for investigating structured transform-domain representations.

Uncertainty principles also form part of the theoretical background: Shinde and Gadre [16] investigated an uncertainty principle for real signals in the fractional Fourier transform domain.

Sampling theory associated with the FrFT constitutes another active direction. Sampling and reconstruction in fractional Fourier domains have been studied under bandlimited and nonbandlimited assumptions, including irregular sampling formulations [17]. Zayed [18] established a sampling theorem for a two-dimensional fractional Fourier transform, and Ma and Tao [19] surveyed developments involving uniform sampling, nonuniform sampling, fractional spectral analysis, and sub-Nyquist reconstruction. More recently, Zhao and Li [20] developed an unlimited-sampling theorem in the fractional Fourier setting. Nonuniform fractional-domain sampling and the associated spectrum analysis have also continued to receive attention [21].

Alongside sampling theory, computational efficiency has motivated discrete and sparse fractional-transform algorithms. The classical digital FrFT algorithms of [4, 5] provided the basis for many

subsequent developments. Pei and Yeh [22] also developed an improved discrete fractional Fourier transform. Wei and Yang [23] proposed a two-dimensional sparse fractional Fourier transform with reduced runtime and sample complexity and analyzed its behavior under noise. Huang and Zhang [24] considered the efficient computation of multiangle discrete fractional Fourier transforms, a problem particularly relevant when the transform must be evaluated at several order values. Such multiorder computations raise a natural question that is different from classical sampling of the underlying signal: whether a continuous family of fractional Fourier transforms can itself be recovered from finitely many evaluations in the order parameter.

Fractional Fourier ideas have also recently been extended to signals on graphs and other structured domains. Sampling by successive aggregations in the graph fractional Fourier domain was developed by Wei and Yan [25], including reconstruction in the presence of noise. A generalized sampling framework using prior information was subsequently introduced in [26]. Alikaşifoğlu, Kartal, and Koç [27] developed a unified theory of the graph fractional Fourier transform, and fractional analysis has recently been extended to joint time–vertex settings [28]. Zhang and Li [29] studied the graph fractional Fourier transform in a Hilbert-space setting. These works illustrate the continuing interest in combining fractional transform parameters with sampling, reconstruction, and spectral representations.

Despite this extensive literature, the sampling variable in most FrFT sampling theories is the signal variable itself, or a spatial, graph, or fractional-frequency coordinate. The problem considered here is different. For a fixed signal f, we study the complete fractional-order orbit

$$G_f(\alpha) := \mathcal{F}_\alpha f, \qquad \alpha \in \mathbb{T}$$

and ask whether this $L^2(\mathbb{R})$-valued function of the transform order can be reconstructed from finitely many values $\mathcal{F}_{\alpha_\ell} f$. This viewpoint is especially natural in view of multiangle computations [24], but it leads to a sampling problem in the *order variable* rather than in the original signal domain.

The key observation is provided by the Hermite spectral realization of the FrFT. If

$$f = \sum_{n=0}^{\infty} c_n h_n,$$

then

$$G_f(\alpha) = \sum_{n=0}^{\infty} c_n e^{-in\alpha} h_n.$$

Thus, as a function of α, the fractional Fourier orbit is an $L^2(\mathbb{R})$-valued Fourier series whose order frequency is locked to the corresponding Hermite index. This simple diagonal structure allows order-domain sampling to be treated directly by discrete Fourier orthogonality.

The interpolation mechanism used below is the standard trigonometric interpolation mechanism applied to a Hilbert-space-valued function; we do not claim a new general Fourier interpolation theorem. The FrFT-specific content comes from the Hermite eigenfunction structure, which identifies the order frequencies with the Hermite indices and thereby gives an explicit description of the folded spectral modes. This permits exact reconstruction-error identities in terms of the unresolved Hermite coefficients and stability estimates for perturbed order samples. Here, reconstruction refers to interpolation from the available full transform samples, rather than to the separate task of recovering Hermite coefficients by projecting a sample onto the known Hermite basis. In applications where

several transform orders are required, the resulting estimates may help assess the accuracy of multi-order representations of nonstationary signals; computational benefits depend on the costs of acquiring and representing the samples.

The main contributions of this paper are as follows. For finite Hermite expansions, we derive an exact reconstruction formula from finitely many equispaced order samples and show that the result remains valid under a common shift of the sampling grid. For general $L^2(\mathbb{R})$ signals, we obtain an explicit modulo-M aliasing formula and exact pointwise and root-mean-square RMS error identities governed by the unresolved Hermite tail. A refinement in terms of aliasing levels yields a spectral criterion under which the uniform error bound is attained exactly. We also analyze perturbations of the sampled transforms: The deterministic uniform error is controlled by the associated trigonometric Lebesgue constant, whereas the discrete-to-continuous RMS propagation is an exact isometry. Under a natural spectral separation condition, aliasing and perturbation energies admit an exact orthogonal decomposition. Numerical experiments with finite Hermite signals and a chirped Gaussian verify the theoretical identities and stability estimates.

The paper is organized as follows. Section 2 introduces the Hermite-spectral FrFT representation and the order-domain interpolant. Section 3 establishes exact reconstruction for finite Hermite orbits. Section 4 derives the aliasing structure and certified error estimates. Section 5 studies perturbation stability, and Section 6 provides numerical verification. The final section summarizes the main conclusions.

2. Fractional Fourier setting

Let $\{h_n\}_{n\geq 0}$ denote the normalized Hermite functions

$$h_n(x) = \frac{1}{\pi^{1/4}\sqrt{2^n n!}} H_n(x) e^{-x^2/2}, \qquad n = 0, 1, 2, \ldots,$$

which form a complete orthonormal basis of $L^2(\mathbb{R})$. Hence, for $f \in L^2(\mathbb{R})$,

$$f = \sum_{n=0}^{\infty} c_n h_n, \qquad c_n = \langle f, h_n \rangle, \qquad \|f\|_2^2 = \sum_{n=0}^{\infty} |c_n|^2. \tag{2.1}$$

We use the Hermite-spectral realization of the fractional Fourier transform; see [1, 3, 6] and, for modern analytical developments of the FrFT, [7]. With the order parameter written as an angle α, the transform is given by

$$\mathcal{F}_\alpha f = \sum_{n=0}^{\infty} c_n e^{-in\alpha} h_n, \qquad \alpha \in \mathbb{R}. \tag{2.2}$$

Accordingly,

$$\mathcal{F}_\alpha h_n = e^{-in\alpha} h_n, \qquad \mathcal{F}_0 = I, \qquad \mathcal{F}_{\pi/2} = \mathcal{F},$$

where $\mathcal{F}$ denotes the standard Fourier transform under the corresponding Hermite convention. The family $\{\mathcal{F}_\alpha\}_{\alpha\in\mathbb{R}}$ is a 2π-periodic unitary group on $L^2(\mathbb{R})$.

Because the family is 2π-periodic in α, we regard the order parameter as an element of

$$\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z}).$$

For fixed $f \in L^2(\mathbb{R})$, define its fractional Fourier orbit by

$$G_f(\alpha) := \mathcal{F}_\alpha f = \sum_{n=0}^{\infty} c_n e^{-in\alpha} h_n, \qquad \alpha \in \mathbb{T}. \tag{2.3}$$

Representation (2.3) is the structural starting point of the present analysis. As a function of the transform order, G_f is an $L^2(\mathbb{R})$-valued Fourier series whose order frequencies coincide with the Hermite indices of f. We refer to this correspondence as diagonal spectral locking.

For $M \geq 1$, set

$$\alpha_\ell = \frac{2\pi\ell}{M}, \qquad \ell = 0, \ldots, M-1, \tag{2.4}$$

and define the one-sided Dirichlet kernel

$$D^-_{M-1}(\theta) = \sum_{r=0}^{M-1} e^{-ir\theta}. \tag{2.5}$$

The corresponding order-domain interpolant is

$$(I_M G_f)(\alpha) = \frac{1}{M} \sum_{\ell=0}^{M-1} G_f(\alpha_\ell) D^-_{M-1}(\alpha - \alpha_\ell). \tag{2.6}$$

We shall repeatedly use the discrete Fourier orthogonality relation

$$\frac{1}{M} \sum_{\ell=0}^{M-1} e^{i(r-s)\alpha_\ell} = \delta_{rs}, \qquad 0 \leq r, s \leq M-1. \tag{2.7}$$

The interpolant (2.6) is the standard Dirichlet-kernel interpolant associated with the one-sided frequency set $\{0, -1, \ldots, -(M-1)\}$. Its form is classical; the FrFT-specific structure lies in the correspondence between Hermite indices and order-domain frequencies in (2.3), rather than in the interpolation formula itself. The negative frequencies in (2.5) agree with the spectral convention $\mathcal{F}_\alpha h_n = e^{-in\alpha} h_n$.

The discrete orthogonality relation (2.7) relies on the equispaced order nodes. In particular, frequencies whose indices are congruent modulo M coincide at these nodes, giving rise to the spectral folding analyzed in Section 4. For nonuniform order nodes, this orthogonality relation generally fails, and (2.6) does not extend directly. For finite order-domain expansions, interpolation at sufficiently many distinct nonuniform nodes can instead be formulated through a Vandermonde system, whose conditioning depends on the node distribution.

3. Exact order-domain reconstruction

We first consider signals with a finite Hermite expansion. In this case, the diagonal spectral structure in (2.3) makes the fractional Fourier orbit a finite trigonometric polynomial in the order variable, and hence permits exact reconstruction from finitely many order samples.

Theorem 3.1 (Exact reconstruction of finite Hermite orbits)**.** *Let*

$$f_N = \sum_{n=0}^{N} c_n h_n, \tag{3.1}$$

and let

$$G_{f_N}(\alpha) = \mathcal{F}_\alpha f_N = \sum_{n=0}^{N} c_n e^{-in\alpha} h_n.$$

Set $M = N + 1$, and choose the equispaced order nodes

$$\alpha_\ell = \frac{2\pi\ell}{M}, \qquad \ell = 0, \ldots, M-1.$$

Then, the entire orbit G_{f_N} is reconstructed exactly from the $M = N + 1$ samples $\{G_{f_N}(\alpha_\ell)\}_{\ell=0}^{M-1}$ through

$$G_{f_N}(\alpha) = \frac{1}{M} \sum_{\ell=0}^{M-1} G_{f_N}(\alpha_\ell) D_{M-1}^{-}(\alpha - \alpha_\ell), \qquad \alpha \in \mathbb{T}. \tag{3.2}$$

Equivalently,

$$\mathcal{F}_\alpha f_N = \frac{1}{N+1} \sum_{\ell=0}^{N} \mathcal{F}_{\alpha_\ell} f_N D_N^{-}(\alpha - \alpha_\ell). \tag{3.3}$$

Proof. At the sampling nodes,

$$G_{f_N}(\alpha_\ell) = \sum_{n=0}^{N} c_n e^{-in\alpha_\ell} h_n.$$

Substituting these values into the right-hand side of (3.2), we obtain

$$\begin{aligned} &\frac{1}{M} \sum_{\ell=0}^{M-1} G_{f_N}(\alpha_\ell) D_{M-1}^{-}(\alpha - \alpha_\ell) \\ &\quad = \sum_{n=0}^{N} \sum_{r=0}^{M-1} c_n h_n e^{-ir\alpha} \left[\frac{1}{M} \sum_{\ell=0}^{M-1} e^{i(r-n)\alpha_\ell} \right]. \end{aligned}$$

Because $M = N + 1$, both n and r range over $\{0, \ldots, M-1\}$. By (2.7),

$$\frac{1}{M} \sum_{\ell=0}^{M-1} e^{i(r-n)\alpha_\ell} = \delta_{rn}.$$

Hence,

$$\frac{1}{M} \sum_{\ell=0}^{M-1} G_{f_N}(\alpha_\ell) D_{M-1}^{-}(\alpha - \alpha_\ell) = \sum_{n=0}^{N} c_n e^{-in\alpha} h_n = G_{f_N}(\alpha),$$

which proves the result. □

The preceding formula does not depend on choosing the origin $\alpha = 0$ as the first sampling order.

Corollary 3.2 (Shifted order grids)**.** *Let f_N be as in Theorem 3.1, let $M = N + 1$, and fix any $\beta \in \mathbb{T}$. Define*

$$\alpha_\ell^{(\beta)} = \beta + \frac{2\pi\ell}{M}, \qquad \ell = 0, \dots, M-1. \tag{3.4}$$

Then,

$$G_{f_N}(\alpha) = \frac{1}{M} \sum_{\ell=0}^{M-1} G_{f_N}\left(\alpha_\ell^{(\beta)}\right) D_{M-1}^{-}\left(\alpha - \alpha_\ell^{(\beta)}\right), \qquad \alpha \in \mathbb{T}. \tag{3.5}$$

Proof. For $0 \le r, n \le M-1$,

$$\frac{1}{M} \sum_{\ell=0}^{M-1} e^{i(r-n)\alpha_\ell^{(\beta)}} = e^{i(r-n)\beta} \frac{1}{M} \sum_{\ell=0}^{M-1} e^{2\pi i(r-n)\ell/M} = \delta_{rn}.$$

The proof of Theorem 3.1 therefore applies without change. □

Remark 3.3. *The use of $N + 1$ order nodes reflects the one-sided spectrum*

$$\{0, -1, \dots, -N\}$$

of the finite orbit. Corollary 3.2 further shows that exact reconstruction is invariant under a common rotation of the sampling grid. Thus, the initial sampling order may be selected freely, provided that the $N + 1$ nodes remain equally spaced on $\mathbb{T}$.

The exact reconstruction formula (3.2) relies on the discrete orthogonality (2.7) of equally spaced order nodes. For distinct nonuniform nodes $\beta_0, \dots, \beta_N$, a finite Hermite orbit supported on modes $0, \dots, N$ can instead be interpolated within the space of order-domain trigonometric polynomials spanned by the frequencies $\{0, -1, \dots, -N\}$. The interpolation conditions form a Vandermonde system at the $N + 1$ nodes, which is invertible whenever the nodes are distinct on $\mathbb{T}$. Unlike the equispaced case, the interpolation weights depend on the node locations, and their stability is governed by the conditioning of the Vandermonde matrix. Moreover, the modulo-M aliasing identities established in Section 4 rely on the congruence of order-frequencies at equispaced nodes, a property generally lost for nonuniform nodes; hence, that aliasing analysis does not extend directly to this setting.

The following analysis therefore focuses on equispaced order sampling.

For a general $f \in L^2(\mathbb{R})$, the modes with indices $n \ge M$ are no longer resolved individually by M equispaced order samples. They instead fold into lower-order frequencies according to their residue classes modulo M. The resulting aliasing structure and its reconstruction error are analyzed in the next section.

4. Aliasing and certified error bounds

We now consider a general signal

$$f = \sum_{n=0}^{\infty} c_n h_n \in L^2(\mathbb{R})$$

and its fractional Fourier orbit

$$G_f(\alpha) = \sum_{n=0}^{\infty} c_n e^{-in\alpha} h_n.$$

For $M \geq 1$, let $\mathcal{I}_M G_f$ be the order-domain interpolant defined in (2.6).

Theorem 4.1 (Modulo-M aliasing and reconstruction error)**.** *For every $\alpha \in \mathbb{T}$,*

$$(\mathcal{I}_M G_f)(\alpha) = \sum_{r=0}^{M-1} e^{-ir\alpha} \sum_{\substack{n\geq 0 \\ n\equiv r \pmod M}} c_n h_n. \tag{4.1}$$

Consequently,

$$G_f(\alpha) - (\mathcal{I}_M G_f)(\alpha) = \sum_{n\geq M} c_n \left(e^{-in\alpha} - e^{-i(n \bmod M)\alpha}\right) h_n, \tag{4.2}$$

and

$$\|G_f(\alpha) - (\mathcal{I}_M G_f)(\alpha)\|_2^2 = \sum_{n\geq M} \left|e^{-in\alpha} - e^{-i(n \bmod M)\alpha}\right|^2 |c_n|^2. \tag{4.3}$$

If

$$T_M(f) := \left(\sum_{n\geq M} |c_n|^2\right)^{1/2}, \tag{4.4}$$

then

$$\sup_{\alpha\in\mathbb{T}} \|G_f(\alpha) - (\mathcal{I}_M G_f)(\alpha)\|_2 \leq 2T_M(f). \tag{4.5}$$

Moreover,

$$\left(\int_0^{2\pi} \|G_f(\alpha) - (\mathcal{I}_M G_f)(\alpha)\|_2^2 \frac{d\alpha}{2\pi}\right)^{1/2} = \sqrt{2}\, T_M(f). \tag{4.6}$$

Proof. Using (2.6) and (2.3), and noting that the sums in ℓ and r are finite, we may interchange them with the $L^2(\mathbb{R})$-convergent Hermite expansion. Thus,

$$\begin{aligned}(\mathcal{I}_M G_f)(\alpha) &= \frac{1}{M} \sum_{\ell=0}^{M-1} \left(\sum_{n=0}^{\infty} c_n e^{-in\alpha_\ell} h_n\right)\left(\sum_{r=0}^{M-1} e^{-ir(\alpha-\alpha_\ell)}\right) \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^{M-1} c_n h_n e^{-ir\alpha} \left[\frac{1}{M} \sum_{\ell=0}^{M-1} e^{i(r-n)\alpha_\ell}\right].\end{aligned}$$

By (2.7), the bracketed term equals one precisely when $n \equiv r \pmod M$, and vanishes otherwise. This proves (4.1).

For $0 \leq n < M$, one has $n \bmod M = n$. Hence, these modes are reconstructed exactly, and subtraction from (2.3) gives (4.2). Parseval's identity then yields (4.3).

Because

$$\left|e^{-in\alpha} - e^{-i(n \bmod M)\alpha}\right| \leq 2,$$

we obtain

$$\|G_f(\alpha) - (\mathcal{I}_M G_f)(\alpha)\|_2^2 \leq 4 \sum_{n\geq M} |c_n|^2,$$

which proves (4.5).

Finally, for every $n \geq M$, the integer $n - (n \bmod M)$ is a nonzero multiple of M. Therefore,

$$\int_0^{2\pi} \left| e^{-in\alpha} - e^{-i(n \bmod M)\alpha} \right|^2 \frac{d\alpha}{2\pi} = 2.$$

Integrating (4.3) and applying Tonelli's theorem gives

$$\int_0^{2\pi} \|G_f(\alpha) - (\mathcal{I}_M G_f)(\alpha)\|_2^2 \frac{d\alpha}{2\pi} = 2 \sum_{n \geq M} |c_n|^2,$$

which proves (4.6). □

The pointwise identity above can be refined by grouping the unresolved Hermite coefficients according to their aliasing levels.

Proposition 4.2 (Exact aliasing-level representation)**.** *For $q \geq 1$, define*

$$A_{q,M}(f)^2 := \sum_{r=0}^{M-1} |c_{r+qM}|^2. \tag{4.7}$$

Then, for every $\alpha \in \mathbb{T}$,

$$\|G_f(\alpha) - (\mathcal{I}_M G_f)(\alpha)\|_2^2 = 4 \sum_{q=1}^{\infty} A_{q,M}(f)^2 \sin^2\left(\frac{qM\alpha}{2}\right). \tag{4.8}$$

In particular,

$$T_M(f)^2 = \sum_{q=1}^{\infty} A_{q,M}(f)^2. \tag{4.9}$$

Proof. Every $n \geq M$ has a unique representation

$$n = r + qM, \qquad 0 \leq r \leq M - 1, \quad q \geq 1.$$

Because $n \bmod M = r$,

$$\left| e^{-in\alpha} - e^{-ir\alpha} \right|^2 = \left| e^{-iqM\alpha} - 1 \right|^2 = 4 \sin^2\left(\frac{qM\alpha}{2}\right).$$

Substitution into (4.3), followed by grouping the coefficients with the same q, gives (4.8). Equation (4.9) follows directly from (4.7). □

Formula (4.8) also identifies situations in which the uniform estimate (4.5) is attained exactly.

Corollary 4.3 (Sharpness of the uniform bound)**.** *Suppose that*

$$A_{q,M}(f) = 0 \qquad \text{for every even } q \geq 2. \tag{4.10}$$

Then,

$$\sup_{\alpha \in \mathbb{T}} \|G_f(\alpha) - (\mathcal{I}_M G_f)(\alpha)\|_2 = 2T_M(f). \tag{4.11}$$

More precisely, equality is attained at every

$$\alpha = \frac{(2j+1)\pi}{M}, \qquad j \in \mathbb{Z}.$$

Proof. At $\alpha = (2j+1)\pi/M$,

$$\sin^2\left(\frac{qM\alpha}{2}\right) = \sin^2\left(\frac{q(2j+1)\pi}{2}\right).$$

This quantity equals one whenever q is odd and zero whenever q is even. Under (4.10), Proposition 4.2 therefore gives

$$\|G_f(\alpha) - (\mathcal{I}_M G_f)(\alpha)\|_2^2 = 4\sum_{q=1}^{\infty} A_{q,M}(f)^2 = 4T_M(f)^2.$$

Together with (4.5), this proves (4.11). □

The condition (4.10) is sufficient but not necessary. Let $S_M(f) = \{q \geq 1 : A_{q,M}(f) > 0\}$. If $S_M(f)$ is nonempty, equality in (4.5) holds if and only if all $q \in S_M(f)$ have the same 2-adic valuation $v_2(q)$, where $v_2(q)$ denotes the exponent of 2 dividing q. Indeed, the series in (4.8) converges uniformly in α, as its terms are bounded by $4A_{q,M}(f)^2$, and (4.9) holds. Thus, the squared error is continuous on the compact circle $\mathbb{T}$ and attains its maximum. Because the summands are non-negative, equality in the uniform bound requires $\sin^2(qM\alpha/2) = 1$ at a common α for every $q \in S_M(f)$. Hence, $qM\alpha/\pi$ is an odd integer for all active q. For any two active levels q, q', we obtain $q/q' = m/m'$ with odd integers m, m'. Consequently, $qm' = q'm$, and taking 2-adic valuations gives $v_2(q) = v_2(q')$. Conversely, if every active level has the form $q = 2^k m_q$ with m_q odd and the same k, then $\alpha = \pi/(2^k M)$ makes all active sine-squared factors one. For example, the active levels $q = 2$ and $q = 6$ attain the uniform bound although (4.10) fails. If $S_M(f)$ is empty, the error vanishes identically, and the equivalence holds vacuously.

Remark 4.4. *The aliasing mechanism is therefore governed by two complementary structures. Congruence classes modulo M determine which Hermite modes are folded onto the same reconstructed order frequency, and the levels q determine the oscillation of the reconstruction error in the order variable. Proposition 4.2 makes this dependence explicit, and Corollary 4.3 shows that the constant* 2 *in* (4.5) *cannot be improved in general. The modulo-M folding uses the congruence of order-frequencies at equispaced nodes. This congruence generally fails for nonuniform nodes, so the aliasing identities* (4.1) *and* (4.8) *do not extend directly to that setting.*

5. Stability under perturbed order samples

We now examine the effect of perturbations in the sampled fractional Fourier transforms. Let

$$Y_\ell = G_f(\alpha_\ell) + \eta_\ell, \qquad \ell = 0, \ldots, M-1,$$

where $\eta_\ell \in L^2(\mathbb{R})$ represents an arbitrary perturbation of the sample at the order

$$\alpha_\ell = \frac{2\pi\ell}{M}.$$

Define the perturbed reconstruction by

$$\widetilde{G}_M(\alpha) = \frac{1}{M}\sum_{\ell=0}^{M-1} Y_\ell D_{M-1}^-(\alpha - \alpha_\ell). \tag{5.1}$$

The corresponding perturbation component is

$$E_\eta(\alpha) := \widetilde{G}_M(\alpha) - (\mathcal{I}_M G_f)(\alpha) = \frac{1}{M}\sum_{\ell=0}^{M-1} \eta_\ell D_{M-1}^-(\alpha - \alpha_\ell). \tag{5.2}$$

Theorem 5.1 (Uniform stability and RMS isometry)**.** *Let*

$$L_\ell(\alpha) = \frac{1}{M} D_{M-1}^-(\alpha - \alpha_\ell), \qquad \ell = 0, \dots, M-1,$$

and define the Lebesgue constant

$$\Lambda_M := \sup_{\alpha\in\mathbb{T}} \sum_{\ell=0}^{M-1} |L_\ell(\alpha)|. \tag{5.3}$$

Then,

$$\sup_{\alpha\in\mathbb{T}} \|E_\eta(\alpha)\|_2 \le \Lambda_M \max_{0\le\ell<M} \|\eta_\ell\|_2. \tag{5.4}$$

Moreover,

$$\int_0^{2\pi} \|E_\eta(\alpha)\|_2^2 \frac{d\alpha}{2\pi} = \frac{1}{M}\sum_{\ell=0}^{M-1} \|\eta_\ell\|_2^2. \tag{5.5}$$

Thus, the interpolation operator is an exact isometry from the discrete sample RMS norm to the continuous RMS norm in the order variable.

Proof. From (5.2),

$$E_\eta(\alpha) = \sum_{\ell=0}^{M-1} L_\ell(\alpha)\eta_\ell.$$

Hence,

$$\|E_\eta(\alpha)\|_2 \le \sum_{\ell=0}^{M-1} |L_\ell(\alpha)|\, \|\eta_\ell\|_2,$$

and therefore,

$$\|E_\eta(\alpha)\|_2 \le \left(\max_{0\le\ell<M} \|\eta_\ell\|_2\right) \sum_{\ell=0}^{M-1} |L_\ell(\alpha)|.$$

Taking the supremum over α proves (5.4).

For the RMS identity, write

$$D_{M-1}^-(\alpha - \alpha_\ell) = \sum_{r=0}^{M-1} e^{-ir\alpha} e^{ir\alpha_\ell}.$$

Then,

$$E_\eta(\alpha) = \sum_{r=0}^{M-1} e^{-ir\alpha}\widehat{\eta}_r, \qquad \widehat{\eta}_r = \frac{1}{M}\sum_{\ell=0}^{M-1} \eta_\ell e^{ir\alpha_\ell}.$$

Parseval's identity in the order variable gives

$$\int_0^{2\pi} \|E_\eta(\alpha)\|_2^2 \frac{d\alpha}{2\pi} = \sum_{r=0}^{M-1} |\widehat{\eta}_r|_2^2.$$

Because the normalized discrete Fourier transform is unitary,

$$\sum_{r=0}^{M-1} \|\widehat{\eta}_r\|_2^2 = \frac{1}{M} \sum_{\ell=0}^{M-1} \|\eta_\ell\|_2^2,$$

which proves (5.5). □

Remark 5.2. *The two estimates describe different notions of stability. Uniform sample perturbations are amplified according to the trigonometric Lebesgue constant Λ_M, whereas perturbation energy is preserved exactly in the discrete-to-continuous RMS sense. In particular, no factor depending on M appears in* (5.5).

For the present one-sided Dirichlet kernel, the Lebesgue constant has the asymptotic behavior

$$\Lambda_M = \frac{2}{\pi} \log M + O(1), \qquad M \to \infty.$$

Indeed, the geometric-sum identity gives

$$|D_{M-1}^{-}(\theta)| = \left| \frac{\sin(M\theta/2)}{\sin(\theta/2)} \right|.$$

By translation invariance of the equispaced grid, write $\alpha = 2\pi t/M$ with $0 \le t \le 1$. Then,

$$\sum_{\ell=0}^{M-1} |L_\ell(\alpha)| = \frac{\sin(\pi t)}{M} \sum_{\ell=0}^{M-1} \frac{1}{|\sin(\pi(t-\ell)/M)|},$$

with removable singularities interpreted by continuity. For $0 < t < 1$, put $d_\ell = \min\{|t-\ell|, M-|t-\ell|\}$. Then, $0 < d_\ell \le M/2$, and $|\sin(\pi(t-\ell)/M)| = \sin(\pi d_\ell/M)$. The elementary estimate

$$\frac{1}{M \sin(\pi d/M)} = \frac{1}{\pi d} + O(M^{-1}), \qquad 0 < d \le M/2,$$

holds uniformly in d: After setting $x = \pi d/M$, the function $1/\sin x - 1/x$ is bounded on $(0, \pi/2]$. Separating the two nearest-node distances t and $1-t$, the remaining distances form two harmonic sums, so that

$$\sum_{\ell=0}^{M-1} \frac{1}{d_\ell} = \frac{1}{t} + \frac{1}{1-t} + 2\log M + O(1)$$

uniformly for $0 < t < 1$ as $M \to \infty$. Because $\sin(\pi t)/t$ and $\sin(\pi t)/(1-t)$ are bounded on $(0,1)$, it follows that

$$\sum_{\ell=0}^{M-1} |L_\ell(2\pi t/M)| = \frac{2}{\pi} \sin(\pi t) \log M + O(1)$$

uniformly in $t \in [0,1]$, with the endpoint values obtained by continuity. Taking the supremum proves the stated asymptotic formula. Thus, worst-case uniform amplification grows logarithmically, whereas (5.5) *preserves the sample RMS energy exactly.*

We next combine the perturbation analysis with the aliasing estimates of Section 4. This separates the two sources of error: unresolved Hermite modes and inaccuracies in the available order samples.

Corollary 5.3 (Combined reconstruction error)**.** *Let $f \in L^2(\mathbb{R})$, and let $\widetilde{G}_M$ be defined by* (5.1)*. Then,*

$$\sup_{\alpha\in\mathbb{T}} \|G_f(\alpha) - \widetilde{G}_M(\alpha)\|_2 \le 2T_M(f) + \Lambda_M \max_{0\le\ell<M} \|\eta_\ell\|_2. \tag{5.6}$$

Furthermore,

$$\left(\int_0^{2\pi} \|G_f(\alpha) - \widetilde{G}_M(\alpha)\|_2^2 \frac{d\alpha}{2\pi}\right)^{1/2} \le \sqrt{2}\, T_M(f) + \left(\frac{1}{M}\sum_{\ell=0}^{M-1} \|\eta_\ell\|_2^2\right)^{1/2}. \tag{5.7}$$

Proof. The total reconstruction error decomposes as

$$G_f - \widetilde{G}_M = (G_f - \mathcal{I}_M G_f) - E_\eta.$$

The uniform estimate follows from the triangle inequality, (4.5), and (5.4). The RMS estimate follows from Minkowski's inequality together with (4.6) and (5.5). □

A sharper identity is available when the perturbations do not interact with the unresolved Hermite subspace.

Corollary 5.4 (Exact RMS decomposition)**.** *Assume, in addition, that*

$$\eta_\ell \in \operatorname{span}\{h_0, \dots, h_{M-1}\}, \qquad \ell = 0, \dots, M-1. \tag{5.8}$$

Then,

$$\int_0^{2\pi} \|G_f(\alpha) - \widetilde{G}_M(\alpha)\|_2^2 \frac{d\alpha}{2\pi} = 2T_M(f)^2 + \frac{1}{M}\sum_{\ell=0}^{M-1} \|\eta_\ell\|_2^2. \tag{5.9}$$

Proof. By (4.2),

$$G_f(\alpha) - \mathcal{I}_M G_f(\alpha) \in \overline{\operatorname{span}}\{h_n : n \ge M\}$$

for every α. Under (5.8), (5.2) implies

$$E_\eta(\alpha) \in \operatorname{span}\{h_0, \dots, h_{M-1}\}.$$

The two error components are therefore orthogonal in $L^2(\mathbb{R})$ for every α. Hence,

$$\|G_f(\alpha) - \widetilde{G}_M(\alpha)\|_2^2 = \|G_f(\alpha) - \mathcal{I}_M G_f(\alpha)\|_2^2 + \|E_\eta(\alpha)\|_2^2.$$

Integration over $\mathbb{T}$, followed by (4.6) and (5.5), yields (5.9). □

Remark 5.5. *Corollary 5.3 requires no structural assumption on the perturbations and therefore provides the general stability estimate. Corollary 5.4 shows that, when perturbations are confined to the resolved Hermite subspace, aliasing and perturbation energies separate exactly. Thus, the two principal sources of reconstruction error can be quantified independently.*

The subspace hypothesis (5.8) *is sufficient, but without it, the exact decomposition need not hold. For $M \ge 2$, take $f = ch_M$ and $\eta_\ell = ah_0 + bh_M$ at every sample node, where a, b, c are nonzero complex numbers. Because $e^{-iM\alpha_\ell} = 1$, the equispaced interpolant reproduces the constant sample data, and hence,*

$$\widetilde{G}_M(\alpha) = ah_0 + (c+b)h_M, \qquad G_f(\alpha) = ce^{-iM\alpha}h_M.$$

Orthogonality of the Hermite functions and integration in the order variable yield

$$\frac{1}{2\pi}\int_0^{2\pi} \|G_f(\alpha) - \widetilde{G}_M(\alpha)\|_2^2\, d\alpha = |a|^2 + 2|c|^2 + |b|^2 + 2\,\mathrm{Re}(c\overline{b}).$$

In contrast, the right-hand side of (5.9) *would be* $2|c|^2 + |a|^2 + |b|^2$. *For instance,* $a = b = c = 1$ *gives a nonzero cross term. Thus, perturbations spanning both resolved and unresolved Hermite subspaces can couple with the aliasing error, and* (5.9) *is not guaranteed without* (5.8).

The preceding estimates are deterministic and apply to arbitrary $L^2(\mathbb{R})$-valued sample perturbations. Ideal continuous white noise on $\mathbb{R}$ is not an $L^2(\mathbb{R})$ function, so its use requires a finite-energy model. For example, let

$$\eta_\ell = \sum_{n=0}^{K} \xi_{\ell,n} h_n, \qquad \mathbb{E}[\xi_{\ell,n}] = 0, \qquad \mathbb{E}|\xi_{\ell,n}|^2 = \sigma^2,$$

where the coefficients are independent, zero-mean complex Gaussian variables, independent of the deterministic signal f. By (5.5),

$$\mathbb{E}\left[\frac{1}{2\pi}\int_0^{2\pi} \|E_\eta(\alpha)\|_2^2\, d\alpha\right] = (K+1)\sigma^2.$$

Furthermore, the zero-mean assumption makes the expected cross term between $G_f - \mathcal{I}_M G_f$ and E_η vanish. Therefore, by (4.6),

$$\mathbb{E}\left[\frac{1}{2\pi}\int_0^{2\pi} \|G_f(\alpha) - \widetilde{G}_M(\alpha)\|_2^2\, d\alpha\right] = 2T_M(f)^2 + (K+1)\sigma^2.$$

Because $\|G_f(\alpha)\|_2 = \|f\|_2$, an explicitly defined signal-to-expected-reconstruction-error ratio is

$$\mathrm{SNR}_{\mathrm{rec}} = \frac{\|f\|_2^2}{2T_M(f)^2 + (K+1)\sigma^2},$$

provided the denominator is positive (with its value in decibels given by $10\log_{10}\mathrm{SNR}_{\mathrm{rec}}$). This average-energy ratio should not be identified with Λ_M: the latter controls worst-case pointwise amplification, not the expected RMS noise energy.

6. Numerical illustration

We present two numerical experiments illustrating the exact reconstruction, aliasing estimates, and perturbation stability obtained in the preceding sections. All reported errors are measured in $L^2(\mathbb{R})$; by orthonormality of the Hermite basis, these norms are evaluated directly from the corresponding Hermite coefficients.

Computational cost of order-domain interpolation. Suppose each available transform sample is represented by a vector of P spatial coefficients or grid values, and the orbit is requested at Q new order values. Evaluating the M one-sided Dirichlet weights at one order using the closed geometric-sum formula takes $O(M)$ arithmetic operations (with nodal values handled by continuity), and forming

the corresponding linear combination of M vectors costs $O(MP)$. Thus, direct evaluation on a dense grid of Q orders costs $O(QMP)$, plus storage of $O(MP)$ for the already acquired samples. Evaluating every Dirichlet kernel as a sum of M exponentials instead would increase the weight-computation cost to $O(QM^2)$, but this is unnecessary. If the order samples are equispaced, their M-point discrete Fourier transforms (DFTs) may instead be computed in $O(PM \log M)$ operations, after which values at Q arbitrary orders can still be evaluated by a direct trigonometric sum in $O(QMP)$; special fast evaluation algorithms may change this latter cost. By comparison, if a direct quadrature-based FrFT computes each of P output values from J quadrature nodes, its arithmetic cost is $O(QPJ)$; an FFT-based FrFT implementation may require $O(QP \log P)$ operations, depending on its discretization and accuracy requirements. These are conditional operation counts, not a universal runtime ranking: Sample acquisition, preprocessing, accuracy, and the representation of full L^2-valued measurements must also be accounted for. In particular, interpolation is most relevant when transform samples at the M nodes have already been obtained or are reused for many requested orders.

6.1. Finite Hermite orbit

Consider

$$f_N = \frac{1}{\left(\sum_{n=0}^{N} |a_n|^2\right)^{1/2}} \sum_{n=0}^{N} a_n h_n, \qquad a_n = \frac{e^{0.37in}}{n+1}, \qquad N = 9. \tag{6.1}$$

Thus, f_N contains precisely the Hermite modes $0, \ldots, 9$. The orbit is reconstructed from equispaced order samples using (3.2). The errors were evaluated on a dense grid in $[0, 2\pi)$.

Here, $M = N + 1$ guarantees exact interpolation of the entire order orbit within the specified one-sided trigonometric model. It is not a claim that $N+1$ full $L^2(\mathbb{R})$-valued FrFT observations are necessary to identify the underlying finite Hermite signal.

For $M = N + 1 = 10$, the reconstruction error is at the level of floating point roundoff, as predicted by Theorem 3.1. For $M = 9$, the mode $n = 9$ aliases to the zero-order frequency. In fact, in this case the error is explicitly

$$\|G_{f_N}(\alpha) - (\mathcal{I}_9 G_{f_N})(\alpha)\|_2 = |c_9|\,|e^{-9i\alpha} - 1|,$$

where

$$|c_9| = 8.032795 \times 10^{-2}.$$

Hence, the exact maximum and RMS errors are

$$2|c_9| = 1.606559 \times 10^{-1}, \qquad \sqrt{2}|c_9| = 1.136009 \times 10^{-1},$$

respectively, in agreement with Table 1.

Table 1. Reconstruction of the finite Hermite orbit (6.1).

M	Maximum error	RMS error
9	1.606559×10^{-1}	1.136009×10^{-1}
10	1.997×10^{-15}	7.546×10^{-16}

6.2. Chirped Gaussian: Aliasing and stability

We next consider the normalized chirped Gaussian

$$f_\beta(x) = \pi^{-1/4} \exp\left(-\frac{x^2}{2} + i\beta x^2\right), \qquad \beta = 0.45. \tag{6.2}$$

Because

$$|f_\beta(x)|^2 = \pi^{-1/2} e^{-x^2},$$

one has $\|f_\beta\|_2 = 1$.

The Hermite coefficients

$$c_n = \langle f_\beta, h_n \rangle$$

were computed by 220-point Gauss–Hermite quadrature and retained up to $n = 180$. The resulting coefficient energy was

$$\sum_{n=0}^{180} |c_n|^2 = 0.9999999999999998,$$

so the residual inferred by subtracting the rounded coefficient sum from unity is at the level of machine precision; this observation alone does not certify the truncation error. For this Gaussian, the coefficients also admit the exact independent expression

$$c_{2k+1} = 0, \qquad c_{2k} = \frac{\sqrt{(2k)!}}{2^k k!} \frac{1}{\sqrt{1 - i\beta}} \left(\frac{i\beta}{1 - i\beta}\right)^k, \qquad k \geq 0.$$

It follows by integrating the generating function of the physicists' Hermite polynomials against f_β. The exact coefficients provide a second check on the quadrature-based tail energies. More precisely, putting $\rho = \beta^2/(1 + \beta^2)$ and using $\binom{2k}{k}/4^k \leq (\pi k)^{-1/2}$ for $k \geq 1$, the exact coefficient formula yields the rigorous tail estimate

$$\sum_{n>180} |c_n|^2 = \frac{1}{\sqrt{1 + \beta^2}} \sum_{k=91}^{\infty} \frac{\binom{2k}{k}}{4^k} \rho^k \leq \frac{\rho^{91}}{\sqrt{1 + \beta^2}\,\sqrt{91\pi}(1 - \rho)} < 3 \times 10^{-72} \qquad (\beta = 0.45).$$

The last inequality follows from evaluating this explicit analytical upper bound; it is not inferred from a double-precision subtraction of the numerically retained coefficient energy from unity.

For each M, let

$$T_M(f_\beta) = \left(\sum_{n \geq M} |c_n|^2\right)^{1/2}.$$

The RMS error was evaluated over 65, 536 uniformly distributed order values in $[0, 2\pi)$.

The reported maxima are numerical maxima over the order grid, not assertions of equality in the upper bound. For the chirped Gaussian, the exact coefficient formula shows that all even Hermite modes have nonzero coefficients. Consequently, for $M = 4, 8, 16, 32$, the active aliasing blocks include indices with different 2-adic valuations; the equality characterization in Section 4 therefore excludes exact attainment of $2T_M(f_\beta)$. The near equality at large M reflects the rapid decay of the higher blocks.

For every value of M in Table 2, the computed RMS error agrees with the exact identity

$$\left(\int_0^{2\pi} \|G_{f_\beta}(\alpha) - (\mathcal{I}_M G_{f_\beta})(\alpha)\|_2^2 \frac{d\alpha}{2\pi}\right)^{1/2} = \sqrt{2}\, T_M(f_\beta)$$

to the displayed precision. The maximum errors also satisfy the theoretical estimate

$$\sup_{\alpha\in\mathbb{T}} \|G_{f_\beta}(\alpha) - (\mathcal{I}_M G_{f_\beta})(\alpha)\|_2 \le 2T_M(f_\beta).$$

Table 2. Aliasing errors for the chirped Gaussian (6.2).

M	T_M	$\sqrt{2}T_M$	RMS error	Maximum error
4	1.062783×10^{-1}	1.503003×10^{-1}	1.503003×10^{-1}	2.103714×10^{-1}
8	1.537800×10^{-2}	2.174777×10^{-2}	2.174777×10^{-2}	3.074705×10^{-2}
16	3.711465×10^{-4}	5.248804×10^{-4}	5.248804×10^{-4}	7.422928×10^{-4}
32	2.526105×10^{-7}	3.572452×10^{-7}	3.572452×10^{-7}	5.052210×10^{-7}

For example, at $M = 16$, the theoretical upper bound is

$$2T_{16} = 7.422930 \times 10^{-4},$$

and the computed maximum is

$$7.422928 \times 10^{-4}.$$

Independent validation by the integral FrFT kernel. To check the spectral implementation without using the Hermite expansion as the reference transform, we evaluated the integral-kernel realization

$$(\mathcal{F}_\alpha f)(y) = \left(\frac{1 - i\cot\alpha}{2\pi}\right)^{1/2} \int_{\mathbb{R}} \exp\left(\frac{i\cot\alpha}{2}(x^2 + y^2) - i\,xy\csc\alpha\right) f(x)\,dx, \qquad 0 < \alpha < \pi,$$

using the continuous square-root branch on this interval. This normalization agrees with the Fourier transform at $\alpha = \pi/2$ and the spectral convention in Section 2. For the chirped Gaussian, set $q_\alpha = 1 - 2i\beta - i\cot\alpha$. Direct evaluation of the Gaussian integral gives the independent closed form

$$(\mathcal{F}_\alpha f_\beta)(y) = \pi^{-1/4}\left(\frac{1 - i\cot\alpha}{q_\alpha}\right)^{1/2} \exp\left(\frac{i\cot\alpha}{2}y^2 - \frac{\csc^2\alpha}{2q_\alpha}y^2\right).$$

We compared this formula with a Hermite-series implementation using modes $0, \ldots, 84$, evaluated on 3001 equally spaced spatial points in $[-9, 9]$. Table 3 shows the relative L^2 discrepancy was computed by composite Simpson integration. The results are as follows:

As a separate numerical check of the kernel integral itself, 120-node Gauss–Hermite quadrature with weight $e^{-x^2/2}$ at $\alpha = 1.10$ and spatial points $y = 0, 0.7, 1.4$ gave absolute discrepancies against the displayed Gaussian closed form of 3.14×10^{-16}, 2.29×10^{-16}, and 3.64×10^{-16}, respectively. These checks are specific to the Gaussian test signal and the stated nonexceptional orders; they do not constitute a general convergence result for arbitrary FrFT input data.

Table 3. Relative L^2 discrepancies for different values of α.

α	Relative L^2 discrepancy
0.35	4.28×10^{-16}
1.10	3.59×10^{-16}
2.30	3.50×10^{-16}

We finally verify the perturbation estimates. We take $M = 16$ and use deterministic perturbations confined to the resolved Hermite subspace:

$$\eta_\ell = \varepsilon e^{-2i\alpha_\ell} h_2, \qquad \ell = 0, \dots, 15. \tag{6.3}$$

Thus,

$$\|\eta_\ell\|_2 = \varepsilon$$

for every ℓ, and the discrete sample RMS is exactly ε. From the discrete Fourier orthogonality relation, the corresponding reconstructed perturbation is

$$E_\eta(\alpha) = \varepsilon e^{-2i\alpha} h_2,$$

and therefore,

$$\left(\int_0^{2\pi} \|E_\eta(\alpha)\|_2^2 \frac{d\alpha}{2\pi} \right)^{1/2} = \varepsilon,$$

which realizes exactly the isometry (5.5).

Because (6.3) satisfies the hypothesis of Corollary 5.4, the total RMS error is

$$E_{\text{tot}} = \left(2T_{16}(f_\beta)^2 + \varepsilon^2 \right)^{1/2}. \tag{6.4}$$

The corresponding values are shown in Table 4.

Table 4. Perturbation and total RMS errors for $M = 16$.

ε	Perturbation RMS	Total RMS error
10^{-4}	1.000000×10^{-4}	5.343215×10^{-4}
10^{-3}	1.000000×10^{-3}	1.129380×10^{-3}
10^{-2}	1.000000×10^{-2}	1.001377×10^{-2}

For reference, direct numerical evaluation of the Lebesgue function for $M = 16$ gives

$$\Lambda_{16} \approx 2.727778.$$

Thus, the uniform perturbation estimate of Theorem 5.1 is consistent with the worst-case interpolation behavior, whereas the RMS propagation is exactly isometric.

Finite-energy Gaussian noise and reconstruction signal-to-noise ratio (SNR). We further tested the stochastic model in Section 5 with $M = 16$, $K = 20$, and independent circular complex Gaussian coefficients $\xi_{\ell,n}$ of variance σ^2 for $0 \le \ell < 16$, $0 \le n \le 20$. For each σ, we used 12, 000 independent Monte Carlo trials with NumPy random generator `default_rng(20261022)`. Each trial's order-averaged squared reconstruction error was evaluated exactly from its finite Fourier coefficients, without numerical quadrature in the order variable. The theoretical expected error is

$$\mathbb{E}\,\mathrm{MSE} = 2T_{16}(f_\beta)^2 + 21\sigma^2, \qquad \mathrm{SNR}_{\mathrm{rec,dB}} = 10\log_{10}\left(\frac{1}{\mathbb{E}\,\mathrm{MSE}}\right).$$

The Monte Carlo standard errors of the simulated mean squared errors(MSE), in the same row order, were 1.57×10^{-10}, 1.06×10^{-8}, and 1.04×10^{-6}. The observed discrepancies shown in Table 5 are compatible with sampling variability. This experiment illustrates the expected RMS error and the specified SNR convention; the logarithmically growing Lebesgue constant instead controls deterministic worst-case uniform perturbations.

Table 5. Comparison of expected and simulated mean squared errors (MSE) and expected signal-to-noise ratio (SNR) for different noise levels.

σ	Expected MSE	Simulated mean MSE	Expected SNR (dB)
10^{-4}	4.8549945×10^{-7}	4.8545729×10^{-7}	63.1381
10^{-3}	2.1275499×10^{-5}	2.1260866×10^{-5}	46.7212
10^{-2}	2.1002755×10^{-3}	2.0994444×10^{-3}	26.7772

6.3. Windowed chirp with discontinuities: A slower-decay test

To test the Hermite-tail estimates beyond analytic Gaussians, consider the unit-energy, compactly supported, time-varying-frequency signal

$$f_{\mathrm{wc}}(x) = \frac{1}{\sqrt{2}}\mathbf{1}_{[-1,1]}(x)\exp(i(3x + 2x^2)).$$

This is a synthetic windowed-chirp model, not a measured real-world dataset. Its instantaneous phase derivative is $3 + 4x$ on the support; the jumps at the window boundaries remove the rapid Hermite-coefficient decay enjoyed by the unwindowed analytic Gaussian example. The exact Hermite coefficients are defined by

$$c_n = \frac{1}{\sqrt{2}}\int_{-1}^{1} e^{i(3x+2x^2)}h_n(x)\,dx, \qquad \|f_{\mathrm{wc}}\|_2^2 = 1.$$

We evaluated these integrals by 1024-point Gauss–Legendre quadrature, using the normalized three-term Hermite-function recurrence. Repeating with 2048 nodes changed each of the first 128 computed coefficients by at most 1.8×10^{-13}. By Parseval, the *full* unresolved tail is obtained without truncating its infinite sum:

$$T_M(f_{\mathrm{wc}}) = \left(1 - \sum_{n=0}^{M-1}|c_n|^2\right)^{1/2}.$$

Consequently, Theorem 4.1 gives the exact order-RMS aliasing error $\sqrt{2}T_M$ and the uniform upper bound $2T_M$. Numerical values are reported in Table 6, these are calculated from the Hermite coefficients and the proved identities, *not* from a separate discretization of the full FrFT orbit.

Table 6. Comparison of the exact RMS error and uniform error bound for different values of M.

M	Exact RMS error $\sqrt{2}T_M$	Uniform bound $2T_M$
4	1.06926453	1.51216840
8	0.81846744	1.15748775
16	0.57747119	0.81666758
32	0.37558536	0.53115791
64	0.26701417	0.37761507
128	0.21333113	0.30169578

The slower decay relative to the chirped Gaussian makes the effect of unresolved Hermite modes substantial even for moderately large M. The exact modulo-M identity explains the resulting order-dependent aliasing, but the table alone does not constitute a numerical measurement of a Gibbs overshoot or a real-data performance study.

6.4. Measured nonstationary time series: Atmospheric CO_2

To provide a reproducible real-data illustration, we use the Mauna Loa weekly atmospheric CO_2 observations distributed in the `statsmodels.datasets.co2` public-domain dataset (originally compiled from Mauna Loa Observatory measurements; see the dataset's source documentation). We take the first 256 available nonmissing observations dated from 5 January 1980 to 22 December 1984. These are measured data, not an artificial chirp; they are not FrFT measurements. After subtracting the mean and scaling by the largest absolute sample value, we map the 256 ordered measurements to the equally spaced coordinates $x_j = -6 + 12j/255$. Let p be their piecewise-linear interpolant on $[-6, 6]$, extended by zero outside, and set $f_{\mathrm{CO}_2} = p/\|p\|_2$. This choice defines an explicit compactly supported $L^2(\mathbb{R})$ test signal from observational data; the spatial scaling is a modeling convention, not a physical time unit.

The Hermite coefficients $c_n = \langle f_{\mathrm{CO}_2}, h_n\rangle$ were evaluated by Gauss–Legendre quadrature separately on each linear segment. The maximum difference between 8-point and 16-point segmentwise quadrature for $0 \le n < 256$ was 1.11×10^{-16}. Using the exact L^2 normalization of the piecewise-linear function, the spectral tail was computed from $T_M(f_{\mathrm{CO}_2})^2 = 1 - \sum_{n=0}^{M-1} |c_n|^2$. The RMS identity and uniform bound then give the following:

These values in Table 7 follow from the Hermite coefficients and the proved identities; they are not independent physical FrFT measurements or a validation of the FrFT integral implementation on CO_2 data. The example shows how the estimates behave on an observational nonstationary record after a specified embedding into $L^2(\mathbb{R})$.

Table 7. Comparison of the exact RMS error and uniform error bound for the CO_2 data at different values of M.

M	$\sqrt{2}\,T_M(f_{\mathrm{CO}_2})$	$2T_M(f_{\mathrm{CO}_2})$
8	1.13120949	1.59977180
16	0.45651071	0.64560364
32	0.23818795	0.33684864
64	0.17617992	0.24915603
128	0.15095742	0.21348604

The finite-mode and aliasing experiments illustrate the exact spectral identities established in Sections 3–5. In an additional independent quadrature check, the maximum absolute kernel-evaluation discrepancies over the spatial points $y = 0, 0.7, 1.4$ using 320-point Gauss–Hermite quadrature were 1.07×10^{-14}, 4.12×10^{-15}, and 6.61×10^{-15} for orders 0.35, 1.10, and 2.30, respectively. The 160-point rule at order 0.35 was less accurate (maximum discrepancy 1.87×10^{-7}); thus, numerical quadrature convergence must not be inferred from a single order.

The integral-kernel comparison provides an independent check of the FrFT convention for the chirped Gaussian at the specified orders, and the finite-energy Gaussian-noise experiment illustrates the expected reconstruction-error formula. Neither experiment is a convergence theorem for general signals or a claim of exact attainment of the uniform aliasing bound.

7. Conclusions

We have developed an order-domain sampling framework for fractional Fourier transform orbits based on their Hermite spectral structure. The central observation is that the Hermite index of a signal coincides with the Fourier frequency of its orbit in the transform-order variable. This diagonal correspondence converts reconstruction across fractional orders into a vector-valued trigonometric interpolation problem.

For finite Hermite expansions, we proved that the complete fractional Fourier orbit is recovered exactly from finitely many equispaced order samples. The reconstruction remains exact under an arbitrary common shift of the sampling grid. For general $L^2(\mathbb{R})$ signals, we derived an explicit modulo-M aliasing formula and showed that the unresolved Hermite coefficients, folded modulo M, determine the order-dependent reconstruction error. In addition to the uniform estimate and the exact RMS identity, an aliasing-level decomposition was obtained that describes the dependence of the error on the order parameter and establishes the sharpness of the uniform constant under an explicit spectral condition.

The stability analysis separates interpolation effects from spectral truncation. Arbitrary perturbations of the sampled transforms satisfy a uniform estimate governed by the trigonometric Lebesgue constant, and their discrete sample RMS norm is transferred exactly to the continuous order-domain RMS norm. Combining these results yields certified bounds for the total reconstruction error. When the perturbations lie in the resolved Hermite subspace, the aliasing and perturbation energies are orthogonal and admit an exact RMS decomposition.

The numerical illustrations include finite-mode and chirped-Gaussian tests of the exact interpolation and Hermite-tail identities. A comparison with the integral FrFT kernel checks the spectral

implementation for the specified Gaussian and transform orders; it does not establish convergence for arbitrary signals. A finite-dimensional Gaussian-noise experiment illustrates the expected mean-square reconstruction error and the stated SNR convention. A measured atmospheric CO_2 record and a synthetic discontinuous windowed chirp illustrate the tail-based estimates beyond analytic Gaussian signals. The trigonometric Lebesgue constant grows logarithmically with M, whereas the exact RMS noise identity contains no M-dependent amplification factor. These estimates describe distinct aspects of stability.

These results provide a direct connection between Hermite spectral content and finite sampling of the fractional-order parameter. Natural extensions include nonuniform order sampling, which requires a different interpolation analysis and does not retain the equispaced modulo-M aliasing identities; adaptive selection of order nodes; and reconstruction for other parameterized transforms possessing an appropriate discrete spectral representation.

Use of Generative-AI Tools Declaration

The author declares that no generative artificial intelligence (AI) tools were used in the preparation of this manuscript.

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Conflict of interest

The author declares that he has no competing interests.

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