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Research article

Rational-type contractions in controlled metric spaces and their application to neutron transport

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Abstract: Fixed-point theory has evolved through the generalization of underlying spaces and the development of broader contractive conditions. Motivated by these directions, this manuscript establishes novel fixed-point theorems for modified rational (ψ, φ)-contractions in the setting of controlled metric spaces. The proposed framework extends existing rational-type contraction results by combining the defining properties of ψ and φ with iterative techniques to obtain existence and uniqueness results under weaker assumptions. The applicability of the theoretical findings is demonstrated through an application to a neutron transport integral equation, where the existence and uniqueness of the solution are established. Furthermore, several illustrative examples are presented to validate the proposed results and demonstrate the effectiveness of the developed theoretical framework.

Keywords: fixed-point; integral equations; neutron transport; controlled metric spaces
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1. Introduction and preliminaries

Integral equations play a fundamental role in modeling and analyzing complex physical systems, particularly in nuclear engineering, where they provide powerful mathematical frameworks for describing particle transport phenomena. The importance of integral equations lies in their ability to capture global system behavior through nonlocal operators, making them essential for understanding

steady-state distributions in various scientific and engineering applications. Integral equations of neutron-transport type have attracted significant attention due to their critical applications in nuclear reactor physics, radiation shielding, and medical physics.

Karapinar et al. [1] investigated integral equations using fixed-point theory, while Dahhouch and Marzouki [2] and Zahid et al. [3] studied fixed-point approaches to Volterra-type integral problems. Furthermore, Shah et al. [4] and Aldwoah et al. [5] explored generalized (ψ, ϕ)-contractions and their applications to integral equations and related mathematical models.

The neutron transport phenomenon in homogeneous media at steady-state conditions is mathematically characterized by the following linear Boltzmann equation [9]:

$$\frac{1}{\mu_n}\frac{\partial\rho(\mathbb{b},\mu_n)}{\partial\mathbb{b}} + \Sigma_t(\mathbb{b})\rho(\mathbb{b},\mu_n) = \int_{-1}^{1}\Sigma_s(\mathbb{b},\mu_n' \to \mu_n)\rho(\mathbb{b},\mu_n')d\mu_n' + S(\mathbb{b},\mu_n),$$

where the following hold:

- $\rho(\mathbb{b},\mu_n)$ represents the angular flux of a neutron;
- $\Sigma_t(\mathbb{b})$ denotes the cross-section (total);
- $S(\mathbb{b},\mu_n)$ is the source of the neutron;
- $\Sigma_s(\mathbb{b},\mu_n' \to \mu_n)$ represents the cross-section of differential scattering.

Assuming isotropic scattering and planar symmetry, the original integro-differential equation simplifies to the Peierls integral formulation:

$$\gamma(x) = \lambda\int_0^1 K(x,y)\gamma(y)dy + S(x),$$

where $\gamma(x)$ denotes the scalar neutron flux density. This transformation to a purely integral formulation creates an ideal mathematical foundation for applying fixed-point methods.

The application of fixed-point theory has emerged as a fundamental approach for analyzing integral equations in nuclear science. This mathematical framework provides rigorous methods for demonstrating solution existence and uniqueness for neutron transport problems, thereby guaranteeing physical consistency and operational reliability in nuclear applications. Recent advancements in fixed-point techniques within controlled metric spaces and related generalized frameworks represent natural extensions of classical approaches, particularly valuable for modeling complex neutron transport phenomena with heterogeneous geometries and multiscale characteristics.

The implications of fixed-point theory are particularly significant in nuclear reactor physics and safety engineering. Fixed-point principles enable the verification of solution properties for neutron transport equations, which model both steady-state and dynamic reactor behavior under various operational scenarios. These integral formulations originate from the mathematical modeling of neutron interaction processes including scattering, absorption, and fission events, playing crucial roles in reactor design optimization and safety assessment [6–8].

The utility of fixed-point theory extends throughout mathematical physics for addressing integral problems in particle transport. Through appropriate contractivity conditions, fixed-point methods enable the investigation of solution properties including stability, asymptotic behavior, and regularity features. Neutron transport integral equations of the form

$$\gamma(x) = \lambda\int_0^1 K(x,y)\gamma(y)dy + S(x)$$

find widespread application in nuclear engineering, radiation treatment planning, and particle physics, where they provide mathematically rigorous descriptions of complex physical systems [9–11].

Fixed-point theory ensures that iterative numerical procedures for neutron transport equations will converge to unique physical solutions, providing reliable computational methods in nuclear engineering applications. It is this mathematical foundation that allows for the development of advanced simulation codes for reactor physics, radiation protection, and nuclear medicine, demonstrating the profound connection between pure mathematical theory and applied engineering solution.

In 1969, Boyd and Wong [12] developed a novel class of contraction known as φ-contractions, which generalized the classical Banach contraction principle. Subsequently, in 1997, Guerre-Delabriere and Alber in [13] proposed the idea of weak φ-contractions and extended their applicability to Hilbert spaces. However, Rhoades [14] later investigated that the results established by Alber [13] are valid, as well, in the setting of complete metric spaces , as elaborated below.

Theorem 1.1. *[14] Let us assume that $(\mathscr{B}_m, m_d)$ denote complete metric spaces (CMSes) with $H : \mathscr{B}_m \to \mathscr{B}_m$ defined as a mapping. Suppose that for every $\mathbb{g}, y \in \mathscr{B}_m$,*

$$m_d(H\mathbb{g}, Hy) \leq m_d(\mathbb{g}, y) - \varphi(m_d(\mathbb{g}, y)), \tag{1.1}$$

and $\varphi : [0, \infty) \to [0, \infty)$ is defined as nondecreasing and with no discontinuity and $\varphi(0) = 0$ with $\varphi(u) > 0$ for all $u > 0$. Then, H admits a unique fixed point (UFP).

Furthermore, Dutta et al. [15] presented a generalized version of Theorem 1.1, which is stated below:

Theorem 1.2. *[15] Suppose $(\mathscr{B}_m, m_d)$ denote (CMSes) and assume $H : \mathscr{B}_m \to \mathscr{B}_m$ is a mapping. Consider for all $\mathbb{g}, y \in \mathscr{B}_m$,*

$$\psi(m_d(H\mathbb{g}, Hy)) \leq \psi(m_d(\mathbb{g}, y)) - \varphi(m_d(\mathbb{g}, y)), \tag{1.2}$$

where

1. *$\psi : [0, \infty) \to [0, \infty)$ denote functions which are monotonic, continuous, and nondecreasing along $\psi(u) = 0$ iff $u = 0$.*
2. *$\varphi : [0, \infty) \to [0, \infty)$ denote functions that are lower-semicontinuous along $\varphi(u) = 0 \Leftrightarrow u = 0$.*

Then, H admits a UFP.

For the mappings H and S on the metric space $(\mathscr{B}_m, m_d)$, we consider the below notions:

1.
$$\mathscr{N}(H\mathbb{g}, Hy) = \max\left\{m_d(\mathbb{g}, y), m_d(y, Hy)\frac{1 + m_d(\mathbb{g}, H\mathbb{g})}{1 + m_d(\mathbb{g}, y)}\right\};$$

2.
$$\mathscr{M}(H\mathbb{g}, Sy) = \max\left\{m_d(\mathbb{g}, y), m_d(\mathbb{g}, H\mathbb{g}), m_d(y, Hy), \frac{m_d(y, H\mathbb{g}) + m_d(\mathbb{g}, Hy)}{2}\right\};$$

3.

$$\mathscr{M}_1(H\mathbb{g}, S\,y) = \max\left\{m_d(\mathbb{g}, y), m_d(\mathbb{g}, H\mathbb{g}), m_d(y, Hy), \frac{m_d(y, H\mathbb{g}) + m_d(\mathbb{g}, Hy)}{2}, \right.$$
$$\left. \frac{m_d(\mathbb{g}, H\mathbb{g}) + m_d(y, Hy)}{2}, m_d(y, Hy)\frac{1 + m_d(\mathbb{g}, H\mathbb{g})}{1 + m_d(\mathbb{g}, y)}, m_d(\mathbb{g}, H\mathbb{g})\frac{1 + m_d(y, H\mathbb{g})}{1 + m_d(\mathbb{g}, y)}\right\};$$

4.

$$\mathscr{M}_2(H\mathbb{g}, Hy) = \max\left\{m_d(\mathbb{g}, y), m_d(\mathbb{g}, H\mathbb{g}), m_d(y, Hy), \frac{m_d(\mathbb{g}, H\mathbb{g}) + m_d(\mathbb{g}, Hy)}{2}, \right.$$
$$\left. \frac{m_d(\mathbb{g}, Hy) + m_d(y, Hy)}{2}, m_d(y, Hy)\frac{1 + m_d(\mathbb{g}, H\mathbb{g})}{1 + m_d(\mathbb{g}, y)}, m_d(\mathbb{g}, H\mathbb{g})\frac{1 + m_d(y, Hy)}{1 + m_d(\mathbb{g}, y)}\right\}.$$

Furthermore, in 2009, Q. Zhang et al. [16] established the following extension of Theorem 1.1.

Theorem 1.3. *[16] Let us consider $(\mathscr{B}_m, m_d)$ as a CMS and assume that $S, H : \mathscr{B}_m \to \mathscr{B}_m$ denotes two mappings. Suppose that for all $\mathbb{g}, y \in \mathscr{B}_m$,*

$$m_d(Hy, S\,\mathbb{g}) \leq \mathscr{M}(Hy, S\,\mathbb{g}) - \varphi(\mathscr{M}(Hy, S\,\mathbb{g})), \tag{1.3}$$

and φ is defined in Theorem 1.2. Then, both the maps will admit a unique and common fixed-point.

Additionally, Doric et al. [17] derived a corresponding common fixed-point theorem for a pair of mappings, thereby broadening the scope of the previously mentioned findings.

Theorem 1.4. *[17] Let us assume $(\mathscr{B}_m, m_d)$ as a CMS and assume that $H, S : \mathscr{B}_m \to \mathscr{B}_m$ denotes two mappings. Consider for every $\mathbb{g}, y \in \mathscr{B}_m$,*

$$\psi(m_d(Hy, S\,\mathbb{g})) \leq \psi(\mathscr{M}(Hy, S\,\mathbb{g})) - \varphi(\mathscr{M}(Hy, S\,\mathbb{g})), \tag{1.4}$$

whereas ψ and φ are described in Theorem 1.2. Then, both maps will admit a unique common fixed point.

Likewise, He [18] established common fixed theorems in 2017 for pairs of mappings satisfying a generalized weak contraction condition of (ψ, φ)-type within a CMS.

Theorem 1.5. *[18] Let us assume that $(\mathscr{B}_{\mathbb{m}}, m_d)$ denotes a CMS and consider that $H, S : \mathscr{B}_m \to \mathscr{B}_m$ denotes two mappings. Suppose that for every $\mathbb{g}, y \in \mathscr{B}_m$, $\frac{1}{2}\min\{m_d(\mathbb{g}, H\mathbb{g}), m_d(y, S\,y)\} \leq m_d(\mathbb{g}, y)$ implies*

$$\psi(m_d(H\mathbb{g}, S\,y)) \leq \psi(\mathscr{M}(H\mathbb{g}, S\,y)) - \varphi(\mathscr{M}(H\mathbb{g}, S\,y)), \tag{1.5}$$

whereas ψ and φ are described in Theorem 1.2. Then, both maps will admit a unique common fixed point.

More recently in 2020, researchers [19] established the following result for generalized (ψ, φ)-Suzuki weak contractions via a rational-type expression framework.

Theorem 1.6. *[19] Let us assume that $\mathscr{B}_m$ denotes a CMS and consider that $H : \mathscr{B}_m \to \mathscr{B}_m$ denotes a mapping. Consider that for every $\mathbb{g}, y \in \mathscr{B}_m$,*

$$\frac{1}{2} m_d(y, Hy) \leq m_d(y, \mathbb{g})$$

implies

$$\psi(m_d(Hy, H\mathbb{g})) \leq \psi(\mathscr{N}(Hy, H\mathbb{g})) - \varphi(\mathscr{N}(Hy, H\mathbb{g})), \tag{1.6}$$

whereas ψ and φ are described in Theorem 1.2. Then, H will admit a unique common fixed-point

Most recently, Arya et al. [20] established results for a common fixed-point of the generalized (ψ, φ)-rational contractive mappings with rational-type expressions in a CMS.

Theorem 1.7. *[20] Let us assume $\mathscr{B}_m$ denotes a CMS and assume that $H, S : \mathscr{B}_m \to \mathscr{B}_m$ denotes two mappings. Consider that for every $\mathbb{g}, y \in \mathscr{B}_m$,*

$$\psi(m_d(H\mathbb{g}, Sy)) \leq \psi(\mathscr{M}_1(H\mathbb{g}, Sy)) - \varphi(\mathscr{M}_1(H\mathbb{g}, Sy)), \tag{1.7}$$

whereas ψ and φ are defined in Theorem 1.2. Then, both the maps will admit a unique common fixed point.

The contraction principle has undergone substantial generalizations through various extensions of the underlying metric space structure. Around 1989, the well-known Bakhtin [21] alongside Czerwik [22] forwarded the idea for b-metric spaces by variation in the triangular inequality through the incorporation of a constant $s \geq 1$. Pursuing this line of inquiry, Kamran [23] established extended b-metric spaces where rather than the constant $s \geq 1$, a function is used, and hence, it is a more general context. This seminal extension created a huge amount of research activity in extended b-metric spaces. For instance, E. Karapınar et al. [24] established results on cyclic orbital with cyclic orbitals F-contractions and their applications. Additionally, Wasfi in [25, 26] discussed new types of contraction mappings in such a context. Similarly, Javed with Abdeljawad [27] worked on fixed-point theorems in the context of an orthogonal extended b-metric. Building upon these developments, a further generalization was introduced by Mlaiki et al. in [28], who conceived the structure of controlled metric-type spaces. In this innovative framework, the control function operates distinctly on each term of the triangular inequality, offering a broader and more versatile structure that encompasses the previous extensions.

Remark 1.8. *From this point onward, the notation $\mathbb{m}_c$ will be used exclusively for the controlled metric, whereas $\mathbb{m}_d$ will denote the metric on an ordinary metric space whenever required.*

Definition 1.9. *[28] Let $\mathbb{G} \neq \varnothing$ denote a set (nonempty); the structure defined as $\mathbb{m}_e : \mathbb{G} \times \mathbb{G} \to [0, +\infty)$ is labeled as a controlled metric space. Assume that the conditions below hold true:*

1. $\mathbb{m}_c(w, o) = 0$ *means* $w = o \ \forall \ w, o \in \mathbb{G}$.
2. $\mathbb{m}_c(w, o) = \mathbb{m}_c(o, w) \ \forall \ w, o \in \mathbb{G}$.
3. $\mathbb{m}_c(w, o) \leq \alpha(w, p)\mathbb{m}_c(w, p) + \alpha(p, o)\mathbb{m}_c(p, o)$.

Definition 1.10. *[28] Suppose that $(\mathbb{G}, \mathbb{m}_c)$ as a controlled metric space. A sequence $s_n \subset \mathbb{G}$ is labeled as follows:*

(a) ***Convergent*** *to a point $q \in \mathbb{G}$ when for each $0 <\ni 0$, there is an $N \in \mathbb{N}$ such that $\mathbb{m}_c(s_n, q) <\ni$ for every $n \geq N$. We denote this by $s_n \to q$ or $\lim n \to \infty s_n = q$.*

(b) ***Cauchy*** *when for each $\ni > 0$, there exists $N \in \mathbb{N}$ such that $\mathbb{m}_c(s_m, s_n) <\ni$ for all $m, n \geq N$.*

Definition 1.11. *[28] Let $(\mathbb{G}, \mathbb{m}_c)$ denote a controlled metric type space; then, it is deeded as complete if every Cauchy sequence converges in $\mathbb{G}$.*

Example 1.12. *[28] Take $\mathbb{G} = \{0, 1, 2\}$. Let $\mathbb{m}_c$ be described as:*

$$\mathbb{m}_c(0, 0) = \mathbb{m}_c(1, 1) = \mathbb{m}_c(2, 2) = 0$$

and

$$\mathbb{m}_c(0, 1) = \mathbb{m}_c(1, 0) = 1, \quad \mathbb{m}_c(0, 2) = \mathbb{m}_c(2, 0) = \frac{1}{2}, \quad \mathbb{m}_c(1, 2) = \mathbb{m}_c(2, 1) = \frac{2}{5}.$$

Select $\sigma : \mathbb{G} \times \mathbb{G} \to [1, \infty)$ as symmetric and described by:

$$\sigma(0, 0) = \sigma(1, 1) = \sigma(2, 2) = \sigma(0, 2) = 1, \quad \sigma(1, 2) = \frac{5}{4}, \quad \sigma(0, 1) = \frac{11}{10}.$$

It is trivial to observe that $\mathbb{m}_c$ is a controlled metric type.
Noting that

$$\mathbb{m}_c(0, 1) = 1 > \frac{99}{100} = \sigma(0, 1)[\mathbb{m}_c(0, 2) + \mathbb{m}_c(2, 1)],$$

$\mathbb{m}_c$ turns out to be not extended b-metric by taking the similar function $\sigma = \theta$.

This manuscript introduces modified (ψ, φ)-rational contractions in the structure of controlled metric-type spaces. The (ψ, φ)-contractions provide a significant extension of traditional contraction mappings, enabling broader analysis of convergence and fixed-point existence in more comprehensive settings. The derived results are applied to ensure the existence and uniqueness of solutions to the neutron transport equation, which models particle behavior in nuclear reactors and radiation shielding. Additionally, examples are provided to strengthen the practicality of our established results.

1.1. ***Comparison with existing literature***

The proposed results differ substantially from the existing literature on generalized (ψ, φ)-contractions and rational-type contractions. In particular, the novelty of the present manuscript can be summarized as follows:

- Dutta et al. [15] introduced generalized (ψ, ϕ)-contractions in complete metric spaces without incorporating rational expressions or control functions.
- The work in [19] established generalized (ψ, φ)-Suzuki weak contractions involving the rational expression $\mathscr{N}(\cdot, \cdot)$ under a Suzuki-type admissibility condition in metric spaces.
- Arya et al. [20] considered common fixed-point results for generalized (ψ, φ)-rational contractions using two self-mappings and the rational expression $\mathscr{M}_1(\cdot, \cdot)$.
- In contrast, the present work introduces a modified rational expression $\mathscr{M}_2(\cdot, \cdot)$ together with an additional control parameter $\xi \in (0, 1)$ and a new asymptotic condition involving the control function σ. Consequently, our framework is developed in controlled metric spaces, which properly generalize several existing metric-type structures.

- Furthermore, unlike [19, 20], our approach combines modified rational (ψ, φ)-contractions with the geometry of controlled metric spaces and demonstrates their applicability to neutron-transport models.

Table 1 summarizes the differences between the existing fixed-point results and the present study in terms of the underlying spaces, mappings, rational expressions, and applications.

Table 1. Comparison of the present results with existing literature.

Reference	Space	Mapping	Rational term	Application
[15]	Metric	Single	None	No
[19]	Metric	Single	$\mathscr{N}$	No
[20]	Metric	Two maps	$\mathscr{M}_1$	No
Present work	Controlled metric	Single	$\mathscr{M}_2$	Neutron transport

2. Main results

We now leverage the concepts introduced in the previous section to prove new fixed-point theorems. Our primary finding is the following.

Theorem 2.1. *Let $\mathbb{G}$ denote complete controlled metric spaces. Suppose that $H : \mathbb{G} \to \mathbb{G}$ denotes the mapping. For every $\mathbb{g}, \mathbb{b} \in \mathbb{G}$, $\xi \in (0, 1)$, and*

$$\psi(\mathbb{m}_c(H\mathbb{g}, H\mathbb{b})) \leq \psi(\xi \mathscr{M}_2(H\mathbb{g}, H\mathbb{b})) - \varphi(\mathscr{M}_2(H\mathbb{g}, H\mathbb{b})), \tag{2.1}$$

whereas Theorem 1.2 is defined by ψ and φ.
Suppose that

$$\limsup_{i\to\infty} \sup_{m>i} \left[\frac{\sigma(\mathbb{b}_{i+1}, \mathbb{b}_{i+2})}{\sigma(\mathbb{b}_i, \mathbb{b}_{i+1})} \sigma(\mathbb{b}_{i+1}, \mathbb{b}_m) \right] < \frac{1}{\xi}. \tag{2.2}$$

In addition, presume that, for each $\mathbb{b} \in \mathbb{G}$, we have that

$$\lim_{n\to\infty} \sigma(\mathbb{b}_n, \mathbb{b}) \quad \textit{and} \quad \lim_{n\to\infty} \sigma(\mathbb{b}, \mathbb{b}_n) \quad \textit{exist and are finite}.$$

Consequently, H admits a UFP.

Proof. The sequence $\{\mathbb{b}_n\}$ is defined by the recurrence relation $\mathbb{b}_{n+1} = H\mathbb{b}_n$ for all $n \in \mathbb{N}$. When there exists an index $n_0 \in \mathbb{N}$ such that $\mathbb{b}_{n_0+1} = \mathbb{b}_{n_0}$, then $\mathbb{b}_{n_0}$ is a fixed-point of H, and the proof is done. Therefore, we may assume that $\mathbb{b}_{n+1} \neq \mathbb{b}_n$ for all $n \in \mathbb{N}$. Now, if $\mathbb{g} = \mathbb{b}_n$, and $y = \mathbb{b}_{n-1}$, then by 2.1,

$$\psi\left(\mathbb{m}_c\left(H\mathbb{b}_n, H\mathbb{b}_{n-1}\right)\right) \leq \psi\left(\xi \mathscr{M}_2\left(H\mathbb{b}_n, H\mathbb{b}_{n-1}\right)\right) - \varphi\left(\mathscr{M}_2\left(H\mathbb{b}_n, H\mathbb{b}_{n-1}\right)\right),$$

where

$$\mathscr{M}_2(\mathbb{b}_n, \mathbb{b}_{n-1}) = \max\Bigg\{ \mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n-1}), \mathbb{m}_c(\mathbb{b}_n, H\mathbb{b}_n), \mathbb{m}_c(\mathbb{b}_{n-1}, H\mathbb{b}_{n-1}),$$
$$\frac{\mathbb{m}_c(\mathbb{b}_n, H\mathbb{b}_n) + \mathbb{m}_c(\mathbb{b}_n, H\mathbb{b}_{n-1})}{2}, \frac{\mathbb{m}_c(\mathbb{b}_n, H\mathbb{b}_{n-1}) + \mathbb{m}_c(\mathbb{b}_{n-1}, H\mathbb{b}_{n-1})}{2},$$

$$\mathbb{m}_c(\mathbb{b}_{n-1}, H\mathbb{b}_{n-1})\left(\frac{1+\mathbb{m}_c(\mathbb{b}_n, H\mathbb{b}_n)}{1+\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n-1})}\right), \mathbb{m}_c(\mathbb{b}_n, H\mathbb{b}_n)\left(\frac{1+\mathbb{m}_c(\mathbb{b}_{n-1}, H\mathbb{b}_{n-1})}{1+\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n-1})}\right)\Bigg\}.$$

$$\begin{aligned}\mathscr{M}_2(\mathbb{b}_n, \mathbb{b}_{n-1}) = \max\Bigg\{&\mathbb{m}_c(\mathbb{b}_{n-1}, \mathbb{b}_n), \mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1}), \mathbb{m}_c(\mathbb{b}_{n-1}, \mathbb{b}_n),\\ &\frac{\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1})}{2}, \frac{\mathbb{m}_c(\mathbb{b}_{n-1}, \mathbb{b}_n)}{2}, \mathbb{m}_c(\mathbb{b}_{n-1}, \mathbb{b}_n)\left(\frac{1+\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1})}{1+\mathbb{m}_c(\mathbb{b}_{n-1}, \mathbb{b}_n)}\right), \mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1})\Bigg\}.\end{aligned}$$

$$\mathscr{M}_2(\mathbb{b}_n, \mathbb{b}_{n-1}) = \max\left\{\mathbb{m}_c(\mathbb{b}_{n-1}, \mathbb{b}_n), \mathbb{m}_c(\mathbb{b}_{n-1}, \mathbb{b}_n)\left(\frac{1+\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1})}{1+\mathbb{m}_c(\mathbb{b}_{n-1}, \mathbb{b}_n)}\right), \mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1})\right\}.$$

Because

$$A_n\frac{1+B_n}{1+A_n} \leq \max\{A_n, B_n\}.$$

Thus,

$$\mathscr{M}_2(\mathbb{b}_n, \mathbb{b}_{n-1}) = \max\left\{\mathbb{m}_c(\mathbb{b}_{n-1}, \mathbb{b}_n), \mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1})\right\}.$$

From (2.3), we have

$$\psi(\mathbb{m}_c(\mathbb{b}_{n+1}, \mathbb{b}_n)) \leq \psi(\xi \max\{\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1}), \mathbb{m}_c(\mathbb{b}_{n-1}, \mathbb{b}_n)\}) - \varphi(\max\{\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1}), \mathbb{m}_c(\mathbb{b}_{n-1}, \mathbb{b}_n)\}). \quad (2.3)$$

Suppose, on the contrary, that

$$\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1}) > \mathbb{m}_c(\mathbb{b}_{n-1}, \mathbb{b}_n).$$

Then, the maximum in (2.3) is attained at $\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1})$, and therefore,

$$\psi(\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1})) \leq \psi(\xi\{\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1})\}) - \varphi(\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1})).$$

Because $\varphi(t) > 0$ for every $t > 0$, we obtain

$$\psi(\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1})) < \psi(\xi\{\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1})\}).$$

On the other hand, $\xi \in (0, 1)$ implies

$$(\xi\{\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1})\}) < \mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1}),$$

and because ψ is nondecreasing, it follows that

$$\psi(\xi\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1})) \leq \psi(\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1})),$$

which is a contradiction. Hence,

$$\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1}) \leq \mathbb{m}_c(\mathbb{b}_{n-1}, \mathbb{b}_n).$$

Consequently,

$$\psi(\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1})) \leq \psi(\xi\mathbb{m}_c(\mathbb{b}_{n-1}, \mathbb{b}_n)) - \varphi(\mathbb{m}_c(\mathbb{b}_{n-1}, \mathbb{b}_n)).$$

Because φ is non-negative,

$$\psi(\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1})) \leq \psi(\xi\mathbb{m}_c(\mathbb{b}_{n-1}, \mathbb{b}_n)). \tag{2.4}$$

Employing the defining property of ψ, it follows that for all $n \in \mathbb{N}$,

$$\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1}) \leq \xi\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n-1}).$$

Inductively,

$$\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1}) \leq \xi^n\mathbb{m}_c(\mathbb{b}_0, \mathbb{b}_1). \tag{2.5}$$

Because

$$\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1}) \leq \mathbb{m}_c(\mathbb{b}_{n-1}, \mathbb{b}_n),$$

the sequence $\{\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1})\}_{n\geq 0}$ is nonincreasing and bounded below by zero. Hence, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1}) = \lim_{n\to\infty} \mathbb{m}_c(\mathbb{b}_{n-1}, \mathbb{b}_n) = r. \tag{2.6}$$

From the previous inequality, we have

$$\psi(\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1})) \leq \psi(\xi\mathbb{m}_c(\mathbb{b}_{n-1}, \mathbb{b}_n)) - \varphi(\mathbb{m}_c(\mathbb{b}_{n-1}, \mathbb{b}_n)).$$

Taking the limit superior on both sides and using

$$\limsup_{n\to\infty}(u_n - v_n) \leq \limsup_{n\to\infty} u_n - \liminf_{n\to\infty} v_n,$$

we obtain

$$\psi(r) \leq \psi(\xi r) - \liminf_{n\to\infty} \varphi(\mathbb{m}_c(\mathbb{b}_{n-1}, \mathbb{b}_n)).$$

Here, the continuity of ψ and the lower semicontinuity of φ yield

$$\psi(r) \leq \psi(\xi r) - \varphi(r).$$

Because $\xi \in (0, 1)$, and ψ is nondecreasing,

$$\psi(\xi r) \leq \psi(r).$$

Therefore,

$$\psi(r) \leq \psi(r) - \varphi(r),$$

which implies $\varphi(r) \leq 0$. Because φ is non-negative, we have $\varphi(r) = 0$. By the defining property of φ, this gives $r = 0$. Consequently,

$$\lim_{n\to\infty} \mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1}) = 0.$$

Now, for some $m > n$ and 1.9, we have

$$\begin{aligned}\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_m) &\leq \sigma(\mathbb{b}_n, \mathbb{b}_{n+1})\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1}) + \sigma(\mathbb{b}_{n+1}, \mathbb{b}_m)\mathbb{m}_c(\mathbb{b}_{n+1}, \mathbb{b}_m)\\ &\leq \sigma(\mathbb{b}_n, \mathbb{b}_{n+1})\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1}) + \sigma(\mathbb{b}_{n+1}, \mathbb{b}_m)\sigma(\mathbb{b}_{n+1}, \mathbb{b}_{n+2})\mathbb{m}_c(\mathbb{b}_{n+1}, \mathbb{b}_{n+2})\end{aligned}$$

$$
\begin{aligned}
&+ \sigma(\mathbb{b}_{n+1}, \mathbb{b}_m)\sigma(\mathbb{b}_{n+2}, \mathbb{b}_m)\mathbb{m}_c(\mathbb{b}_{n+2}, \mathbb{b}_m) \\
&\leq \sigma(\mathbb{b}_n, \mathbb{b}_{n+1})\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1}) + \sigma(\mathbb{b}_{n+1}, \mathbb{b}_m)\sigma(\mathbb{b}_{n+1}, \mathbb{b}_{n+2})\mathbb{m}_c(\mathbb{b}_{n+1}, \mathbb{b}_{n+2}) \\
&\quad + \sigma(\mathbb{b}_{n+1}, \mathbb{b}_m)\sigma(\mathbb{b}_{n+2}, \mathbb{b}_m)\sigma(\mathbb{b}_{n+2}, \mathbb{b}_{n+3})\mathbb{m}_c(\mathbb{b}_{n+2}, \mathbb{b}_{n+3}) \\
&\quad + \sigma(\mathbb{b}_{n+1}, \mathbb{b}_m)\sigma(\mathbb{b}_{n+2}, \mathbb{b}_m)\sigma(\mathbb{b}_{n+3}, \mathbb{b}_m)\mathbb{m}_c(\mathbb{b}_{n+3}, \mathbb{b}_m) \\
&\leq \quad \cdots \\
&\leq \sigma(\mathbb{b}_n, \mathbb{b}_{n+1})\mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\sigma(\mathbb{b}_j, \mathbb{b}_m)\right)\sigma(\mathbb{b}_i, \mathbb{b}_{i+1})\mathbb{m}_c(\mathbb{b}_i, \mathbb{b}_{i+1}) \\
&\quad + \prod_{k=n+1}^{m-1}\sigma(\mathbb{b}_k, \mathbb{b}_m)\mathbb{m}_c(\mathbb{b}_{m-1}, \mathbb{b}_m) \\
&\leq \sigma(\mathbb{b}_n, \mathbb{b}_{n+1})\xi^n\mathbb{m}_c(\mathbb{b}_0, \mathbb{b}_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\sigma(\mathbb{b}_j, \mathbb{b}_m)\right)\sigma(\mathbb{b}_i, \mathbb{b}_{i+1})\xi^i\mathbb{m}_c(\mathbb{b}_0, \mathbb{b}_1) \\
&\quad + \prod_{i=n+1}^{m-1}\sigma(\mathbb{b}_i, \mathbb{b}_m)\xi^{m-1}\mathbb{m}_c(\mathbb{b}_0, \mathbb{b}_1) \\
&\leq \sigma(\mathbb{b}_n, \mathbb{b}_{n+1})\xi^n\mathbb{m}_c(\mathbb{b}_0, \mathbb{b}_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\sigma(\mathbb{b}_j, \mathbb{b}_m)\right)\sigma(\mathbb{b}_i, \mathbb{b}_{i+1})\xi^i\mathbb{m}_c(\mathbb{b}_0, \mathbb{b}_1) \\
&\quad + \left(\prod_{i=n+1}^{m-1}\sigma(\mathbb{b}_i, \mathbb{b}_m)\right)\xi^{m-1}\sigma(\mathbb{b}_{m-1}, \mathbb{b}_m)\mathbb{m}_c(\mathbb{b}_0, \mathbb{b}_1) \\
&= \sigma(\mathbb{b}_n, \mathbb{b}_{n+1})\xi^n\mathbb{m}_c(\mathbb{b}_0, \mathbb{b}_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\sigma(\mathbb{b}_j, \mathbb{b}_m)\right)\sigma(\mathbb{b}_i, \mathbb{b}_{i+1})\xi^i\mathbb{m}_c(\mathbb{b}_0, \mathbb{b}_1) \\
&\leq \sigma(\mathbb{b}_n, \mathbb{b}_{n+1})\xi^n\mathbb{m}_c(\mathbb{b}_0, \mathbb{b}_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=0}^{i}\sigma(\mathbb{b}_j, \mathbb{b}_m)\right)\sigma(\mathbb{b}_i, \mathbb{b}_{i+1})\xi^i\mathbb{m}_c(\mathbb{b}_0, \mathbb{b}_1).
\end{aligned}
$$

For each m, define

$$
a_{i,m} = \left(\prod_{j=0}^{i}\sigma(\mathbb{b}_j, \mathbb{b}_m)\right)\sigma(\mathbb{b}_i, \mathbb{b}_{i+1})\xi^i.
$$

Then,

$$
\frac{a_{i+1,m}}{a_{i,m}} = \xi\frac{\sigma(\mathbb{b}_{i+1}, \mathbb{b}_{i+2})}{\sigma(\mathbb{b}_i, \mathbb{b}_{i+1})}\sigma(\mathbb{b}_{i+1}, \mathbb{b}_m).
$$

By (2.2),

$$
\limsup_{i\to\infty}\sup_{m>i}\left[\xi\frac{\sigma(\mathbb{b}_{i+1}, \mathbb{b}_{i+2})}{\sigma(\mathbb{b}_i, \mathbb{b}_{i+1})}\sigma(\mathbb{b}_{i+1}, \mathbb{b}_m)\right] < 1.
$$

Hence, there exist $q \in (0, 1)$ and $N \in \mathbb{N}$ such that

$$
a_{i+1,m} \leq q\, a_{i,m}, \qquad i \geq N, \quad m > i + 1.
$$

For the fixed index N, the assumption

$$
\lim_{m\to\infty}\sigma(\mathbb{b}_j, \mathbb{b}_m)
$$

exists and is finite for every fixed j. Therefore, each sequence $\{\sigma(\mathbb{b}_j, \mathbb{b}_m)\}_{m\geq 1}$ is bounded. Because only finitely many factors occur in $a_{N,m}$, there exists a constant $C_N > 0$ such that

$$a_{N,m} \leq C_N$$

for every $m > N$. Repeated application of $a_{i+1,m} \leq q a_{i,m}$ therefore gives

$$a_{i,m} \leq C_N q^{i-N}, \qquad i \geq N, \quad m > i.$$

Consequently,

$$\begin{aligned}\sum_{i=n+1}^{m-1} a_{i,m} &\leq C_N \sum_{i=n+1}^{m-1} q^{i-N} \\ &\leq \frac{C_N q^{n+1-N}}{1-q} \longrightarrow 0 \qquad \text{as } n \to \infty,\end{aligned}$$

uniformly with respect to $m > n$. It remains to consider the first term

$$\xi^n \sigma(\mathbb{b}_n, \mathbb{b}_{n+1}).$$

Set

$$c_n = \xi^n \sigma(\mathbb{b}_n, \mathbb{b}_{n+1}).$$

Then,

$$\frac{c_{n+1}}{c_n} = \xi \frac{\sigma(\mathbb{b}_{n+1}, \mathbb{b}_{n+2})}{\sigma(\mathbb{b}_n, \mathbb{b}_{n+1})}.$$

Because $\sigma(\mathbb{b}_{n+1}, \mathbb{b}_m) \geq 1$, Condition (2.2) implies

$$\limsup_{n\to\infty} \frac{c_{n+1}}{c_n} < 1.$$

Hence,

$$\lim_{n\to\infty} \xi^n \sigma(\mathbb{b}_n, \mathbb{b}_{n+1}) = 0.$$

Therefore,

$$\lim_{n,m\to\infty} \mathbb{m}_c(\mathbb{b}_n, \mathbb{b}_m) = 0.$$

Therefore, $\{\mathbb{b}_n\}$ is a Cauchy sequence in the complete controlled metric space $(\mathbb{G}, \mathbb{m}_c)$. Hence, there exists some $g \in \mathbb{G}$ such that

$$\lim_{n\to\infty} \mathbb{m}_c(\mathbb{b}_n, g) = 0.$$

Thus,

$$\mathbb{b}_n \longrightarrow g.$$

We shall now show that g is a fixed-point of H.

Utilizing (2.1), we have

$$\psi\left(\mathbb{m}_c\left(Hg, H\mathbb{b}_n\right)\right) \leq \psi\left(\xi \mathscr{M}_2\left(Hg, H\mathbb{b}_n\right)\right) - \varphi\left(\mathscr{M}_2\left(Hg, H\mathbb{b}_n\right)\right)$$

$$
\begin{aligned}
= \ & \psi\Bigg(\xi \max\Bigg\{\mathbb{m}_c(g,\mathbb{b}_n), \mathbb{m}_c(g,Hg), \mathbb{m}_c(\mathbb{b}_n,H\mathbb{b}_n),\\
& \frac{\mathbb{m}_c(g,Hg)+\mathbb{m}_c(g,H\mathbb{b}_n)}{2}, \frac{\mathbb{m}_c(g,H\mathbb{b}_n)+\mathbb{m}_c(\mathbb{b}_n,H\mathbb{b}_n)}{2},\\
& \mathbb{m}_c(\mathbb{b}_n,H\mathbb{b}_n)\left(\frac{1+\mathbb{m}_c(g,Hg)}{1+\mathbb{m}_c(g,\mathbb{b}_n)}\right), \mathbb{m}_c(g,Hg)\left(\frac{1+\mathbb{m}_c(\mathbb{b}_n,H\mathbb{b}_n)}{1+\mathbb{m}_c(g,\mathbb{b}_n)}\right)\Bigg\}\Bigg)\\
& -\varphi\Bigg(\max\Bigg\{\mathbb{m}_c(g,\mathbb{b}_n), \mathbb{m}_c(g,Hg), \mathbb{m}_c(\mathbb{b}_n,H\mathbb{b}_n),\\
& \frac{\mathbb{m}_c(g,Hg)+\mathbb{m}_c(g,H\mathbb{b}_n)}{2}, \frac{\mathbb{m}_c(g,H\mathbb{b}_n)+\mathbb{m}_c(\mathbb{b}_n,H\mathbb{b}_n)}{2},\\
& \mathbb{m}_c(\mathbb{b}_n,H\mathbb{b}_n)\left(\frac{1+\mathbb{m}_c(g,Hg)}{1+\mathbb{m}_c(g,\mathbb{b}_n)}\right), \mathbb{m}_c(g,Hg)\left(\frac{1+\mathbb{m}_c(\mathbb{b}_n,H\mathbb{b}_n)}{1+\mathbb{m}_c(g,\mathbb{b}_n)}\right)\Bigg\}\Bigg).
\end{aligned}
$$

Letting $n \to \infty$, we have $\psi(\mathbb{m}_c(g,Hg)) \leq \psi(\xi\mathbb{m}_c(g,Hg)) - \varphi(\mathbb{m}_c(g,Hg))$, and as a result, $g = Hg$.

To verify that the fixed-point is unique, assume by contradiction that there exists a second fixed-point w of H such that $w \neq g$, and we get

$$
\begin{aligned}
\psi(\mathbb{m}_c(g,w)) &= \psi(\mathbb{m}_c(Hg,Hw))\\
&\leq \psi(\xi\mathscr{M}_2(Hg,Hw)) - \varphi(\mathscr{M}_2(Hg,Hw))\\
&= \psi(\mathbb{m}_c(g,w)) - \varphi(\mathbb{m}_c(g,w)),
\end{aligned}
$$

$\varphi(\mathbb{m}_c(w,g)) = 0$.
We come up with $w = g$, and the proof is finished. □

Theorem 2.2. *Let us assume that $\mathbb{G}$ represents a complete controlled metric space. Suppose that $H : \mathbb{G} \to \mathbb{G}$ denotes the mapping. Let for every $\mathbb{g}, \mathbb{b} \in \mathbb{G}$, $\xi \in (0,1)$, and*

$$\psi(\mathbb{m}_c(H\mathbb{g},H\mathbb{b})) \leq \psi(\xi\mathscr{N}(H\mathbb{g},H\mathbb{b})) - \varphi(\mathscr{N}(H\mathbb{g},H\mathbb{b})), \tag{2.7}$$

and Theorem 1.2 is defined by ψ and φ.
Suppose that

$$\limsup_{i\to\infty} \sup_{m>i}\left[\frac{\sigma(\mathbb{b}_{i+1},\mathbb{b}_{i+2})}{\sigma(\mathbb{b}_i,\mathbb{b}_{i+1})}\sigma(\mathbb{b}_{i+1},\mathbb{b}_m)\right] < \frac{1}{\xi}. \tag{2.8}$$

In addition, presume that, for each $\mathbb{b} \in \mathbb{G}$, we have that

$$\lim_{n\to\infty}\sigma(\mathbb{b}_n,\mathbb{b}) \quad \textit{and} \quad \lim_{n\to\infty}\sigma(\mathbb{b},\mathbb{b}_n) \quad \textit{exist and are finite}.$$

Then, H admits a UFP.

Proof. The proof follows analogously from the previous theorem. □

We now present some essential consequences of the previous theorems. Our first corollary is the following.

Corollary 2.3. *Let us consider $\mathbb{G}$ as a complete controlled metric space. Suppose that $H : \mathbb{G} \to \mathbb{G}$ denotes the mapping. Let for every $\mathbb{g}, \mathbb{b} \in \mathbb{G}$, $\xi \in (0,1)$, and*

$$\mathbb{m}_c(H\mathbb{g},H\mathbb{b}) \leq \xi\mathscr{M}_2(H\mathbb{g},H\mathbb{b}) - \varphi(\mathscr{M}_2(H\mathbb{g},H\mathbb{b})), \tag{2.9}$$

whereas Theorem 1.2 is defined byψ and φ.
Assume that

$$\limsup_{i\to\infty} \sup_{m>i} \left[\frac{\sigma(\mathbb{b}_{i+1}, \mathbb{b}_{i+2})}{\sigma(\mathbb{b}_i, \mathbb{b}_{i+1})}\sigma(\mathbb{b}_{i+1}, \mathbb{b}_m)\right] < \frac{1}{\xi}. \tag{2.10}$$

Moreover, assume that for every $\mathbb{b} \in \mathbb{G}$, we have that

$$\lim_{n\to\infty} \sigma(\mathbb{b}_n, \mathbb{b}) \quad \textit{and} \quad \lim_{n\to\infty} \sigma(\mathbb{b}, \mathbb{b}_n) \quad \textit{exist and are finite}.$$

Then, H admits a UFP.

Proof. The proof is quite straightforward by putting $\psi = I$(Identity) in the Theorem 2.1. □

Corollary 2.4. *Consider that $\mathbb{G}$ denotes a complete controlled metric space. Suppose that $H : \mathbb{G} \to \mathbb{G}$ denotes the mapping. Let for every $\mathbb{g}, \mathbb{b} \in \mathbb{G}$, $\xi \in (0, 1)$, and*

$$\mathbb{m}_c(H\mathbb{g}, H\mathbb{b}) \leq \xi \mathbb{m}_c(\mathbb{g}, \mathbb{b}) - \varphi(\mathbb{m}_c(\mathbb{g}, \mathbb{b})), \tag{2.11}$$

while Theorem is 1.2 defined by ψ and φ.
Suppose that

$$\limsup_{i\to\infty} \sup_{m>i} \left[\frac{\sigma(\mathbb{b}_{i+1}, \mathbb{b}_{i+2})}{\sigma(\mathbb{b}_i, \mathbb{b}_{i+1})}\sigma(\mathbb{b}_{i+1}, \mathbb{b}_m)\right] < \frac{1}{\xi}. \tag{2.12}$$

Moreover, consider that for every $\mathbb{b} \in \mathbb{G}$, we have that

$$\lim_{n\to\infty} \sigma(\mathbb{b}_n, \mathbb{b}) \quad \textit{and} \quad \lim_{n\to\infty} \sigma(\mathbb{b}, \mathbb{b}_n) \quad \textit{exist and are finite}.$$

Then, H admits a UFP.

Proof. We prove this is trivial by putting $\mathscr{M}_2(H\mathbb{g}, H\mathbb{b}) = \mathbb{m}_c(\mathbb{g}, \mathbb{b})$ in Theorem 2.1. □

Corollary 2.5. *Suppose that $\mathbb{G}$ denotes a complete controlled metric space. Suppose that $H : \mathbb{G} \to \mathbb{G}$ denotes the mapping. Let for every $\mathbb{g}, \mathbb{b} \in \mathbb{G}$, $\xi \in (0, 1)$, and*

$$\psi(\mathbb{m}_c(H\mathbb{g}, H\mathbb{b})) \leq \psi(\xi \mathbb{m}_c(\mathbb{g}, \mathbb{b})) - \varphi(\mathbb{m}_c(\mathbb{g}, \mathbb{b})), \tag{2.13}$$

whereas Theorem 1.2 is defined by ψ and φ.
Suppose that

$$\limsup_{i\to\infty} \sup_{m>i} \left[\frac{\sigma(\mathbb{b}_{i+1}, \mathbb{b}_{i+2})}{\sigma(\mathbb{b}_i, \mathbb{b}_{i+1})}\sigma(\mathbb{b}_{i+1}, \mathbb{b}_m)\right] < \frac{1}{\xi}. \tag{2.14}$$

Moreover, consider that for every $\mathbb{b} \in \mathbb{G}$, we have that

$$\lim_{n\to\infty} \sigma(\mathbb{b}_n, \mathbb{b}) \quad \textit{and} \quad \lim_{n\to\infty} \sigma(\mathbb{b}, \mathbb{b}_n) \quad \textit{exist and are finite}.$$

Then, H admits a UFP.

Proof. The proof is trivial by taking $\mathscr{M}_2(H\mathbb{g}, H\mathbb{b}) = d(\mathbb{g}, \mathbb{b})$ and $\psi = I$ in the previous result (2.1). □

Remark 2.6. *The result (2.5) is the consequence (1.1) of [14] in controlled metric spaces.*

Remark 2.7. *The result* (2.4) *is the consequence (1.2) of [15] in controlled metric spaces.*

Example 2.8. *Suppose that* $\mathbb{G} = \{0, 1, 2\}$. *Let* $\mathbb{m}_c : \mathbb{G} \times \mathbb{G} \to [0, \infty)$ *be symmetric and defined as*

$$\mathbb{m}_c(0, 0) = \mathbb{m}_c(1, 1) = \mathbb{m}_c(2, 2) = 0,$$

$$\mathbb{m}_c(0, 1) = \mathbb{m}_c(1, 0) = 1, \qquad \mathbb{m}_c(0, 2) = \mathbb{m}_c(2, 0) = \frac{3}{5},$$

and

$$\mathbb{m}_c(1, 2) = \mathbb{m}_c(2, 1) = \frac{1}{2}.$$

Take $\sigma : \mathbb{G} \times \mathbb{G} \to [1, \infty)$ *as a symmetric function defined by*

$$\sigma(0, 0) = \sigma(1, 1) = \sigma(2, 2) = \sigma(0, 2) = 1,$$

$$\sigma(1, 2) = \frac{6}{5}, \qquad \sigma(0, 1) = \frac{4}{3}.$$

Clearly, $(\mathbb{G}, \mathbb{m}_c, \sigma)$ *is a controlled metric space. Define the self-map* $H : \mathbb{G} \to \mathbb{G}$ *by*

$$H0 = 1, \qquad H1 = 2, \qquad H2 = 2.$$

Choose

$$\xi = \frac{2}{3},$$

and define $\psi, \varphi : [0, \infty) \to [0, \infty)$ *by*

$$\psi(t) = t, \qquad \varphi(t) = \frac{t}{6}.$$

For all $\mathbb{g}, \mathbb{b} \in \mathbb{G}$, *we verify that*

$$\mathbb{m}_c(H\mathbb{g}, H\mathbb{b}) \leq \frac{1}{2}\mathscr{M}_2(H\mathbb{g}, H\mathbb{b}).$$

On the other hand,

$$\begin{aligned}
&\psi\left(\xi\mathscr{M}_2(H\mathbb{g}, H\mathbb{b})\right) - \varphi\left(\mathscr{M}_2(H\mathbb{g}, H\mathbb{b})\right) \\
&= \frac{2}{3}\mathscr{M}_2(H\mathbb{g}, H\mathbb{b}) - \frac{1}{6}\mathscr{M}_2(H\mathbb{g}, H\mathbb{b}) \\
&= \frac{1}{2}\mathscr{M}_2(H\mathbb{g}, H\mathbb{b}).
\end{aligned}$$

Therefore,

$$\psi\left(\mathbb{m}_c(H\mathbb{g}, H\mathbb{b})\right) \leq \psi\left(\xi\mathscr{M}_2(H\mathbb{g}, H\mathbb{b})\right) - \varphi\left(\mathscr{M}_2(H\mathbb{g}, H\mathbb{b})\right),$$

which satisfies Condition (2.1). Hence, all the conditions of Theorem 2.1 are satisfied, and therefore, H admits a unique fixed-point. Further,

$$H2 = 2$$

is the unique fixed-point.

Example 2.9. *Let*

$$\mathbb{G} = \{0, 1, 2\},$$

and define

$$\mathbb{m}_c(\mathbb{g}, \mathbb{b}) = |\mathbb{g} - \mathbb{b}|, \qquad \mathbb{g}, \mathbb{b} \in \mathbb{G}.$$

Clearly, $(\mathbb{G}, \mathbb{m}_c)$ *is a complete metric space and, consequently, a complete controlled metric space with*

$$\sigma(\mathbb{g}, \mathbb{b}) = 1$$

for all $\mathbb{g}, \mathbb{b} \in \mathbb{G}$.

Define the self-map

$$H : \mathbb{G} \longrightarrow \mathbb{G}$$

by

$$H(0) = 0, \qquad H(1) = 0, \qquad H(2) = 1.$$

Choose

$$\xi = \frac{4}{5}, \qquad \psi(t) = t, \qquad \varphi(t) = \frac{t}{10}.$$

Then,

$$\psi(\xi t) - \varphi(t) = \frac{4}{5}t - \frac{1}{10}t = \frac{7}{10}t.$$

Using the definition of $\mathscr{M}_2$ *associated with the self-map H, the only pairs for*

$$\mathbb{m}_c(H\mathbb{g}, H\mathbb{b}) \neq 0$$

are

$$(0, 2), \qquad (2, 0), \qquad (1, 2), \qquad (2, 1).$$

For $(\mathbb{g}, \mathbb{b}) = (0, 2)$ *and* $(2, 0)$*, we obtain*

$$\mathbb{m}_c(H0, H2) = \mathbb{m}_c(0, 1) = 1$$

and

$$\mathscr{M}_2(0, 2) = \mathscr{M}_2(2, 0) = 2.$$

Therefore,

$$\mathbb{m}_c(H0, H2) = 1 \leq \frac{7}{10}(2) = \frac{7}{5}.$$

Similarly, for $(\mathbb{g}, \mathbb{b}) = (1, 2)$ *and* $(2, 1)$*,*

$$\mathbb{m}_c(H1, H2) = \mathbb{m}_c(0, 1) = 1,$$

whereas

$$\mathscr{M}_2(1, 2) = \mathscr{M}_2(2, 1) = \frac{3}{2}.$$

Hence,

$$\mathbb{m}_c(H1, H2) = 1 \leq \frac{7}{10}\left(\frac{3}{2}\right) = \frac{21}{20}.$$

For all remaining pairs,

$$\mathbb{m}_c(H\mathbb{g}, H\mathbb{b}) = 0,$$

and therefore, the required contractive inequality holds trivially.

Consequently, for every $\mathbb{g}, \mathbb{b} \in \mathbb{G}$,

$$\psi\left(\mathbb{m}_c(H\mathbb{g}, H\mathbb{b})\right) \leq \psi\left(\xi \mathscr{M}_2(\mathbb{g}, \mathbb{b})\right) - \varphi\left(\mathscr{M}_2(\mathbb{g}, \mathbb{b})\right).$$

Thus, the modified rational (ψ, φ)*-contractive condition of Theorem 2.1 is satisfied.*

Moreover, because $\sigma = 1$,

$$\limsup_{i\to\infty} \sup_{m>i} \left[\frac{\sigma(\mathbb{b}_{i+1}, \mathbb{b}_{i+2})}{\sigma(\mathbb{b}_i, \mathbb{b}_{i+1})} \sigma(\mathbb{b}_{i+1}, \mathbb{b}_m)\right] = 1,$$

Because

$$1 < \frac{5}{4} = \frac{1}{\xi},$$

the control-function condition is also satisfied. The remaining limits involving σ *clearly exist and are finite.*

Hence, all the assumptions of Theorem 2.1 are fulfilled, and H has a unique fixed-point. Indeed,

$$H(0) = 0,$$

so the unique fixed-point is

$$u = 0.$$

We now show that the classical Banach contraction principle cannot be applied to this mapping. For the pair $1, 2 \in \mathbb{G}$,

$$\mathbb{m}_c(H1, H2) = \mathbb{m}_c(0, 1) = 1,$$

whereas

$$\mathbb{m}_c(1, 2) = 1.$$

Thus,

$$\frac{\mathbb{m}_c(H1, H2)}{\mathbb{m}_c(1, 2)} = 1.$$

Consequently, there does not exist any constant $k \in (0, 1)$ *such that*

$$\mathbb{m}_c(H\mathbb{g}, H\mathbb{b}) \leq k\, \mathbb{m}_c(\mathbb{g}, \mathbb{b})$$

for all $\mathbb{g}, \mathbb{b} \in \mathbb{G}$.

Therefore, H is not a Banach contraction, whereas it satisfies the modified rational (ψ, φ)*-contractive condition of Theorem 2.1. This example demonstrates that the proposed theorem can apply to mappings for which the classical Banach contraction principle is not applicable.*

Example 2.10. *Let* $\mathbb{G} = \{a, b, c, d\}$. *Define a symmetric controlled metric* $\mathbb{m}_c : \mathbb{G} \times \mathbb{G} \to [0, \infty)$ *by the table*

$\mathbb{m}_c$	a	b	c	d
a	0	$\frac{3}{4}$	$\frac{2}{3}$	1
b	$\frac{3}{4}$	0	$\frac{7}{10}$	$\frac{9}{10}$
c	$\frac{2}{3}$	$\frac{7}{10}$	0	$\frac{4}{5}$
d	1	$\frac{9}{10}$	$\frac{4}{5}$	0

(that is, for example, $\mathbb{m}_c(a, b) = 3/4$, $\mathbb{m}_c(c, d) = 4/5$, *etc.). Take the control function* $\sigma : \mathbb{G}\times\mathbb{G} \to [1, \infty)$ *(symmetric) by*

$$\sigma(\mathbb{b}, y) = \begin{cases} 1, & \mathbb{b} = y, \\ \frac{11}{10}, & \{\mathbb{b}, y\} = \{a, b\}, \\ \frac{5}{4}, & \{\mathbb{b}, y\} = \{b, c\}, \\ \frac{9}{8}, & \{\mathbb{b}, y\} = \{c, d\}, \\ 1, & \textit{otherwise.} \end{cases}$$

One checks that $(\mathbb{G}, \mathbb{m}_c)$ *is a complete controlled metric space; because* $\mathbb{G}$ *is finite, every Cauchy sequence is eventually constant.*

Define $H : \mathbb{G} \to \mathbb{G}$ *by*

$$H(a) = d, \qquad H(b) = d, \qquad H(c) = d, \qquad H(d) = d.$$

Thus, d *is a fixed-point for* H, *and for every starting point, the orbit* $\{\mathbb{b}_n\}$ *becomes constant from the first iterate onward:* $\mathbb{b}_1 = d$, $\mathbb{b}_2 = d, \ldots$

Choose the auxiliary functions

$$\psi(t) = t, \qquad \varphi(t) = \tfrac{1}{4}t, \qquad \xi = \tfrac{1}{2}.$$

We now verify the hypotheses of Theorem 2.1.

For arbitrary $\mathbb{g}, \mathbb{b} \in \mathbb{G}$ *we have* $H\mathbb{g} = H\mathbb{b} = d$, *so*

$$\psi(\mathbb{m}_c(H\mathbb{g}, H\mathbb{b})) = \psi(\mathbb{m}_c(d, d)) = \psi(0) = 0.$$

On the other side, $\mathscr{M}_2(H\mathbb{g}, H\mathbb{b})$ *is the maximum of several expressions formed from distances involving* $H\mathbb{g} = d$, *and* $H\mathbb{b} = d$. *When both arguments are* d, *all pairwise distances appearing in the max are* ≥ 0, *and in particular, when* $\mathbb{g} \neq \mathbb{b}$, *at least one entry of the maximum is strictly positive (for example,* $\mathbb{m}_c(\mathbb{g}, \mathbb{b}) > 0$ *enters the definition of* $\mathscr{M}_2$*). Hence, for every choice of* $\mathbb{g}, \mathbb{b}$, *we obtain*

$$0 = \psi(\mathbb{m}_c(H\mathbb{g}, H\mathbb{b})) \leq \psi(\mathscr{M}_2(H\mathbb{g}, H\mathbb{b})) - \varphi(\mathscr{M}_2(H\mathbb{g}, H\mathbb{b}))$$

because the right-hand side equals $\mathscr{M}_2 - \frac{1}{4}\mathscr{M}_2 = \frac{3}{4}\mathscr{M}_2 \geq 0$. *When* $\mathbb{g} = \mathbb{b} = d$, *both sides are* 0, *so* (2.1) *holds in every case.*

Fix any initial point $\mathbb{b}_0 \in \mathbb{G}$, *and form the orbit* $\mathbb{b}_n = H^n(\mathbb{b}_0)$. *As observed above, the orbit satisfies* $\mathbb{b}_1 = d$, $\mathbb{b}_2 = d, \ldots$, *so for all sufficiently large* i, *we have* $\mathbb{b}_i = \mathbb{b}_{i+1} = \mathbb{b}_m = d$. *Therefore,*

$$\lim_{i\to\infty} \frac{\sigma(\mathbb{b}_{i+1}, \mathbb{b}_{i+2})}{\sigma(\mathbb{b}_i, \mathbb{b}_{i+1})}\, \sigma(\mathbb{b}_{i+1}, \mathbb{b}_m) = \frac{\sigma(d, d)}{\sigma(d, d)}\sigma(d, d) = 1.$$

Hence,

$$\limsup_{i\to\infty} \sup_{m>i} \left[\frac{\sigma(\mathbb{b}_{i+1}, \mathbb{b}_{i+2})}{\sigma(\mathbb{b}_i, \mathbb{b}_{i+1})} \sigma(\mathbb{b}_{i+1}, \mathbb{b}_m) \right] = 1.$$

With our choice $\xi = \frac{1}{2}$*, we have* $1 < 1/\xi = 2.$*, so the strict inequality in* (2.2) *is satisfied.*

For any fixed $\mathbb{b} \in \mathbb{G}$*, the sequence* $\sigma(\mathbb{b}_n, \mathbb{b})$ *is eventually constant because* $\mathbb{b}_n = d$ *for all large n. Thus, the limits* $\lim_{n\to\infty} \sigma(\mathbb{b}_n, \mathbb{b})$ *and* $\lim_{n\to\infty} \sigma(\mathbb{b}, \mathbb{b}_n)$ *exist and are finite.*

All conditions for Theorem 2.1 hold, so H admits a UFP. Because $H(\mathbb{b}) = d$ *for every* $\mathbb{b} \in \mathbb{G}$*, the UFP is d.*

3. Application

Neutron-transport equations are of critical importance in nuclear engineering and reactor physics, as they accurately model particle behavior in nuclear reactors, radiation shielding, and medical physics [9–11]. Fixed-point theory (FPT) provides essential mathematical tools to prove the existence and uniqueness of solutions to these integral equations, ensuring physical predictability and operational safety. The application demonstrates how abstract mathematical frameworks directly support the design and safety analysis of nuclear systems, bridging theoretical mathematics with practical engineering solutions for energy production and radiation management.

3.1. Application to the steady-state neutron transport

The neutron transport phenomenon in homogeneous media at steady-state conditions is mathematically characterized by the linear Boltzmann equation

$$\frac{1}{\mu_n} \frac{\partial \rho(\mathbb{b}, \mu_n)}{\partial \mathbb{b}} + \Sigma_t(\mathbb{b})\rho(\mathbb{b}, \mu_n) = \int_{-1}^{1} \Sigma_s(\mathbb{b}, \mu_n' \to \mu_n)\rho(\mathbb{b}, \mu_n')d\mu_n' + S(\mathbb{b}, \mu_n),$$

where the following hold:

- The function $\rho(\mathbf{b}, \mu_n)$ denotes the angular neutron flux.
- The coefficient $\Sigma_t(\mathbf{b})$ is the total cross-section.
- The kernel $\Sigma_s(\mathbf{b}, \mu_n' \to \mu_n)$ represents the differential scattering cross-section.
- The term $S(\mathbf{b}, \mu_n)$ describes the neutron source.

Assuming both isotropic scattering and plane symmetry, the general transport equation simplifies to the well-known Peierls integral equation:

$$\mathscr{V}(\mathbb{b}) = \lambda \int_0^1 K(\mathbb{b}, y)\mathscr{V}(y)dy + S(\mathbb{b}), \tag{3.1}$$

where the following hold:

- The scalar flux is defined as $\mathscr{V}(\mathbb{b}) = \int_{-1}^{1} \rho(\mathbb{b}, \mu_n)d\mu_n$.
- The transport kernel is given by $K(\mathbb{b}, y) = (1/2)E_1(|\mathbb{b} - y|)$.
- $E_1(z) = \int_1^\infty (e^{-zt}/t)\, dt$ denotes the exponential integral function.
- The scattering ratio is defined as $\lambda = \Sigma_s/\Sigma_t$:

Let $\mathbb{G} = C([0,1], \mathbb{R})$ denote the space of real-valued continuous functions on $[0,1]$:

$$\mathbb{m}_c(\mathbb{b}(p), y(p)) = \sup_{p\in[0,1]} |\mathbb{b}(p) - y(p)|^2, \tag{3.2}$$

for all $\mathbb{b}, y \in \mathbb{G}$. Note that $(\mathbb{G}, \mathbb{m}_c, \sigma)$ is a controlled metric space with the function $\sigma : \mathbb{G}\times\mathbb{G} \longrightarrow [1,\infty)$ defined as

$$\sigma(\mathbb{b}, y) = 2.$$

To analyze this problem, we formulate it in terms of an integral operator. Consider the neutron-transport operator $H : C([0,1]) \to C([0,1])$, defined by

$$H\mathscr{V}(\mathbb{b}) = \lambda \int_0^1 K(\mathbb{b}, y)\mathscr{V}(y)dy + S(\mathbb{b}).$$

Before stating the application theorem, we explain the role of its assumptions. The uniform bound on the kernel controls the norm of the integral operator, whereas the restriction on the scattering ratio ensures that the contraction generated by the operator is sufficiently strong. These assumptions allow the transport operator to satisfy the modified rational (ψ, φ)-contractive condition of Theorem 2.1.

Next, we will proceed to prove our result.

Theorem 3.1. *Let* $\mathbb{G} = C([0,1], \mathbb{R})$ *be endowed with*

$$\mathbb{m}_c(\mathscr{V}_1, \mathscr{V}_2) = \|\mathscr{V}_1 - \mathscr{V}_2\|_\infty^2,$$

and let $H : \mathbb{G} \to \mathbb{G}$ *be the transport operator defined by*

$$(H\mathscr{V})(\mathbb{b}) = \lambda \int_0^1 K(\mathbb{b}, y)\mathscr{V}(y)\, dy + S(\mathbb{b}).$$

Assume that

$$I := \sup_{\mathbb{b}\in[0,1]} \int_0^1 |K(\mathbb{b}, y)|\, dy < \infty$$

and that there exists $\xi \in \left(0, \frac{1}{2}\right)$ *such that*

$$q := |\lambda|^2 I^2 < \xi.$$

Then, the transport Eq (3.1) *admits a unique solution,* $\mathscr{V}^* \in C([0,1], \mathbb{R})$.

Proof. Let

$$\mathbb{G} = C([0,1], \mathbb{R}),$$

and define the operator $H : \mathbb{G} \to \mathbb{G}$ by

$$(H\mathscr{V})(\mathbb{b}) = \lambda \int_0^1 K(\mathbb{b}, y)\mathscr{V}(y)\, dy + S(\mathbb{b}). \tag{3.3}$$

Because K, S, and $\mathscr{V}$ are continuous, the function $H\mathscr{V}$ is continuous on $[0,1]$. Hence, H is a self-mapping on $\mathbb{G}$.

Moreover, $\mathscr{V} \in \mathbb{G}$ is a solution of the transport Eq (3.1) if and only if

$$H\mathscr{V} = \mathscr{V}.$$

Therefore, it is enough to prove that H admits a unique fixed-point.

For $\mathscr{V}_1, \mathscr{V}_2 \in \mathbb{G}$, we have

$$\begin{aligned}&(H\mathscr{V}_1)(\mathbb{b}) - (H\mathscr{V}_2)(\mathbb{b})\\ &= \lambda \int_0^1 K(\mathbb{b}, y)(\mathscr{V}_1(y) - \mathscr{V}_2(y))\, dy,\end{aligned}$$

because the source term $S(\mathbb{b})$ cancels. Consequently,

$$\begin{aligned}|(H\mathscr{V}_1)(\mathbb{b}) - (H\mathscr{V}_2)(\mathbb{b})| &\leq |\lambda| \int_0^1 |K(\mathbb{b}, y)|\, |\mathscr{V}_1(y) - \mathscr{V}_2(y)|\, dy\\ &\leq |\lambda| \left(\int_0^1 |K(\mathbb{b}, y)|\, dy \right) \|\mathscr{V}_1 - \mathscr{V}_2\|_\infty.\end{aligned}$$

Taking the supremum over $\mathbb{b} \in [0, 1]$, we obtain

$$\|H\mathscr{V}_1 - H\mathscr{V}_2\|_\infty \leq |\lambda| I \|\mathscr{V}_1 - \mathscr{V}_2\|_\infty,$$

where

$$I = \sup_{\mathbb{b} \in [0,1]} \int_0^1 |K(\mathbb{b}, y)|\, dy.$$

Squaring both sides gives

$$\|H\mathscr{V}_1 - H\mathscr{V}_2\|_\infty^2 \leq |\lambda|^2 I^2 \|\mathscr{V}_1 - \mathscr{V}_2\|_\infty^2.$$

Using the controlled metric

$$\mathbb{m}_c(\mathscr{V}_1, \mathscr{V}_2) = \|\mathscr{V}_1 - \mathscr{V}_2\|_\infty^2,$$

we obtain

$$\mathbb{m}_c(H\mathscr{V}_1, H\mathscr{V}_2) \leq q\, \mathbb{m}_c(\mathscr{V}_1, \mathscr{V}_2), \tag{3.4}$$

where

$$q = |\lambda|^2 I^2.$$

By assumption,

$$q < \xi < 1.$$

Because the rational expression $\mathscr{M}_2(H\mathscr{V}_1, H\mathscr{V}_2)$ contains $\mathbb{m}_c(\mathscr{V}_1, \mathscr{V}_2)$ as one of its terms, we have

$$\mathbb{m}_c(\mathscr{V}_1, \mathscr{V}_2) \leq \mathscr{M}_2(H\mathscr{V}_1, H\mathscr{V}_2).$$

Therefore, from (3.4),

$$\mathbb{m}_c(H\mathscr{V}_1, H\mathscr{V}_2) \leq q\, \mathscr{M}_2(H\mathscr{V}_1, H\mathscr{V}_2).$$

Now, define

$$\psi(t) = t$$

and

$$\varphi(t) = (\xi - q)t, \qquad t \geq 0.$$

Because $q < \xi$, we have $\xi - q > 0$. Hence, φ is continuous and non-negative, and

$$\varphi(t) = 0 \quad \Longleftrightarrow \quad t = 0.$$

Also,

$$\begin{aligned} q\,\mathscr{M}_2(H\mathscr{V}_1, H\mathscr{V}_2) &= \xi\mathscr{M}_2(H\mathscr{V}_1, H\mathscr{V}_2) - (\xi - q)\mathscr{M}_2(H\mathscr{V}_1, H\mathscr{V}_2) \\ &= \psi(\xi\mathscr{M}_2(H\mathscr{V}_1, H\mathscr{V}_2)) - \varphi(\mathscr{M}_2(H\mathscr{V}_1, H\mathscr{V}_2))\,. \end{aligned}$$

Thus,

$$\psi(\mathbb{m}_c(H\mathscr{V}_1, H\mathscr{V}_2)) \leq \psi(\xi\mathscr{M}_2(H\mathscr{V}_1, H\mathscr{V}_2)) - \varphi(\mathscr{M}_2(H\mathscr{V}_1, H\mathscr{V}_2))\,.$$

Hence, H satisfies the modified rational (ψ, φ)-contractive condition of Theorem 2.1.

In the present space, the control function is taken as

$$\sigma(\mathscr{V}_1, \mathscr{V}_2) = 2$$

for all $\mathscr{V}_1, \mathscr{V}_2 \in \mathbb{G}$. Therefore,

$$\begin{aligned} \limsup_{i\to\infty} \sup_{m>i} \left[\frac{\sigma(\mathbb{b}_{i+1}, \mathbb{b}_{i+2})}{\sigma(\mathbb{b}_i, \mathbb{b}_{i+1})}\sigma(\mathbb{b}_{i+1}, \mathbb{b}_m)\right] &= \sup_{m\geq 1} \lim_{i\to\infty} 2 \\ &= 2. \end{aligned}$$

Because $\xi \in \left(0, \frac{1}{2}\right)$,

$$2 < \frac{1}{\xi}.$$

Hence, the supremum condition required in Theorem 2.1 is satisfied.

Therefore, all the hypotheses of Theorem 2.1 are satisfied. Consequently, H admits a unique fixed-point $\mathscr{V}^* \in \mathbb{G}$. Because the fixed-points of H are exactly the solutions of the transport equation, it follows that (3.1) possesses a unique solution,

$$\mathscr{V}^* \in C([0, 1], \mathbb{R}).$$

□

Example 3.2. *Consider the neutron transport integral Eq* (3.1) *on* $[0, 1]$ *with*

$$K(\mathbb{b}, y) = \frac{1}{2}e^{-|\mathbb{b}-y|}, \qquad S(\mathbb{b}) = \sin(\pi\mathbb{b}) + 1, \qquad \mathbb{b}, y \in [0, 1].$$

Let

$$\mathbb{G} = C([0, 1], \mathbb{R}),$$

and define

$$\mathbb{m}_c(\mathscr{V}_1, \mathscr{V}_2) = \sup_{p\in[0,1]} |\mathscr{V}_1(p) - \mathscr{V}_2(p)|^2.$$

Because

$$|\mathscr{V}_1(p) - \mathscr{V}_3(p)|^2 \leq 2|\mathscr{V}_1(p) - \mathscr{V}_2(p)|^2 + 2|\mathscr{V}_2(p) - \mathscr{V}_3(p)|^2,$$

we obtain

$$\mathbb{m}_c(\mathscr{V}_1, \mathscr{V}_3) \leq 2\mathbb{m}_c(\mathscr{V}_1, \mathscr{V}_2) + 2\mathbb{m}_c(\mathscr{V}_2, \mathscr{V}_3).$$

Thus, $(\mathbb{G}, \mathbb{m}_c, \sigma)$ *is a complete controlled metric space with*

$$\sigma(\mathscr{V}_1, \mathscr{V}_2) = 2.$$

Define the transport operator $H : \mathbb{G} \to \mathbb{G}$ *by*

$$(H\mathscr{V})(\mathbb{b}) = \lambda \int_0^1 K(\mathbb{b}, y)\mathscr{V}(y)\, dy + S(\mathbb{b}). \tag{3.5}$$

We verify the hypotheses of Theorem 3.1.

(1) Estimate of the kernel integral. *For every* $\mathbb{b} \in [0, 1]$*, we have*

$$\begin{aligned}\int_0^1 |K(\mathbb{b}, y)|\, dy &= \frac{1}{2}\int_0^1 e^{-|\mathbb{b}-y|}\, dy \\ &= \frac{1}{2}\left(\int_0^{\mathbb{b}} e^{-(\mathbb{b}-y)}\, dy + \int_{\mathbb{b}}^1 e^{-(y-\mathbb{b})}\, dy\right) \\ &= \frac{1}{2}\left(2 - e^{-\mathbb{b}} - e^{-(1-\mathbb{b})}\right).\end{aligned}$$

The maximum occurs at $\mathbb{b} = 1/2$*. Therefore,*

$$I := \sup_{\mathbb{b}\in[0,1]} \int_0^1 |K(\mathbb{b}, y)|\, dy = 1 - e^{-1/2} \approx 0.393469.$$

Consequently,

$$I^2 = (1 - e^{-1/2})^2 \approx 0.154818.$$

(2) Choice of the scattering ratio and contraction parameter. *Choose*

$$\lambda = 0.9,$$

and set

$$\xi = I^2 = (1 - e^{-1/2})^2.$$

Then,

$$q := |\lambda|^2 I^2 = 0.81 I^2 \approx 0.125403.$$

Hence,

$$0 < q < \xi \approx 0.154818 < 1.$$

Thus, the principal assumption of Theorem 3.1 is satisfied.

(3) Contractive estimate for the transport operator. *For any* $\mathscr{V}_1, \mathscr{V}_2 \in \mathbb{G}$ *and* $\mathbb{b} \in [0, 1]$*, we obtain*

$$|(H\mathscr{V}_1)(\mathbb{b}) - (H\mathscr{V}_2)(\mathbb{b})|$$

$$= |\lambda| \left| \int_0^1 K(\mathbb{b}, y)(\mathscr{V}_1(y) - \mathscr{V}_2(y))\, dy \right|$$
$$\leq |\lambda| \int_0^1 |K(\mathbb{b}, y)|\, dy\, \|\mathscr{V}_1 - \mathscr{V}_2\|_\infty.$$

Taking the supremum over $\mathbb{b} \in [0, 1]$ *and squaring yields*

$$\begin{aligned} \mathbb{m}_c(H\mathscr{V}_1, H\mathscr{V}_2) &= \|H\mathscr{V}_1 - H\mathscr{V}_2\|_\infty^2 \\ &\leq |\lambda|^2 I^2 \|\mathscr{V}_1 - \mathscr{V}_2\|_\infty^2 \\ &= q\, \mathbb{m}_c(\mathscr{V}_1, \mathscr{V}_2). \end{aligned}$$

Because

$$\mathscr{M}_2(H\mathscr{V}_1, H\mathscr{V}_2) \geq \mathbb{m}_c(\mathscr{V}_1, \mathscr{V}_2),$$

it follows that

$$\mathbb{m}_c(H\mathscr{V}_1, H\mathscr{V}_2) \leq q\, \mathscr{M}_2(H\mathscr{V}_1, H\mathscr{V}_2).$$

(4) Verification of the (ψ, φ)***-contractive condition.*** *Define*

$$\psi(t) = t$$

and

$$\varphi(t) = (\xi - q)t.$$

For the present choices,

$$\xi - q = I^2 - 0.81I^2 = 0.19I^2 \approx 0.029415 > 0.$$

Therefore, φ *is continuous and non-negative and satisfies*

$$\varphi(t) = 0 \quad \Longleftrightarrow \quad t = 0.$$

Moreover,

$$\begin{aligned} \psi(\mathbb{m}_c(H\mathscr{V}_1, H\mathscr{V}_2)) &\leq q\, \mathscr{M}_2(H\mathscr{V}_1, H\mathscr{V}_2) \\ &= \xi \mathscr{M}_2(H\mathscr{V}_1, H\mathscr{V}_2) - (\xi - q)\mathscr{M}_2(H\mathscr{V}_1, H\mathscr{V}_2) \\ &= \psi(\xi \mathscr{M}_2(H\mathscr{V}_1, H\mathscr{V}_2)) - \varphi(\mathscr{M}_2(H\mathscr{V}_1, H\mathscr{V}_2)). \end{aligned}$$

Hence, the modified rational (ψ, φ)*-contractive condition is satisfied.*

(5) Verification of the control-function conditions. *Because* $\sigma \equiv 2$*, we have*

$$\limsup_{i\to\infty} \sup_{m>i} \left[\frac{\sigma(\mathbb{b}_{i+1}, \mathbb{b}_{i+2})}{\sigma(\mathbb{b}_i, \mathbb{b}_{i+1})} \sigma(\mathbb{b}_{i+1}, \mathbb{b}_m) \right]$$
$$= \sup_{m\geq 1} \lim_{i\to\infty} \frac{2}{2} \cdot 2 = 2.$$

On the other hand,

$$\frac{1}{\xi} = \frac{1}{(1 - e^{-1/2})^2} \approx 6.459.$$

Therefore,

$$2 < \frac{1}{\xi}.$$

Furthermore, for every fixed $\mathscr{V} \in \mathbb{G}$,

$$\lim_{n\to\infty} \sigma(\mathscr{V}_n, \mathscr{V}) = \lim_{n\to\infty} \sigma(\mathscr{V}, \mathscr{V}_n) = 2,$$

and these limits are finite.

(6) Conclusion. *All the assumptions of Theorem 3.1 are satisfied for*

$$K(\mathbb{b}, y) = \frac{1}{2} e^{-|\mathbb{b}-y|}, \qquad S(\mathbb{b}) = \sin(\pi\mathbb{b}) + 1, \qquad \lambda = 0.9.$$

Consequently, the operator H possesses a unique fixed-point,

$$\mathscr{V}^* \in C([0, 1], \mathbb{R}).$$

Therefore, the neutron transport Eq (3.1) *admits a unique continuous scalar flux solution* $\mathscr{V}^*$.

Visual interpretation can be done with the help of Figure 1.

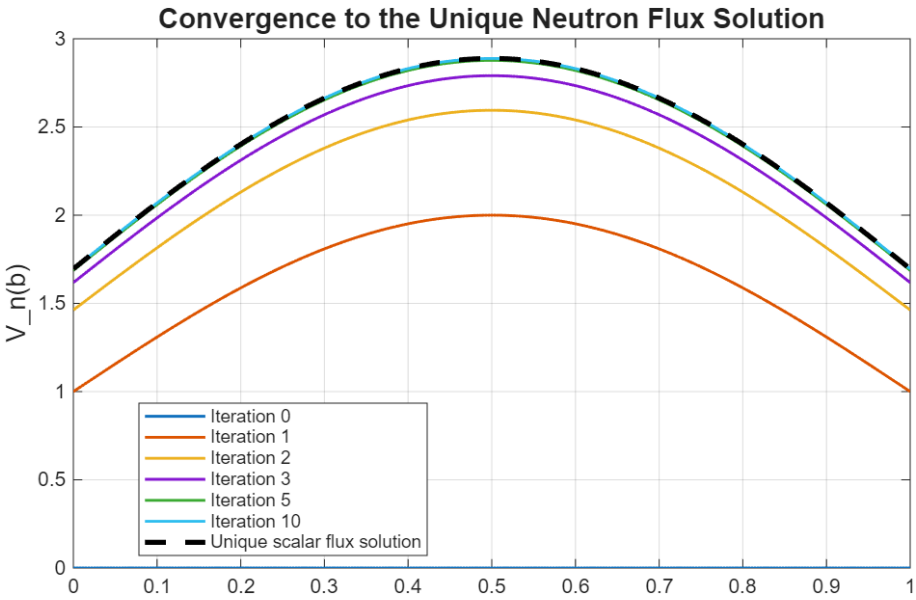


Figure 1. Convergence of the Picard iterates $\mathscr{V}_{n+1} = H\mathscr{V}_n$ toward the unique scalar neutron flux solution $\mathscr{V}^*$ for $\lambda = 0.9$.

4. Conclusions

This paper introduces modified rational (ψ, φ)-contractions in controlled metric spaces and establishes new fixed-point theorems under suitable contractive conditions. The applicability of the developed theory is demonstrated through the neutron transport integral equation and several illustrative examples. The obtained results extend existing rational-type contraction principles in controlled metric spaces and provide a useful framework for studying nonlinear integral equations.

Future research may extend the proposed framework to double controlled metric spaces and investigate applications to systems of integral equations, nuclear engineering models, and other nonlinear mathematical models.

Authors contributions

All authors contributed equally to the conceptualization, methodology, and writing of the manuscript. All authors have equal contribution and have agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflict of interest in this paper.

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