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Research article

Geometric structure of quasi-conformally flat relativistic bulk viscous fluid string spacetimes and Q-soliton dynamics in f($\mathbb{R}, \mathbb{T}$)-gravity

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Abstract: This article investigates relativistic bulk viscous fluid string spacetime with **f**($\mathbb{R}, \mathbb{T}$)-gravity. A quasi-conformal flat relativistic bulk viscous fluid string spacetime is studied in a linear **f**($\mathbb{R}, \mathbb{T}$) = $\mathbb{R} + 2\alpha\mathbb{T}$-gravity model. The solution yields a total pressure equal to the negative of the energy density ($\bar{P} = -\rho$) satisfying the **NEC, WEC, DEC** while violating **SEC**. Moreover, this article also explores **Q**-soliton on a relativistic bulk viscous fluid string spacetime with **f**($\mathbb{R}, \mathbb{T}$)-gravity and exhibits shrinking, steady, and expanding behavior with some specific vector fields.

Keywords: energy conditions; **Q**-solitons; vector field; quasi-conformal curvature tensor
Mathematics Subject Classification: 53C25, 53E20, 53Z05

1. Introduction

The most effective approach to the study of the universe large scale structure is the theory of general relativity (**GR**). In the absence of dark energy, **GR** fails to account the acceleration of the early universe and late-time acceleration. **GR** is insufficient to elucidate gravitational dynamics in this regime, and it needs to be modified to attain theories that accommodate inflation and Dark Energy. Indeed, **GR** is efficient when it describes phenomena across both extra galactic and galactic range. This refined modification to the Einstein field equations (**EFEs**) provides a theoretical framework for describing dynamics of cosmological models [1, 2].

To precisely match observational data, **EFE**s are considered as appropriate when dark matter is modeled as a cold and pressure-less component [3]. A mysterious component of the universe is dark energy which plays a significant role in cosmic energy density, and it is believed to be the major cause of its accelerated expansion.

Inspired by this approach, various mathematicians and physicists have extensively studied **GR** within the context of Einstein-Hilbert action and its extensions to modified gravity theories. In particular, $\mathbf{f}(\mathbb{T})$-theory [4] and $\mathbf{f}(\mathbb{R})$-gravity [5] are the widely studied modified theories, where $\mathbb{T}$ represents the trace of the energy momentum tensor and $\mathbb{R}$ denotes the Ricci scalar. Such models constitute the extensions of the solitonic structures beyond the $\mathbf{f}(\mathbb{R},\mathbb{T})$-gravity and play a vital role in advancing our understanding of the quantum aspects of gravity [6, 7]. **GR** can be broadened to $\mathbf{f}(\mathbb{R})$-gravity through the replacement of geometric part of Einstein-Hilbert action with an arbitrary function $\mathbf{f}(\mathbb{R})$. The corresponding field equations possess higher-order curvature terms and additional dynamical degrees of freedom, providing an insightful framework for explaining the properties of massive neutron stars [8, 9].

Nevertheless, $\mathbf{f}(\mathbb{R})$-gravity cannot accommodate some cosmological models, including stable stellar configuration, and exhibits limitations with solar system constraints [10–12]. To overcome the drawbacks of existing modified gravity models, $\mathbf{f}(\mathbb{R},\mathbb{T})$-gravity was developed, in which the gravitational Lagrangian $\mathbf{f}$ is an arbitrary function of $\mathbb{R}$ and $\mathbb{T}$ [13]. This model has effectively described several cosmological and astrophysical characteristics of the universe.

In **GR**, a spacetime is regarded as a Lorentzian manifold in which the study of the nature of vector fields serves as an appropriate framework for investigating cosmological models. Let $(\mathbf{M}^N, \mathbf{g})$ represent a Lorentzian manifold with $(N > 3)$, and it is characterized as perfect fluid spacetime if its Ricci tensor grabs the form,

$$\mathbf{S} = m\mathbf{g} + n\gamma \otimes \gamma,$$

where γ is 1-form and m, n are scalar fields. A perfect fluid is served as an idealized model of matter distribution.

Let $(\mathbf{M}^N, \mathbf{g})$ be a pseudo-Riemannian or non-flat Riemannian manifold of dimension N, which is said to be a **generalized quasi-Einstein manifold** $(\mathbf{GQE})_\mathbf{N}$ [14] if its Ricci tensor takes the following relation,

$$\mathbf{S} = m\mathbf{g} + n\gamma \otimes \gamma + o\delta \otimes \delta, \tag{1.1}$$

where m, n, and o are scalars with a condition $m, n \neq 0$, and γ, δ are non-vanishing 1-forms, and ξ and ς are unit time-like and unit space-like vector fields, which satisfy $\mathbf{g}(\mathcal{D}, \xi) = \gamma(\mathcal{D}), \gamma(\xi) = -1$ and $\mathbf{g}(\mathcal{D}, \varsigma) = \delta(\mathcal{D}), \delta(\varsigma) = 1$ and $\mathbf{g}(\xi, \varsigma) = 0\ \forall\ \mathcal{D} \in \chi(\mathbf{M})$. Moreover, ξ and ς serve as the generators of the manifold $(\mathbf{M}^N, \mathbf{g})$. In addition, (1.1) represents a perfect fluid spacetime when o tends to be zero.

2. Relativistic bulk viscous fluid string spacetime ($\mathbb{BVFSS}$) with $\mathbf{f}(\mathbb{R},\mathbb{T})$-gravity

Siddiqi et al. [15, 16] proposed the concept of relativistic $\mathbb{BVFSS}$ to investigate relativistic viscous fluids in the presence of cosmic string distributions. Due to their importance in explaining early-universe dynamics and exotic forms of matter, these spacetimes have been widely studied in cosmology, modified gravity theories [17, 18], and astrophysics [19, 20].

A relativistic $\mathbb{BVFSS}$ is a class of imperfect fluid applicable to relativistic fluid environments, particularly in the presence of intense gravitational interactions approaching the speed of light. The inclusion of bulk viscosity introduces dissipative effects that opposes uniform contraction or expansion of the fluid. Compared with perfect fluid, imperfect fluids account for dissipative process that make them well suited for describing entropy generating phenomena, such as the inflationary phase and the late-time acceleration of the universe.

In various high-energy physical theories, such as cosmological models and string theories with topological defects, spacetime is expected to contain string-like configurations. Incorporating these models with bulk viscosity enhances their physical description, providing a more realistic analysis of anisotropies, cosmic structure formation, and non-equilibrium dynamics of the universe.

The energy momentum tensor of a relativistic bulk viscous fluid string spacetime T [15,21] is given by,

$$T(\mathcal{D},\mathcal{E}) = (\bar{P}+\rho)\gamma(\mathcal{D})\gamma(\mathcal{E}) - \bar{P}\mathbf{g}(\mathcal{D},\mathcal{E}) - \lambda\delta(\mathcal{D})\delta(\mathcal{E}), \tag{2.1}$$

$$\bar{P} = P - 3\theta\mathbf{H}(=\epsilon\rho), \tag{2.2}$$

where P is the isotropic pressure, $\bar{P}$ is the total pressure, λ is the string tension density, ρ is the rest energy density, $\mathbf{H}$ is the Hubble parameter, θ is the coefficient of bulk viscosity, and the term $3\theta\mathbf{H}$ is generally referred to as bulk viscous pressure.

In this regard, the expression $\gamma(\mathcal{D}) = \mathbf{g}(\mathcal{D},\xi)$ denotes 1-form induced by the time-like velocity vector ξ and $\mathbf{g}(\xi,\xi) = -1$. Similarly, $\delta(\mathcal{D}) = \mathbf{g}(\mathcal{D},\zeta)$ denotes 1-form induced by the space-like velocity vector ς, $\mathbf{g}(\zeta,\zeta) = 1$, and $\mathbf{g}(\xi,\zeta) = 0$, respectively.

Remark 2.1. *In view of* (2.2)*, the equation of the state relating the total pressure and energy density is expressed as,*

$$\bar{P} = \epsilon\rho, \qquad P = \epsilon_0\rho, \qquad \text{where} \qquad \epsilon = \epsilon_0 - \bar{\Omega}, \qquad (0 \le \epsilon \le 1).$$

Consider a relativistic bulk viscous fluid string spacetime governed by the linear model of $\mathbf{f}(\mathbb{R},\mathbb{T})$-gravity [13,22,23], and we have,

$$\mathbf{f}(\mathbb{R},\mathbb{T}) = \mathbb{R} + 2\alpha\mathbb{T}, \tag{2.3}$$

where $\mathbb{R}$ is the Ricci scalar, $\mathbb{T}$ is trace of energy-momentum tensor, and α is constant. The modified Einstein-Hilbert action is given by,

$$\mathbb{H} = \frac{1}{16\pi}\int [\mathbf{f}(\mathbb{R},\mathbb{T}) + \mathcal{L}_m]\sqrt{(-\mathbf{g})}d^4x, \tag{2.4}$$

where $\mathcal{L}_m$ represents the Lagrangian matter of the scalar field. The stress-energy tensor is written by,

$$T_{ij} = -\frac{2}{\sqrt{-\mathbf{g}}}\frac{\partial(\sqrt{-\mathbf{g}}\mathcal{L}_m)}{\partial\mathbf{g}^{ij}}. \tag{2.5}$$

In this framework, the Lagrangian matter depends only on $\mathbf{g}_{ij}$ without its derivatives. Accordingly, the field equation of $\mathbf{f}(\mathbb{R},\mathbb{T})$-gravity is written as,

$$\begin{aligned}\mathbf{f}_{\mathbb{R}}(\mathbb{R},\mathbb{T})\mathbf{S}(\mathcal{D},\mathcal{E}) - \frac{1}{2}\mathbf{f}(\mathbb{R},\mathbb{T})\mathbf{g}(\mathcal{D},\mathcal{E}) + (\mathbf{g}(\mathcal{D},\mathcal{E})\nabla_c^c - \nabla_{\mathcal{D}}\nabla_{\mathcal{E}})\mathbf{f}_{\mathbb{R}}(\mathbb{R},\mathbb{T})\\ = 8\pi T(\mathcal{D},\mathcal{E}) - \mathbf{f}_{\mathbb{T}}(\mathbb{R},\mathbb{T})T(\mathcal{D},\mathcal{E}) - \mathbf{f}_{\mathbb{T}}(\mathbb{R},\mathbb{T})\phi(\mathcal{D},\mathcal{E}),\end{aligned} \tag{2.6}$$

where $\mathbf{f}_{\mathbb{R}}$ and $\mathbf{f}_{\mathbb{T}}$ represent $\frac{\partial\mathbf{f}}{\partial\mathbb{R}}$, $\frac{\partial\mathbf{f}}{\partial\mathbb{T}}$ and ∇_c^c is the d'Alembert operator, respectively. Moreover, ϕ is given by,

$$\phi(\mathcal{D},\mathcal{E}) = -2T(\mathcal{D},\mathcal{E}) + \mathbf{g}(\mathcal{D},\mathcal{E})\mathcal{L}_m - 2\mathbf{G}\frac{\partial^2\mathcal{L}_m}{\partial\mathbf{G}^2}, \tag{2.7}$$

where **G** represents the inverse metric corresponding to **g**. Since the Lagrangian matter is not uniquely defined, we choose $\mathcal{L}_m = \bar{P}$. Consequently, the energy momentum tensor of bulk viscous fluid string spacetime is given in (2.1).

Then (2.7) leads to,

$$\phi(\mathcal{D},\mathcal{E}) = \bar{P}\mathbf{g}(\mathcal{D},\mathcal{E}) - 2T(\mathcal{D},\mathcal{E}). \tag{2.8}$$

From (2.3) and (2.6), we have,

$$\begin{aligned}\mathbf{S}(\mathcal{D},\mathcal{E}) &= \frac{1}{2}(\mathbb{R}+2\alpha\mathbb{T})\mathbf{g}(\mathcal{D},\mathcal{E})+) + (\mathbf{g}(\mathcal{D},\mathcal{E})\nabla_c^c - \nabla_{\mathcal{D}}\nabla_{\mathcal{E}})\mathbf{f}_{\mathbb{R}}(\mathbb{R},\mathbb{T})\\ &\quad + 8\pi T(\mathcal{D},\mathcal{E}) - 2\alpha T(\mathcal{D},\mathcal{E}) - 2\alpha\phi(\mathcal{D},\mathcal{E})\\ &= [\frac{\mathbb{R}}{2} + \alpha\mathbb{T}]\mathbf{g}(\mathcal{D},\mathcal{E}) + 8\pi T(\mathcal{D},\mathcal{E}) - 2\alpha T(\mathcal{D},\mathcal{E}) - 2\alpha\phi(\mathcal{D},\mathcal{E}).\end{aligned} \tag{2.9}$$

Now making use of (2.1) and (2.7) in the preceding equation, we have,

$$\begin{aligned}\mathbf{S}(\mathcal{D},\mathcal{E}) &= \Big[\frac{\mathbb{R}}{2} + \alpha\mathbb{T} - 4\alpha\bar{P} - 8\pi\bar{P}\Big]\mathbf{g}(\mathcal{D},\mathcal{E})\\ &\quad + \Big[8\pi\bar{P} + 8\pi\rho + 2\alpha\bar{P} + 2\alpha\rho\Big]\gamma(\mathcal{D})\gamma(\mathcal{E}) - ((8\pi+2\alpha)\lambda)\delta(\mathcal{D})\delta(\mathcal{E}).\end{aligned} \tag{2.10}$$

Now, contracting the last relation we have,

$$\mathbb{R} = \bar{P}[40\pi + 38\alpha] + [\rho+\lambda][8\pi + 6\alpha]. \tag{2.11}$$

Now, comparing (1.1) with (2.10), we have,

Theorem 2.2. *A four-dimensional relativistic bulk viscous fluid string spacetime* $(\mathbf{M}^4, \mathbf{g})$ *in* $\mathbf{f}(\mathbb{R},\mathbb{T})$*-gravity is a generalized quasi-Einstein spacetime.*

Also, we have,

Corollary 2.3. *In a relativistic* $\mathbb{BVFSS}$ *with* $\mathbf{f}(\mathbb{R},\mathbb{T})$*-gravity, the scalar curvature is expressed as,*

$$\mathbb{R} = \bar{P}[40\pi + 38\alpha] + [\rho+\lambda][8\pi + 6\alpha].$$

3. Quasi-conformal flat curvature tensor on a relativistic $\mathbb{BVFSS}$ with $\mathbf{f}(\mathbb{R},\mathbb{T})$-gravity

Yano et al. [24] formulated the notion of the quasi-conformal curvature tensor, and it is expressed as,

$$\begin{aligned}\mathbb{C}^*(\mathcal{D},\mathcal{E})\mathcal{F} &= a\mathbf{R}(\mathcal{D},\mathcal{E})\mathcal{F} + b\Big[\mathbf{S}(\mathcal{E},\mathcal{F})\mathcal{D} - \mathbf{S}(\mathcal{D},\mathcal{F})\mathcal{E} + \mathbf{g}(\mathcal{E},\mathcal{F})\mathbb{Q}\mathcal{D} - \mathbf{g}(\mathcal{D},\mathcal{F})\mathbb{Q}\mathcal{E}\Big]\\ &\quad - \frac{\mathbb{R}}{4}\Big[\frac{a}{3} + 2b\Big]\Big[\mathbf{g}(\mathcal{E},\mathcal{F})\mathcal{D} - \mathbf{g}(\mathcal{D},\mathcal{F})\mathcal{E}\Big],\end{aligned} \tag{3.1}$$

where $\mathbb{R}$ is the scalar curvature, a, b are constants, and $\mathbb{Q}$ denotes the Ricci operator.

From the preceding relation, we have,

$$\begin{aligned}\mathbb{C}'^{*}(\mathcal{D},\mathcal{E},\mathcal{F},\mathcal{H}) &= a\mathbf{R}'(\mathcal{D},\mathcal{E},\mathcal{F},\mathcal{H})\\ &+ b\Big[\mathbf{S}(\mathcal{E},\mathcal{F})\mathbf{g}(\mathcal{D},\mathcal{H}) - \mathbf{S}(\mathcal{D},\mathcal{F})\mathbf{g}(\mathcal{E},\mathcal{H})\\ &+ \mathbf{g}(\mathcal{E},\mathcal{F})\mathbf{S}(\mathcal{D},\mathcal{H}) - \mathbf{g}(\mathcal{D},\mathcal{F})\mathbf{S}(\mathcal{E},\mathcal{H})\Big]\\ &- \frac{\mathbb{R}}{4}\Big[\frac{a}{3}+2b\Big]\Big[\mathbf{g}(\mathcal{E},\mathcal{F})\mathbf{g}(\mathcal{D},\mathcal{H}) - \mathbf{g}(\mathcal{D},\mathcal{F})\mathbf{g}(\mathcal{E},\mathcal{H})\Big], \end{aligned} \tag{3.2}$$

where $\mathbf{R}'(\mathcal{D},\mathcal{E},\mathcal{F},\mathcal{H}) = \mathbf{g}(\mathbf{R}(\mathcal{D},\mathcal{E})\mathcal{F},\mathcal{H})$ and $\mathbb{C}'^{*}(\mathcal{D},\mathcal{E},\mathcal{F},\mathcal{H}) = \mathbf{g}(\mathbb{C}'^{*}(\mathcal{D},\mathcal{E})\mathcal{F},\mathcal{H})$.

If $\mathbb{C}'^{*}(\mathcal{D},\mathcal{E},\mathcal{F},\mathcal{H}) = 0$ in the preceding equation, we have,

$$\begin{aligned} a\mathbf{R}(\mathcal{D},\mathcal{E},\mathcal{F},\mathcal{H}) &+ b\Big[\mathbf{S}(\mathcal{E},\mathcal{F})\mathbf{g}(\mathcal{D},\mathcal{H}) - \mathbf{S}(\mathcal{D},\mathcal{F})\mathbf{g}(\mathcal{E},\mathcal{H})\\ &+ \mathbf{g}(\mathcal{E},\mathcal{F})\mathbf{S}(\mathcal{D},\mathcal{H}) - \mathbf{g}(\mathcal{D},\mathcal{F})\mathbf{S}(\mathcal{E},\mathcal{H})\Big]\\ &- \frac{\mathbb{R}}{4}\Big[\frac{a}{3}+2b\Big]\Big[\mathbf{g}(\mathcal{E},\mathcal{F})\mathbf{g}(\mathcal{D},\mathcal{H}) - \mathbf{g}(\mathcal{D},\mathcal{F})\mathbf{g}(\mathcal{E},\mathcal{H})\Big] = 0. \end{aligned} \tag{3.3}$$

Contracting $\mathcal{D}$ and $\mathcal{H}$ in the previous relation, we get,

$$(a+2b)\mathbf{S}(\mathcal{E},\mathcal{F}) = (a+2b)\frac{\mathbb{R}}{4}\mathbf{g}(\mathcal{E},\mathcal{F}).$$

Finally,

$$\mathbf{S}(\mathcal{E},\mathcal{F}) = \frac{\mathbb{R}}{4}\mathbf{g}(\mathcal{E},\mathcal{F}). \tag{3.4}$$

Equating (3.4) and (2.10), we have,

$$\begin{aligned}\frac{\mathbb{R}}{4}\mathbf{g}(\mathcal{D},\mathcal{E}) &= \Big[\frac{1}{2}\mathbb{R} - 4\bar{P}(\alpha+2\pi) + \alpha\mathbb{T}\Big]\mathbf{g}(\mathcal{D},\mathcal{E})\\ &+ \Big[8\pi\bar{P} + 8\pi\rho + 2\alpha\bar{P} + 2\alpha\rho\Big]\gamma(\mathcal{D})\gamma(\mathcal{E}) - \Big[(8\pi+2\alpha)\lambda\Big]\delta(\mathcal{D})\delta(\mathcal{E}). \end{aligned} \tag{3.5}$$

Equating left hand side and right hand side, we get,

$$\Big[8\pi\bar{P} + 8\pi\rho + 2\alpha\bar{P} + 2\alpha\rho\Big] = 0, \tag{3.6}$$

$$\Big[(8\pi+2\alpha)\lambda\Big] = 0, \tag{3.7}$$

$$\Big[\frac{\mathbb{R}}{2} - 4\bar{P}(\alpha+2\pi) + \alpha\mathbb{T}\Big] = \frac{\mathbb{R}}{4}. \tag{3.8}$$

From (3.6), taking $\bar{P}$ and ρ as common, we get, $(8\pi+2\alpha)(\bar{P}+\rho) = 0$. If $(8\pi+2\alpha) \neq 0$, then $\bar{P}+\rho = 0 \implies \bar{P} = -\rho$. From (3.7), if $(8\pi+2\alpha) \neq 0$, then $\lambda = 0$.

Remark 3.1. *Hence $\lambda = 0$, shows that the string tension contribution disappears in the quasi-conformally flat case considered here. This result is therefore, considered as a limiting case of relativistic $\mathbb{BVFSS}$.*

Theorem 3.2. *A quasi-conformal flat curvature tensor with relativistic $\mathbb{BVFSS}$ admitting $\mathbf{f}(\mathbb{R},\mathbb{T})$-gravity expresses the vacuum dominated era.*

Similarly, from (3.8), substituting our obtained results and using $\mathbb{T} = -4\bar{P}$, we get,

$$\bar{P} = \frac{\mathbb{R}}{32(\alpha + \pi)} \tag{3.9}$$

and

$$\rho = -\frac{\mathbb{R}}{32(\alpha + \pi)}. \tag{3.10}$$

Also, we get,

Corollary 3.3. *The corresponding effective equation of state parameter is obtained as*

$$\epsilon_{eff} = \frac{\bar{P}}{\rho} = -1.$$

3.1. Energy condition analysis

The standard energy conditions [25] are parametrized by energy density ρ, and total pressure $\bar{P}$ is given by,

$$\textbf{NEC} \Longleftrightarrow \rho + \bar{P} \geq 0, \tag{3.11}$$

$$\textbf{WEC} \Longleftrightarrow \rho + \bar{P} \geq 0 \quad \text{and} \quad \rho \geq 0, \tag{3.12}$$

$$\textbf{DEC} \Longleftrightarrow \rho \pm \bar{P} \geq 0 \quad \text{and} \quad \rho \geq 0, \tag{3.13}$$

$$\textbf{SEC} \Longleftrightarrow \rho + 3\bar{P} \geq 0 \quad \text{and} \quad \rho + \bar{P} \geq 0. \tag{3.14}$$

Here, **SEC**, **NEC**, **WEC**, and **DEC** denote the strong energy condition, null energy condition, weak energy condition, and dominant energy condition, respectively. Using the explicit expressions for the effective pressure and energy density obtained from the quasi-conformal flatness condition, the energy conditions are examined within the parameter domain considered in the present analysis. From Figure 1, the energy density cannot be negative for $\mathbb{R} \in [-2.3, -2]$ and $\alpha \in [1, 1.3]$. In our case, **NEC** is saturated, and the energy density is positive from our result, which implies that **WEC** is satisfied. From Figure 2, $\rho - \bar{P}$ is strictly positive for $\mathbb{R} \in [-2.3, -2]$ and $\alpha \in [1, 1.3]$, which conveys that **DEC** is satisfied. For Figure 3, $\rho + 3\bar{P}$, which is negative for $\mathbb{R} \in [-2.3, -2]$ and $\alpha \in [1, 1.3]$, shows that **SEC** is violated.

The obtained violation of the **SEC** is also relevant from cosmological point of view. In the Friedmann-Lemaître-Robertson-Walker framework, the Friedmann acceleration equation is given by,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3\bar{P}), \tag{3.15}$$

which shows that the accelerating expansion of the universe ($\ddot{a} > 0$) is associated with violation of **SEC**. Therefore, the **SEC** violation obtained in our model is consistent with the behavior associated with the late-time accelerated expansion of the universe.

From the above result, relativistic $\mathbb{BVFSS}$ satisfies the equation of parameter $\epsilon = -1$ and can be interpreted as exhibiting vacuum energy or dark energy-like behavior within the linear $\mathbf{f}(\mathbb{R}, \mathbb{T})$-framework.

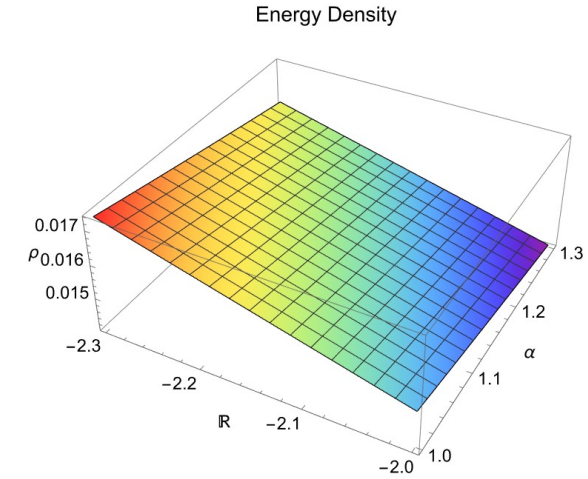


Figure 1. Evolution of ρ associated with $\mathbb{R}$ and α.

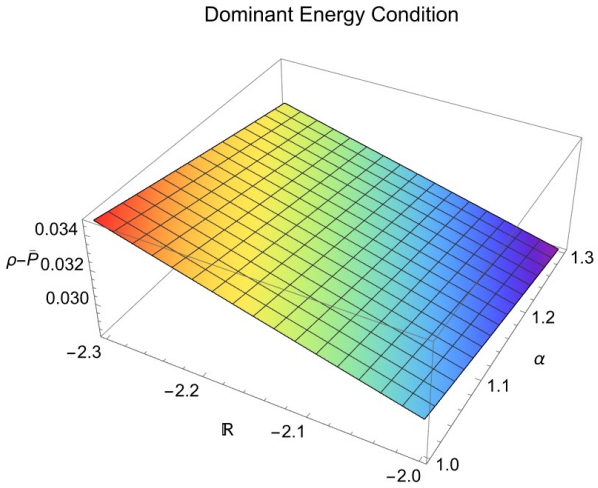


Figure 2. Evolution of $\rho - \bar{P}$ associated with $\mathbb{R}$ and α.

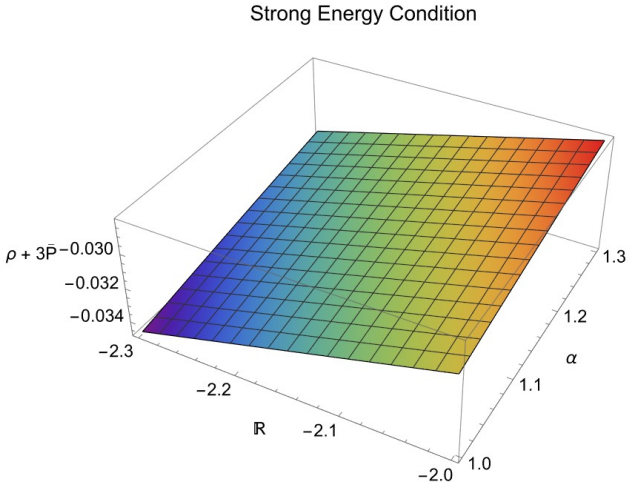


Figure 3. Evolution of $\rho + 3\bar{P}$ associated with $\mathbb{R}$ and α.

4. Q-solitons on a relativistic $\mathbb{BVFSS}$ with $\mathbf{f}(\mathbb{R}, \mathbb{T})$-gravity

The Ricci flow theory [26], proposed by Hamilton in 1982 [27], has played a vital role in the expansion of differential geometry and non-linear geometric partial differential equations. As a notable generalization of the Ricci flow, Udriste [28] proposed the Riemann flow, and it is given by,

$$\frac{\partial}{\partial t}\bar{\mathbf{g}}(t) = -2\mathbf{R}\mathbf{g}(t), \tag{4.1}$$

where $\mathbf{R}$ is a $(0,4)$-type Riemann curvature tensor and $\bar{\mathbf{g}} = \frac{1}{2}\mathbf{g}\cdot\mathbf{g}$ represents the Kulkarni-Nomizu product [29], and it is written as,

$$\begin{aligned}(\bar{\omega}\cdot\bar{\theta})(\mathcal{D},\mathcal{E},\mathcal{F},\mathcal{H}) = \bar{\omega}(\mathcal{D},\mathcal{H})\bar{\theta}(\mathcal{E},\mathcal{F}) + \bar{\omega}(\mathcal{E},\mathcal{F})\bar{\theta}(\mathcal{D},\mathcal{H})\\ - \bar{\omega}(\mathcal{D},\mathcal{F})\bar{\theta}(\mathcal{E},\mathcal{H}) - \bar{\omega}(\mathcal{E},\mathcal{H})\bar{\theta}(\mathcal{D},\mathcal{F}),\end{aligned} \tag{4.2}$$

where $\bar{\omega}$, $\bar{\theta}$ are covariant $(0,2)$-tensors and $\mathcal{D},\mathcal{E},\mathcal{F},\mathcal{H} \in \chi(\mathbf{M})$, respectively.

A Riemann soliton is a distinguished self-similar solution of a Riemann flow on a Riemannian manifold $(\mathbf{M}^N, \mathbf{g})$, evolving under diffeomorphisms and dilations, and it is defined by,

$$\mathbf{g}\bigodot\mathscr{L}_{\mathcal{V}}\mathbf{g} + \varphi\mathbf{g}\bigodot\mathbf{g} + 2\mathbf{R} = 0, \tag{4.3}$$

where φ is scalar and $\mathscr{L}_{\mathcal{V}}\mathbf{g}$ is the Lie derivative of the vector field $\mathcal{V}$. Moreover, the above soliton equation is said to be shrinking if $\varphi < 0$, steady if $\varphi = 0$, and expanding if $\varphi > 0$, respectively. Also, if $\mathcal{V}$=gradΦ, then the foregoing equation becomes,

$$\mathbf{g}\bigodot\nabla^2\Phi + \varphi\mathbf{g}\bigodot\mathbf{g} + 2\mathbf{R} = 0, \tag{4.4}$$

where Φ is a smooth function and it is known as the gradient Riemann soliton.

A specific $(0,4)$-tensor introduced by Mantica and Suh [30] is described as $\mathbf{Q}$-tensor, and it is given by,

$$\begin{aligned}\mathbf{Q}(\mathcal{D},\mathcal{E},\mathcal{F},\mathcal{H}) = \mathbf{R}(\mathcal{D},\mathcal{E},\mathcal{F},\mathcal{H})\\ - \frac{\Psi}{N-1}\Big[\mathbf{g}(\mathcal{E},\mathcal{F})\mathbf{g}(\mathcal{D},\mathcal{H}) - \mathbf{g}(\mathcal{D},\mathcal{F})\mathbf{g}(\mathcal{E},\mathcal{H})\Big],\end{aligned} \tag{4.5}$$

where Ψ is a scalar function and $\mathcal{D},\mathcal{E},\mathcal{F},\mathcal{H} \in \chi(\mathbf{M})$. Such generalized curvature tensors are extensively studied in contact geometry and recent cosmological models.

Inspired by the notion of Riemann solitons, Gaurab Mitra et al. [31] introduced the concept of $\mathbf{Q}$-soliton obtained by replacing the Riemann curvature tensor $\mathbf{R}$ with $\mathbf{Q}$-tensor. The corresponding soliton equation on $(\mathbf{M}^N, \mathbf{g})$ is given by,

$$\mathbf{g}\bigodot\mathscr{L}_{\mathcal{V}}\mathbf{g} + \varphi\mathbf{g}\bigodot\mathbf{g} + 2\mathbf{Q} = 0, \tag{4.6}$$

where $\mathscr{L}_{\mathcal{V}}\mathbf{g}$ is the Lie derivative with respect to vector field $\mathcal{V}$ and φ is scalar. According to φ, the above soliton equation is said to be shrinking if $\varphi < 0$, expanding if $\varphi > 0$, and steady if $\varphi = 0$, respectively. Moreover, if $\mathcal{V}$=gradΦ, then the gradient $\mathbf{Q}$ soliton relation is given by,

$$\mathbf{g}\bigodot\nabla^2\Phi + \varphi\mathbf{g}\bigodot\mathbf{g} + 2\mathbf{Q} = 0, \tag{4.7}$$

where Φ is a smooth function.

Kuhnel et al. [32] proposed the concept of the conformal killing vector field. The conformal vector field $\mathcal{V}$ satisfies,

$$\mathcal{L}_{\mathcal{V}}\mathbf{g}(\mathcal{D},\mathcal{E}) = 2\Omega\mathbf{g}(\mathcal{D},\mathcal{E}), \tag{4.8}$$

where $\Omega \in C^{\infty}(\mathbf{M})$ and $\mathcal{D},\mathcal{E} \in \chi(\mathbf{M})$.

Making use of (4.2) and (4.5) in (4.6) and considering it in $(\mathbf{M}^4, \mathbf{g})$ we have,

$$\begin{aligned}
&2\mathbf{R}(\mathcal{D},\mathcal{E},\mathcal{F},\mathcal{H}) - \frac{2\Psi}{3}\Big[\mathbf{g}(\mathcal{E},\mathcal{F})\mathbf{g}(\mathcal{D},\mathcal{H}) - \mathbf{g}(\mathcal{D},\mathcal{F})\mathbf{g}(\mathcal{E},\mathcal{H})\Big] \\
&= -2\varphi\Big[\mathbf{g}(\mathcal{D},\mathcal{H})\mathbf{g}(\mathcal{E},\mathcal{F}) - \mathbf{g}(\mathcal{D},\mathcal{F})\mathbf{g}(\mathcal{E},\mathcal{H})\Big] - \Big[\mathbf{g}(\mathcal{D},\mathcal{H})\mathcal{L}_{\mathcal{V}}\mathbf{g}(\mathcal{E},\mathcal{F}) + \mathbf{g}(\mathcal{E},\mathcal{F})\mathcal{L}_{\mathcal{V}}\mathbf{g}(\mathcal{D},\mathcal{H})\Big] \\
&\quad + \Big[\mathbf{g}(\mathcal{D},\mathcal{F})\mathcal{L}_{\mathcal{V}}\mathbf{g}(\mathcal{E},\mathcal{H}) + \mathbf{g}(\mathcal{E},\mathcal{H}\mathcal{L}_{\mathcal{V}}\mathbf{g}(\mathcal{D},\mathcal{F})\Big].
\end{aligned}$$

Contracting $\mathcal{D}$ and $\mathcal{H}$ in the above equation, we have,

$$2\mathbf{S}(\mathcal{E},\mathcal{F}) = -\Big[2div\mathcal{V} + 6\varphi - 2\Psi\Big]\mathbf{g}(\mathcal{E},\mathcal{F}) - 2\mathcal{L}_{\mathcal{V}}\mathbf{g}(\mathcal{E},\mathcal{F}). \tag{4.9}$$

Here, the vector field $\mathcal{V}$ is a conformal killing vector field, so the preceding equation will be written as,

$$2\Omega\mathbf{g}(\mathcal{E},\mathcal{F}) + \mathbf{S}(\mathcal{E},\mathcal{F}) + \Big[div\mathcal{V} + 3\varphi - \Psi\Big]\mathbf{g}(\mathcal{E},\mathcal{F}) = 0. \tag{4.10}$$

Again, contracting the preceding equation, we get,

$$8\Omega + \mathbb{R} + 4div\mathcal{V} + 12\varphi - 4\Psi = 0.$$

Making use of (2.11) in the above relation, we get,

$$\varphi = \frac{4\Psi - 8\Omega - \bar{P}[40\pi + 38\alpha] - [\rho + \lambda][8\pi + 6\alpha] - 4div\mathcal{V}}{12}. \tag{4.11}$$

Hence, we have,

Theorem 4.1. *Within the context of relativistic* $\mathbb{BVFSS}$ *with* $\mathbf{f}(\mathbb{R},\mathbb{T})$*-gravity admitting* $\mathbf{Q}$*-soliton, where the vector field* $\mathcal{V}$ *is conformal killing, the soliton is shrinking if*

$$\frac{4\Psi - 8\Omega - \bar{P}[40\pi + 38\alpha] - [\rho + \lambda][8\pi + 6\alpha] - 4div\mathcal{V}}{12} < 0,$$

and steady if

$$\frac{4\Psi - 8\Omega - \bar{P}[40\pi + 38\alpha] - [\rho + \lambda][8\pi + 6\alpha] - 4div\mathcal{V}}{12} = 0,$$

and expanding if

$$\frac{4\Psi - 8\Omega - \bar{P}[40\pi + 38\alpha] - [\rho + \lambda][8\pi + 6\alpha] - 4div\mathcal{V}}{12} > 0,$$

respectively.

Corollary 4.2. *If* $\Psi = 0$ *in* (4.11)*, then the above result satisfies with the Riemann soliton with the same conditions.*

Remark 4.3. *As in the previous section, the quasi-conformal flatness condition leads us to certain restrictions such as $\lambda = 0$ and $\bar{P} = -\rho$. So, implying this in our obtained result will provide an additional insights into the corresponding soliton structure.*

From (4.11), we have,

$$\varphi = \frac{4\Psi - 8\Omega + 32\rho(\pi + \alpha) - 4div\mathcal{V}}{12}. \tag{4.12}$$

Since ($\lambda = 0$) corresponds to the vanishing of the string tension, the relativistic $\mathbb{BVFSS}$ consequently reduces to a relativistic bulk viscous fluid spacetime $\mathbb{BVFS}$ and obeys our soliton nature.

Definition 4.4. *Let $(\mathbf{M}^N, \mathbf{g})$ be a Riemannian manifold with a unit vector field $\mathcal{V}$, which is known as an anti-torqued vector field [21, 33] if it satisfies the upcoming relation,*

$$\nabla_{\mathcal{D}}\mathcal{V} = \varsigma(\mathcal{D} - \gamma(\mathcal{D})\mathcal{V}), \tag{4.13}$$

where γ is the one-form associated with anti-torqued vector field $\mathcal{V}$, $\varsigma \in C^\infty(\mathbf{M})$ and $\mathcal{D} \in \chi(\mathbf{M})$. The preceding condition imparts to the flow of ξ with the rigid geometric structure. Moreover, the flow lines of $\mathcal{V}$ are geodesics, and the non-killing nature is completely governed by ς, which determines the expansion and shear of the orthogonal distribution to $\mathcal{V}$. The existence of an anti-torqued vector field endows a well-defined geometric structure on a spacetime manifold by restricting the covariant derivative of the vector field and alters the nature of curvature tensors. Consequently, this geometric framework determines the evolution of the soliton and may generate new cosmological models endowed with modified gravity theories.

For a $\mathbb{BVFSS}$ admitting anti-torqued vector field $\mathcal{V}$, the upcoming relations will hold:

$$(\nabla_{\mathcal{D}}\gamma)(\mathcal{E}) = \varsigma[\mathbf{g}(\mathcal{D}, \mathcal{E}) - \gamma(\mathcal{D})\gamma(\mathcal{E})], \tag{4.14}$$

$$\gamma(\nabla_{\mathcal{V}}\mathcal{V}) = -2\varsigma, \nabla_{\mathcal{V}}\mathcal{V} = 2\varsigma\mathcal{V}, \tag{4.15}$$

$$\begin{aligned}\mathbf{R}(\mathcal{D}, \mathcal{E})\mathcal{V} = {} & \varsigma^2[\gamma(\mathcal{D})\mathcal{E} - \gamma(\mathcal{E})\mathcal{D}] + \mathcal{D}(\varsigma)[\mathcal{E} - \gamma(\mathcal{E})\mathcal{V}] \\ & - \mathcal{E}(\varsigma)[\mathcal{D} - \gamma(\mathcal{D})\mathcal{V}],\end{aligned} \tag{4.16}$$

$$\mathbf{R}(\mathcal{D}, \mathcal{V})\mathcal{V} = \varsigma^2[\mathcal{D} + \gamma(\mathcal{D})\mathcal{V}] + 2\mathcal{D}(\varsigma)\mathcal{V} - \mathcal{V}(\varsigma)[\mathcal{D} - \gamma(\mathcal{D})\mathcal{V}], \tag{4.17}$$

$$\begin{aligned}\gamma(\mathbf{R}(\mathcal{D}, \mathcal{E})\mathcal{F}) = {} & \varsigma^2[\gamma(\mathcal{E})\mathbf{g}(\mathcal{D}, \mathcal{F}) - \gamma(\mathcal{D})\mathbf{g}(\mathcal{E}, \mathcal{F})] \\ & - \mathcal{D}(\varsigma)[\mathbf{g}(\mathcal{E}, \mathcal{F}) - \gamma(\mathcal{E})\gamma(\mathcal{F})] + \mathcal{E}(\varsigma)[\mathbf{g}(\mathcal{D}, \mathcal{F}) - \gamma(\mathcal{D})\gamma(\mathcal{F})],\end{aligned} \tag{4.18}$$

$$(\mathscr{L}_{\mathcal{V}}\mathbf{g})(\mathcal{D}, \mathcal{E}) = 2\varsigma[\mathbf{g}(\mathcal{D}, \mathcal{E}) - \gamma(\mathcal{D})\gamma(\mathcal{E})]. \tag{4.19}$$

Making use of (4.13) in $\mathbf{R}(\mathcal{D}, \mathcal{E})\varsigma = \nabla_{\mathcal{D}}\nabla_{\mathcal{E}}\varsigma - \nabla_{\mathcal{E}}\nabla_{\mathcal{D}}\varsigma - \nabla_{[\mathcal{D},\mathcal{E}]}\varsigma$ and by simplification, we obtain (4.16). Moreover, (4.17) and (4.18) come from (4.16). Here, differentiating $\mathbf{g}$ with respect to ς and using (4.14), we get (4.19).

From (4.9) where the vector field is anti-torqued vector field and by (4.19), we have,

$$2\varsigma[\mathbf{g}(\mathcal{D}, \mathcal{E}) - \gamma(\mathcal{D})\gamma(\mathcal{E})] + \mathbf{S}(\mathcal{D}, \mathcal{E}) + [div\mathcal{V} + 3\varphi - \Psi]\mathbf{g}(\mathcal{D}, \mathcal{E}) = 0. \tag{4.20}$$

Contracting the preceding equation, we have,

$$6\varsigma + \mathbb{R} + 4div\mathcal{V} + 12\varphi - 4\Psi = 0. \tag{4.21}$$

Using the value of (2.11) in above equation, we get,

$$\varphi = \frac{4\Psi - 4div\mathcal{V} - \bar{P}[40\pi + 38\alpha] - [\rho + \lambda][8\pi + 6\alpha] - 6\varsigma}{12}. \tag{4.22}$$

Hence, we state,

Theorem 4.5. *In the framework relativistic $\mathbb{BVFSS}$ with $\mathbf{f}(\mathbb{R}, \mathbb{T})$-gravity admitting $\mathbf{Q}$-soliton, where the vector field $\mathcal{V}$ is an anti-torqued vector field, the soliton is shrinking if*

$$\frac{4\Psi - 4div\mathcal{V} - \bar{P}[40\pi + 38\alpha] - [\rho + \lambda][8\pi + 6\alpha] - 6\varsigma}{12} < 0,$$

and steady if

$$\frac{4\Psi - 4div\mathcal{V} - \bar{P}[40\pi + 38\alpha] - [\rho + \lambda][8\pi + 6\alpha] - 6\varsigma}{12} = 0,$$

and expanding if

$$\frac{4\Psi - 4div\mathcal{V} - \bar{P}[40\pi + 38\alpha] - [\rho + \lambda][8\pi + 6\alpha] - 6\varsigma}{12} > 0.$$

Hence,

Corollary 4.6. *If $\Psi = 0$ in above relation, then the foregoing result satisfies with Riemann soliton structure.*

Remark 4.7. *As in the previous section, for a quasi-conformal flatness condition, we will carry forward our limitations to this condition, thus we get,*

$$\varphi = \frac{4\Psi - 4div\mathcal{V} + 32\rho[\pi + \alpha] - 6\varsigma}{12}.$$

So, the relativistic $\mathbb{BVFSS}$ reduces to the relativistic $\mathbb{BVFS}$ and satisfies the soliton nature.

Now, by applying the characterization of the Lie derivative, we have,

$$\mathscr{L}_{\mathcal{V}}\mathbf{g}(\mathcal{D}, \mathcal{E}) = \mathbf{g}(\nabla_{\mathcal{D}}\mathcal{V}, \mathcal{E}) + \mathbf{g}(\nabla_{\mathcal{E}}\mathcal{V}, \mathcal{D}). \tag{4.23}$$

By using the above relation in (4.9), we have,

$$\mathbf{g}(\nabla_{\mathcal{D}}\mathcal{V}, \mathcal{E}) + \mathbf{g}(\nabla_{\mathcal{E}}\mathcal{V}, \mathcal{D}) + \mathbf{S}(\mathcal{D}, \mathcal{E}) + [div\mathcal{V} + 3\varphi - \Psi] = 0. \tag{4.24}$$

Inserting (2.10) in the above relation, we get,

$$\begin{aligned}\mathbf{g}(\nabla_{\mathcal{D}}\mathcal{V}, \mathcal{E}) + \mathbf{g}(\nabla_{\mathcal{E}}\mathcal{V}, \mathcal{D}) + n\gamma(\mathcal{D})\gamma(\mathcal{E}) + o\delta(\mathcal{D})\delta(\mathcal{E}) \\ + [div\mathcal{V} + 3\varphi - \Psi + m]\mathbf{g}(\mathcal{D}, \mathcal{E}) = 0.\end{aligned} \tag{4.25}$$

The exterior derivative of ω leads to,

$$2(d\omega)(\mathcal{D}, \mathcal{E}) = \mathbf{g}(\nabla_{\mathcal{D}}\mathcal{V}, \mathcal{E}) - \mathbf{g}(\nabla_{\mathcal{E}}\mathcal{V}, \mathcal{D}), \tag{4.26}$$

where $\omega(\mathcal{D}) = \mathbf{g}(\mathcal{D}, \mathcal{V})$. The skew-symmetric of $d\omega$ is $\mathbf{g}(\mathcal{D}, A\mathcal{E}) = -\mathbf{g}(A\mathcal{D}, \mathcal{E})$, so $d\omega(\mathcal{D}, \mathcal{E}) = -\mathbf{g}(A\mathcal{D}, \mathcal{E})$. Hence, the (4.26) will be,

$$\mathbf{g}(\nabla_{\mathcal{D}}\mathcal{V}, \mathcal{E}) - \mathbf{g}(\nabla_{\mathcal{E}}\mathcal{V}, \mathcal{D}) = -2\mathbf{g}(A\mathcal{D}, \mathcal{E}). \tag{4.27}$$

Mixing (4.27) with (4.25), we have,

$$\nabla_{\mathcal{D}}\mathcal{V} = -A\mathcal{D} - \frac{1}{2}\Big[div\mathcal{V} + 3\varphi - \Psi + m\Big]\mathcal{D} - \frac{1}{2}n\gamma(\mathcal{D})\xi - \frac{1}{2}o\delta(\mathcal{D})\zeta. \tag{4.28}$$

By taking the covariant derivative with $\mathcal{E}$ in the above equation, we have,

$$\begin{aligned}\nabla_{\mathcal{E}}\nabla_{\mathcal{D}}\mathcal{V} =& -\nabla_{\mathcal{E}}(A\mathcal{D}) - \frac{1}{2}(\mathcal{E}(div\mathcal{V} + 3\varphi - \Psi + m))\mathcal{D}\\ &- \frac{1}{2}(div\mathcal{V} + 3\varphi - \Psi + m)\nabla_{\mathcal{E}}\mathcal{D} - \frac{1}{2}\gamma(\mathcal{D})\xi\mathcal{E}n\\ &- \frac{1}{2}n\nabla_{\mathcal{E}}(\gamma(\mathcal{D})\xi) - \frac{1}{2}\delta(\mathcal{D})\zeta\mathcal{E}o - \frac{1}{2}o\nabla_{\mathcal{E}}(\delta(\mathcal{D})\zeta.\end{aligned} \tag{4.29}$$

Interchanging $\mathcal{D}$ and $\mathcal{E}$ in the preceding relation, we get,

$$\begin{aligned}\nabla_{\mathcal{D}}\nabla_{\mathcal{E}}\mathcal{V} =& -\nabla_{\mathcal{D}}(A\mathcal{E}) - \frac{1}{2}(\mathcal{D}(div\mathcal{V} + 3\varphi - \Psi + m))\mathcal{E}\\ &- \frac{1}{2}(div\mathcal{V} + 3\varphi - \Psi + m)\nabla_{\mathcal{D}}\mathcal{E} - \frac{1}{2}\gamma(\mathcal{E})\xi\mathcal{D}n\\ &- \frac{1}{2}n\nabla_{\mathcal{D}}(\gamma(\mathcal{E})\xi) - \frac{1}{2}\delta(\mathcal{E})\zeta\mathcal{D}o - \frac{1}{2}o\nabla_{\mathcal{D}}(\delta(\mathcal{E})\zeta.\end{aligned} \tag{4.30}$$

Similarly,

$$\nabla_{[\mathcal{D},\mathcal{E}]}\mathcal{V} = -A[\mathcal{D}, \mathcal{E}] - \frac{1}{2}[div\mathcal{V} + 3\varphi - \Psi + m][\mathcal{D}, \mathcal{E}] - \frac{1}{2}m\gamma([\mathcal{D}, \mathcal{E}])\xi - \frac{1}{2}o\delta([\mathcal{D}, \mathcal{E}])\zeta. \tag{4.31}$$

Using (4.29)–(4.31), in the Riemann curvature tensor formula, we have,

$$\begin{aligned}\mathbf{R}(\mathcal{D}, \mathcal{E})\mathcal{V} =& (\nabla_{\mathcal{E}}A)\mathcal{D} - (\nabla_{\mathcal{D}}A)\mathcal{E} - \frac{1}{2}(\mathcal{D}div\mathcal{V} + 3\mathcal{D}\varphi - \mathcal{D}\Psi + \mathcal{D}m)\mathcal{E}\\ &+ \frac{1}{2}(\mathcal{E}div\mathcal{V} + 3\mathcal{E}\varphi - \mathcal{E}\Psi + \mathcal{E}m)\mathcal{D} - \frac{1}{2}\gamma(\mathcal{E})\xi\mathcal{D}n + \frac{1}{2}\gamma(\mathcal{D})\xi\mathcal{E}n\\ &+ \frac{1}{2}n(\gamma(\mathcal{D})\mathcal{E} - \gamma(\mathcal{E})\mathcal{D}) - \frac{1}{2}\delta(\mathcal{D})(\mathcal{E}n)\zeta\\ &+ \frac{1}{2}o\Big(\delta(\mathcal{D})\nabla_{\mathcal{E}}\zeta - \delta(\mathcal{E})\nabla_{\mathcal{D}}\zeta + \mathbf{g}(\mathcal{D}, \nabla_{\mathcal{E}}\zeta)\zeta - \mathbf{g}(\mathcal{E}, \nabla_{\mathcal{D}}\zeta)\zeta\Big).\end{aligned} \tag{4.32}$$

By taking the inner product with $\mathcal{F}$, we have,

$$\begin{aligned}\mathbf{g}(\mathbf{R}(\mathcal{D},\mathcal{E})\mathcal{V},\mathcal{H}) &= \mathbf{g}((\nabla_{\mathcal{E}}A)\mathcal{D},\mathcal{H}) - \mathbf{g}((\nabla_{\mathcal{D}}A)\mathcal{E},\mathcal{H})\\ &\quad - \frac{1}{2}\Big(\mathcal{D}div\mathcal{V} + 3\mathcal{D}\varphi - \mathcal{D}\Psi + \mathcal{D}m\Big)\mathbf{g}(\mathcal{E},\mathcal{H})\\ &\quad + \frac{1}{2}\Big(\mathcal{E}div\mathcal{V} + 3\mathcal{E}\varphi - \mathcal{E}\Psi + \mathcal{E}m\Big)\mathbf{g}(\mathcal{D},\mathcal{H})\\ &\quad - \frac{1}{2}\gamma(\mathcal{E})\gamma(\mathcal{H})\mathcal{D}n + \frac{1}{2}\gamma(\mathcal{D})\gamma(\mathcal{H})\mathcal{E}n\\ &\quad + \frac{1}{2}n(\gamma(\mathcal{D})\mathbf{g}(\mathcal{E},\mathcal{H}) - \gamma(\mathcal{E})\mathbf{g}(\mathcal{D},\mathcal{H})\\ &\quad - \frac{1}{2}\delta(\mathcal{E})\delta(\mathcal{H})\mathcal{D}o + \frac{1}{2}\delta(\mathcal{D})\delta(\mathcal{H})\mathcal{E}o\\ &\quad + \frac{1}{2}o\Big(\delta(\mathcal{H}\mathbf{g}(\nabla_{\mathcal{D}}\zeta,\mathcal{E}) - \delta(\mathcal{H})\mathbf{g}(\nabla_{\mathcal{E}}\zeta,\mathcal{D}). \end{aligned} \tag{4.33}$$

As $d\omega$ is closed and $\nabla_{\mathcal{H}}A$ is skew self-adjoint,

$$\mathbf{g}(\mathcal{D},(\nabla_{\mathcal{H}}A)\mathcal{E}) + \mathbf{g}(\mathcal{E},(\nabla_{\mathcal{D}}A)\mathcal{H}) + \mathbf{g}(\mathcal{H},(\nabla_{\mathcal{E}}A)\mathcal{D}) = 0. \tag{4.34}$$

Now, contracting (4.33), we get,

$$\begin{aligned}\mathbf{S}(\mathcal{E},\mathcal{V}) &= -(divA)\mathcal{E} + \frac{3}{2}\Big(\mathcal{E}m + \mathcal{E}div\mathcal{V} + 3\mathcal{E}\varphi - \mathcal{E}\Psi\Big)\\ &\quad - \frac{1}{2}\gamma(\mathcal{E})\xi n - \frac{1}{2}\mathcal{E}n - \frac{3}{2}n\gamma(\mathcal{E}) - \frac{1}{2}\delta(\mathcal{E})\zeta o\\ &\quad + \frac{1}{2}\mathcal{E}o - \frac{1}{2}o(div\zeta)\delta(\mathcal{E}). \end{aligned} \tag{4.35}$$

Now applying (2.10) in (4.35), we get,

$$\begin{aligned}(divA)\mathcal{H} &= -m\mathbf{g}(\mathcal{H},\mathcal{D}) - n\gamma(\mathcal{H})\gamma(\mathcal{D}) - o\delta(\mathcal{H})\delta(\mathcal{D})\\ &\quad + \frac{3}{2}\Big(\mathcal{H}m + \mathcal{H}div\mathcal{D} + 3\mathcal{H}\varphi - \mathcal{H}\Psi\Big)\\ &\quad - \frac{1}{2}\gamma(\mathcal{H})\xi n - \frac{1}{2}\mathcal{H}n - \frac{3}{2}n\gamma(\mathcal{H}) - \frac{1}{2}\delta(\mathcal{H})\zeta o\\ &\quad + \frac{1}{2}\mathcal{H}o - \frac{1}{2}o(div\zeta)\delta(\mathcal{H}), \end{aligned} \tag{4.36}$$

where

$$m = \Big[\tfrac{\mathbb{R}}{2} + \alpha(-5\bar{P} - \rho - \lambda) - 4\alpha\bar{P} - 8\pi\bar{P}\Big],\quad n = \Big[8\pi\bar{P} + 8\pi\rho + 2\alpha\bar{P} + 2\alpha\rho\Big]$$

and

$$o = -((8\pi + 2\alpha)\lambda).$$

Also, from (4.28), we have,

$$\begin{aligned}\nabla_{\mathcal{D}}|\mathcal{V}|^2 &= 2\mathbf{g}(\nabla_{\mathcal{D}}\mathcal{V},\mathcal{V}),\\ &= -2\mathbf{g}(A\mathcal{D},\mathcal{V}) - n\gamma(\mathcal{D})\gamma(\mathcal{V}) - o\delta(\mathcal{D})\delta(\mathcal{V})\\ &\quad -(div\mathcal{V} + 3\varphi - \Psi + m)\mathbf{g}(\mathcal{D},\mathcal{V}).\end{aligned} \tag{4.37}$$

Substituting (2.10) and (4.10), we have,

$$(\mathscr{L}_{\mathcal{V}}\mathbf{g})(\mathcal{D},\mathcal{V}) = -n\gamma(\mathcal{D})\gamma(\mathcal{V}) - o\delta(\mathcal{D})\delta(\mathcal{V}) - (m + 3\varphi - \Psi + div\mathcal{V})\mathbf{g}(\mathcal{D},\mathcal{V}). \tag{4.38}$$

Making use of (4.38) in (4.37), we have,

$$\nabla_{\mathcal{D}}|\mathcal{V}|^2 + 2\mathbf{g}(A\mathcal{D},\mathcal{V}) - \mathscr{L}_{\mathcal{V}}\mathbf{g}(\mathcal{D},\mathcal{V}) = 0. \tag{4.39}$$

Hence, from (4.36) and (4.37), we state,

Theorem 4.8. *Let ω represent the metric dual* 1*-form with respect to vector field $\mathcal{V}$, and the tensor field A is expressed by $d\omega(\mathcal{D},\mathcal{E}) = \mathbf{g}(\mathcal{D},A\mathcal{E})\ \forall\ \mathcal{D},\mathcal{E}\in\chi(\mathbf{M})$. Within the context of relativistic $\mathbb{BVFSS}$ with $\mathbf{f}(\mathbb{R},\mathbb{T})$-gravity admitting $\mathbf{Q}$-soliton, then the upcoming relations will satisfy:*

$$\begin{aligned}(divA)\mathcal{H} &= -m\mathbf{g}(\mathcal{H},\mathcal{D}) - n\gamma(\mathcal{H})\gamma(\mathcal{D}) - o\delta(\mathcal{H})\delta(\mathcal{D})\\ &\quad + \frac{3}{2}\Big(\mathcal{H}m + \mathcal{H}div\mathcal{D} + 3\mathcal{H}\varphi - \mathcal{H}\Psi\Big)\\ &\quad - \frac{1}{2}\gamma(\mathcal{H})\xi n - \frac{1}{2}\mathcal{H}n - \frac{3}{2}n\gamma(\mathcal{H}) - \frac{1}{2}\delta(\mathcal{H})\zeta o\\ &\quad + \frac{1}{2}\mathcal{H}o - \frac{1}{2}o(div\zeta)\delta(\mathcal{H}),\end{aligned}$$

and

$$\nabla_{\mathcal{D}}|\mathcal{V}|^2 + 2\mathbf{g}(A\mathcal{D},\mathcal{V}) - \mathscr{L}_{\mathcal{V}}\mathbf{g}(\mathcal{D},\mathcal{V}) = 0.$$

5. Conclusions and future directions

In this present investigation, a relativistic bulk viscous fluid string spacetime has been analyzed within the framework of the linear $\mathbf{f}(\mathbb{R},\mathbb{T})$-gravity model under the quasi-conformal flatness condition. Explicit expressions for the energy density and effective pressure have been derived, and the graphical study reveals that the null, weak and dominant energy conditions are fulfilled, whereas the strong energy condition is violated, indicating the possibility of an accelerating cosmological phase.

In addition, the existence of $\mathbf{Q}$-solitons on a relativistic bulk viscous fluid string spacetime within the linear framework of $\mathbf{f}(\mathbb{R},\mathbb{T})$-gravity, together with specific vector fields, exhibits shrinking, steady, and expanding behavior. As the continuation of the present work, the proposed framework may be extended to other modified theories of gravity, and it can be further employed to investigate traversable Morris-Thorne wormhole geometries admitting geometric soliton structures.

Author contributions

All authors made equal contributions to this work. All authors have read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare that no generative artificial intelligence (AI) tools were used in the writing, preparation, or development of this manuscript.

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Conflict of interest

The authors confirm that they have no competing interests or conflicts of interest related to this manuscript.

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