AIMS Mathematics, 11(11): 36660–36678.
DOI:10.3934/math.20261417
Received: 09 August 2026
Revised: 09 October 2026
Accepted: 30 October 2026
Published: 05 November 2026

https://www.aimspress.com/journal/Math

Research article

Depth of powers of binomial edge ideals of complete tripartite graphs

Hong Wang[1], **Zhongming Tang**[2,*] **and Chengyao Jiang**[2]

[1] School of Mathematics and Statistics, Suzhou University of Technology, Changshu 215500, China
[2] Department of Mathematics, Soochow (Suzhou) University, Suzhou 215006, China

* **Correspondence:** Email: zmtang@suda.edu.cn.

Abstract: Let $G = K_{n_1,n_2,n_3}$ be a complete tripartite graph on $[n]$ with $n > 3$ and J_G its binomial edge ideal in the polynomial ring $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ over a field K. For any $t \geq 2$, the depth(S/J_G^t) is computed explicitly, which is 3 for large t.

Keywords: depth; binomial edge ideal; tripartite graph
Mathematics Subject Classification: 13C15, 05E40

1. Introduction

Let I be an ideal of a commutative Noetherian ring R. By Brodmann's famous result [1, Theorem 2], the depth(R/I^t) is a constant for large powers t; that is, there exist integers $t_0 > 0$ and $c \geq 0$ such that depth$(R/I^t) = c$ for all $t \geq t_0$. Given an ideal I, it is a challenging task to find such t_0 and c.

Let G be a simple graph. The binomial edge ideal J_G of G was first introduced in [2] and [3] independently. Since then, many researches on binomial edge ideals of graphs have been done; see [4]. One interesting problem in this area is the computation of the depth of these ideals. However, only a few results on the depth of powers of binomial edge ideals have been obtained, such as in [5]. The depths of powers of the binomial edge ideal of a complete bipartite graph were obtained in [6]. The aim of this paper is to compute the depth of powers of binomial edge ideals of complete tripartite graphs.

Let $G = (V(G), E(G))$ be a finite simple graph with $V(G) = [n] := \{1, 2, \ldots, n\}$ and $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be a polynomial ring in $2n$ variables over a field K. Then, the binomial edge ideal of G, denoted by J_G, is the ideal of S generated by all binomials:

$$f_{ij} := x_i y_j - x_j y_i, 1 \leq i < j \leq n, \{i, j\} \in E(G).$$

When G is complete, J_G is just $I_2(X)$ where X is the generic matrix:

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}.$$

In this case, the depth of the powers of J_G has a specific property: $\operatorname{depth}(S/J_G) = n + 1$, and $\operatorname{depth}(S/J_G^t) = 3$ for any $t \geq 2$; see [7]. When $G = K_{n_1,n_2}$ is a complete bipartite graph, the depth of S/J_G has been obtained in [8]. For any $t \geq 2$, it is proved in [6] that $\operatorname{depth}(S/J_G^t) = n + 1$ if $n_1 = 1$ or $n_2 = 1$, and 4 if $n_1, n_2 \geq 2$. We are interested in the depth of powers of the binomial edge ideals of complete tripartite graphs $\operatorname{depth}(S/J^t_{K_{n_1,n_2,n_3}})$. By Brodmann's result [1], this depth is a constant for large powers t. Different from the case of bipartite graphs, this depth is 3. The difficult challenge part of the paper is to identity the smallest t_0 such that $\operatorname{depth}(S/J^t_{K_{n_1,n_2,n_3}}) = 3$ for all $t \geq t_0$.

After preparing some auxiliary results in Section 2, we find the powers whose depth is 3 in Section 3. Section 4 is devoted to computing the depth of small powers. The paper's challenge lies in Section 5, where the left important value of depths are computed.

2. Preliminaries

Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K, $\mathfrak{m} = (x_1, \ldots, x_n)$ its homogeneous maximal ideal, and M a finitely generated S-module. The depth of M, denoted by $\operatorname{depth}(M)$, is the length of a maximal M-sequence contained in $\mathfrak{m}$. The following well-known depth lemma is used frequently to compute the depth of some modules.

Lemma 2.1. *(Depth lemma) Let* $0 \to M \to N \to L \to 0$ *be a short exact sequence of finitely generated* S*-modules. Then,*

(1) $\operatorname{depth}(M) \geq \min\{\operatorname{depth}(N), \operatorname{depth}(L) + 1\}$*, and the equality holds if* $\operatorname{depth}(N) \neq \operatorname{depth}(L)$*,*
(2) $\operatorname{depth}(N) \geq \min\{\operatorname{depth}(M), \operatorname{depth}(L)\}$*, and the equality holds if* $\operatorname{depth}(M) \neq \operatorname{depth}(L) + 1$.

Lemma 2.2. *Let* I *be an ideal of* S*. If* $I = I_1 \cap I_2$*, then*

$$\operatorname{depth}(S/I) \geq \min\{\operatorname{depth}(S/I_1), \operatorname{depth}(S/I_2), \operatorname{depth}(S/(I_1 + I_2)) + 1\},$$

and the equality holds if $\min\{\operatorname{depth}(S/I_1), \operatorname{depth}(S/I_2)\} \neq \operatorname{depth}(S/(I_1 + I_2))$.

Proof. The result follows by applying Lemma 2.1 to the short exact sequence:

$$0 \longrightarrow \frac{S}{I} \longrightarrow \frac{S}{I_1} \oplus \frac{S}{I_2} \longrightarrow \frac{S}{I_1 + I_2} \longrightarrow 0.$$

□

Set interval $[m, n] = \{m, m + 1, \ldots, n\}$ for any two positive integers m, n with $m \leq n$. Especially, write $[m]$ for $[1, m]$.

Let $G = ([n], E(G))$ be a simple graph on $[n]$ and $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be a polynomial ring in $2n$ variables over a field K. The binomial edge ideal of G, denoted by J_G, is the ideal of S generated by all binomials:

$$f_{ij} := x_i y_j - x_j y_i,\ i < j, \{i, j\} \in E(G).$$

Note that each generated element f_{ij} of J_G is exactly a 2-minor of the matrix

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}.$$

We call G a complete graph on $[n]$, denoted by K_n, if its edge set $E(G) = \{\{i, j\} \mid i, j \in [n],\ i \neq j\}$. Then, $J_{K_n} = (f_{ij} \mid 1 \leq i < j \leq n)$. The following result is well-known.

Lemma 2.3. *([7, Theorem 2.1])* $\operatorname{depth}(S/J_{K_n}) = n + 1$, *and for any* $t \geq 2$,

$$\operatorname{depth}(S/J^t_{K_n}) = 3.$$

Let G be the complete r-partite graph $K_{n_1,\ldots,n_r}$; that is, $V(G)$ is the disjoint union of $V_i := [(\sum_{t=1}^{i-1} n_t) + 1, \sum_{t=1}^{i} n_t]$; $i = 1, 2, \ldots, r$; and $E(G) = \{\{a, b\} \mid a \text{ and } b \text{ are not in the same } V_i\}$. We sometimes denote $K_{n_1,\ldots,n_r}$ by $K_{V_1,\ldots,V_r}$. If $r = 1$, then $J_G = 0$. In the following, we always assume that $r > 1$. When $n_1 = n_2 = \cdots = n_r = 1$, $G = K_{1,1,\ldots,1}$ is a complete graph.

Let I be a nonzero ideal of a ring R. The tth symbolic power of I, denoted by $I^{(t)}$, is defined as

$$I^{(t)} = \cap_{\mathfrak{p}\in \operatorname{Min}(I)}(I^t R_{\mathfrak{p}} \cap R),$$

where $\operatorname{Min}(I)$ is the set of minimal prime ideals of I. Assume that $I = Q_1 \cap Q_2 \cap \cdots \cap Q_m$ is an irredundant primary decomposition of I; then, it is well-known that

$$I^{(t)} = \bigcap_{\sqrt{Q_i}\in \operatorname{Min}(I)} Q_i^{(t)}.$$

Let $T \subseteq [n]$. Denote the complete graph on T by K_T, and set the ideal $P_T = (x_i, y_i \mid i \in T)$ of S. If $T = [m]$ for some $m \leq n$, then we write P_m for $P_{[m]}$. Set $P_\emptyset = S$ by convention. For an ideal I of a ring R, put $I^i = R$ for $i \leq 0$ by convention.

Let $G = K_{n_1,\ldots,n_r}$ be a complete multipartite graph on $[n]$. Then, by [9, Theorem 4.3], $J^t_G = J^{(t)}_G$ for any $t \geq 1$. On the other hand, assume that $n_1 \leq n_2 \leq \cdots \leq n_r$, $r > 1$, and $n_r > 1$; that is, G is not a complete graph. Furthermore, assume that $n_1 = n_2 = \cdots = n_s = 1$ and that $n_{s+1} > 1$. Then, by [9, Lemma 2.2], all the associated prime ideals of J_G are minimal and exactly $J_{K_n}, P_i, i = s+1, \ldots, r$, where $P_i = (x_j, y_j \mid j \notin V_i)$. It is clear that $P^{(t)}_i = P^t_i$. On the other hand, $J^{(t)}_{K_n} = J^t_{K_n}$ is well-known (see [10, Theorem 10.4]). Thus, it follows that $J^t_G = J^t_{K_n} \cap P^t_{s+1} \cap P^t_{s+2} \cap \cdots \cap P^t_r$. Then,

$$J^t_G = J^t_{K_n} \cap P^t_{\overline{V}_1} \cap \cdots \cap P^t_{\overline{V}_r},$$

where $\overline{V}_i = [n] \setminus V_i$, $i = 1, \ldots, r$.

We will use this equality to compute the $\operatorname{depth}(S/J^t_G)$.

The following property will be used frequently.

Lemma 2.4. *([11, Proposition 4.9]) Let* T *be a subset of* $[n]$, $1 \leq a < b \leq n$, $1 \leq c < d \leq n$, *and* $\{a, b\} \nsubseteq [c, d]$. *Then, for any* $t, s \geq 1$,

(1) $(P^s_T + J_{K_{[c,d]}}) : f_{ab} = P^{s-|\{a,b\}\cap T|}_T + J_{K_{[c,d]}}$,
(2) $J^t_{K_n} : f_{ab} = J^{t-1}_{K_n}$,
(3) $(J^t_{K_n} + P^s_T) : f_{ab} = J^{t-1}_{K_n} + P^{s-|\{a,b\}\cap T|}_T$,
(4) $(J^t_{K_n} + P^s_T + J_{K_{[c,d]}}) : f_{ab} = J^{t-1}_{K_n} + P^{s-|\{a,b\}\cap T|}_T + J_{K_{[c,d]}}$,

where $P^i_T = S$ *for* $i \leq 0$ *by convention.*

When $r = 2$, the following three lemmas are proved in [6]. The same arguments hold for $r > 2$.

Lemma 2.5. *([6, Corollary 2.12]) Let* $T_1, \ldots, T_r$ *be subsets of* $[n]$, *and* $1 \leq a \leq b \leq n$. *Then, for any* $t, t_i \geq 1, i = 1, \ldots, r$, *one has*

$$(J^t_{K_n} \cap P^{t_1}_{T_1} \cap \cdots \cap P^{t_r}_{T_r}) + J_{K_{[a,b]}} = (J^t_{K_n} + J_{K_{[a,b]}}) \cap (P^{t_1}_{T_1} + J_{K_{[a,b]}}) \cap \cdots \cap (P^{t_r}_{T_r} + J_{K_{[a,b]}}).$$

Lemma 2.6. *([6, Corollary 2.14]) Let $T_1, \ldots, T_r$ be subsets of $[n]$, $1 \le a < b \le n$, and $\{a, b\} \nsubseteq [c, d]$. Then, for any $t, t_1, \ldots, t_r \ge 1$, one has*

$$((J_{K_n}^t \cap P_{T_1}^{t_1} \cap \cdots \cap P_{T_r}^{t_r}) + J_{K_{[c,d]}}) : f_{ab} = (J_{K_n}^{t-1} \cap P_{T_1}^{t_1'} \cap \cdots \cap P_{T_r}^{t_r'}) + J_{K_{[c,d]}},$$

where $t_i' = t_i - |\{a, b\} \cap T_i|$, $i = 1, \ldots, r$.

Lemma 2.7. *([6, Lemma 2.16]) Let $T_1, \ldots, T_r$ be subsets of $[n]$, $m \ge 1$, $n - m \ge 2$, and $1 \le a < b < n$. Then, for any $t \ge 1$ and $j = b + 1, b + 2, \ldots, n$,*

(1) $J_{K[a,b]} + (f_{a,b+1}, \ldots, f_{a,j}) = J_{K_{[a,j]}} \cap (J_{K_{[a,b]}} + (x_a, y_a))$;
(2) $(J_{K_n}^t \cap P_{T_1}^{t_1} \cap \cdots \cap P_{T_r}^{t_r}) + J_{K_{[a,b]}} + (f_{a,b+1}, \ldots, f_{a,j}) = ((J_{K_n}^t \cap P_{T_1}^{t_1} \cap \cdots \cap P_{T_r}^{t_r}) + J_{K_{[a,j]}}) \cap ((J_{K_n}^t \cap P_{T_1}^{t_1} \cap \cdots \cap P_{T_r}^{t_r}) + J_{K_{[a,b]}} + (x_a, y_a))$;
(3) $((J_{K_n}^t \cap P_{T_1}^{t_1} \cap \cdots \cap P_{T_r}^{t_r}) + J_{K_{[a,b]}} + (f_{a,b+1}, \ldots, f_{a,j-1})) : f_{a,j} = (J_{K_n}^{t-1} \cap P_{T_1}^{t_1'} \cap \cdots \cap P_{T_r}^{t_r'}) + J_{K_{[a,j-1]}}$, *where $t_i' = t_i - |\{a, b\} \cap T_i|$, $i = 1, \ldots, r$.*

Remark 2.1. *In the above equalities of Lemmas 2.5–2.7, when $J_{K_{[a,b]}}$ is replaced by $J_{K_{S_1}} + \cdots + J_{K_{S_l}}$ with $S_i \subseteq [n], i = 1, \ldots, l$, the results remain true provided the union of Gröbner bases of $J_{K_{S_i}}$, $i = 1, \ldots, l$, is a Gröbner basis of $J_{K_{S_1}} + \cdots + J_{K_{S_l}}$ (cf. [6, Section 2]). We will use this remark in the discussions of Section 5.*

Lemma 2.8. *Let $T_1, \ldots, T_r, T$ be subsets of $[n]$. Then,*

(1) $(J_{K_n}^t \cap P_{T_1}^{t_1} \cap \cdots \cap P_{T_r}^{t_r}) + (J_{K_n}^t \cap P_T^{t'}) = J_{K_n}^t$ *if $t' \le t$, and $t_i \le t, T_i \cup T = [n], i = 1, \ldots, r$;*
(2) $(J_{K_n}^t \cap P_{T_1}^{t_1} \cap \cdots \cap P_{T_r}^{t_r}) + J_{K_T} = (J_{K_n}^t \cap P_{T_1}^{t_1} \cap \cdots \cap P_{T_{r-1}}^{t_{r-1}}) + J_{K_T}$ *if $t_r \le t$, and $T_r \cup T = [n]$;*
(3) $(J_{K_n}^t \cap P_{T_1}^{t_1} \cap \cdots \cap P_{T_r}^{t_r}) + P_T^{t'} = J_{K_n}^t + P_T^{t'}$ *if $t' \le t$, and $t_i \le t, T_i \cup T = [n], i = 1, \ldots, r$.*

Proof. (1) $(J_{K_n}^t \cap P_{T_1}^{t_1} \cap \cdots \cap P_{T_r}^{t_r}) + (J_{K_n}^t \cap P_T^{t'}) \supseteq (J_{K_n}^t \cap P_{\overline{T}}^t) + (J_{K_n}^t \cap P_T^t) = J_{K_n}^t$; the last equality follows by [11, Lemma 5.2].

(2) $(J_{K_n}^t \cap P_{T_1}^{t_1} \cap \cdots \cap P_{T_r}^{t_r}) + J_{K_T} \supseteq J_{K_n}^t \cap P_{T_1}^{t_1} \cap \cdots \cap P_{T_{r-1}}^{t_{r-1}} \cap P_{\overline{T}}^t + J_{K_T} = (J_{K_n}^t \cap P_{\overline{T}}^t + J_{K_T}) \cap P_{T_1}^{t_1} \cap \cdots \cap P_{T_{r-1}}^{t_{r-1}} + J_{K_T}$, and $J_{K_n}^t \cap P_{\overline{T}}^t + J_{K_T} = J_{K_n}^t + J_{K_T}$. Then, the equality follows.

(3) $(J_{K_n}^t \cap P_{T_1}^{t_1} \cap \cdots \cap P_{T_r}^{t_r}) + P_T^{t'} = (J_{K_n}^t \cap P_{T_1}^{t_1} \cap \cdots \cap P_{T_r}^{t_r}) + (J_{K_n}^t \cap P_T^{t'}) + P_T^{t'} = J_{K_n}^t + P_T^{t'}$, and the last equality follows by (1). □

In [6], when using induction to prove that $\operatorname{depth}(S/J_{K_{m,n-m}}^t) = 4$, the crucial result is $\operatorname{depth}(S/(J_{K_{m,n-m}}^t + J_{K_{[a,b]}})) \ge 4$; that is, $\operatorname{depth}(S/((J_{K_n}^t \cap P_{[m]}^t \cap P_{[m+1,n]}^t) + J_{K_{[a,b]}})) \ge 4$. The same argument can be applied to get the following

Lemma 2.9. *Let $m, n - m \ge 2$. Then, for $t \ge 1$; $s = t$ or $t - 1$; and $1 \le a \le m$, $a < b \le n$,*

$$\operatorname{depth}(S/((J_{K_n}^t \cap P_{[m]}^s \cap P_{[m+1,n]}^t) + J_{K_{[a,b]}})) \ge 4.$$

For $r \ge 3$, the depth of $S/J_{K_{n_1,\ldots,n_r}}$ is known.

Lemma 2.10. *([12, Theorems 3.4 and 4.3]) Let $1 = n_1 = \cdots = n_s < n_{s+1} \le \cdots \le n_r$. Then,* $\operatorname{depth}(S/J_{K_{n_1,\ldots,n_r}}) = n_{s+1} + 2$.

Remark 2.2. *Our method of computing* $\operatorname{depth}(S/J^t_{K_{n_1,n_2,n_3}})$ *is using induction. This induction has two aspects. One is on the power t, and the other is on the number of vertices n. Taking* f_{n_1,n_1+1}*, we have a short exact sequence*

$$0 \longrightarrow \frac{S}{J^t_{K_{n_1,n_2,n_3}} : f_{n_1,n_1+1}} \xrightarrow{\cdot f_{n_1,n_1+1}} \frac{S}{J^t_{K_{n_1,n_2,n_3}}} \longrightarrow \frac{S}{J^t_{K_{n_1,n_2,n_3}} + (f_{n_1,n_1+1})} \longrightarrow 0.$$

The colon ideal $J^t_{K_{n_1,n_2,n_3}} : f_{n_1,n_1+1} = J^{t-1}_{K_n} \cap P^{t-1}_{\overline{V}_1} \cap P^{t-1}_{\overline{V}_2} \cap P^{t-2}_{\overline{V}_3}$ *has smaller powers by Lemma 2.6. On the other hand, to estimate the more general* $\operatorname{depth}(S/(J^t_{K_{n_1,n_2,n_3}} + J_{K_{[a,b]}}))$*, we will ask the help of the sequence* $f_{a,b+1}, \ldots, f_{a,n}$ *which will induce* $n-b$ *short exact sequences:*

$$\begin{aligned}
0 \to \frac{S}{(J^t_{K_{n_1,n_2,n_3}} + J_{K_{[a,b]}}) : f_{a,b+1}} &\xrightarrow{\cdot f_{a,b+1}} \frac{S}{J^t_{K_{n_1,n_2,n_3}} + J_{K_{[a,b]}}} \to \frac{S}{J^t_{K_{n_1,n_2,n_3}} + J_{K_{[a,b]}} + J_{b+1}} \to 0,\\
0 \to \frac{S}{(J^t_{K_{n_1,n_2,n_3}} + J_{K_{[a,b]}} + J_{b+1}) : f_{a,b+2}} &\xrightarrow{\cdot f_{a,b+2}} \frac{S}{J^t_{K_{n_1,n_2,n_3}} + J_{K_{[a,b]}} + J_{b+1}} \to \frac{S}{J^t_{K_{n_1,n_2,n_3}} + J_{K_{[a,b]}} + J_{b+2}} \to 0,\\
&\vdots\\
0 \to \frac{S}{(J^t_{K_{n_1,n_2,n_3}} + J_{K_{[a,b]}} + J_{n-1}) : f_{a,n}} &\xrightarrow{\cdot f_{a,n}} \frac{S}{J^t_{K_{n_1,n_2,n_3}} + J_{K_{[a,b]}} + J_{n-1}} \to \frac{S}{J^t_{K_{n_1,n_2,n_3}} + J_{K_{[a,b]}} + J_n} \to 0,
\end{aligned}$$

where $J_j = (f_{a,b+1}, \ldots, f_{a,j})$*,* $j = b+1, \ldots, n$*. Then, by Lemma 2.7(3), the colon ideal* $(J^t_{K_{n_1,n_2,n_3}} + J_{K_{[a,b]}} + J_{j-1}) : f_{a,j} = (J^{t-1}_{K_n} \cap P^{t'_1}_{\overline{V}_1} \cap P^{t'_2}_{\overline{V}_2} \cap P^{t'_3}_{\overline{V}_3}) + J_{K_{[a,j-1]}}$ *has smaller powers. Note that, by Lemma 2.7(2), the last ideal is*

$$J^t_{K_{n_1,n_2,n_3}} + J_{K_{[a,b]}} + J_n = (J^t_{K_{n_1,n_2,n_3}} + J_{K_{[a,n]}}) \cap (J^t_{K_{n_1,n_2,n_3}} + J_{K_{[a,b]}} + (x_a, y_a)),$$

where $a \in V_1$*,* $J^t_{K_{n_1,n_2,n_3}} + J_{K_{[a,n]}} = (J^t_{K_n} \cap P^t_{\overline{V}_1}) + J_{K_{[a,n]}}$ *by Lemma 2.8(2), whose depth can be easily obtained, and the difficult part* $J^t_{K_{n_1,n_2,n_3}} + J_{K_{[a,b]}} + (x_a, y_a)$ *reduces the number n by cutting a. When using induction, the staring point is the basic step whose ideals have small powers, and another important point is the inductive step where the truth for smaller powers should be confirmed.*

3. Large powers

By Brodmann's result [1, Theorem 2], $\operatorname{depth}(S/J^t_{K_{n_1,n_2,n_3}})$ is a constant for large t. We will show that this constant is 3.

Lemma 3.1. *Let* $1 \le a \le b \le n$ *and* $t \ge 1$*. Then,*

$$\operatorname{depth}(S/(J^t_{K_n} + J_{K_{[a,b]}})) \ge 3.$$

Proof. We may assume that $a = 1$. If $b = n$, the result is clear.

Let us use induction on $t \ge 1$ and $n \ge 2$ together to prove that $\operatorname{depth}(S/(J^t_{K_n} + J_{K_b})) \ge 3$ for $1 \le b < n$. When $t = 1$ or $n = 2$, the result is clearly true. Now, assume that $t > 1$ and $n > 2$, and the result is true for $t-1$ and $n-1$. Taking the sequence $f_{1,b+1}, \ldots, f_{1,n}$, we have short exact sequences:

$$0 \longrightarrow \frac{S}{J^{t-1}_{K_n} + J_{K_{j-1}}} \xrightarrow{\cdot f_{1,j}} \frac{S}{J^t_{K_n} + J_{K_b} + J_{j-1}} \longrightarrow \frac{S}{J^t_{K_n} + J_{K_b} + J_j} \longrightarrow 0,$$

$j = b+1, \ldots, n$, where $J_j = (f_{1,b+1}, \ldots, f_{1,j})$, $j = b+1, \ldots, n$, $J_b = 0$. The last ideal is $J^t_{K_n} + J_{K_b} + J_n = J_{K_n} \cap (J^t_{K_n} + J_{K_b} + (x_1, y_1))$. By an induction hypothesis on t, $\mathrm{depth}(S/(J^{t-1}_{K_n} + J_{K_j})) \geq 3$. Because $\mathrm{depth}(S/(J^t_{K_n} + J_{K_b} + (x_1, y_1))) = \mathrm{depth}(S_1/(J^t_{K_{n-1}} + J_{K_b})) \geq 3$, where $S_1 = K[x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}]$, by an induction hypothesis on n, it follows from Lemma 2.2 that $\mathrm{depth}(S/(J_{K_n} \cap (J^t_{K_n} + J_{K_b} + (x_1, y_1)))) \geq 3$. The proof is completed. □

The following similar result will be used in Section 4.

Lemma 3.2. *(1) Let $0 \leq m < n$ and $n - m \geq 2$. Then,* $\mathrm{depth}(S/(J_{K_n} + P^2_m)) \geq 3$, *and for $1 \leq a \leq m \leq b < n$,* $\mathrm{depth}(S/(J^3_{K_n} + P^2_m + J_{K[a,b]})) \geq 3$;

(2) Let $n - m \geq 2$ and $1 \leq a \leq m \leq b < n$. Then, for $t = 1, 2$, $\mathrm{depth}(S/(P^t_m + J_{K[a,b]})) \geq 4$.

Proof. (1) We first prove that $\mathrm{depth}(S/(J_{K_n} + P^2_m)) \geq 3$. From the short exact sequence

$$0 \to \frac{S}{J_{K_n} \cap P^2_m} \to \frac{S}{J_{K_n}} \oplus \frac{S}{P^2_m} \to \frac{S}{J_{K_n} + P^2_m} \to 0,$$

$\mathrm{depth}(S/J_{K_n}) = n+1$, $\mathrm{depth}(S/P^2_m) = 2(n-m)$, and applying the Depth lemma, we see that it is enough to show that $\mathrm{depth}(S/(J_{K_n} \cap P^2_m)) \geq 4$.

Using the sequence $f_{1,m+1}, \ldots, f_{1,n}$ to $S/(J_{K_n} \cap P^2_m)$, we have short exact sequences

$$0 \longrightarrow \frac{S}{P_m + J_{K_{j-1}}} \xrightarrow{\cdot f_{1,j}} \frac{S}{(J_{K_n} \cap P^2_m) + J_{j-1}} \longrightarrow \frac{S}{(J_{K_n} \cap P^2_m) + J_j} \longrightarrow 0,$$

$j = m+1, \ldots, n$, where $J_j = (f_{1,m+1}, \ldots, f_{1,j})$, $j = m+1, \ldots, n$, $J_m = 0$. We have the last ideal $(J_{K_n} \cap P^2_m) + J_n = (J_{K_n} \cap P^2_m) + (f_{1,2} \ldots, f_{1,n}) = (J_{K_n} \cap P^2_m) + J_n \cap (x_1, y_1) = J_{K_n} \cap (P^2_m + (x_1, y_1))$, whose factor ring has at least depth 3 by Lemma 2.2. On the other hand, note that, for $j = m+1, \ldots, n$, $\mathrm{depth}(S/(P_m + J_{K_{j-1}})) = 4$ if $n - m = 2$ and $2(n-j+1) + (j-1-m) + 1 > 4$ if $n - m > 2$. Then, $\mathrm{depth}(S/(J_{K_n} \cap P^2_m)) \geq 4$ follows.

Now, let us use induction on $m \geq 0$ to prove that $\mathrm{depth}(S/(J^3_{K_n} + P^2_m + J_{K[a,b]})) \geq 3$. When $m = 0$, it follows by Lemma 3.1. For $m > 0$, using the sequence $f_{1,m+1}, \ldots, f_{1,n}$, we have short exact sequences:

$$0 \longrightarrow \frac{S}{J^2_{K_n} + P_m + J_{K_{j-1}}} \xrightarrow{\cdot f_{1,j}} \frac{S}{J^3_{K_n} + P^2_m + J_{K[a,b]} + J_{j-1}} \longrightarrow \frac{S}{J^3_{K_n} + P^2_m + J_{K[a,b]} + J_j} \longrightarrow 0,$$

$j = m+1, \ldots, n$, where $J_j = (f_{1,m+1}, \ldots, f_{1,j})$; $j = m+1, \ldots, n$; and $J_m = 0$. $\mathrm{depth}(S/(J^2_{K_n} + P_m + J_{K_{j-1}})) \geq 3$, $j = m+1, \ldots, n$, follows by Lemma 3.1. On the other hand, we have the last ideal

$$\begin{aligned} J^3_{K_n} + P^2_m + J_{K[a,b]} + J_n &= J^3_{K_n} + P^2_m + J_{K[a,b]} + (f_{1,m+1}, \ldots, f_{1,n}) \\ &= J^3_{K_n} + P^2_m + J_{K[a,b]} + (f_{1,2}, \ldots, f_{1,n}) \\ &= J^3_{K_n} + P^2_m + J_{K[a,b]} + J_{K_n} \cap (x_1, y_1) \\ &= (J_{K_n} + P^2_m) \cap (J^3_{K_n} + P^2_m + J_{K[a,b]} + (x_1, y_1)), \end{aligned}$$

which has required depth because of $\mathrm{depth}(S/(J_{K_n} + P^2_m)) \geq 3$ and the induction hypothesis on m.

(2) When $t = 1$, $\mathrm{depth}(S/(P_m + J_{K[a,b]})) = \mathrm{depth}(K[x_i, y_i \mid i = m+1, \ldots, n]/J_{K_{[m+1,b]}}))$, and the result is clear from Lemma 2.3. When $t = 2$, the argument is similar to (1). □

Lemma 3.3. *([6, Lemma 3.1]) Let $m \geq 1$, $n - m \geq 2$, and $1 \leq a \leq b < n$. Then,*

$$\operatorname{depth}(S/((J_{K_n} \cap P_m) + J_{K_{[a,b]}})) = n - m + 2,$$

and, for any $1 \leq a < n$, $\operatorname{depth}(S/((J_{K_n} \cap P_m) + J_{K_{[a,n]}})) \geq 4$.

Lemma 3.4. *Let $1 \leq m < n$, $1 \leq a \leq b \leq n$, and $t \geq s \geq 1$. Then,*

$$\operatorname{depth}(S/((J^t_{K_n} \cap P^s_m) + J_{K_{[a,b]}})) \geq 3.$$

Proof. By induction on $t \geq 1$ and $n \geq 2$ together, when $n = 2$, the result is clear. If $t = 1$, it follows by Lemma 3.3. Now, assume that $t > 1$, and $n > 2$, and the result is true for $t - 1$ and $n - 1$. Note that we may assume that $a \leq m + 1$.

When $a \leq m$, using the sequence $f_{a,b+1}, \ldots, f_{a,n}$, we have short exact sequences

$$0 \longrightarrow \frac{S}{(J^{t-1}_{K_n} \cap P^{s_j}_m) + J_{K_{[a,j-1]}}} \xrightarrow{\cdot f_{a,j}} \frac{S}{(J^t_{K_n} \cap P^s_m) + J_{K_{[a,b]}} + J_{j-1}} \longrightarrow \frac{S}{(J^t_{K_n} \cap P^s_m) + J_{K_{[a,b]}} + J_j} \longrightarrow 0,$$

$j = b+1, \ldots, n$, where $J_j = (f_{a,b+1}, \ldots, f_{a,j})$; $j = b+1, \ldots, n$; $J_b = 0$; and $s_j = s-2$ if $j \leq m$ and $s-1$ if $j > m$. The last ideal is $(J^t_{K_n} \cap P^s_m) + J_{K_{[a,b]}} + J_n = (J^t_{K_n} + J_{K_{[a,n]}}) \cap ((J^t_{K_n} \cap P^s_m) + J_{K_{[a,b]}} + (x_a, y_a))$. Then, the depth of colon ideals follows by an induction hypothesis on t, and the depth of $(J^t_{K_n} \cap P^s_m) + J_{K_{[a,b]}} + (x_a, y_a)$ follows by an induction hypothesis on n.

When $a = m + 1$, from the short exact sequence

$$0 \to \frac{S}{(J^{t-1}_{K_n} \cap P^{s-1}_m) + J_{K_{[m+1,b]}}} \xrightarrow{\cdot f_{m,b}} \frac{S}{(J^t_{K_n} \cap P^s_m) + J_{K_{[m+1,b]}}} \to \frac{S}{(J^t_{K_n} \cap P^s_m) + J_{K_{[m+1,b]}} + (f_{m,b})} \to 0,$$

and

$$\begin{aligned} & (J^t_{K_n} \cap P^s_m) + J_{K_{[m+1,b]}} + (f_{m,b}) \\ = \; & (J^t_{K_n} \cap P^s_m) + J_{K_{[m+1,b]}} + J_{K_{[m,b]}} \cap (x_b, y_b) \\ = \; & ((J^t_{K_n} \cap P^s_m) + J_{K_{[m,b]}}) \cap ((J^t_{K_n} \cap P^s_m) + J_{K_{[m+1,b]}} + (x_b, y_b)), \end{aligned}$$

we get the required result because the depth of $S/((J^{t-1}_{K_n} \cap P^{s-1}_m) + J_{K_{[m+1,b]}})$ follows by an induction hypothesis on t. The depth of $S/((J^t_{K_n} \cap P^s_m) + J_{K_{[m+1,b]}} + (x_b, y_b))$ follows by induction hypothesis on n, and the depth of $S/((J^t_{K_n} \cap P^s_m) + J_{K_{[m,b]}})$ follows by the case $a \leq m$. □

Proposition 3.1. *Assume $t \geq 1$, $1 \leq a \leq b \leq n$, and $T_1, \ldots, T_r$ pairwise disjoint intervals of $[n]$. Then, for $t_i \leq t$, $i = 1, \ldots, r$,*

$$\operatorname{depth}(S/((J^t_{K_n} \cap P^{t_1}_{\overline{T}_1} \cap \cdots \cap P^{t_r}_{\overline{T}_r}) + J_{K_{[a,b]}})) \geq 3.$$

Proof. By induction on $r \geq 1$, when $r = 1$, the result follows by Lemma 3.4. Set $I_r = J^t_{K_n} \cap P^{t_1}_{\overline{T}_1} \cap \cdots \cap P^{t_r}_{\overline{T}_r}$. Then, $I_{r-1} + (J^t_{K_n} \cap P^{t_r}_{\overline{T}_r}) = J^t_{K_n}$ by Lemma 2.8(1). Applying Lemma 2.2 to the short exact sequence

$$0 \longrightarrow \frac{S}{I_r + J_{K_{[a,b]}}} \longrightarrow \frac{S}{I_{r-1} + J_{K_{[a,b]}}} \oplus \frac{S}{(J^t_{K_n} \cap P^{t_r}_{\overline{T}_r}) + J_{K_{[a,b]}}} \longrightarrow \frac{S}{J^t_{K_n} + J_{K_{[a,b]}}} \longrightarrow 0,$$

we get the result by induction hypothesis and Lemma 3.1. □

Proposition 3.2. *Let $t > 2$, if there exist $f_{a_1,b_1}, f_{a_2,b_2}, \ldots, f_{a_p,b_p}$ in $J_{K_{n_1,\ldots,n_r}}$ such that*

$$J^t_{K_{n_1,\ldots,n_r}} : \prod_{i=1}^{p} f_{a_i,b_i} = J^2_{K_n},$$

then

$$\operatorname{depth}(S/J^t_{K_{n_1,\ldots,n_r}}) = 3.$$

Proof. Note that $J^t_{K_{n_1,\ldots,n_r}} : \prod_{i=1}^{p} f_{a_i,b_i} = (J^t_{K_{n_1,\ldots,n_r}} : \prod_{i=1}^{p-1} f_{a_i,b_i}) : f_{a_p,b_p}$, and if I has a form $J^t_{K_n} \cap P^{t_1}_{\overline{T}_1} \cap \cdots \cap P^{t_r}_{\overline{T}_r}$ and for some $f_{a,b}$, $S/(I : f_{a,b})$ has depth 3, then from the short exact sequence

$$0 \longrightarrow S/(I : f_{a,b}) \xrightarrow{\cdot f_{a,b}} S/I \longrightarrow S/(I, f_{a,b}) \longrightarrow 0,$$

and $\operatorname{depth}(S/(I, f_{a,b})) \geq 3$ by Proposition 3.1, we see that $\operatorname{depth}(S/I) = 3$.

Then, from $\operatorname{depth}(S/J^2_{K_n}) = 3$, we get the result inductively. □

The following theorem shows when $\operatorname{depth}(S/J^t_{K_{n_1,n_2,n_3}}) = 3$.

Theorem 3.1. *Let $n = n_1 + n_2 + n_3 > 3$. Then,*

$$\operatorname{depth}(S/J^t_{K_{n_1,n_2,n_3}}) = 3,$$

when one of following conditions is satisfied:

(1) $n_1 = n_2 = 1, n_3 \geq 2$, and $t \geq 4$;
(2) $n_1 = 1, n_2, n_3 \geq 2$, and $t \geq 6$;
(3) $n_i \geq 2, i = 1, 2, 3$, and $t \geq 8$.

Proof. The following equalities give answers for the above cases from Proposition 3.2:

(1) $J^t_{K_{1,1,n_3}} : f^{t-2}_{12} = J^2_{K_n}$ for $t \geq 4$;
(2) $J^t_{K_{1,n_2,n_3}} : f^{t-4}_{12} f^2_{1,n_2+2} = J^2_{K_n}$ for $t \geq 6$;
(3) $J^t_{K_{n_1,n_2,n_3}} : f^2_{1,n_1+1} f^2_{1,n_1+n_2+1} f^{t-6}_{n_1+1,n_1+n_2+1} = J^2_{K_n}$ for $t \geq 8$. □

Then, for the computation of $\operatorname{depth}(S/J^t_{K_{n_1,n_2,n_3}})$, the left cases are

(1) $n_1 = n_2 = 1, n_3 \geq 2$, and $t = 2, 3$;
(2) $n_1 = 1, n_2, n_3 \geq 2$, and $t = 2, 3, 4, 5$;
(3) $n_i \geq 2, i = 1, 2, 3$, and $t = 2, 3, 4, 5, 6, 7$.

4. Small powers

When using induction on t to compute the depth of $S/J^t_{K_{n_1,n_2,n_3}}$, the basic step is the depth with small t, which is the topic of this section. We will consider the ideal $(J^t_{K_n} \cap P^{t_1}_{\overline{T}_1} \cap \cdots \cap P^{t_r}_{\overline{T}_r}) + J_{K_{[a,b]}}$ where $r \leq 3$ and $t, t_1, \ldots, t_r$ are small. Firstly, Let $r = 1$, and then let $r = 2$, $r = 3$.

Lemma 4.1. *([6, Lemma 4.1]) Let $1 \leq m < n$, $m \leq b < n$. Then, for $s = t$ or $t - 1$,*

$$\operatorname{depth}(S/((J^t_{K_n} \cap P^s_m) + J_{K_b})) \geq n - m + 2.$$

Proof. When $s = t$, the result is just [6, Lemma 4.1]. When $s = t - 1$, the same argument there also holds. □

Lemma 4.2. *Let* $1 \le a \le m \le b \le m+1$ *and* $(t, s) = (2,1), (2,2), (3,3)$. *Then,*

$$\operatorname{depth}(S/((J^t_{K_n} \cap P^s_m) + J_{K_{[a,b]}})) \ge n - m + 2.$$

Proof. By induction on $m \ge 1$ and $(t, s) \in \{(2,1), (2,2), (3,3)\}$, when $m = 1$ or $a = 1$, it follows by Lemma 4.1. Now, assume that $m > 1$ and that $a > 1$. By the sequence $f_{a-1,a}, \ldots, f_{1,a}$, we have short exact sequences:

$$0 \longrightarrow \frac{S}{(J^{t-1}_{K_n} \cap P^{s-2}_m) + J_{K_{[j+1,b]}}} \xrightarrow{\cdot f_{j,a}} \frac{S}{(J^t_{K_n} \cap P^s_m) + J_{K_{[a,b]}} + J_{j+1}} \longrightarrow \frac{S}{(J^t_{K_n} \cap P^s_m) + J_{K_{[a,b]}} + J_j} \longrightarrow 0,$$

$j = a-1, a-2, \ldots, 1$, where $J_j = (f_{j,a}, \ldots, f_{a-1,a})$, $j = 1, , \ldots, a-1$, and $J_a = 0$. The last ideal $(J^t_{K_n} \cap P^s_m) + J_{K_{[a,b]}} + J_1$ is the intersection of $(J^t_{K_n} \cap P^s_m) + J_{K_b}$ and $(J^t_{K_n} \cap P^s_m) + J_{K_{[a+1,b]}} + (x_a, y_a)$, where $J_{K_{[a+1,b]}} = 0$ when $a = m$. By an induction hypothesis on m, the depth of the factor ring of this last ideal is greater than or equal to $n - m + 2$. Then, we obtain the result inductively for $(t, s) = (2,1), (2,2)$, and $(3,3)$. □

Lemma 4.3. *Let* $m \ge 1, n - m \ge 2$. *Then,* $\operatorname{depth}(S/(J^t_{K_n} \cap P^t_m)) = n - m + 2$, $t = 1, 2, 3$.

Proof. When $t = 1$, it follows by Lemma 3.3. For $t = 2$ and then 3, applying Lemma 4.2 to the following short exact sequence:

$$0 \to \frac{S}{J^{t-1}_{K_n} \cap P^{t-1}_m} \xrightarrow{\cdot f_{m,m+1}} \frac{S}{J^t_{K_n} \cap P^t_m} \to \frac{S}{(J^t_{K_n} \cap P^t_m) + (f_{m,m+1})} \to 0,$$

we obtain the required results. □

Then, we can prove the following main result of this section.

Theorem 4.1. *Let* $n_3 > 1$. *Then,*

$$\operatorname{depth}(S/J^t_{K_{1,1,n_3}}) = \begin{cases} n_3 + 2, & \text{if } t = 1, 2, 3, \\ 3, & \text{if } t \ge 4. \end{cases}$$

Proof. It follows by Lemma 4.3 and Theorem 3.1(1). □

The above theorem solved the case (1) in Theorem 3.1. For the other two cases, we need some preparations.

Lemma 4.4. *Let* T_1, T_2 *be two disjoint intervals of* $[n]$ *with* $|T_i| \ge 2$, $i = 1, 2$. *Then, for* $t = 2, 3$,

$$\operatorname{depth}(S/(J^t_{K_n} \cap P^t_{\overline{T}_1} \cap P^t_{\overline{T}_2})) = 4.$$

Proof. By Lemma 4.3, $\operatorname{depth}(S/(J^t_{K_n} \cap P^t_{\overline{T}_i})) = |T_i| + 2 \ge 4$, $i = 1, 2$. Then, as $\operatorname{depth}(S/J^t_{K_n}) = 3$ by Lemma 2.3, we obtain the result by applying the Depth lemma to the following short exact sequence

$$0 \longrightarrow \frac{S}{J^t_{K_n} \cap P^t_{\overline{T}_1} \cap P^t_{\overline{T}_2}} \longrightarrow \frac{S}{J^t_{K_n} \cap P^t_{\overline{T}_1}} \oplus \frac{S}{J^t_{K_n} \cap P^t_{\overline{T}_2}} \longrightarrow \frac{S}{J^t_{K_n}} \longrightarrow 0.$$

□

Proposition 4.1. *Let $n_1 \geq 1$ and $n_2, n_3 \geq 2$. Then, for $t = 2, 3$,*

$$\operatorname{depth}(S/J^t_{K_{n_1,n_2,n_3}}) = 4.$$

Proof. Let $V_i = [(\sum_{t=1}^{i-1} n_t) + 1, \sum_{t=1}^{i} n_t], i = 1, 2, 3$. If $n_1 = 1$, then $J^t_{K_{n_1,n_2,n_3}} = J^t_{K_n} \cap P^t_{\overline{V}_2} \cap P^t_{\overline{V}_3}$, and the result follows by Lemma 4.4.

Assume that $n_1 > 1$. Then, $J^t_{K_{n_1,n_2,n_3}} = J^t_{K_n} \cap P^t_{\overline{V}_1} \cap P^t_{\overline{V}_2} \cap P^t_{\overline{V}_3}$, and there is a short exact sequence

$$0 \longrightarrow \frac{S}{J^t_{K_{n_1,n_2,n_3}}} \longrightarrow \frac{S}{J^t_{K_n} \cap P^t_{\overline{V}_1} \cap P^t_{\overline{V}_2}} \oplus \frac{S}{J^t_{K_n} \cap P^t_{\overline{V}_3}} \longrightarrow \frac{S}{J^t_{K_n}} \longrightarrow 0.$$

The result follows again by Lemma 4.4 and Depth lemma, as in the proof of Lemma 4.4. □

For further discussions, we need to improve Lemma 4.2.

Lemma 4.5. *Let $(t, s) = (2, 1), (2, 2), (3, 3)$; $1 \leq a \leq m < b < n$; and $n - m \geq 2$. Then,*

$$\operatorname{depth}(S/((J^t_{K_n} \cap P^s_m) + J_{K_{[a,b]}})) \geq 4.$$

Proof. By induction on $b \geq m + 1$, when $b = m + 1$, it follows by Lemma 4.2. For $b > m + 1$, similar to the proof of Lemma 4.2, by the sequence $f_{a-1,b}, \ldots, f_{1,b}$, we have short exact sequences:

$$0 \longrightarrow \frac{S}{(J^{t-1}_{K_n} \cap P^{s-1}_m) + J_{K_{[j+1,b]}}} \xrightarrow{\cdot f_{j,b}} \frac{S}{(J^t_{K_n} \cap P^s_m) + J_{K_{[a,b]}} + J_{j+1}} \longrightarrow \frac{S}{(J^t_{K_n} \cap P^s_m) + J_{K_{[a,b]}} + J_j} \longrightarrow 0,$$

$j = a - 1, a - 2, \ldots, 1$, where $J_j = (f_{j,b}, \ldots, f_{a-1,b})$, $j = 1, \ldots, a - 1$, and $J_a = 0$. The last ideal is $(J^t_{K_n} \cap P^s_m) + J_{K_{[a,b]}} + J_1 = ((J^t_{K_n} \cap P^s_m) + J_{K_b}) \cap ((J^t_{K_n} \cap P^s_m) + J_{K_{[a,b-1]}} + (x_b, y_b))$. Then, we get the results by using the same argument as in the proof of Lemma 4.2 and noting that when $t = s = 2$, the depth of the colon ideals follows by Lemma 3.3. □

Corollary 4.1. *Let $n = n_1 + n_2 + n_3$, $V_i = [(\sum_{t=1}^{i-1} n_t) + 1, \sum_{t=1}^{i} n_t]$. Then, for $1 \leq t \leq 3$,*

(1) $\operatorname{depth}(S/((J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^t_{\overline{V}_3}) + J_{K_{[a,b]}})) \geq 4$ *for* $n_1, n_3 \geq 2, n_2 \geq 0, a \in V_1, b \in V_3$;

(2) $\operatorname{depth}(S/((J^3_{K_n} \cap P^3_{\overline{V}_2} \cap P^t_{\overline{V}_3}) + J_{K_{[a,b]}})) \geq 4$ *for* $n_1 \geq 0, n_2, n_3 \geq 2, a \in V_2, b \in V_3$.

Proof. (1) Assume first that $t = 3$. Note that $(J^3_{K_n} \cap P^3_{\overline{V}_1}) + (J^3_{K_n} \cap P^3_{\overline{V}_3}) = J^3_{K_n}$ by Lemma 2.8(1). On the other hand, by Lemma 4.5, $\operatorname{depth}(S/((J^3_{K_n} \cap P^3_{\overline{V}_1}) + J_{K_{[a,b]}})) \geq 4$, and $\operatorname{depth}(S/((J^3_{K_n} \cap P^3_{\overline{V}_3}) + J_{K_{[a,b]}})) \geq 4$, and by Lemma 3.1, $\operatorname{depth}(S/(J^3_{K_n} + J_{K_{[a,b]}})) \geq 3$. Then, the result follows by applying Lemma 2.2 to the following short exact sequence:

$$0 \to \frac{S}{(J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^3_{\overline{V}_3}) + J_{K_{[a,b]}}} \to \frac{S}{(J^3_{K_n} \cap P^3_{\overline{V}_1}) + J_{K_{[a,b]}}} \oplus \frac{S}{(J^3_{K_n} \cap P^3_{\overline{V}_3}) + J_{K_{[a,b]}}} \to \frac{S}{J^3_{K_n} + J_{K_{[a,b]}}} \to 0.$$

For $t = 1, 2$, we use the short exact sequence

$$0 \to \frac{S}{(J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^t_{\overline{V}_3}) + J_{K_{[a,b]}}} \to \frac{S}{(J^3_{K_n} \cap P^3_{\overline{V}_1}) + J_{K_{[a,b]}}} \oplus \frac{S}{P^t_{\overline{V}_3} + J_{K_{[a,b]}}} \to \frac{S}{J^3_{K_n} + P^t_{\overline{V}_3} + J_{K_{[a,b]}}} \to 0,$$

and the result follows by $\operatorname{depth}(S/(P^t_{\overline{V}_3} + J_{K_{[a,b]}})) \geq 4$ and $\operatorname{depth}(S/(J^3_{K_n} + P^t_{\overline{V}_3} + J_{K_{[a,b]}})) \geq 3$ by virtue of Lemma 3.2.

(2) Similar to (1). □

Lemma 4.6. *Let $n = n_1 + n_2 + n_3$; $V_i = [(\sum_{t=1}^{i-1} n_t) + 1, \sum_{t=1}^{i} n_t]$; $i = 1, 2, 3$; $n_1 \geq 1$; $n_2, n_3 \geq 2$; $t_3 = 1$ or 2. Then,*

(1) $\operatorname{depth}(S/((J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^3_{\overline{V}_2} \cap P^{t_3}_{\overline{V}_3}) + J_{K_{[a,b]}})) \geq 4$, $a \in V_1, b \in V_2$;
(2) $\operatorname{depth}(S/(J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^3_{\overline{V}_2} \cap P^{t_3}_{\overline{V}_3})) = 4$.

Proof. As $P^{t_3}_{\overline{V}_3} + J_{K_{[a,b]}} = P^{t_3}_{\overline{V}_3}$,

$$(J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^3_{\overline{V}_2} \cap P^{t_3}_{\overline{V}_3}) + J_{K_{[a,b]}} = ((J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^3_{\overline{V}_2}) + J_{K_{[a,b]}}) \cap P^{t_3}_{\overline{V}_3},$$

and, by Lemma 2.8(3),

$$((J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^2_{\overline{V}_2}) + J_{K_{[a,b]}}) + P^{t_3}_{\overline{V}_3} = J^3_{K_n} + P^{t_3}_{\overline{V}_3}.$$

Then, we get (1) by Lemma 2.2 and note that when $t_3 = 2$, $\operatorname{depth}(S/(J^3_{K_n} + P^{t_3}_{\overline{V}_3})) \geq 3$ by Lemma 3.2. Omitting $J_{K_{[a,b]}}$ in the above argument, we obtain (2). □

Then, we can prove the following

Proposition 4.2. *Let $n_1 \geq 1$, $n_2, n_3 \geq 2$. Then,*

$$\operatorname{depth}(S/J^4_{K_{n_1,n_2,n_3}}) = 4.$$

Proof. Let $V_i = [(\sum_{t=1}^{i-1} n_t) + 1, \sum_{t=1}^{i} n_t]$, $i = 1, 2, 3$. We have a short exact sequence,

$$0 \to \frac{S}{J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^3_{\overline{V}_2} \cap P^2_{\overline{V}_3}} \xrightarrow{\cdot f_{n_1,n_1+1}} \frac{S}{J^4_{K_{n_1,n_2,n_3}}} \to \frac{S}{J^4_{K_{n_1,n_2,n_3}} + (f_{n_1,n_1+1})} \to 0.$$

Because $\operatorname{depth}(S/(J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^3_{\overline{V}_2} \cap P^2_{\overline{V}_3})) \geq 4$ by Lemma 4.6(2), it is enough to prove that $\operatorname{depth}(S/(J^4_{K_{n_1,n_2,n_3}} + (f_{n_1,n_1+1}))) \geq 4$.

By the sequence $f_{n_1,n_1+2}, \ldots, f_{n_1,n}$, we have short exact sequences:

$$0 \longrightarrow \frac{S}{(J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^{t_j}_{\overline{V}_2} \cap P^{s_j}_{\overline{V}_3}) + J_{K_{[n_1,j]}}} \xrightarrow{\cdot f_{n_1,j}} \frac{S}{J^4_{K_{n_1,n_2,n_3}} + J_{j-1}} \longrightarrow \frac{S}{J^4_{K_{n_1,n_2,n_3}} + J_j} \longrightarrow 0,$$

$j = n_1 + 2, \ldots, n$, where $J_j = (f_{n_1,n_1+1}, \ldots, f_{n_1,j})$, $j = n_1 + 1, \ldots, n$, and $t_j = 3$ if $j \in V_2$ and 0 if $j \in V_3$, $s_j = 2$ if $j \in V_2$, and 3 if $j \in V_3$. Note that $\operatorname{depth}(S/((J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^3_{\overline{V}_2} \cap P^2_{\overline{V}_3}) + J_{K_{[n_1,j]}})) \geq 4$, $j \in V_2$ by Lemma 4.6(1), and $\operatorname{depth}(S/((J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^3_{\overline{V}_3}) + J_{K_{[n_1,j]}})) \geq 4$, $j \in V_3$, by Corollary 4.1(1). The last ideal is $J^4_{K_{n_1,n_2,n_3}} + J_n = (J^4_{K_n} \cap P^4_{\overline{V}_1}) + J_{K_{[n_1,n]}}) \cap (J^4_{K_{n_1,n_2,n_3}} + (x_{n_1}, y_{n_1}))$.

Let us use induction on $n_1 \geq 1$ to prove that $\operatorname{depth}(S/(J^4_{K_{n_1,n_2,n_3}} + (x_{n_1}, y_{n_1}))) \geq 4$, then the result follows. When $n_1 = 1$, $J^4_{K_{n_1,n_2,n_3}} + (x_{n_1}, y_{n_1})$ induces to $S/J^4_{K_{n_2,n_3}}$, whose depth is 4 by [6, Theorem 4.7]. When $n_1 > 1$, $\operatorname{depth}(S/(J^4_{K_{n_1,n_2,n_3}} + (x_{n_1}, y_{n_1}))) = \operatorname{depth}(S_1/J^4_{K_{n_1-1,n_2,n_3}})$, where $S_1 = K[x_2, \ldots, x_n; y_2, \ldots, y_n]$. Then, by an induction hypothesis, we get the required result. □

5. The case $n_1 \geq 1$ and $n_2, n_3 \geq 2$

Assuming that $n_1 \geq 1$ and that $n_2, n_3 \geq 2$, let us compute the depth$(S/J^t_{K_{n_1,n_2,n_3}})$. By Theorem 3.1 and Proposition 4.2, the left cases are $t = 5, 6, 7$. We will see that when $n_1 = 1$, and $n_1 \geq 2$, the results are totally different.

Firstly, let us prove that depth$(S/J^5_{K_{n_1,n_2,n_3}}) = 4$. Applying the Depth lemma to the short exact sequence

$$0 \to \frac{S}{J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^3_{\overline{V}_3}} \xrightarrow{\cdot f_{n_1,n_1+1}} \frac{S}{J^5_{K_{n_1,n_2,n_3}}} \to \frac{S}{J^5_{K_{n_1,n_2,n_3}} + (f_{n_1,n_1+1})} \to 0,$$

it is enough to show that depth$(S/(J^5_{K_{n_1,n_2,n_3}} + J_{K_{[a,b]}})) \geq 4$ for $a \in V_1, b \in V_2$ and depth$(S/(J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^3_{\overline{V}_3})) = 4$. Similarly, in order to get that depth$(S/(J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^3_{\overline{V}_3})) = 4$, it is enough to show that depth$(S/((J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^3_{\overline{V}_3}) + J_{K_{[a,b]}})) \geq 4$ for $a \in V_1, b \in V_2$, and depth$(S/(J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^3_{\overline{V}_2} \cap P_{\overline{V}_3})) = 4$, which is true by Lemma 4.6(2). In order to obtain the two inequalities, we need the following lemma.

Lemma 5.1. *Let $n = n_1 + n_2 + n_3$ and $V_i = [(\sum_{t=1}^{i-1} n_t) + 1, \sum_{t=1}^{i} n_t]$, $i = 1, 2, 3$. Then, for $(t, s) = (3, 2), (4, 3), (4, 4)$, one has*

(1) depth$(S/((J^t_{K_n} \cap P^t_{\overline{V}_2} \cap P^s_{\overline{V}_3}) + J_{K_b})) \geq 4$, *for $n_1 \geq 0, n_2, n_3 \geq 2$, $n_1 \leq b < n_1 + n_2$;*

(2) depth$(S/((J^t_{K_n} \cap P^t_{\overline{V}_1} \cap P^s_{\overline{V}_3}) + J_{K_{[a,b]}})) \geq 4$, *for $n_1, n_3 \geq 2, n_2 \geq 0$, $1 < a \leq n_1$, $n_1 + n_2 \leq b < n$.*

Proof. (1) By induction on $n_1 \geq 0$, if $n_1 = 0$, the case $(3, 2)$ follows by Corollary 4.1 and the other cases follows by Lemma 2.9. Now, assume that $n_1 > 0$ and that the results are true for $n_1 - 1$. Using the sequence $f_{1,b+1}, \ldots, f_{1,n}$, we have short exact sequences:

$$0 \to \frac{S}{(J^{t-1}_{K_n} \cap P^{t_j}_{\overline{V}_2} \cap P^{s_j}_{\overline{V}_3}) + J_{K_{j-1}}} \xrightarrow{\cdot f_{1,j}} \frac{S}{(J^t_{K_n} \cap P^t_{\overline{V}_2} \cap P^s_{\overline{V}_3}) + J_{K_b} + J_{j-1}} \to \frac{S}{(J^t_{K_n} \cap P^t_{\overline{V}_2} \cap P^s_{\overline{V}_3}) + J_{K_b} + J_j} \to 0,$$

$j = b + 1, \ldots, n$, where $J_j = (f_{1,b+1}, \ldots, f_{1,j})$; $j = b + 1, \ldots, n$; $J_b = 0$; and $t_j = t - 1$ if $j \leq n_1 + n_2$ and 0 if $j > n_1 + n_2$, and $s_j = s - 2$ if $j \leq n_1 + n_2$ and $s - 1$ if $j > n_1 + n_2$. The last ideal is $(J^t_{K_n} \cap P^t_{\overline{V}_2} \cap P^s_{\overline{V}_3}) + J_{K_b} + J_n = J_{K_n} \cap ((J^t_{K_n} \cap P^t_{\overline{V}_2} \cap P^s_{\overline{V}_3}) + J_{K_b} + (x_1, y_1))$.

For $b + 1 \leq j \leq n_1 + n_2$, $S/((J^{t-1}_{K_n} \cap P^{t_j}_{\overline{V}_2} \cap P^{s_j}_{\overline{V}_3}) + J_{K_{j-1}}) = S/((J^{t-1}_{K_n} \cap P^{t-1}_{\overline{V}_2} \cap P^{s-2}_{\overline{V}_3}) + J_{K_{j-1}})$, whose depth is greater than or equal to 4 by Lemma 4.5 when $(t, s) = (3, 2)$, and by Corollary 4.1 when $(t, s) = (4, 3), (4, 4)$.

For $j > n_1 + n_2$, $S/((J^{t-1}_{K_n} \cap P^{t_j}_{\overline{V}_2} \cap P^{s_j}_{\overline{V}_3}) + J_{K_{j-1}}) = S/((J^{t-1}_{K_n} \cap P^{s-1}_{\overline{V}_3}) + J_{K_{j-1}})$ whose depth is greater than or equal to 4 by Lemma 4.1.

Then, it is enough to show that the depth of the last ideal depth$(S/((J^t_{K_n} \cap P^t_{\overline{V}_2} \cap P^s_{\overline{V}_3}) + J_{K_b} + J_n)) \geq 4$, which is guaranteed by depth$(S/((J^t_{K_n} \cap P^t_{\overline{V}_2} \cap P^s_{\overline{V}_3}) + J_{K_b} + (x_1, y_1)) \geq 4$, and this latter inequality follows by induction hypothesis on n_1.

(2) The case $(3, 2)$ follows by Corollary 4.1. Let us use induction on $b \geq n_1 + n_2$ to prove the other two cases. If $b = n_1 + n_2$, exchanging the positions of V_1 and V_2, (2) is just (1). Now, assume that $b > n_1 + n_2$ and that the results are true for $b - 1$. Using the sequence $f_{a-1,b}, \ldots, f_{1,b}$, we have short exact sequences:

$$0 \to \frac{S}{(J^{t-1}_{K_n} \cap P^{t-1}_{\overline{V}_1} \cap P^{s-1}_{\overline{V}_3}) + J_{K_{[j+1,b]}}} \xrightarrow{\cdot f_{j,b}} \frac{S}{(J^t_{K_n} \cap P^t_{\overline{V}_1} \cap P^s_{\overline{V}_3}) + J_{K_{[a,b]}} + J_{j+1}} \to \frac{S}{(J^t_{K_n} \cap P^t_{\overline{V}_1} \cap P^s_{\overline{V}_3}) + J_{K_{[a,b]}} + J_j} \to 0,$$

$j = a-1, a-2, \ldots, 1$, where $J_j = (f_{j,b}, \ldots, f_{a-1,b})$, $j = 1, \ldots, a-1$. The last ideal, $(J^t_{K_n} \cap P^t_{\overline{V}_1} \cap P^s_{\overline{V}_3}) + J_{K_{[a,b]}} + J_1$, is the intersection of $(J^t_{K_n} \cap P^s_{\overline{V}_3}) + J_{K_b}$ and $(J^t_{K_n} \cap P^t_{\overline{V}_1} \cap P^s_{\overline{V}_3}) + J_{K_{[a,b-1]}} + (x_b, y_b)$.

$\operatorname{depth}(S/((J^{t-1}_{K_n} \cap P^{t-1}_{\overline{V}_1} \cap P^{s-1}_{\overline{V}_3}) + J_{K_{[j+1,b]}})) \geq 4$ follows by Corollary 4.1 for $(t,s) = (4,3), (4,4)$, and $\operatorname{depth}(S/((J^t_{K_n} \cap P^t_{\overline{V}_1} \cap P^s_{\overline{V}_3}) + J_{K_{[a,b]}} + J_1)) \geq 4$ follows by $\operatorname{depth}(S/((J^t_{K_n} \cap P^s_{\overline{V}_3}) + J_{K_b})) \geq 4$ and $\operatorname{depth}(S/((J^t_{K_n} \cap P^t_{\overline{V}_1} \cap P^s_{\overline{V}_3}) + J_{K_{[a,b-1]}} + (x_b, y_b))) \geq 4$, where the former one follows by Lemma 4.1, and the latter one follows by an induction hypothesis on b. Then, the result follows. □

Note that when $(t,s) = (2,2), (3,3)$, the above results are given in Corollary 4.1.

We need two steps to obtain that $\operatorname{depth}(S/J^5_{K_{n_1,n_2,n_3}}) = 4$.

Proposition 5.1. *Let $n = n_1+n_2+n_3$ and $V_i = [(\sum_{t=1}^{i-1} n_t)+1, \sum_{t=1}^{i} n_t]$, $i = 1,2,3$. Assume that $n_1, n_2 \geq 1$, $(n_1, n_2) \neq (1,1)$, and $n_2, n_3 \geq 2$. Then,*

(1) $\operatorname{depth}(S/((J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^3_{\overline{V}_3}) + J_{K_{[a,b]}})) \geq 4$, $1 < a \leq n_1$, $n_1 \leq b < n_1+n_2$;

(2) $\operatorname{depth}(S/(J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^3_{\overline{V}_3})) = 4$.

Proof. Firstly Note that there is a symmetry between n_1 and n_2 (or V_1 and V_2).

By Lemma 4.6(2), $\operatorname{depth}(S/(J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^3_{\overline{V}_2} \cap P_{\overline{V}_3})) = 4$. Then, from the short exact sequence

$$0 \to \frac{S}{J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^3_{\overline{V}_2} \cap P_{\overline{V}_3}} \xrightarrow{\cdot f_{n_1,n_1+1}} \frac{S}{J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^3_{\overline{V}_3}} \to \frac{S}{(J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^3_{\overline{V}_3}) + (f_{n_1,n_1+1})} \to 0,$$

by the depth lemma, we see that (1) implies (2).

Note that by the sequence $f_{a,b+1}, \ldots, f_{a,n}$, we have short exact sequences:

$$0 \longrightarrow \frac{S}{(J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^{t_j}_{\overline{V}_2} \cap P^{s_j}_{\overline{V}_3}) + J_{K_{[a,j-1]}}} \xrightarrow{\cdot f_{a,j}} \frac{S}{(J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^3_{\overline{V}_3}) + J_{K_{[a,b]}} + J_{j-1}}$$
$$\longrightarrow \frac{S}{(J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^3_{\overline{V}_3}) + J_{K_{[a,b]}} + J_j} \longrightarrow 0,$$

$j = b+1, \ldots, n$, where $J_j = (f_{a,b+1}, \ldots, f_{a,j})$, $j = b+1, \ldots, n$, $J_b = 0$, and $t_j = 3$ if $j \leq n_1+n_2$ and 0 if $j > n_1+n_2$, and $s_j = 1$ if $j \leq n_1+n_2$ and 2 if $j > n_1+n_2$, and the last ideal, $(J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^3_{\overline{V}_3}) + J_{K_{[a,b]}} + J_n$, is the intersection of $(J^4_{K_n} \cap P^4_{\overline{V}_1}) + J_{K_{[a,n]}}$ and $(J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^3_{\overline{V}_3}) + J_{K_{[a+1,b]}} + (x_a, y_a)$.

When $j \leq n_1+n_2$, $\operatorname{depth}(S/((J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^{t_j}_{\overline{V}_2} \cap P^{s_j}_{\overline{V}_3}) + J_{K_{[a,j-1]}})) = \operatorname{depth}(S/(J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^3_{\overline{V}_2} \cap P_{\overline{V}_3}) + J_{K_{[a,j-1]}})) \geq 4$ follows by Lemma 4.6(1); when $j > n_1+n_2$, $\operatorname{depth}(S/((J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^{t_j}_{\overline{V}_2} \cap P^{s_j}_{\overline{V}_3}) + J_{K_{[a,j-1]}})) = \operatorname{depth}(S/((J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^2_{\overline{V}_3}) + J_{K_{[a,j-1]}}) \geq 4$ follows by Lemma 5.1. Then, it remains to estimate the depth of the last ideal, which reduces to the estimation of the depth of $S/((J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^3_{\overline{V}_3}) + J_{K_{[a+1,b]}} + (x_a, y_a))$, as $\operatorname{depth}(S/((J^4_{K_n} \cap P^4_{\overline{V}_1}) + J_{K_{[a,n]}})) \geq 4$ is known by Lemma 4.1.

Now, let us use induction on $n_1 \geq 1$ to prove (1) and (2). When $n_1 = 1$, the ideal $(J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^3_{\overline{V}_3}) + J_{K_{[a+1,b]}} + (x_a, y_a)$ reduces to $(J^4_{K_{[2,n]}} \cap P^4_{\overline{V}_2} \cap P^3_{\overline{V}_3}) + J_{K_{[2,b]}} + (x_1, y_1)$, which is about a bipartite graph. Note that the same result is true provided $n_2 = 1$.

Now, assume that $n_1 \geq 2$, $n_2 \geq 2$ and that the results (1) and (2) are true for $n_1 - 1$. Exchanging the positions of V_1 and V_2, we may assume that the results (1) and (2) are also true for $n_2 - 1$. When $a = n_1$, exchanging the positions of V_1 and V_2, we may assume that $b = n_1 + 1$. Then, for the ideal

$(J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^3_{\overline{V}_3}) + J_{K_{[a+1,b]}} + (x_a, y_a)$, if $a < n_1$, we use an induction hypothesis of (1) on n_1; if $a = n_1$ (in this case, the ideal becomes $(J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^3_{\overline{V}_3}) + (x_a, y_a)$), we use an induction hypothesis of (2) on n_1. Then, (1) and (2) hold.

This way of using an induction hypothesis of (1) and (2) alternately will be applied again in the following proofs. □

Proposition 5.2. *Let $n_1 \geq 1$ and $n_2, n_3 \geq 2$. One has*

(1) $\operatorname{depth}(S/(J^5_{K_{n_1,n_2,n_3}} + J_{K_{[a,b]}})) \geq 4$, $1 < a \leq n_1$, $n_1 \leq b < n_1 + n_2$;
(2) $\operatorname{depth}(S/J^5_{K_{n_1,n_2,n_3}}) = 4$.

Proof. Let $V_i = [(\sum_{t=1}^{i-1} n_t) + 1, \sum_{t=1}^{i} n_t]$, $i = 1, 2, 3$. Note that by the sequence $f_{a,b+1}, \ldots, f_{a,n}$, we have short exact sequences:

$$0 \to \frac{S}{(J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^{t_j}_{\overline{V}_2} \cap P^{s_j}_{\overline{V}_3}) + J_{K_{[a,j-1]}}} \xrightarrow{\cdot f_{a,j}} \frac{S}{J^5_{K_{n_1,n_2,n_3}} + J_{K_{[a,b]}} + J_{j-1}} \to \frac{S}{J^5_{K_{n_1,n_2,n_3}} + J_{K_{[a,b]}} + J_j} \to 0,$$

$j = b+1, \ldots, n$, where $J_j = (f_{a,b+1}, \ldots, f_{a,j})$, $j = b+1, \ldots, n$, $J_b = 0$, and $t_j = 4$ if $j \leq n_1 + n_2$ and 0 if $j > n_1 + n_2$, and $s_j = 3$ if $j \leq n_1 + n_2$ and 4 if $j > n_1 + n_2$. The last ideal, $J^5_{K_{n_1,n_2,n_3}} + J_{K_{[a,b]}} + J_n$, is the intersection of $(J^5_{K_n} \cap P^5_{\overline{V}_1}) + J_{K_{[a,n]}}$ and $J^5_{K_{n_1,n_2,n_3}} + J_{K_{[a+1,b]}} + (x_a, y_a)$.

That depths $S/((J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^3_{\overline{V}_3}) + J_{K_{[a,j-1]}})$ and $S/((J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_3}) + J_{K_{[a,j-1]}})$ are greater than and equal to 4 are given by Proposition 5.1 and Lemma 5.1. Applying the same argument as in Proposition 5.1, we get the required depth of the last ideal. The proof is complete. □

Theorem 5.1. *Let $n_2, n_3 \geq 2$. Then,*

$$\operatorname{depth}(S/J^t_{K_{1,n_2,n_3}}) = \begin{cases} 4, & \text{if } t = 2, 3, 4, 5, \\ 3, & \text{if } t \geq 6. \end{cases}$$

Proof. It follows by Propositions 4.1, 4.2, and 5.2(2) and Theorem 3.1(2). □

In the rest of this section, we will show that $\operatorname{depth}(S/J^t_{K_{n_1,n_2,n_3}}) = 4$ provided $n_i \geq 2, i = 1, 2, 3$ and $t = 6, 7$. When using induction to reduce n_1, n_2, n_3, we should not reduce them to 1 because in this case, the related depth is 3 by Theorem 5.1. That is, the basic step of induction should be $n_i = 2$.

We first try to prove that $\operatorname{depth}(S/J^6_{K_{n_1,n_2,n_3}}) = 4$. Similar to the proof of the depth of $S/J^5_{K_{n_1,n_2,n_3}}$, applying the depth lemma to the short exact sequence

$$0 \to \frac{S}{J^5_{K_n} \cap P^5_{\overline{V}_1} \cap P^5_{\overline{V}_2} \cap P^4_{\overline{V}_3}} \xrightarrow{\cdot f_{n_1,n_1+1}} \frac{S}{J^6_{K_{n_1,n_2,n_3}}} \to \frac{S}{J^6_{K_{n_1,n_2,n_3}} + (f_{n_1,n_1+1})} \to 0,$$

it is enough to show that $\operatorname{depth}(S/(J^6_{K_{n_1,n_2,n_3}} + J_{K_{[a,b]}})) \geq 4$ for $a \in V_1, b \in V_2$ and $\operatorname{depth}(S/(J^5_{K_n} \cap P^5_{\overline{V}_1} \cap P^5_{\overline{V}_2} \cap P^4_{\overline{V}_3})) = 4$. Similarly, in order to get that $\operatorname{depth}(S/(J^5_{K_n} \cap P^5_{\overline{V}_1} \cap P^5_{\overline{V}_2} \cap P^4_{\overline{V}_3})) = 4$, it is enough to show that $\operatorname{depth}(S/(J^5_{K_n} \cap P^5_{\overline{V}_1} \cap P^5_{\overline{V}_2} \cap P^4_{\overline{V}_3} + J_{K_{[a,b]}})) \geq 4$ for $a \in V_1, b \in V_2$ and $\operatorname{depth}(S/(J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^2_{\overline{V}_3}))=4$, and so on. Although $\operatorname{depth}(S/(J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^3_{\overline{V}_2})) = 4$ by Lemma 4.4, it remains to show that $\operatorname{depth}(S/(J^t_{K_n} \cap P^t_{\overline{V}_1} \cap P^t_{\overline{V}_2} \cap P^s_{\overline{V}_3} + J_{K_{[a,b]}})) \geq 4$ for $a \in V_1, b \in V_2$ provided $(t, s) = (6, 6), (5, 4), (4, 2)$.

When proving that $\text{depth}(S/J^7_{K_{n_1,n_2,n_3}}) = 4$, we need that $\text{depth}(S/(J^t_{K_n} \cap P^t_{\overline{V}_1} \cap P^t_{\overline{V}_2} \cap P^s_{\overline{V}_3} + J_{K_{[a,b]}})) \geq 4$ for $a \in V_1, b \in V_2$ provided $(t, s) = (7, 7), (6, 5), (5, 3), (4, 1)$.

On the other hand, when proving these inequalities, two inductions will be applied. One is on the indices t and s, and another is on n_1, n_2, n_3. As mentioned above, the basic step of the latter induction is $n_i = 2$. Furthermore, to reach this basic step, one need must to reduce the number of Ps in $J^t_{K_n} \cap P^t_{\overline{V}_1} \cap P^t_{\overline{V}_2} \cap P^s_{\overline{V}_3}$ from 3 to 2. Hence, we need to show the following result, which is crucial to the following discussions.

Lemma 5.2. *Let $m, n - m \geq 2$. If $m = 2$, or $n - m = 2$, then,*

(1) $\text{depth}(S/((J^4_{K_n} \cap P^4_{[m+1,n]} \cap P^2_m) + J_{K_{[a,b]}})) \geq 4$, $1 < a \leq m$, $m \leq b < n$;
(2) $\text{depth}(S/((J^4_{K_n} \cap P^4_{[m+1,n]} \cap P_m) + J_{K_{[a,b]}})) \geq 4$, $1 < a \leq m$, $m \leq b < n$.

Proof. Let us prove (1); (2) is similar.

Assume that $m = 2$, then $a = 2$. First using the sequence $f_{2,b+1}, \ldots, f_{2,n}$, we have short exact sequences:

$$0 \to \frac{S}{(J^3_{K_n} \cap P^3_{[3,n]} \cap P_2) + J_{K_{[2,j-1]}}} \xrightarrow{\cdot f_{2,j}} \frac{S}{(J^4_{K_n} \cap P^4_{[3,n]} \cap P^2_2) + J_{K_{[2,b]}} + J_{j-1}} \to \frac{S}{(J^4_{K_n} \cap P^4_{[3,n]} \cap P^2_2) + J_{K_{[2,b]}} + J_j} \to 0,$$

$j = b + 1, \ldots, n$, where $J_j = (f_{2,b+1}, \ldots, f_{2,j})$, $j = b + 1, \ldots, n$, and $J_b = 0$. We get colon ideals $(J^3_{K_n} \cap P^3_{[3,n]} \cap P_2) + J_{K_{[2,b']}}$, $b \leq b' < n$, whose depth follows by Corollary 4.1, and the last ideal is $(J^4_{K_n} \cap P^4_{[3,n]} \cap P^2_2) + J_{K_{[2,b]}} + J_{K_{\{2\},[3,n]}}$. Second using the sequence $f_{1,3}, \ldots, f_{1,n}$ to the ideal $(J^4_{K_n} \cap P^4_{[3,n]} \cap P^2_2) + J_{K_{[2,b]}} + J_{K_{\{2\},[3,n]}}$, we have short exact sequences:

$$0 \to \frac{S}{(J^3_{K_n} \cap P^3_{[3,n]} \cap P_2) + J_{K_{[2,b]}} + J_{K_{\{2\},[3,n]}} + J_{K_{\{1\}\cup[3,j-1]}}} \xrightarrow{\cdot f_{1,j}} \frac{S}{(J^4_{K_n} \cap P^4_{[3,n]} \cap P^2_2) + J_{K_{[2,b]}} + J_{K_{\{2\},[3,n]}} + J_{j-1}}$$
$$\to \frac{S}{(J^4_{K_n} \cap P^4_{[3,n]} \cap P^2_2) + J_{K_{[2,b]}} + J_{K_{\{2\},[3,n]}} + J_j} \to 0,$$

$j = 3, \ldots, n$, where $J_j = (f_{1,3}, \ldots, f_{1,j})$, $j = 3, \ldots, n$, and $J_2 = 0$. By Remark 2.1, the colon ideals have the forms:

$$(J^3_{K_n} \cap P^3_{[3,n]} \cap P_2) + J_{K_{[2,b]}} + J_{K_{\{2\},[3,n]}} + J_{K_{\{1\}\cup[3,j]}}, 3 \leq j < n,$$

whose factor ring's depth is greater than or equal to 4 by the following Sublemma 5.1. The last ideal becomes $(J^4_{K_n} \cap P^4_{[3,n]} \cap P^2_2) + J_{K_{[2,b]}} + J_{K_{2,n-2}} = (J^4_{K_n} \cap P^4_{[3,n]} \cap P^2_2) + J_{K_n} \cap P_2 \cap P_{[3,n]} + J_{K_{[2,b]}} = J_{K_{2,n-2}} + J_{K_{[2,b]}}$ which is about a bipartite graph and has required depth as mentioned before Lemma 2.9. The result follows.

When $n - m = 2$, then $b = n - 1$, just first using the sequence $f_{a-1,n-1}, \ldots, f_{1,n-1}$ and then $f_{n-2,n}, \ldots, f_{1,n}$; we get the same result by the above argument. □

Sublemma 5.1. *Let $n = n_1 + n_2 + n_3$ and $V_i = [(\sum_{t=1}^{i-1} n_t) + 1, \sum_{t=1}^{i} n_t]$, $i = 1, 2, 3$. Assuming that $n_1 = 2$, then*

(1) $\text{depth}(S/((J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^3_{\overline{V}_2}) + J_{K_{[2,b]}} + J_{K_{\{2\},[3,n]}} + J_{K_{\{1\}\cup[3,j]}})) \geq 4$ *for* $n_2 \geq 2, n_3 \geq 0$, *and* $2 \leq b < n_2 + 2, 3 \leq j < n_2 + 2$;
(2) $\text{depth}(S/((J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P_{\overline{V}_3}) + J_{K_{[2,b]}} + J_{K_{\{2\},[3,n]}} + J_{K_{\{1\}\cup[3,j]}})) \geq 4$ *for* $n_2 \geq 0, n_3 \geq 2$, *and* $2 \leq b < n, n_2 + 2 \leq j < n$.

Proof. We need to modify the proof of Corollary 4.1. Comparing that proof, the crucial result for the proof here is that $\operatorname{depth}(S/((J^3_{K_n} \cap P^3_{\overline{V}_1}) + J_{K_{[a,b]}} + J_{K_{\{2\},[3,n]}} + J_{K_{\{1\}\cup[3,j]}})) \geq 4$, which can be however obtained as in the proof of Lemma 4.5 by using the sequence $f_{1,j+1}, \ldots, f_{1,n}$. □

Lemma 5.3. *Let $m, n - m \geq 2$. Then,*

(1) $\operatorname{depth}(S/((J^4_{K_n} \cap P^4_{[m+1,n]} \cap P^2_m) + J_{K_{[a,b]}})) \geq 4$, $1 < a \leq m$, $m \leq b < n$;
(2) $\operatorname{depth}(S/(J^4_{K_n} \cap P^4_{[m+1,n]} \cap P^2_m)) = 4$;
(3) $\operatorname{depth}(S/((J^4_{K_n} \cap P^4_{[m+1,n]} \cap P_m) + J_{K_{[a,b]}})) \geq 4$, $1 < a \leq m$, $m \leq b < n$;
(4) $\operatorname{depth}(S/(J^4_{K_n} \cap P^4_{[m+1,n]} \cap P_m)) = 4$.

Proof. Let us use induction on $m \geq 2$ to prove (1) and (2) together; (3) and (4) are similar. The basic step is Lemma 5.2. Now, assume that $m > 2$ and that the results are true for $m - 1$. By the sequence $f_{a,b+1}, \ldots, f_{a,n}$, we have short exact sequences:

$$0 \to \frac{S}{(J^3_{K_n} \cap P^3_{[m+1,n]} \cap P_m) + J_{K_{[a,j-1]}}} \xrightarrow{\cdot f_{a,j}} \frac{S}{(J^4_{K_n} \cap P^4_{[m+1,n]} \cap P^2_m) + J_{K_{[a,b]}} + J_{j-1}} \to \frac{S}{(J^4_{K_n} \cap P^4_{[m+1,n]} \cap P^2_m) + J_{K_{[a,b]}} + J_j} \to 0,$$

$j = b+1, \ldots, n$, where $J_j = (f_{a,b+1}, \ldots, f_{a,j})$; $j = b+1, \ldots, n$; and $J_b = 0$. We get colon ideals $(J^3_{K_n} \cap P^3_{[m+1,n]} \cap P_m) + J_{K_{[a,b']}}$, $b \leq b' < n$, whose depths are given in Corollary 4.1, and the last ideal is $(J^4_{K_n} \cap P^4_{[m+1,n]} \cap P^2_m) + J_{K_{[a+1,b]}} + (x_a, y_a)$. Then, using the induction hypothesis as in Proposition 5.1, the result follows. □

Lemma 5.4. *Let $n = n_1 + n_2 + n_3$, $n_i \geq 2$ and $V_i = [(\sum_{t=1}^{i-1} n_t) + 1, \sum_{t=1}^{i} n_t]$, $i = 1, 2, 3$. If $n_1 = 2$, then for $2 \leq b < n_2 + 2$,*

(1) $\operatorname{depth}(S/((J^t_{K_n} \cap P^t_{\overline{V}_1} \cap P^t_{\overline{V}_2} \cap P^s_{\overline{V}_3}) + J_{K_{[2,b]}})) \geq 4, (t, s) = (4.2), (5, 4), (6, 6)$;
(2) $\operatorname{depth}(S/((J^t_{K_n} \cap P^t_{\overline{V}_1} \cap P^t_{\overline{V}_2} \cap P^s_{\overline{V}_3}) + J_{K_{[2,b]}})) \geq 4, (t, s) = (4.1), (5, 3), (6, 5), (7, 7)$.

Proof. (1) can be proved inductively for $(4.2), (5, 4), (6, 6)$; (2) is similar. The case $(4, 2)$ is the basic step. Let us prove the case $(4, 2)$ and omit the other two.

As in the proof of Lemma 5.2, we need to consider first the sequences $f_{2,b+1}, \ldots, f_{2,n}$. We get colon ideals $(J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^3_{\overline{V}_2}) + J_{K_{[2,b']}}$, $b \leq b' < n_2 + 2$ and $(J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P_{\overline{V}_3}) + J_{K_{[2,b^*]}}$, $n_2 + 2 \leq b^* < n$, whose depths are given by Corollary 4.1, and then by the sequence $f_{1,3}, \ldots, f_{1,n}$, we get colon ideals $(J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^3_{\overline{V}_2}) + J_{K_{[2,b]}} + J_{K_{\{2\},[3,n]}} + J_{K_{\{1\}\cup[3,j]}}$, $3 \leq j < n_2 + 2$, and $(J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P_{\overline{V}_3}) + J_{K_{[2,b]}} + J_{K_{\{2\},[3,n]}} + J_{K_{\{1\}\cup[3,j]}}$, $n_2 + 2 \leq j < n$, whose depths follow by the above Sublemma 5.1. The last ideal is equal to

$$\begin{aligned} & ((J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^2_{\overline{V}_3}) + J_{K_{[2,b]}}) + (J_{K_n} \cap P_{V_1} \cap P_{\overline{V}_1}) \\ = \; & ((J_{K_n} \cap P_{\overline{V}_1}) + J_{K_{[2,b]}}) \cap ((J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^2_{\overline{V}_3}) + J_{K_{[2,b]}} + P_{V_1}) \\ = \; & ((J_{K_n} \cap P_{\overline{V}_1}) + J_{K_{[2,b]}}) \cap ((J^4_{K_{[3,n]}} \cap P^4_{\overline{V}_2} \cap P^2_{\overline{V}_3}) + J_{K_{[3,b]}} + P_2), \end{aligned}$$

where the depth of $J^4_{K_{[3,n]}} \cap P^4_{\overline{V}_2} \cap P^2_{\overline{V}_3} + J_{K_{[3,b]}}$ is given by Lemma 5.3. Then the result follows. □

The above lemma is the basic step of the induction on $n_1 \geq 2$ and $n_2 \geq 2$ used in the proof of the following proposition, which is parallel to Propositions 5.1 and 5.2.

Proposition 5.3. *Let $n = n_1 + n_2 + n_3$, $n_i \geq 2$, and $V_i = [(\sum_{t=1}^{i-1} n_t) + 1, \sum_{t=1}^{i} n_t]$, $i = 1, 2, 3$. One has, for $(t, s) = (4, 2), (5, 4), (6, 6)$,*

(1) $\operatorname{depth}(S/((J^t_{K_n} \cap P^t_{\overline{V}_1} \cap P^t_{\overline{V}_2} \cap P^s_{\overline{V}_3}) + J_{K_{[a,b]}}) \geq 4, a \in V_1, b \in V_2$;
(2) $\operatorname{depth}(S/(J^t_{K_n} \cap P^t_{\overline{V}_1} \cap P^t_{\overline{V}_2} \cap P^s_{\overline{V}_3})) = 4$.

and For $(t, s) = (4, 1), (5, 3), (6, 5), (7, 7)$,

(3) $\operatorname{depth}(S/((J^t_{K_n} \cap P^t_{\overline{V}_1} \cap P^t_{\overline{V}_2} \cap P^s_{\overline{V}_3}) + J_{K_{[a,b]}})) \geq 4, a \in V_1, b \in V_2$;
(4) $\operatorname{depth}(S/(J^t_{K_n} \cap P^t_{\overline{V}_1} \cap P^t_{\overline{V}_2} \cap P^s_{\overline{V}_3})) = 4$.

Proof. Let us prove (1) and (2); (3) and (4) are similar.

Applying the depth lemma to the short exact sequence

$$0 \to \frac{S}{J^{t-1}_{K_n} \cap P^{t-1}_{\overline{V}_1} \cap P^{t-1}_{\overline{V}_2} \cap P^{s-2}_{\overline{V}_3}} \xrightarrow{\cdot f_{n_1,n_1+1}} \frac{S}{J^t_{K_n} \cap P^t_{\overline{V}_1} \cap P^t_{\overline{V}_2} \cap P^s_{\overline{V}_3}} \to \frac{S}{(J^t_{K_n} \cap P^t_{\overline{V}_1} \cap P^t_{\overline{V}_2} \cap P^s_{\overline{V}_3}) + (f_{n_1,n_1+1})} \to 0,$$

we see that the results for the case (t, s) depend on the results for the case $(t - 1, s - 2)$, so, we first prove $(4, 2)$, then $(5, 4)$ and $(6, 6)$.

Let us use induction on $n_1 \geq 2$ to prove the case $(4, 2)$. From the short exact sequence

$$0 \to \frac{S}{J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^3_{\overline{V}_2}} \xrightarrow{\cdot f_{n_1,n_1+1}} \frac{S}{J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^2_{\overline{V}_3}} \to \frac{S}{(J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^2_{\overline{V}_3}) + (f_{n_1,n_1+1})} \to 0,$$

and $\operatorname{depth}(S/(J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^3_{\overline{V}_2})) = 4$ by Lemma 4.4, we see that it is enough to show that $\operatorname{depth}(S/((J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^2_{\overline{V}_3}) + J_{K_{[a,b]}})) \geq 4, a \in V_1, b \in V_2$. When $n_1 = 2$, it is Lemma 5.4. Now, assume that $n_1 > 2$ and that results are true for $n_1 - 1$. Exchanging the positions of V_1 and V_2, we may assume that the results are also true for $n_2 - 1$. By the sequence $f_{a,b+1}, \ldots, f_{a,n}$, we have short exact sequences:

$$0 \longrightarrow \frac{S}{(J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^{t_j}_{\overline{V}_2} \cap P^{s_j}_{\overline{V}_3}) + J_{K_{[a,j-1]}}} \xrightarrow{\cdot f_{a,j}} \frac{S}{(J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^2_{\overline{V}_3}) + J_{K_{[a,b]}} + J_{j-1}}$$
$$\longrightarrow \frac{S}{(J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^2_{\overline{V}_3}) + J_{K_{[a,b]}} + J_j} \longrightarrow 0,$$

$j = b+1, \ldots, n$, where $J_j = (f_{a,b+1}, \ldots, f_{a,j})$; $j = b+1, \ldots, n$; $J_b = 0$; and $t_j = 3$ if $j \leq n_1+n_2$ and 0 if $j > n_1+n_2$, and $s_j = 0$ if $j \leq n_1+n_2$ and 1 if $j > n_1+n_2$, and the last ideal, $(J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^2_{\overline{V}_3}) + J_{K_{[a,b]}} + J_n$, is the intersection of $(J^4_{K_n} \cap P^4_{\overline{V}_1}) + J_{K_{[a,n]}}$ and $(J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^2_{\overline{V}_3}) + J_{K_{[a+1,b]}} + (x_a, y_a)$. Let us compare the proof of Proposition 5.1. There are two groups of colon ideals: $S/((J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^3_{\overline{V}_2} \cap P_{\overline{V}_3}) + J_{K_{[a,j-1]}})$, $j \leq n_1 + n_2$, and $S/((J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^2_{\overline{V}_3}) + J_{K_{[a,j-1]}})$, $j > n_1 + n_2$. Here, the two groups are $S/((J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P^3_{\overline{V}_2}) + J_{K_{[a,j-1]}})$, $j \leq n_1 + n_2$ and $S/((J^3_{K_n} \cap P^3_{\overline{V}_1} \cap P_{\overline{V}_3}) + J_{K_{[a,j-1]}})$, $j > n_1 + n_2$; that they have at least depth 4 follows by Corollary 4.1. On the other hand, for the depth of the last ideal, it remains to estimate the depth of $S/((J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^3_{\overline{V}_3}) + J_{K_{[a+1,b]}} + (x_a, y_a))$. Here it is $S/((J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^2_{\overline{V}_3}) + J_{K_{[a+1,b]}} + (x_a, y_a))$. Then, when using induction on $n_1 \geq 2$ to prove (1) and (2), as in the case here, we use an induction hypothesis as the the proof of Proposition 5.1 to get the required results.

We can use similar arguments to prove the cases $(5, 4)$ and $(6, 6)$. Note that in the case $(5, 4)$, the first group of colon ideals are $S/((J^4_{K_n} \cap P^4_{\overline{V}_1} \cap P^4_{\overline{V}_2} \cap P^2_{\overline{V}_3}) + J_{K_{[a,j-1]}})$, $j \leq n_1 + n_2$, whose depths are given

by the case $(4, 2)$, and in the case $(6, 6)$, the groups are $S/((J^5_{K_n} \cap P^5_{\overline{V}_1} \cap P^5_{\overline{V}_2} \cap P^4_{\overline{V}_3}) + J_{K_{[a,j-1]}})$, $j \leq n_1 + n_2$, whose depths are given by the case $(5, 4)$. We omit other details. □

Then, we have $\operatorname{depth}(S/J^6_{K_{n_1,n_2,n_3}}) = \operatorname{depth}(S/J^7_{K_{n_1,n_2,n_3}}) = 4$.

Summarily, we have the following theorem.

Theorem 5.2. *Let $n_i \geq 2$, $i = 1, 2, 3$. Then,*

$$\operatorname{depth}(S/J^t_{K_{n_1,n_2,n_3}}) = \begin{cases} 4 & \text{if } t = 2, 3, \ldots, 7, \\ 3 & \text{if } t \geq 8. \end{cases}$$

Proof. It follows by Propositions 4.1, 4.2, 5.2(2), 5.3(2), and (4) and Theorem 3.1(3). □

Remark 5.1. *Our arguments can also be applied to obtain the depths of $S/J^t_{K_{n_1,\ldots,n_r}}$ with $r \geq 4$. Because it is too technical, the details are omitted here.*

6. Conclusions

Let $G = K_{n_1,n_2,n_3}$ be a complete tripartite graph on $[n]$ with $n > 3$ and J_G its binomial edge ideal in the polynomial ring $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ over a field K. In this paper, for any $t \geq 2$, we obtained the depth of S/J^t_G:

(1) When $n_1 = n_2 = 1, n_3 \geq 2$,

$$\operatorname{depth}(S/J^t_{K_{1,1,n_3}}) = \begin{cases} n_3 + 2, & \text{if } t = 2, 3, \\ 3, & \text{if } t \geq 4. \end{cases}$$

(2) When $n_1 = 1, n_2, n_3 \geq 2$,

$$\operatorname{depth}(S/J^t_{K_{1,n_2,n_3}}) = \begin{cases} 4, & \text{if } t = 2, 3, 4, 5, \\ 3, & \text{if } t \geq 6. \end{cases}$$

(3) When $n_i \geq 2$, $i = 1, 2, 3$,

$$\operatorname{depth}(S/J^t_{K_{n_1,n_2,n_3}}) = \begin{cases} 4, & \text{if } t = 2, 3, \ldots, 7, \\ 3, & \text{if } t \geq 8. \end{cases}$$

Author contributions

All authors contributed equally in this paper. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The second author would like to thank the National Natural Science Foundation of China (No. 11471234) for financial support. All authors would also like to express their sincere thanks to the referees for their valuable comments.

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. M. Brodmann, The asymptotic nature of the analytic spread, *Math. Proc. Cambridge*, **86** (1979), 35–39. https://doi.org/10.1017/S030500410000061X
2. J. Herzog, T. Hibi, F. Hreinsdótir, T. Kahle, J. Rauh, Binomial edge ideals and conditional independence statements, *Adv. Appl. Math.*, **45** (2010), 317–333. https://doi.org/10.1016/j.aam.2010.01.003
3. M. Ohtani, Graphs and ideals generated by some 2-minors, *Commun. Algebra*, **39** (2011), 905–917. https://doi.org/10.1080/00927870903527584
4. J. Herzog, T. Hibi, H. Ohsugi, *Binomial ideals*, Cham: Springer-Verlag, 2018. https://doi.org/10.1007/978-3-319-95349-6
5. V. Ene, G. Rinaldo, N. Terai, Powers of binomial edge ideals with quadratic Gröbner bases, *Nagoya Math. J.*, **246** (2022), 233–255. https://doi.org/10.1017/nmj.2021.1
6. H. Wang, Z. Tang, Depth of powers of binomial edge ideals of complete bipartite graphs, *Commun. Algebra*, **51** (2023), 2472–2483. https://doi.org/10.1080/00927872.2022.2164009
7. L. Robbiano, An algebraic property of $\mathbb{P}^1 \times \mathbb{P}^N$, *Commun. Algebra*, **7** (1979), 641–655. https://doi.org/10.1080/00927877908822367
8. P. Schenzel, S. Zafar, Algebraic properties of the binomial edge ideal of a complete bipartite graph, *Anal. Sti. Univ. Ovidius Co.- Mat.*, **22** (2014), 217–237. https://doi.org/10.2478/auom-2014-0043
9. M. Ohtani, Binomial edge ideals of complete multipartite graphs, *Commun. Algebra*, **41** (2013), 3858–3867. https://doi.org/10.1080/00927872.2012.680219
10. W. Bruns, U. Vetter, *Determinantal rings*, Berlin: Springer-Varlag, 1988. https://doi.org/10.1007/BFb0080378
11. H. Wang, Z. Tang, Regularity of powers of binomial edge ideals of complete multipartite graphs, *Czech. Math. J.*, **73** (2023), 793–810. https://doi.org/10.21136/CMJ.2023.0246-22
12. A. Kumar, R. Sarkar, Depth and extremal Betti number of binomial edge ideals, *Math. Nachr.*, **293** (2020), 1746–1761. https://doi.org/10.1002/mana.201900150



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0)