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Research article

Lacunary statistical p-upward quasi-Cauchy sequences and directional block analysis

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Abstract: We study lacunary statistically p-upward quasi-Cauchy sequences, in which one-sided p-step changes are controlled through their exceptional proportions on lacunary blocks, together with the associated compactness and continuity notions. The analysis shows that the parameters p and θ play different roles at different levels of the theory. The corresponding compactness class is independent of both parameters and consists exactly of the subsets of $\mathbb{R}$ that are bounded from below. By contrast, the sequence classes themselves depend on p: We give exact conditions under which the lacunary p-upward class coincides with its natural-density counterpart and prove that the class associated with p is contained in the class associated with q exactly when p divides q. We also show directly, under the basic condition that the lacunary block lengths tend to infinity, that S_θ-p-upward continuity implies uniform continuity on an arbitrary subset of $\mathbb{R}$; on bounded domains, the converse holds as well. For finite blocks, we interpret the exceptional proportions as directional quantities rather than inferential statistics. They retain information that is lost under symmetric absolute differences, while their transformation and perturbation properties yield preservation results for the corresponding asymptotic sequence classes. A trend–seasonality model gives an explicit description of the interaction between the lag and the seasonal period, while the Mauna Loa atmospheric CO_2 series illustrates the resulting directional profiles and their sensitivity to the lag, threshold, block structure, and treatment of missing observations.

Keywords: lacunary statistical convergence; one-sided sequential continuity; lacunary block density; directional block analysis; p-upward quasi-Cauchy sequences
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1. Introduction

Lacunary statistical convergence describes convergence by measuring the proportion of exceptional indices within blocks whose lengths tend to infinity. Introduced in [1], it provides a blockwise counterpart

to statistical convergence. Let $\theta = (t_m)$ be an increasing sequence of positive integers, and put

$$J_m = (t_{m-1}, t_m], \qquad \ell_m = t_m - t_{m-1}.$$

A sequence (u_n) in $\mathbb{R}$ is said to be lacunary statistically convergent to $L \in \mathbb{R}$, briefly S_θ-convergent to L, if

$$\lim_{m\to\infty} \frac{1}{\ell_m} |\{n \in J_m : |u_n - L| \geq \varepsilon\}| = 0$$

for every $\varepsilon > 0$, where $\ell_m \to \infty$. Related forms and applications of this convergence method have been studied in [2–4].

Sequential methods also provide characterizations of compactness- and continuity-type properties on subsets of $\mathbb{R}$ [5]. In this direction, boundedness is closely related to statistical ward compactness [6, Lemma 2] and lacunary statistical ward compactness [7, Theorem 3]. One-sided versions lead instead to one-sided boundedness characterizations; for example, lacunary statistically upward compact subsets of $\mathbb{R}$ are precisely the subsets that are bounded from below [8, Theorem 1.9]. Related one-sided notions are also considered in [9]. Sequential preservation principles also lead to continuity notions that are often closely related to uniform continuity. Quasi-Cauchy sequence preservation and sequential continuity offer two complementary approaches to this question [10–12].

Three related lines of work are especially relevant to the present study. Lacunary statistically upward half quasi-Cauchy sequences were studied in [8], where the condition is one-sided but involves only consecutive terms. Lacunary statistically p-quasi-Cauchy sequences were subsequently considered in [4] for an arbitrary positive integer p, with the corresponding p-step condition expressed through the absolute difference. A one-sided p-step condition based on natural density was considered in [13]. Thus, lacunary block density, one-sided differences, and a general step parameter have been investigated in related settings, but the combined effect of these features on the underlying sequence classes and continuity properties has not been determined in these studies.

An important distinction also concerns the assumptions on the lacunary sequence. Writing $q_m = t_m/t_{m-1}$, the studies [4, 8] impose the condition $\liminf_{m\to\infty} q_m > 1$ throughout their theoretical development. In contrast, our basic results require only $\ell_m \to \infty$. The stronger conditions on the lacunary ratios arise in the comparison between the lacunary and natural-density p-upward sequence classes, for which we establish exact necessary and sufficient conditions.

The main contributions of the present work concern the distinct roles of the lacunary structure and the step parameter in the resulting sequence and continuity classes. At the compactness level, the characterization is independent of both p and θ: A subset of $\mathbb{R}$ is S_θ-p-upward compact if and only if it is bounded from below. At the sequence level, however, both parameters play essential roles. We establish exact necessary and sufficient conditions on the lacunary ratios for the inclusion relations between the lacunary and natural-density p-upward sequence classes. We also prove that the inclusion between two lacunary p-upward sequence classes with step parameters p and q holds precisely when p divides q. This characterization yields strict inclusions for distinct steps when one divides the other, and incomparable classes when neither step divides the other. At the continuity level, we prove directly that S_θ-p-upward continuity implies uniform continuity on every subset of $\mathbb{R}$, assuming only $\ell_m \to \infty$. The converse holds on bounded subsets but fails in general on unbounded domains. Moreover, for bounded sequences, the one-sided lacunary p-step condition is equivalent to its symmetric counterpart. Finally, the finite-block directional quantities distinguish changes that cannot be separated by the corresponding

symmetric p-step quantity, and their transformation and perturbation properties also yield consequences for the asymptotic sequence classes. The trend–seasonality analysis further identifies how the arithmetic relation between the lag p and the seasonal period determines the directional exceptional proportions.

The remainder of the paper is organized as follows. Section 2 presents the notation, preliminary results, directional block quantities, and the data and models used in the finite-block illustrations. Section 3 establishes the main results on sequence classes, compactness, and continuity, including their dependence on the lacunary structure and the step parameter. Section 4 develops the finite-block directional analysis, examines its relation to the asymptotic sequence classes, and presents the trend–seasonality and Mauna Loa CO_2 illustrations with sensitivity analyses. Section 5 concludes the paper.

2. Preliminaries and finite-block methods

2.1. Lacunary setting and notation

Let $\theta = (t_m)$ be an increasing sequence of positive integers, and set

$$J_m = (t_{m-1}, t_m], \qquad \ell_m = t_m - t_{m-1}.$$

Throughout Sections 3 and 4, unless explicitly stated otherwise, we assume only that the lacunary block lengths satisfy

$$\ell_m \longrightarrow \infty.$$

When comparisons with natural-density p-upward behavior are considered, we also use the lacunary ratios

$$q_m = \frac{t_m}{t_{m-1}}$$

and impose additional lower or upper bounds on (q_m) only where they are needed. Unless explicitly stated otherwise, no ratio condition is assumed.

The following elementary counting fact will be used repeatedly.

Lemma 2.1 (Periodic block density)**.** *Let $P \subseteq \mathbb{N}$ be periodic with period $r \in \mathbb{N}$, and let*

$$d = \big|P \cap \{1, \dots, r\}\big|.$$

Then

$$\left|\frac{|P \cap J_m|}{\ell_m} - \frac{d}{r}\right| \leq \frac{r}{\ell_m}$$

for every m. Consequently,

$$\frac{|P \cap J_m|}{\ell_m} \longrightarrow \frac{d}{r}.$$

Proof. Write $\ell_m = kr + s$ with $0 \leq s < r$. Partition J_m into k consecutive intervals of length r and one remaining interval of length s. Every interval of r consecutive integers contains exactly d elements of P, so

$$|P \cap J_m| = kd + h$$

for some $0 \leq h \leq s$. Therefore

$$\left||P \cap J_m| - \frac{d}{r}\ell_m\right| = \left|h - \frac{d}{r}s\right| \leq s < r.$$

Division by ℓ_m gives the stated estimate. ∎

2.2. Directional block quantities

For a real sequence $X = (X_n)$, a lag $p \in \mathbb{N}$, and a threshold $\varepsilon > 0$, define

$$D_{m,p}^{X,+}(\varepsilon) = \frac{1}{\ell_m} \left| \left\{ n \in J_m : X_n - X_{n+p} \geq \varepsilon \right\} \right|.$$

For directional comparison, define also

$$D_{m,p}^{X,-}(\varepsilon) = \frac{1}{\ell_m} \left| \left\{ n \in J_m : X_{n+p} - X_n \geq \varepsilon \right\} \right|$$

and

$$D_{m,p}^{X,\mathrm{sym}}(\varepsilon) = \frac{1}{\ell_m} \left| \left\{ n \in J_m : |X_n - X_{n+p}| \geq \varepsilon \right\} \right|.$$

For $\varepsilon > 0$, the two directional exceptional sets are disjoint, and hence

$$D_{m,p}^{X,\mathrm{sym}}(\varepsilon) = D_{m,p}^{X,+}(\varepsilon) + D_{m,p}^{X,-}(\varepsilon). \tag{2.1}$$

Only blocks for which all required p-step comparisons are available are included in the finite-data illustration.

2.3. Synthetic trend–seasonality model

To obtain an analytically tractable reference case, we consider

$$X_n = an + b \sin\left(\frac{2\pi n}{T}\right), \qquad a > 0,$$

where a represents the linear trend, b the seasonal amplitude, and T the seasonal period. The monthly prototype uses

$$T = 12, \qquad a = 0.1, \qquad b = 3.$$

The lags $p = 1, 3, 6, 12$ correspond respectively to one-month, three-month, half-year, and annual comparisons; the full lag profile $p = 1, \ldots, 24$ is also examined below.

2.4. Empirical data and preprocessing

The empirical illustration uses the weekly Mauna Loa atmospheric CO_2 series distributed with the `statsmodels` data collection [14]. The underlying atmospheric record is associated with the long-running Scripps measurements described by Keeling and Whorf [15]. Weekly observations from March 1958 through December 2001 were aggregated to calendar-month means. Five monthly values (June 1958, October 1958, and February–April 1964) were missing after aggregation and were replaced by linear interpolation, resulting in a series of 526 monthly observations. All finite-block calculations were performed in Python.

For the baseline block analysis, we use dyadic endpoints

$$t_m = 2^m, \qquad J_m = (2^{m-1}, 2^m], \qquad \ell_m = 2^{m-1}.$$

The value $\varepsilon = 1$ ppm is used as a reference threshold rather than as an estimated or theoretically optimal value. Threshold sensitivity is examined at $0.25, 0.50, 1$, and 2 ppm, lag sensitivity is examined over $p = 1, \ldots, 24$, and geometric block structures with ratios $1.5, 2$, and 3 are compared.

3. Main results

We first formulate the lacunary-block counterpart of the natural-density one-sided p-step condition considered in [13].

Definition 3.1. A sequence (u_n) in $\mathbb{R}$ is called lacunary statistically p-upward quasi-Cauchy, briefly S_θ-p-upward quasi-Cauchy, if for every $\varepsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{\ell_m} \left|\left\{n \in J_m : u_n - u_{n+p} \geq \varepsilon\right\}\right| = 0.$$

The class of all such sequences is denoted by $\Delta_p S_\theta^+$.

Every S_θ-convergent sequence and every lacunary statistically p-quasi-Cauchy sequence belongs to $\Delta_p S_\theta^+$. In particular, every slowly oscillating, Cauchy, or convergent sequence belongs to this class. Moreover, $\Delta_p S_\theta^+$ is closed under addition and multiplication by positive scalars. These statements follow from the corresponding exceptional-set inclusions and the fact that the boundary contribution of any fixed shift is $O(p/\ell_m)$.

Definition 3.2. A subset A of $\mathbb{R}$ is said to be lacunary statistically p-upward compact, or simply S_θ-p-upward compact, if every sequence in A admits an S_θ-p-upward quasi-Cauchy subsequence.

Theorem 3.3. *A subset of $\mathbb{R}$ is S_θ-p-upward compact if and only if it is bounded from below.*

Proof. Let A be bounded from below, and let (u_n) be an arbitrary sequence in A. If (u_n) is bounded above, then it is bounded and hence has a convergent subsequence by the Bolzano–Weierstrass theorem. Every convergent sequence is S_θ-p-upward quasi-Cauchy, so in this case the required subsequence exists.

Assume now that (u_n) is unbounded above. We construct a subsequence (u_{n_j}) inductively. Choose $n_1 \in \mathbb{N}$. Having chosen n_j, the tail $(u_n)_{n>n_j}$ is still unbounded above, and therefore one can choose $n_{j+1} > n_j$ such that

$$u_{n_{j+1}} > u_{n_j} + j.$$

It follows that, for every $j \in \mathbb{N}$ and every fixed positive integer p,

$$u_{n_j} - u_{n_{j+p}} < 0.$$

Consequently, for each $\varepsilon > 0$,

$$\left\{j \in J_m : u_{n_j} - u_{n_{j+p}} \geq \varepsilon\right\} = \varnothing$$

for every m, and hence (u_{n_j}) is S_θ-p-upward quasi-Cauchy. Thus A is S_θ-p-upward compact.

Conversely, suppose that A is not bounded from below. Choose $u_1 \in A$. Since A is unbounded below, after u_j has been chosen, one can select $u_{j+1} \in A$ so that

$$u_{j+1} < u_j - 1.$$

Let (u_{j_s}) be any subsequence of (u_j). Since $j_{s+p} \geq j_s + p$, repeated use of the preceding inequality gives

$$u_{j_s} - u_{j_{s+p}} > j_{s+p} - j_s \geq p \geq 1$$

for every $s \in \mathbb{N}$. Hence, with $\varepsilon = 1$,

$$\frac{1}{\ell_m}\left|\left\{s \in J_m : u_{j_s} - u_{j_{s+p}} \geq 1\right\}\right| = 1$$

for every m. Thus no subsequence of (u_j) is S_θ-p-upward quasi-Cauchy, and A is not S_θ-p-upward compact. ■

Theorem 3.3 also shows that the compactness class itself does not distinguish either p or θ: For every admissible θ and every $p \in \mathbb{N}$, the S_θ-p-upward compact subsets of $\mathbb{R}$ are exactly the subsets that are bounded from below. This agrees with the one-step lacunary upward compactness characterization in [8, Theorem 1.9], while the symmetric lacunary p-step theory yields ordinary boundedness [4, Theorem 2.1]. Both of those lacunary theories are developed under the standing condition $\liminf q_m > 1$; by contrast, Theorem 3.3 holds for arbitrary p under the basic assumption $\ell_m \to \infty$. Thus directionality determines the compactness characterization, while the roles of p and θ reappear at the level of the sequence classes.

For comparison, let $\Delta_p S^+$ denote the class of real sequences (u_n) satisfying

$$\lim_{N\to\infty} \frac{1}{N}\left|\left\{n \leq N : u_n - u_{n+p} \geq \varepsilon\right\}\right| = 0$$

for every $\varepsilon > 0$.

Theorem 3.4 (Natural-density versus lacunary-density classes)**.** *Let $p \in \mathbb{N}$ be fixed. Then*

$$\Delta_p S^+ \subseteq \Delta_p S_\theta^+ \quad \Longleftrightarrow \quad \liminf_{m\to\infty} q_m > 1,$$

and

$$\Delta_p S_\theta^+ \subseteq \Delta_p S^+ \quad \Longleftrightarrow \quad \limsup_{m\to\infty} q_m < \infty.$$

Consequently,

$$\Delta_p S_\theta^+ = \Delta_p S^+$$

if and only if

$$1 < \liminf_{m\to\infty} q_m \leq \limsup_{m\to\infty} q_m < \infty.$$

Proof. Let (u_n) be a real sequence. Fix $\varepsilon > 0$, and put

$$E_\varepsilon = \{n \in \mathbb{N} : u_n - u_{n+p} \geq \varepsilon\}.$$

Assume first that $\liminf q_m > 1$ and that E_ε has natural density zero. Since

$$\frac{|E_\varepsilon \cap J_m|}{\ell_m} \leq \frac{t_m}{\ell_m}\frac{|E_\varepsilon \cap [1, t_m]|}{t_m}, \qquad \frac{t_m}{\ell_m} = \frac{q_m}{q_m - 1},$$

the coefficient t_m/ℓ_m is eventually bounded, and therefore the lacunary block proportion tends to zero. Hence $\Delta_p S^+ \subseteq \Delta_p S_\theta^+$.

Conversely, suppose that $\limsup q_m < \infty$ and E_ε has lacunary block density zero. Given $\eta > 0$, for all sufficiently large j,

$$|E_\varepsilon \cap J_j| \leq \eta \ell_j.$$

If $t_{m-1} < N \le t_m$, then, with a constant C accounting for finitely many initial blocks,

$$|E_\varepsilon \cap [1, N]| \le C + \eta t_m.$$

Thus

$$\frac{|E_\varepsilon \cap [1, N]|}{N} \le \frac{C}{N} + \eta \frac{t_m}{t_{m-1}}.$$

The upper bound on q_m and the arbitrariness of η give natural density zero, so $\Delta_p S_\theta^+ \subseteq \Delta_p S^+$.

It remains to note that the converse implications are also necessary. Fridy and Orhan's set-theoretic inclusion criteria for natural and lacunary statistical density [1] show that the two zero-density ideals have exactly the ratio conditions stated above. To transfer those criteria to the present one-sided p-step classes, let $E \subseteq \mathbb{N}$ be arbitrary, let $\mathbf{1}_E$ denote the indicator function of E, set $u_1 = \cdots = u_p = 0$, and define recursively

$$u_{n+p} = u_n - \mathbf{1}_E(n).$$

Then

$$u_n - u_{n+p} = \mathbf{1}_E(n),$$

so for every $0 < \varepsilon \le 1$, the exceptional set is exactly E. Any separating set for the two density ideals therefore gives a separating sequence for the corresponding p-upward classes. This proves the necessity and the final equivalence. ∎

Proposition 3.5 (Exact step-parameter hierarchy)**.** *Let $p, q \in \mathbb{N}$. Then*

$$\Delta_p S_\theta^+ \subseteq \Delta_q S_\theta^+ \quad \Longleftrightarrow \quad p \mid q.$$

In particular, if $p < q$ and $p \mid q$, the inclusion is strict, whereas if neither $p \mid q$ nor $q \mid p$, the two classes are incomparable.

Proof. Suppose first that $q = rp$. Then

$$u_n - u_{n+q} = \sum_{j=0}^{r-1} (u_{n+jp} - u_{n+(j+1)p}).$$

If the left-hand side is at least ε, then at least one summand is at least ε/r. For each $j \in \{0, \dots, r-1\}$, shifting the indices by jp can move at most jp indices across the boundary of J_m. Hence

$$\left|\{n \in J_m : u_n - u_{n+q} \ge \varepsilon\}\right| \le r \left|\{n \in J_m : u_n - u_{n+p} \ge \varepsilon/r\}\right| + \frac{pr(r-1)}{2}.$$

After division by ℓ_m and passage to the limit, the desired inclusion follows.

For necessity, suppose that $p \nmid q$, and define the p-periodic sequence

$$u_n = \begin{cases} 1, & p \mid n, \\ 0, & p \nmid n. \end{cases}$$

Then $u_{n+p} = u_n$ for every n, so $(u_n) \in \Delta_p S_\theta^+$. If $p \mid n$, then $p \nmid (n+q)$ and hence $u_n - u_{n+q} = 1$. By Lemma 2.1, the relative frequency of multiples of p in J_m tends to $1/p$. Therefore $(u_n) \notin \Delta_q S_\theta^+$. This proves the equivalence. The strictness and incomparability statements follow immediately. ∎

We now turn to the continuity notion generated by $\Delta_p S_\theta^+$.

Definition 3.6. Let $A \subseteq \mathbb{R}$, and let $g : A \to \mathbb{R}$. The function g is called lacunary statistically p-upward continuous, briefly S_θ-p-upward continuous, on A if $(g(u_n)) \in \Delta_p S_\theta^+$ whenever $(u_n) \in \Delta_p S_\theta^+$ is a sequence in A.

Proposition 3.7. *If g and ϕ are S_θ-p-upward continuous on $A \subseteq \mathbb{R}$, then $g + \phi$ is also S_θ-p-upward continuous on A.*

Proof. For an S_θ-p-upward quasi-Cauchy sequence (u_n) and $\varepsilon > 0$,

$$\begin{aligned}&\{n \in J_m : (g+\phi)(u_n) - (g+\phi)(u_{n+p}) \geq \varepsilon\}\\ \subseteq&\{n \in J_m : g(u_n) - g(u_{n+p}) \geq \varepsilon/2\} \cup \{n \in J_m : \phi(u_n) - \phi(u_{n+p}) \geq \varepsilon/2\}.\end{aligned}$$

Dividing by ℓ_m and passing to the limit proves the assertion. ∎

Multiplication by a positive scalar, composition when domains and ranges are compatible, and restriction to a subset likewise preserve S_θ-p-upward continuity.

Theorem 3.8. *If g is S_θ-p-upward continuous on a subset A of $\mathbb{R}$, then g is S_θ-upward continuous on A.*

Proof. The case $p = 1$ is immediate. Assume $p > 1$, and let (u_n) be an S_θ-upward quasi-Cauchy sequence in A. For each m, write

$$J_m = \{a_m + 1, \ldots, a_m + \ell_m\}, \qquad a_m = t_{m-1},$$

and decompose

$$\ell_m = ps_m + r_m, \qquad 0 \leq r_m < p.$$

Since $\ell_m \to \infty$, one has $s_m \geq 2$ for all sufficiently large m. On the finitely many preceding blocks, define v_n arbitrarily using terms of (u_n). For every remaining block J_m, partition its first ps_m positions into the p arithmetic chains

$$C_{m,r} = \{a_m + r + kp : 0 \leq k \leq s_m - 1\}, \qquad r = 1, \ldots, p,$$

and define

$$v_{a_m+r+kp} = u_{a_m+(r-1)s_m+k+1}, \qquad 0 \leq k \leq s_m - 1.$$

On the remaining r_m positions of J_m, define v_n using arbitrary terms of (u_n). Since the blocks J_m are pairwise disjoint, these prescriptions define a single sequence (v_n) in A.

For

$$n = a_m + r + kp, \qquad 1 \leq r \leq p, \quad 0 \leq k \leq s_m - 2,$$

both n and $n + p$ lie in the same chain, and

$$v_n - v_{n+p} = u_{a_m+(r-1)s_m+k+1} - u_{a_m+(r-1)s_m+k+2}.$$

There are $p(s_m - 1)$ starting indices covered by this correspondence. Hence the number of starting indices in J_m for which this correspondence is unavailable is

$$\ell_m - p(s_m - 1) = p + r_m \leq 2p - 1.$$

Every p-step comparison crossing from J_m into the next lacunary block is already among the remaining starting indices. Indeed, for every starting index n covered by the above correspondence, both n and $n+p$ lie in J_m; hence any p-step comparison crossing into J_{m+1} necessarily starts at one of the remaining indices. Therefore

$$\frac{|\{n \in J_m : v_n - v_{n+p} \geq \varepsilon\}|}{\ell_m} \leq \frac{|\{j \in J_m : u_j - u_{j+1} \geq \varepsilon\}|}{\ell_m} + \frac{p + r_m}{\ell_m}.$$

Thus (v_n) is S_θ-p-upward quasi-Cauchy.

By the S_θ-p-upward continuity of g, $(g(v_n))$ is also S_θ-p-upward quasi-Cauchy. The correspondence above maps these $p(s_m - 1)$ p-step starting indices bijectively onto $p(s_m - 1)$ one-step starting indices in J_m, and for corresponding indices

$$g(v_n) - g(v_{n+p}) = g(u_j) - g(u_{j+1}).$$

Consequently,

$$\frac{|\{j \in J_m : g(u_j) - g(u_{j+1}) \geq \varepsilon\}|}{\ell_m} \leq \frac{|\{n \in J_m : g(v_n) - g(v_{n+p}) \geq \varepsilon\}|}{\ell_m} + \frac{p + r_m}{\ell_m}.$$

Both terms on the right tend to zero, so $(g(u_n))$ is S_θ-upward quasi-Cauchy. ■

Theorem 3.9 (Uniform continuity on arbitrary domains)**.** *Let $A \subseteq \mathbb{R}$. If $g : A \to \mathbb{R}$ is S_θ-p-upward continuous, then g is uniformly continuous on A.*

Proof. Suppose that g is not uniformly continuous. Then there exist $\varepsilon_0 > 0$ and sequences (a_m) and (b_m) in A such that

$$|a_m - b_m| < \frac{1}{m}, \qquad |g(a_m) - g(b_m)| \geq \varepsilon_0.$$

After interchanging a_m and b_m when necessary, we may assume

$$g(a_m) - g(b_m) \geq \varepsilon_0$$

for every m.

For each lacunary block, write

$$\ell_m = 2ps_m + r_m, \qquad 0 \leq r_m < 2p.$$

On the first $2ps_m$ positions of J_m, form successive blocks of p copies of a_m followed by p copies of b_m, and fill the remaining r_m positions with either a_m or b_m. This defines a sequence (v_n) in A. For all p-step comparisons whose two endpoints lie in the same J_m, one has

$$|v_n - v_{n+p}| \leq |a_m - b_m| < \frac{1}{m}.$$

Only the last p starting indices of J_m can have $n + p \notin J_m$. Hence, for every fixed $\varepsilon > 0$ and all sufficiently large m,

$$\frac{|\{n \in J_m : v_n - v_{n+p} \geq \varepsilon\}|}{\ell_m} \leq \frac{p}{\ell_m} \longrightarrow 0.$$

Thus (v_n) is S_θ-p-upward quasi-Cauchy.

In each complete segment of length $2p$, the first p starting indices satisfy

$$g(v_n) - g(v_{n+p}) = g(a_m) - g(b_m) \geq \varepsilon_0.$$

There are ps_m such indices in J_m. Therefore

$$\frac{|\{n \in J_m : g(v_n) - g(v_{n+p}) \geq \varepsilon_0\}|}{\ell_m} \geq \frac{ps_m}{\ell_m} = \frac{1}{2} - \frac{r_m}{2\ell_m} \longrightarrow \frac{1}{2}.$$

Hence $(g(v_n))$ is not S_θ-p-upward quasi-Cauchy, contradicting the assumed continuity of g. ■

Lemma 3.10 (Bounded-sequence reduction)**.** *Let (u_n) be a bounded real sequence. Then $(u_n) \in \Delta_p S_\theta^+$ if and only if*

$$\lim_{m\to\infty} \frac{1}{\ell_m} \left|\left\{n \in J_m : |u_{n+p} - u_n| \geq \varepsilon\right\}\right| = 0$$

for every $\varepsilon > 0$.

Proof. Only the nontrivial implication needs proof. Let D be the diameter of the range of (u_n), and put $d_n = u_{n+p} - u_n$. If $D = 0$, the assertion is immediate. For $\eta, \delta > 0$, set

$$P_m(\eta) = \{n \in J_m : d_n \geq \eta\}, \qquad N_m(\delta) = \{n \in J_m : -d_n \geq \delta\}.$$

The one-sided assumption gives $|N_m(\delta)|/\ell_m \to 0$. Moreover,

$$\sum_{n=t_{m-1}+1}^{t_m} (u_{n+p} - u_n) = \sum_{j=1}^{p} u_{t_m+j} - \sum_{j=1}^{p} u_{t_{m-1}+j},$$

whose absolute value is at most pD. Since $|d_n| \leq D$,

$$\sum_{n\in J_m} d_n \geq \eta|P_m(\eta)| - D|N_m(\delta)| - \delta\ell_m.$$

Consequently,

$$\frac{|P_m(\eta)|}{\ell_m} \leq \frac{pD}{\eta\ell_m} + \frac{D}{\eta}\frac{|N_m(\delta)|}{\ell_m} + \frac{\delta}{\eta}.$$

Taking the upper limit and then letting $\delta \downarrow 0$ gives $|P_m(\eta)|/\ell_m \to 0$. Together with the one-sided assumption, this proves the symmetric condition. ■

Corollary 3.11 (Bounded-domain characterization)**.** *If $A \subseteq \mathbb{R}$ is bounded, then a function $g : A \to \mathbb{R}$ is S_θ-p-upward continuous if and only if it is uniformly continuous.*

Proof. The forward implication is Theorem 3.9. Conversely, let $(u_n) \in \Delta_p S_\theta^+$ be a sequence in bounded A. By Lemma 3.10, (u_n) is lacunary statistically p-quasi-Cauchy in the symmetric sense. Uniform continuity of g therefore makes $(g(u_n))$ symmetric lacunary statistically p-quasi-Cauchy, and hence S_θ-p-upward quasi-Cauchy. ■

Example 3.12 (Failure of the converse on an unbounded domain)**.** Let $A = [0, \infty)$ and $g(x) = -x$. The function g is Lipschitz and hence uniformly continuous. For $u_n = n$,

$$u_n - u_{n+p} = -p,$$

so $(u_n) \in \Delta_p S_\theta^+$. However,

$$g(u_n) - g(u_{n+p}) = p,$$

and therefore the image sequence is not in $\Delta_p S_\theta^+$. Thus uniform continuity does not imply S_θ-p-upward continuity on arbitrary unbounded domains.

Corollary 3.13. *Every S_θ-p-upward continuous function is ordinarily continuous and lacunary statistically continuous.*

Proof. Ordinary continuity follows from Theorem 3.9. If $S_\theta\text{-}\lim u_n = \lambda$ and $\varepsilon > 0$, continuity at λ gives $\delta > 0$ such that

$$|x - \lambda| < \delta \quad \Longrightarrow \quad |g(x) - g(\lambda)| < \varepsilon.$$

Hence

$$\{n \in J_m : |g(u_n) - g(\lambda)| \geq \varepsilon\} \subseteq \{n \in J_m : |u_n - \lambda| \geq \delta\},$$

and the desired lacunary statistical convergence follows. ∎

Standard consequences of uniform continuity also yield the corresponding q-ward and slowly oscillating continuity properties whenever the cited results apply; see [16, Theorem 3.7] and [17].

Theorem 3.14. *The image of an S_θ-p-upward compact subset of $\mathbb{R}$ under an S_θ-p-upward continuous function is again S_θ-p-upward compact.*

Proof. Let $A \subseteq \mathbb{R}$ be S_θ-p-upward compact, and let $g : A \to \mathbb{R}$ be S_θ-p-upward continuous. Take an arbitrary sequence (v_n) in $g(A)$. For each n, choose $u_n \in A$ such that $v_n = g(u_n)$. Since A is S_θ-p-upward compact, (u_n) has an S_θ-p-upward quasi-Cauchy subsequence (u_{n_j}). The image $(g(u_{n_j})) = (v_{n_j})$ is again S_θ-p-upward quasi-Cauchy, so $g(A)$ is S_θ-p-upward compact. ∎

In particular, an S_θ-p-upward continuous function maps every bounded-below subset of its domain to a bounded-below set. Theorem 3.9 also subsumes the corresponding uniform-continuity conclusion on S_θ-p-upward compact sets.

Theorem 3.15. *Let (g_j) be a sequence of S_θ-p-upward continuous functions on a subset A of $\mathbb{R}$. If (g_j) converges uniformly to a function g on A, then g is S_θ-p-upward continuous on A.*

Proof. Let (u_n) be an arbitrary S_θ-p-upward quasi-Cauchy sequence in A, and fix $\varepsilon > 0$. Since (g_j) converges uniformly to g on A, there exists $N \in \mathbb{N}$ such that

$$|g_N(z) - g(z)| < \frac{\varepsilon}{3}$$

for every $z \in A$.

Since g_N is S_θ-p-upward continuous, the sequence $(g_N(u_n))$ is S_θ-p-upward quasi-Cauchy. Hence

$$\lim_{m\to\infty} \frac{1}{\ell_m} \left| \left\{ n \in J_m : g_N(u_n) - g_N(u_{n+p}) \geq \frac{\varepsilon}{3} \right\} \right| = 0.$$

Moreover,

$$\begin{aligned}
&\left\{n \in J_m : g(u_n) - g(u_{n+p}) \geq \varepsilon\right\} \\
&\subseteq \left\{n \in J_m : g(u_n) - g_N(u_n) \geq \frac{\varepsilon}{3}\right\} \cup \left\{n \in J_m : g_N(u_n) - g_N(u_{n+p}) \geq \frac{\varepsilon}{3}\right\} \\
&\quad \cup \left\{n \in J_m : g_N(u_{n+p}) - g(u_{n+p}) \geq \frac{\varepsilon}{3}\right\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|\left\{n \in J_m : g(u_n) - g(u_{n+p}) \geq \varepsilon\right\}\right| \\
&\leq \left|\left\{n \in J_m : g(u_n) - g_N(u_n) \geq \frac{\varepsilon}{3}\right\}\right| \\
&\quad + \left|\left\{n \in J_m : g_N(u_n) - g_N(u_{n+p}) \geq \frac{\varepsilon}{3}\right\}\right| \\
&\quad + \left|\left\{n \in J_m : g_N(u_{n+p}) - g(u_{n+p}) \geq \frac{\varepsilon}{3}\right\}\right|.
\end{aligned}$$

Dividing both sides by ℓ_m and passing to the limit as $m \to \infty$, we obtain

$$\begin{aligned}
&\lim_{m\to\infty} \frac{1}{\ell_m} \left|\left\{n \in J_m : g(u_n) - g(u_{n+p}) \geq \varepsilon\right\}\right| \\
&\leq \lim_{m\to\infty} \frac{1}{\ell_m} \left|\left\{n \in J_m : g(u_n) - g_N(u_n) \geq \frac{\varepsilon}{3}\right\}\right| \\
&\quad + \lim_{m\to\infty} \frac{1}{\ell_m} \left|\left\{n \in J_m : g_N(u_n) - g_N(u_{n+p}) \geq \frac{\varepsilon}{3}\right\}\right| \\
&\quad + \lim_{m\to\infty} \frac{1}{\ell_m} \left|\left\{n \in J_m : g_N(u_{n+p}) - g(u_{n+p}) \geq \frac{\varepsilon}{3}\right\}\right| = 0.
\end{aligned}$$

Thus $(g(u_n))$ is S_θ-p-upward quasi-Cauchy. Consequently, g is S_θ-p-upward continuous on A. ■

Theorem 3.16. *Let $A \subseteq \mathbb{R}$. Denote by $pULSC(A)$ the family of all S_θ-p-upward continuous real-valued functions on A. Then $pULSC(A)$ is complete in the space of continuous functions on A with respect to uniform convergence.*

Proof. Let g be an element of the closure of $pULSC(A)$. Then there exists a sequence (g_j) in $pULSC(A)$ such that (g_j) converges to g on A.

Let (u_n) be an arbitrary S_θ-p-upward quasi-Cauchy sequence in A, and let $\varepsilon > 0$ be given. Since $g_j \to g$ on A, choose $N \in \mathbb{N}$ such that

$$|g_N(u_n) - g(u_n)| < \frac{\varepsilon}{3}$$

for all $n \in \mathbb{N}$.

Since $g_N \in pULSC(A)$, the sequence $(g_N(u_n))$ is S_θ-p-upward quasi-Cauchy. Hence

$$\lim_{m\to\infty} \frac{1}{\ell_m} \left|\left\{n \in J_m : g_N(u_n) - g_N(u_{n+p}) \geq \frac{\varepsilon}{3}\right\}\right| = 0.$$

Moreover,

$$\begin{aligned}&\left\{n \in J_m : g(u_n) - g(u_{n+p}) \geq \varepsilon\right\}\\ \subseteq &\left\{n \in J_m : g(u_n) - g_N(u_n) \geq \frac{\varepsilon}{3}\right\} \cup \left\{n \in J_m : g_N(u_n) - g_N(u_{n+p}) \geq \frac{\varepsilon}{3}\right\}\\ &\cup \left\{n \in J_m : g_N(u_{n+p}) - g(u_{n+p}) \geq \frac{\varepsilon}{3}\right\}.\end{aligned}$$

Therefore,

$$\lim_{m\to\infty} \frac{1}{\ell_m} \left|\left\{n \in J_m : g(u_n) - g(u_{n+p}) \geq \varepsilon\right\}\right| = 0.$$

Thus $(g(u_n))$ is S_θ-p-upward quasi-Cauchy. Hence $g \in pULSC(A)$, and consequently $pULSC(A)$ is closed. Since $pULSC(A)$ is closed in the complete space $C(A, \mathbb{R})$, it is complete. ■

4. Finite-block directional analysis and empirical illustration

The block quantities introduced in Section 2 are the pre-limit proportions appearing in the definition of $\Delta_p S_\theta^+$. Their role is not purely descriptive: Several quantitative transformation and perturbation bounds pass directly to the asymptotic sequence classes.

4.1. Comparison with the symmetric p-step quantity

The symmetric quantity in (2.1) is the finite-block counterpart of the symmetric lacunary statistically p-quasi-Cauchy condition studied in [4]. Identity (2.1) shows that the symmetric quantity combines the two directions, whereas $D_{m,p}^{X,+}$ and $D_{m,p}^{X,-}$ retain them separately.

This loss of directional information can occur in the simplest examples. Let $X_n = n$ and $Y_n = -n$. For every $0 < \varepsilon \leq p$ and every block J_m,

$$D_{m,p}^{X,\mathrm{sym}}(\varepsilon) = D_{m,p}^{Y,\mathrm{sym}}(\varepsilon) = 1,$$

while

$$D_{m,p}^{X,+}(\varepsilon) = 0, \qquad D_{m,p}^{Y,+}(\varepsilon) = 1.$$

Thus the symmetric quantity cannot distinguish persistent increases from persistent decreases of the same magnitude, while the directional quantity does. The two quantities therefore answer different questions; the one-sided quantity is useful when the sign of a change is part of the phenomenon under study.

4.2. Transformation and deterministic perturbation properties

Proposition 4.1 (Positive affine equivariance)**.** *Let $Y_n = aX_n + b$ with $a > 0$ and $b \in \mathbb{R}$. Then, for every m, p and $\varepsilon > 0$,*

$$D_{m,p}^{Y,+}(\varepsilon) = D_{m,p}^{X,+}\left(\frac{\varepsilon}{a}\right).$$

Proof. Since $Y_n - Y_{n+p} = a(X_n - X_{n+p})$, the two exceptional events are identical after rescaling the threshold. ■

Proposition 4.2 (Nondecreasing Lipschitz domination)**.** *Let g be nondecreasing and L-Lipschitz, with $L > 0$, on a set containing all observations X_n. Then*

$$D_{m,p}^{g(X),+}(\varepsilon) \le D_{m,p}^{X,+}\left(\frac{\varepsilon}{L}\right).$$

Proof. If $g(X_n) - g(X_{n+p}) \ge \varepsilon$, monotonicity gives $X_n \ge X_{n+p}$, and the Lipschitz estimate gives $X_n - X_{n+p} \ge \varepsilon/L$. Hence the transformed exceptional set is contained in the original exceptional set at the rescaled threshold. ∎

Proposition 4.3 (Deterministic perturbation bound)**.** *Let (e_n) be a real sequence, set $Y_n = X_n + e_n$, and, for fixed p, define*

$$\eta_{m,p} = \sup_{n \in J_m} |e_n - e_{n+p}|.$$

For each $m \in \mathbb{N}$, whenever $\varepsilon > \eta_{m,p}$,

$$D_{m,p}^{Y,+}(\varepsilon) \le D_{m,p}^{X,+}(\varepsilon - \eta_{m,p})$$

and

$$D_{m,p}^{X,+}(\varepsilon) \le D_{m,p}^{Y,+}(\varepsilon - \eta_{m,p}).$$

Proof. If $Y_n - Y_{n+p} \ge \varepsilon$, then

$$X_n - X_{n+p} = (Y_n - Y_{n+p}) - (e_n - e_{n+p}) \ge \varepsilon - \eta_{m,p}.$$

This proves the first inequality; the second follows by interchanging X and Y. ∎

Corollary 4.4 (Consequences for the asymptotic classes)**.** *The following statements hold.*

(i) For every $a > 0$ and $b \in \mathbb{R}$,

$$(X_n) \in \Delta_p S_\theta^+ \quad \Longleftrightarrow \quad (aX_n + b) \in \Delta_p S_\theta^+.$$

(ii) If g is nondecreasing and Lipschitz on a set containing the range of (X_n), then

$$(X_n) \in \Delta_p S_\theta^+ \quad \Longrightarrow \quad (g(X_n)) \in \Delta_p S_\theta^+.$$

(iii) If $\eta_{m,p} \to 0$, where $\eta_{m,p}$ is defined as in Proposition 4.3, then

$$(X_n) \in \Delta_p S_\theta^+ \quad \Longleftrightarrow \quad (X_n + e_n) \in \Delta_p S_\theta^+.$$

In particular, the last equivalence holds if $|e_n - e_{n+p}| \to 0$.

Proof. The first two statements follow by passing to the limit in Propositions 4.1 and 4.2. For (iii), fix $\varepsilon > 0$. If $\eta_{m,p} \to 0$, then $\eta_{m,p} < \varepsilon/2$ for all sufficiently large m, and Proposition 4.3 gives

$$D_{m,p}^{X+e,+}(\varepsilon) \le D_{m,p}^{X,+}(\varepsilon/2).$$

The reverse implication is identical. Finally, $|e_n - e_{n+p}| \to 0$ implies $\eta_{m,p} \to 0$ because the lower endpoints of J_m tend to infinity. ∎

In particular, if $Z_n = (X_n - \overline{X})/s_X$, then Proposition 4.1 gives

$$D_{m,p}^{Z,+}\left(\frac{\varepsilon}{s_X}\right) = D_{m,p}^{X,+}(\varepsilon).$$

4.3. An exactly solvable trend–seasonality family

Consider

$$X_n = an + b\sin\left(\frac{2\pi n}{T}\right), \qquad a > 0, \quad b > 0, \tag{4.1}$$

where $T \in \mathbb{N}$ is the seasonal period.

Proposition 4.5 (Exact lag profile and arithmetic dependence)**.** *For $p \in \mathbb{N}$, write*

$$p = kT + r, \qquad 0 \le r < T.$$

Then

$$X_n - X_{n+p} = -ap + b\left[\sin\left(\frac{2\pi n}{T}\right) - \sin\left(\frac{2\pi(n+r)}{T}\right)\right].$$

If $1 \le r < T$, this becomes

$$X_n - X_{n+p} = -ap - 2b\sin\left(\frac{\pi r}{T}\right)\cos\left(\frac{2\pi n}{T} + \frac{\pi r}{T}\right).$$

Consequently,

$$\lim_{m\to\infty} D_{m,p}^{X,+}(\varepsilon) = \rho_p(\varepsilon),$$

where, for $r \neq 0$,

$$\rho_p(\varepsilon) = \frac{1}{T}\left|\left\{1 \le j \le T : -2b\sin\left(\frac{\pi r}{T}\right)\cos\left(\frac{2\pi j}{T} + \frac{\pi r}{T}\right) \ge ap + \varepsilon\right\}\right|.$$

If $T \mid p$, then $\rho_p(\varepsilon) = 0$ for every $\varepsilon > 0$. If T is even and $p \equiv T/2 \pmod{T}$, then

$$X_n - X_{n+p} = -ap + 2b\sin\left(\frac{2\pi n}{T}\right).$$

More generally, with

$$A_r = 2b\sin\left(\frac{\pi r}{T}\right),$$

the condition $ap + \varepsilon > A_r$ implies $\rho_p(\varepsilon) = 0$. For a fixed residue $1 \le r < T$, the sequence $\rho_{r+kT}(\varepsilon)$ is nonincreasing in k and is eventually zero.

Proof. The first identity follows from periodicity, and the second from

$$\sin\alpha - \sin(\alpha+\beta) = -2\sin(\beta/2)\cos(\alpha+\beta/2),$$

with $\alpha = 2\pi n/T$ and $\beta = 2\pi r/T$. The exceptional indicator is T-periodic, so Lemma 2.1 gives the stated limiting proportion. If $r = 0$, the seasonal terms cancel and $X_n - X_{n+p} = -ap < 0$. If $r = T/2$, the identity $\sin(\alpha+\pi) = -\sin\alpha$ gives the displayed half-period formula. The amplitude bound gives $X_n - X_{n+p} \le -ap + A_r$, which proves the vanishing criterion. For $p = r + kT$, the seasonal comparison is unchanged while $-ap$ decreases with k, so the exceptional sets are nested and eventually empty. ■

Remark 4.6. Let $d = \gcd(p, T)$, the greatest common divisor of p and T. The map $j \mapsto j + p \pmod{T}$ decomposes the T seasonal phases into d cycles, each of length T/d. Thus $\gcd(p, T)$ describes the phase-orbit decomposition. The exact exceptional proportion, however, depends more specifically on the residue $p \bmod T$, the trend term ap, and the threshold ε.

For the monthly prototype

$$T = 12, \qquad a = 0.1, \qquad b = 3,$$

the exact limiting proportions are displayed in Table 1, and the lag profile for $\varepsilon = 1$ is shown in Figure 1. The six-month lag compares opposite seasonal phases, whereas the annual lag cancels the seasonal component.

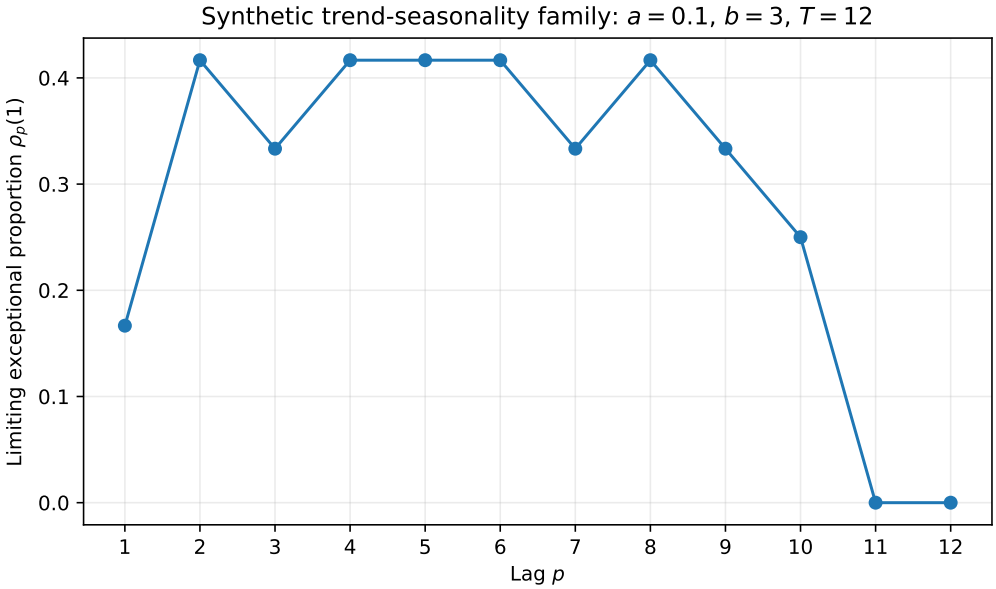


Figure 1. Exact lag profile $\rho_p(1)$ for the trend–seasonality family with $T = 12$, $a = 0.1$, and $b = 3$. The full-period lag removes the seasonal component from the p-step difference.

Table 1. Exact limiting proportions $\rho_p(\varepsilon)$ for $X_n = 0.1n + 3\sin(2\pi n/12)$.

ε	$p = 1$	$p = 3$	$p = 6$	$p = 12$
0.25	0.5000	0.5000	0.4167	0.0000
0.50	0.3333	0.5000	0.4167	0.0000
1.00	0.1667	0.3333	0.4167	0.0000
2.00	0.0000	0.3333	0.4167	0.0000

4.4. Mauna Loa atmospheric CO_2 data

Using the preprocessing described in Section 2, the monthly series has sample mean and sample standard deviation

$$\overline{X} \approx 339.6192, \qquad s_X \approx 17.1165.$$

The calculations below are finite-block illustrations of the directional quantities. For fixed p, ε, and θ, they describe the observed proportions of p-step changes that exceed the prescribed threshold in

the selected direction. They are not estimators of asymptotic class membership, and no probabilistic sampling model, hypothesis test, confidence statement, or convergence-rate claim is attached to them. In particular, a finite series cannot establish

$$D_{m,p}^{X,+}(\varepsilon) \longrightarrow 0 \qquad (m \to \infty).$$

The quantities p, ε, and θ play different roles and are not treated as parameters to be estimated or optimized from the data. The lag p specifies the temporal scale; for monthly data, $p = 1, 3, 6, 12$ correspond to one-month, three-month, half-year, and annual comparisons. The threshold ε specifies the magnitude regarded as relevant; 1 ppm is used as a transparent reference value, and several other thresholds are displayed below. The lacunary sequence determines block localization and resolution. Dyadic blocks provide a simple geometric baseline, and alternative geometric structures are used for sensitivity. No universal optimal choice of p, ε, or θ is asserted.

4.4.1. Dyadic lacunary blocks

For $\varepsilon = 1$ ppm, Table 2 gives the dyadic block proportions for $p = 1, 3, 6, 12$.

Table 2. Values of $D_{m,p}^{X,+}(1)$ for the monthly Mauna Loa CO_2 series.

m	Period	$p = 1$	$p = 3$	$p = 6$	$p = 12$
4	Nov 1958–Jun 1959	0.125	0.375	0.625	0.000
5	Jul 1959–Oct 1960	0.312	0.312	0.375	0.000
6	Nov 1960–Jun 1963	0.219	0.375	0.406	0.000
7	Jul 1963–Oct 1968	0.219	0.281	0.391	0.000
8	Nov 1968–Jun 1979	0.234	0.320	0.398	0.000
9	Jul 1979–Oct 2000	0.262	0.332	0.395	0.000

The six-month lag consistently gives one of the largest directional proportions, whereas

$$D_{m,12}^{X,+}(1) = 0$$

for every complete dyadic block displayed. This is consistent with the half-period/full-period contrast in Proposition 4.5.

4.4.2. Directional versus symmetric variation

In the last complete dyadic block $J_9 = (256, 512]$, Table 3 compares the two directions with the symmetric quantity at $\varepsilon = 1$ ppm.

Table 3. Directional and symmetric block proportions in J_9 at $\varepsilon = 1$ ppm.

p	$D_{9,p}^{X,+}(1)$	$D_{9,p}^{X,-}(1)$	$D_{9,p}^{X,\mathrm{sym}}(1)$
1	0.2617	0.3281	0.5898
3	0.3320	0.5430	0.8750
6	0.3945	0.4922	0.8867
12	0.0000	0.8008	0.8008

For $p = 12$, the symmetric proportion is approximately 0.8008 while the downward directional proportion is zero. Thus large annual changes are common in this block, but none of the corresponding exceedances are decreases of at least 1 ppm in the direction measured by $D^{X,+}_{9,12}(1)$. The symmetric and directional quantities therefore provide different information rather than interchangeable summaries.

4.4.3. Threshold and lag sensitivity

Table 4 evaluates J_9 at four thresholds.

Table 4. Threshold sensitivity in J_9 (Jul 1979–Oct 2000).

ε (ppm)	$p = 1$	$p = 3$	$p = 6$	$p = 12$
0.25	0.3359	0.3672	0.4258	0.0000
0.50	0.3047	0.3516	0.4023	0.0000
1.00	0.2617	0.3320	0.3945	0.0000
2.00	0.0781	0.2656	0.3203	0.0000

The half-year/annual directional contrast persists throughout the displayed threshold range. A broader lag profile is shown in Figure 2 for $p = 1, \ldots, 24$ using the complete block $J_8 = (128, 256]$.

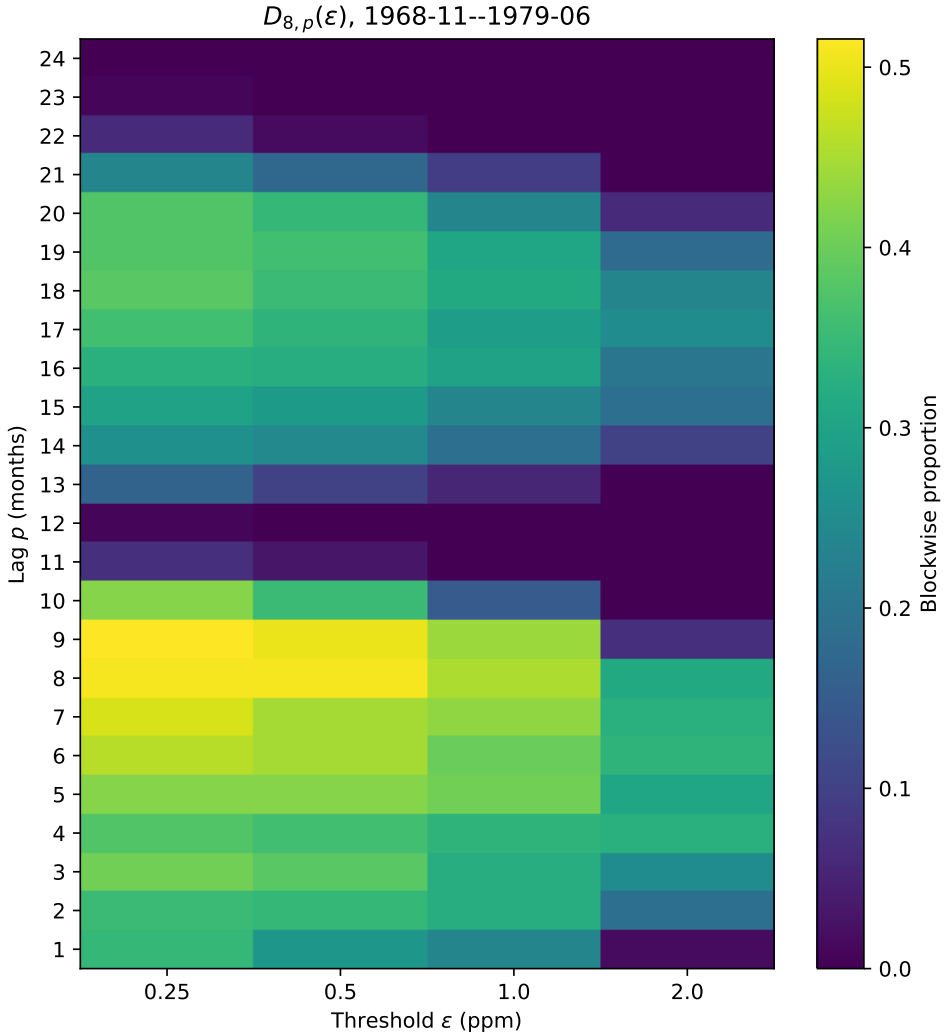


Figure 2. Lag–threshold map $D_{8,p}^{X,+}(\varepsilon)$ for $p = 1, \ldots, 24$ and $\varepsilon \in \{0.25, 0.5, 1, 2\}$ ppm.

4.4.4. Sensitivity to the lacunary block structure

For the geometric structures

$$t_m^{(b)} = \lceil b^m \rceil, \qquad b \in \{1.5, 2, 3\},$$

Table 5 uses the last complete block for which every 12-month comparison remains within the available sample.

Table 5. Finite-block sensitivity to three geometric lacunary structures at $\varepsilon = 1$ ppm.

b	Block length	Period	$D_{m,6}^{X,+}(1)$	$D_{m,12}^{X,+}(1)$
1.5	146	Jul 1982–Aug 1994	0.3973	0.0000
2.0	256	Jul 1979–Oct 2000	0.3945	0.0000
3.0	162	Dec 1964–May 1978	0.4074	0.0000

The exact six-month proportions vary slightly with the chosen blocks, while the qualitative half-year/annual contrast persists in all three displayed structures. This is a finite-block sensitivity observation and not a stationarity claim.

4.4.5. Sensitivity to interpolated observations

Let M denote the set of indices corresponding to the five interpolated monthly values, and define

$$V_{m,p} = \{n \in J_m : n \notin M,\ n + p \notin M\}.$$

For this sensitivity calculation, we use

$$D_{m,p}^{X,+,\mathrm{obs}}(\varepsilon) = \frac{1}{|V_{m,p}|} \left| \left\{ n \in V_{m,p} : X_n - X_{n+p} \geq \varepsilon \right\} \right|.$$

Thus only comparisons whose two endpoints are observed enter the calculation.

Among the complete dyadic blocks in Table 2, only J_6 for $p = 12$ and J_7 are affected. In J_6, removing three annual comparisons leaves the $p = 12$ proportion unchanged at 0. In J_7, the original proportions for $p = 1, 3, 6, 12$,

$$0.2188, \quad 0.2813, \quad 0.3906, \quad 0,$$

become

$$0.2333, \quad 0.2931, \quad 0.3793, \quad 0,$$

after comparisons involving interpolated values are removed. The six-month lag remains the largest among the displayed lags, and the annual directional proportion remains zero. The later blocks J_8 and J_9 contain no relevant comparisons involving interpolated values, so the broad lag profile, threshold sensitivity, and directional-versus-symmetric comparison in those blocks are unchanged.

4.4.6. Standardization and deterministic error interpretation

For the present data, 1 ppm corresponds after standardization to

$$\frac{1}{s_X} \approx 0.0584.$$

Proposition 4.1 therefore shows that standardization leaves the displayed proportions unchanged when the threshold is rescaled consistently. If $|e_n| \leq \delta$, then $\eta_{m,p} \leq 2\delta$, so Proposition 4.3 recovers the bounded-error estimate

$$D_{m,p}^{X+e,+}(\varepsilon) \leq D_{m,p}^{X,+}(\varepsilon - 2\delta), \qquad \varepsilon > 2\delta.$$

These are deterministic threshold-level statements and require no probabilistic error model.

5. Conclusions

Two exact classifications emerge from the present study. Theorem 3.4 determines when replacing natural density by lacunary block density preserves the one-sided p-step sequence class and when the resulting classes differ. Although the corresponding ratio criteria are known in classical density theory,

their precise consequences for the present one-sided p-step classes are established here. Proposition 3.5 further shows that inclusion between classes associated with different steps is governed by divisibility rather than by the numerical order of the steps. In particular, increasing the step does not necessarily yield an inclusion, and classes corresponding to certain distinct steps are incomparable. These results distinguish the respective roles of the density method and the step parameter, providing exact comparisons beyond those available in the related one-step one-sided and symmetric p-step settings.

The mathematical significance of the one-sided condition is also reflected in the compactness and continuity results. In contrast to the symmetric setting, the associated compactness property is characterized by boundedness from below, independently of the step parameter and the lacunary structure. The continuity results also establish that S_θ-p-upward continuity implies uniform continuity, with equivalence on bounded domains and failure of the converse in general, as demonstrated by an explicit counterexample.

The finite-block analysis provides additional mathematical consequences. Its transformation and perturbation estimates yield preservation properties for the corresponding asymptotic sequence classes, extending the significance of the blockwise quantities beyond their descriptive interpretation. For the trend–seasonality family, the exact limiting exceptional proportions are determined by the arithmetic relation between the lag and the seasonal period, together with the contribution of the trend. The Mauna Loa illustration shows how these directional quantities distinguish changes whose opposite directions are combined by the corresponding symmetric quantity. The results therefore provide both exact structural comparisons within lacunary sequence theory and a quantitative interpretation of the directional information retained by the one-sided condition.

Author contributions

Serkan İlter: Conceptualization, methodology, formal analysis, investigation, data curation, validation, writing - original draft, writing - review & editing, supervision, project administration. Seyit Koca: Software, formal analysis, validation, visualization, writing - review & editing. All authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors state no conflict of interest.

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