AIMS Mathematics, 11(11): 36517–36538.
DOI: 10.3934/math.20261412
Received: 12 July 2026
Revised: 13 October 2026
Accepted: 21 October 2026
Published: 05 November 2026

https://www.aimspress.com/journal/Math

Research article

On integers of the form $p + F_{2^k} + F_q$

Yang Gao*

School of Mathematics and Statistics, Huaiyin Normal University, Huaian, Jiangsu 223300, China

* **Correspondence:** Email: dg21210003@smail.nju.edu.cn.

Abstract: In 1934, Romanoff proved that the set of positive integers representable as the sum of a prime and a power of two has a positive lower asymptotic density. Erdős later constructed an infinite arithmetic progression of odd integers, none of which admits such a representation. Let F_n be the Fibonacci sequence. In this paper, we prove that the set of positive integers of the form $p + F_{2^k} + F_q$, where p, q are primes and k is a nonnegative integer, has a positive lower asymptotic density. In the opposite direction, we show that the lower asymptotic density of positive integers that cannot be represented in this form exceeds $\frac{1}{2}$.

Keywords: Romanoff-type problems; primes; asymptotic density; applications of sieve methods
Mathematics Subject Classification: 11P32, 11A41, 11B13

1. Introduction

Romanoff [17] proved that the set of integers of the form $p + 2^n$ has a positive lower asymptotic density, while Erdős [10] used covering systems to construct an infinite arithmetic progression of integers not of this form, showing that the complement also has positive density.

Let F_n denote the n-th Fibonacci number, with $F_0 = 0$, $F_1 = F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$. Replacing powers of two by linear recurrences, Lee [12] established that the set of integers of the form $p + F_k$, where p is prime and k is a nonnegative integer, has a positive lower asymptotic density. Liu and Xue [13] later gave an explicit lower bound, and Wang and Chen [23] proved that the set of positive integers admitting a unique representation of this form has a positive lower asymptotic density. Wang [21] extended the positive density result to Lucas numbers $p + L_n$. In the opposite direction, Šiurys [18] constructed an infinite arithmetic progression of integers not of the form $p^\alpha \pm F_n$.

For sums with two sparse terms, Crocker [5] showed that infinitely many odd integers are not of the form $p + 2^a + 2^b$. Sun and Le [20] and Yuan [26] studied related problems for $c(2^a + 2^b) + p^\alpha$. Pan [16] proved that the set of non-representable positive odd integers for $p + 2^a + 2^b$ has a lower asymptotic density at least $x^{1-\varepsilon}$. Elsholtz, Luca, and Planitzer [8] established positive lower asymptotic density

for the mixed form $p + 2^{2^k} + m!$, and Ding [6] obtained a positive lower asymptotic density result for $p + 2^{a^2} + 2^{b^2}$. Xu and Chen [24] sharpened the latter by showing that a positive proportion of integers have a unique representation of that form, and later [3] extended such results to sums of a prime and several power or Fibonacci terms under a natural summability condition.

Most recently, Xu and Chen [25] studied the square-index case $n = p + F_{k_1^2} + F_{k_2^2}$. They proved that the set of integers representable in this form has positive lower asymptotic density, and showed that both the set of integers with exactly one representation and the set with at least two representations have positive density, while the non-representable set contains an infinite arithmetic progression. Such works typically treat indices of a single type, such as square indices $F_{k_1^2} + F_{k_2^2}$ or exponential indices 2^k. In contrast, we study the mixed-index form $F_{2^k} + F_q$, where one index is a power of two and the other is a prime.

For further developments on Romanoff-type problems, results concerning Romanoff's constant and its density properties can be found in [1, 9]. Extensions to sums of primes and exponential sequences were studied in [2, 4], while related results for linear recurrences were established in [7, 22].

In this paper, we consider integers of the form

$$n = p + F_{2^k} + F_q,$$

where p and q are primes and k is a nonnegative integer.

Theorem 1.1. *The set of positive integers representable as*

$$p + F_{2^k} + F_q,$$

where p and q are primes and k is a nonnegative integer, has positive lower asymptotic density.

In the opposite direction, it is not difficult to find an arithmetic progression of positive even numbers that avoids integers of the form $p + F_{2^k} + F_q$. One can prove that for any nonnegative integer m, the integer $42m+38$ cannot be represented as $p+F_{2^k}+F_q$. Specifically, we set $n = 42m+38$. Suppose that

$$n = p + F_{2^k} + F_q,$$

where p, q are primes and k is a nonnegative integer. Since $2 \mid F_u$ if and only if $3 \mid u$, it follows that F_{2^k} is odd. So $p + F_q$ is also odd by $2 \mid n$. Hence, $p = 2$ or $q = 3$.

Suppose that $k \geq 3$. By $\gcd(F_{2^k}, F_8) = F_8$, we have $F_8 \mid F_{2^k}$. Noting that $F_8 = 21$ and F_{2^k} is odd, we have

$$F_{2^k} \equiv 21 \pmod{42}.$$

So,

$$p + F_q \equiv 38 - 21 \equiv 17 \pmod{42}.$$

If $p = 2$, then $F_q \equiv 15 \pmod{42}$. Thus, $3 \mid F_q$. So, $4 \mid q$, a contradiction. So, $p \geq 3$. Thus, F_q is even and then $q = 3$. It follows that

$$p + 2 \equiv 17 \pmod{42},$$

so, $p \equiv 15 \pmod{42}$. Hence, $3 \mid p$, and then $p = 3$, a contradiction.

When $k = 2$, we have

$$p + F_q \equiv 35 \pmod{42}.$$

If $p = 2$, then $F_q \equiv 33 \pmod{42}$. Thus, $3 \mid F_q$. So, $4 \mid q$, a contradiction. So, $p \geq 3$. Thus, F_q is even and then $q = 3$. It follows that

$$p + 2 \equiv 35 \pmod{42},$$

so, $p \equiv 33 \pmod{42}$. Hence, $3 \mid p$, and then $p = 3$, a contradiction.

When $k = 0$ or $k = 1$, we have

$$p + F_q \equiv 37 \pmod{42}.$$

If $p = 2$, then $F_q \equiv 35 \pmod{42}$. Thus, $7 \mid F_q$. So, $8 \mid q$, a contradiction. So, $p \geq 3$. Thus, F_q is even and then $q = 3$. It follows that

$$p \equiv 35 \pmod{42}.$$

So, $7 \mid p$ and then $p = 7$, a contradiction. This completes the proof.

However, constructing an arithmetic progression of even numbers is of little significance in itself; the specific reason is given below. Overall, the lower asymptotic density of positive even integers representable in this form is 0. More specifically, let n be an even positive integer with $n = p + F_{2^k} + F_q$. Since F_{2^k} is always odd, parity considerations force either $q = 3$, in which case $n = p + F_{2^k} + 2$, or $q \neq 3$, in which case F_q is odd and necessarily $p = 2$. By Lemma 2.1(2) and (4), the number of admissible k with $F_{2^k} \leq x$ is at most $\log_2 L(x) + 1 \ll \log\log x$. Hence, the number of even integers $n \leq x$ admitting such a representation is at most

$$\pi(x)(\log_2 L(x) + 1) + (\log_2 L(x) + 1)\pi(L(x)) \ll \frac{x \log\log x}{\log x} = o(x).$$

Therefore, the lower asymptotic density of integers not representable in this form is at least $\frac{1}{2}$. Hence, to prove that the lower asymptotic density is strictly greater than $\frac{1}{2}$, one needs to construct an infinite arithmetic progression of odd integers.

Theorem 1.2. *There exists an infinite arithmetic progression of positive odd integers, no term of which can be represented as*

$$p + F_{2^k} + F_q,$$

where p and q are primes and k is a nonnegative integer.

This paper is organized as follows: In Section 2, we collect some preliminary lemmas on Fibonacci numbers and sieve estimates. In Section 3, we prove Theorem 1.1, which shows that the representable set has positive lower asymptotic density. In Section 4, we prove Theorem 1.2 by constructing explicit covering systems.

2. Preliminaries

2.1. The fundamental properties of the Fibonacci sequence

The inequalities in the following lemma are elementary; their proofs are omitted. For the congruence properties of the Fibonacci sequence modulo primes, we refer to Somer [19].

Lemma 2.1. *Let $\gamma = \frac{1+\sqrt{5}}{2}$. Then the following statements hold.*

(1) $\left|F_n - \dfrac{\gamma^n}{\sqrt{5}}\right| < \dfrac{1}{2}$ *for all positive integers n.*

(2) *Define* $L(x) = \dfrac{\log(\sqrt{5}\,(x+\frac{1}{2}))}{\log\gamma}$, $x > 1$. *If* $F_n \leqslant x$, *then* $n \leqslant L(x)$.

(3) *Define* $W(x) = \dfrac{\log(\sqrt{5}\,(x-\frac{1}{2}))}{\log\gamma}$, $x > 1$. *If positive integer* $n \leqslant W(x)$, *then* $F_n \leqslant x$.

(4) $2\log x < L(x) < 4\log x \quad (x \geq 3)$; $2\log x < W(x) < 3\log x \quad (x \geq 5)$.

(5) *For any positive integer d, the sequence* $F_n \bmod d$ *is purely periodic: There exists a smallest positive integer* $u(d)$ *(the Pisano period) such that*

$$F_{n+u(d)} \equiv F_n \pmod{d} \quad \textit{for all } n \geq 0.$$

(6) *If p is prime, then in one complete period of the Fibonacci numbers taken modulo p, every residue occurs no more than four times.*

In what follows, we also need to present some properties of the Fibonacci sequence and a table of its values, which will be used repeatedly in Section 4.

For a prime ℓ, let $z(\ell)$ denote the entry point of ℓ in the Fibonacci sequence, that is, the least positive integer z such that

$$F_z \equiv 0 \pmod{\ell}.$$

It is well known that

$$z(\ell) \mid u(\ell), \qquad u(\ell) \in \{z(\ell),\, 2z(\ell),\, 4z(\ell)\}.$$

In Section 4, in the proof of Theorem 1.2, we shall repeatedly use two facts: the standard fact that, for every prime ℓ,

$$\ell \mid F_n \text{ if and only if } z(\ell) \mid n,$$

and the pairing identity that, for every prime ℓ and every odd positive integer r,

$$F_{u(\ell)-r} \equiv F_r \pmod{\ell}.$$

Tables 1 and 2 list the values of $z(\ell)$ and $u(\ell)$ for some primes ℓ; they can be checked by a direct computation.

Table 1. Entry points and Pisano periods (part I).

ℓ	3	5	7	11	23	31	41	47	61	241
$z(\ell)$	4	5	8	10	24	30	20	16	15	120
$u(\ell)$	8	20	16	10	48	30	40	32	60	240

Table 2. Entry points and Pisano periods (part II).

ℓ	769	1103	1601	2161	2521	3041	20641	2207	1087	4481
$z(\ell)$	96	48	80	40	60	80	120	32	64	64
$u(\ell)$	192	96	160	80	120	160	240	64	128	128

Lemma 2.2. *[11, Theorem 328] There exists an absolute constant $c > 0$ such that*

$$\varphi(x) \geq c \frac{x}{\log\log x} \qquad \text{for all } x \geq 3,$$

where φ is Euler's totient function.

Lemma 2.3. *[3, (6.6)] Let $u(d)$ be the minimal positive period of the Fibonacci sequence modulo $d \geq 2$, and set $v(d) = \max\{ u(p) : p \mid d,\ p \text{ prime} \}$. Then for any $\alpha > 0$, the series*

$$\sum_{d=2}^{\infty} \frac{\mu^2(d)}{d\, v(d)^{\alpha}}$$

converges, where μ is the Möbius function.

2.2 Sieve estimates

Lemma 2.4. *(Bombieri–Davenport) Let $h > 0$ be an even integer and let x be sufficiently large. Let $\pi_h(x)$ be the number of primes p such that $p \leq x$ and $p + h$ is prime. Then*

$$\pi_h(x) \ll \frac{x}{\log^2 x} \prod_{p \mid h} \Big(1 + \frac{1}{p}\Big).$$

This estimate can be found in Nathanson [15, Theorem 7.3].

Remark 1. *It is clear that Lemma 2.4 is also true if h is odd.*

Lemma 2.5. *(Brun–Titchmarsh inequality [14]]) Let k be a positive integer and let l be an integer coprime to k. If $x > k$, then the number of primes $p \leq x$ with $p \equiv l \pmod{k}$ satisfies*

$$\pi(x; k, l) \leq \frac{2x}{\phi(k)\log(x/k)},$$

where ϕ is Euler's totient function.

3. Proof of Theorem 1.1

Lemma 3.1. *Let F_n be the Fibonacci sequence. For non-negative integers k_1, k_2 and primes q_1, q_2, the equation*

$$F_{2^{k_1}} + F_{q_1} = F_{2^{k_2}} + F_{q_2}$$

has only the solutions

$$(k_1, q_1) = (k_2, q_2), \qquad (k_1, k_2, q_1, q_2) = (1, 0, q, q), \qquad (k_1, k_2, q_1, q_2) = (0, 1, q, q),$$

where q is an arbitrary prime.

Proof. We first show that if m, n, u, v are distinct positive integers, then

$$F_m + F_n \neq F_u + F_v. \qquad (*)$$

Without loss of generality, assume that

$$m = \max\{m, n, u, v\}.$$

Then $u, v \le m-1$, and hence

$$F_u + F_v \le F_{m-1} + F_{m-2} = F_m.$$

Since $F_n > 0$, we have

$$F_m + F_n > F_m \ge F_u + F_v.$$

Therefore $(*)$ holds.

Now suppose that $F_{2^{k_1}} + F_{q_1} = F_{2^{k_2}} + F_{q_2}$ holds. By the above fact, at least two of $2^{k_1}, q_1, 2^{k_2}, q_2$ are equal. This completes the proof. □

Let $\mathcal{P}$ denote the set of all primes and let $\mathbb{N}_0$ denote the set of nonnegative integers.

Lemma 3.2. *Let* $r(n) = \#\{(p,k,q) \in \mathcal{P}\times\mathbb{N}_0\times\mathcal{P} : n = p + F_{2^k} + F_q\}$ *and* $R(x) = \sum_{n\le x} r(n)$. *Then* $R(x) \asymp x$ *as* $x \to \infty$.

Proof. Denote by $\pi(x)$ the number of primes $p \le x$. It is well known that $\pi(x) \sim x/\log x$. From Lemma 2.1(4), we have $L(x) \asymp \log x$ and $W(x) \asymp \log x$ for large x. Notice $R(x)$ counts triples (p,k,q) with p, q prime, $k \ge 0$, and $p + F_{2^k} + F_q \le x$.

On the one hand, if $p + F_{2^k} + F_q \le x$, then $F_{2^k}, F_q, p \le x$. Combining this with Lemma 2.1(2) and (4), we obtain

$$R(x) \le (\log_2 L(x) + 1)\cdot \pi(L(x))\cdot \pi(x) \ll (\log\log x)\cdot \frac{\log x}{\log\log x}\cdot \frac{x}{\log x} = x.$$

On the other hand, the conditions $p \le x/3$, $F_{2^k} \le x/3$, and $F_q \le x/3$ imply $p + F_{2^k} + F_q \le x$. By Lemma 2.1(3), $F_m \le x/3$ holds whenever $m \le W(x/3)$. Therefore,

$$\#\{k : F_{2^k} \le x/3\} \ge \#\{k : 2^k \le W(x/3)\} \gg \log W(x/3) \asymp \log\log x,$$
$$\#\{q \text{ prime} : F_q \le x/3\} \ge \pi(W(x/3)) \asymp \frac{\log x}{\log\log x}.$$

Together with $\pi(x/3) \asymp x/\log x$, we obtain

$$R(x) \gg (\log\log x)\cdot \frac{\log x}{\log\log x}\cdot \frac{x}{\log x} = x.$$

Thus $R(x) \gg x$ and $R(x) \ll x$, so $R(x) \asymp x$. □

Lemma 3.3. $\sum_{n\le x} r(n)^2 \ll x$.

Proof. It is clear that

$$\sum_{n\le x} r(n)^2 = \#\{(p_1,k_1,q_1,p_2,k_2,q_2) : p_1 + F_{2^{k_1}} + F_{q_1} = p_2 + F_{2^{k_2}} + F_{q_2} \le x\}.$$

Set $h = F_{2^{k_1}} + F_{q_1} - F_{2^{k_2}} - F_{q_2}$, so that $p_2 = p_1 + h$. We classify solutions according to h.

Case I. ($h = 0$) By Lemma 3.1, we obtain that for each pair (k_1, q_1), there are at most O(1) possible pairs (k_2, q_2). Hence the contribution is $O(x)$.

Case II. ($h < 0$) Interchanging (p_1, k_1, q_1) with (p_2, k_2, q_2) replaces h by $-h > 0$, so this case reduces to Case III below by symmetry.

Case III. ($h > 0$) This is the main case. Since $p_2 = p_1 + h$ with h a positive integer, Lemma 2.4 and Remark 1 give

$$\#\{p_1 \le x : p_1 + h \in \mathcal{P}\} \ll \frac{x}{\log^2 x} \prod_{p|h} \Big(1 + \frac{1}{p}\Big).$$

We obtain

$$(\text{contribution of Case III}) \ll \frac{x}{\log^2 x} \sum_{(k_1,q_1,k_2,q_2)}{}' \prod_{p|h} \Big(1 + \tfrac{1}{p}\Big), \tag{1}$$

where $\sum'$ denotes the sum over all quadruples (k_1, q_1, k_2, q_2) with q_1, q_2 prime, $k_1, k_2 \in \mathbb{N}_0$, $F_{2^{k_i}}, F_{q_j} \le x$, and $h = F_{2^{k_1}} + F_{q_1} - F_{2^{k_2}} - F_{q_2} > 0$.

We first dispose of large prime factors of h. Since $h \le 2x$, the number s of distinct prime factors of h exceeding $\log x$ satisfies $(\log x)^s \le h \le 2x$, whence $s \le \log 2x / \log\log x$. Therefore

$$\prod_{\substack{p|h \\ p > \log x}} \Big(1 + \tfrac{1}{p}\Big) \le \Big(1 + \tfrac{1}{\log x}\Big)^{\log 2x/\log\log x} \to 1 \quad (x \to \infty).$$

In view of this, it suffices to show that

$$W := \sum_{(k_1,q_1,k_2,q_2)}{}' \prod_{\substack{p|h \\ p \le \log x}} \Big(1 + \tfrac{1}{p}\Big) \ll \log^2 x. \tag{2}$$

Write $P(d)$ for the largest prime factor of d. We have

$$W = \sum_{(k_1,q_1,k_2,q_2)}{}' \left(1 + \sum_{\substack{d|h \\ d \ge 2,\ P(d) \le \log x}} \frac{\mu^2(d)}{d}\right),$$

where μ is the Möbius function.
It is easy to see that

$$\sum_{(k_1,q_1,k_2,q_2)}{}' 1 \ll \log^2 x.$$

In view of this, it suffices to show that

$$T := \sum_{(k_1,q_1,k_2,q_2)}{}' \sum_{\substack{d|h \\ d \ge 2,\ P(d) \le \log x}} \frac{\mu^2(d)}{d} \ll \log^2 x. \tag{3}$$

Interchanging the order of summation yields

$$T = \sum_{\substack{d \ge 2 \\ P(d) \le \log x}} \frac{\mu^2(d)}{d} \#S_{d,x}, \tag{4}$$

where

$$S_{d,x} = \Big\{(k_1, q_1, k_2, q_2) : F_{2^{k_i}}, F_{q_j} \le x,\ h > 0,\ d \mid h\Big\}.$$

Define

$$H_{d,x} = \Big\{(k_1, q_1, k_2, q_2) : F_{2^{k_i}}, F_{q_j} \le x,\ F_{2^{k_1}} + F_{q_1} \equiv F_{2^{k_2}} + F_{q_2} \pmod{d}\Big\}.$$

Since $S_{d,x} \subseteq H_{d,x}$, it suffices to bound $\#H_{d,x}$. Recall from Lemma 2.3 that $v(d) = \max\{u(p) : p \mid d,\ p \text{ prime}\}$. For each squarefree $d \ge 2$, fix a prime $p_d \mid d$ attaining this maximum, i.e., $u(p_d) = v(d)$.

Now fix a triple (k_1, k_2, q_2) with $F_{2^{k_1}}, F_{2^{k_2}}, F_{q_2} \le x$, and set $l = F_{2^{k_2}} + F_{q_2} - F_{2^{k_1}}$. The condition $F_{2^{k_1}} + F_{q_1} \equiv F_{2^{k_2}} + F_{q_2} \pmod{d}$ becomes $F_{q_1} \equiv l \pmod{d}$, which implies in particular

$$F_{q_1} \equiv l \pmod{p_d}. \tag{5}$$

By Lemma 2.1(6), every residue modulo p_d is attained at most four times in one period. Hence the solutions of (5) lie in the union of at most four residue classes modulo $v(d)$. Since $F_{q_1} \le x$ implies $q_1 \le L(x) < 4\log x$ by Lemma 2.1(2) and (4), we distinguish two subcases.

- *If $v(d) \ge \log x$.* Then $q_1 \le L(x) < 4\log x \le 4\,v(d)$, so each of the four residue classes modulo $v(d)$ contains at most four values of q_1 in $\{2, 3, \dots, 4\,v(d)\}$. This gives at most 16 admissible values of q_1.
- *If $v(d) < \log x$.* Then $L(x) > 2\log x > v(d)$. For each of the (at most four) residue classes modulo $v(d)$: If the representative is not coprime to $v(d)$, the class contains at most one prime; otherwise, by the Brun–Titchmarsh inequality (Lemma 2.5), the number of primes $q_1 \le L(x)$ in that class is at most $2L(x)/(\varphi(v(d))\log(L(x)/v(d)))$. Taking the maximum, we obtain

$$\#\{q_1 \text{ prime} : q_1 \le L(x),\ F_{q_1} \equiv l \pmod{p_d}\} \le 4\max\Big\{1,\ \frac{2L(x)}{\varphi(v(d))\log(L(x)/v(d))}\Big\}. \tag{6}$$

The number of admissible triples (k_1, k_2, q_2) is at most

$$\Big\lfloor \frac{\log L(x)}{\log 2} \Big\rfloor^2 \cdot \pi(L(x)) \ll (\log\log x)^2 \cdot \frac{\log x}{\log\log x} = \log x \cdot \log\log x.$$

Combining with the bounds above, we obtain

$$\#H_{d,x} \ll \log x \cdot \log\log x \times \begin{cases} 16, & \text{if } v(d) \ge \log x, \\ 4\max\Big\{1,\ \dfrac{L(x)}{\varphi(v(d))\log(L(x)/v(d))}\Big\}, & \text{if } v(d) < \log x. \end{cases} \tag{7}$$

Substituting (7) into (4) and using $\max\{1, A\} \le 1 + A$, we find

$$T \ll \log x \cdot \log\log x \Bigg(\sum_{\substack{d \ge 2 \\ P(d) \le \log x}} \frac{\mu^2(d)}{d} + \sum_{\substack{d \ge 2 \\ P(d) \le \log x \\ v(d) < \log x}} \frac{\mu^2(d)}{d} \cdot \frac{L(x)\log\log v(d)}{v(d)\log(L(x)/v(d))} \Bigg), \tag{8}$$

where in the second sum we used Lemma 2.2 in the form $1/\varphi(v(d)) \ll \log\log v(d)/v(d)$. By Mertens' theorem, the first sum is $O(\log\log x)$. Denote the second sum by S; we proceed to bound it.

Fix constants $0 < \varepsilon < 1$ and $0 < \eta < \frac{3}{4}$, and split the range of $v(d)$.

Subcase 1: $v(d) \le (\log x)^{\varepsilon}$. Since $L(x) > 2\log x$ for $x \ge 3$, we have $L(x)/v(d) \ge 2(\log x)^{1-\varepsilon}$, so $\log(L(x)/v(d)) \ge \log 2 + (1-\varepsilon)\log\log x \gg (1-\varepsilon)\log\log x$. It follows that

$$\frac{L(x)}{v(d)\log(L(x)/v(d))} \ll \frac{\log x}{v(d)(1-\varepsilon)\log\log x}.$$

Subcase 2: $(\log x)^{\varepsilon} < v(d) < \log x$. Note that $L(x)/v(d) \le 4\log x/v(d)$. Writing $v(d) = v(d)^{\eta} \cdot v(d)^{1-\eta}$ and using $v(d)^{\eta} > (\log x)^{\varepsilon\eta}$, we obtain

$$\frac{L(x)}{v(d)} \le \frac{4(\log x)^{1-\varepsilon\eta}}{v(d)^{1-\eta}}.$$

Since $L(x)/v(d) > 2\log x/\log x = 2$, we have $\log(L(x)/v(d)) \ge \log 2$, whence

$$\frac{L(x)}{v(d)\log(L(x)/v(d))} \ll \frac{(\log x)^{1-\varepsilon\eta}}{v(d)^{1-\eta}}.$$

Since the Pisano period $u(p) \ge 3$ for every prime p, it follows that $v(d) \ge 3$ for every squarefree $d \ge 2$. Noting that $\log\log t \le t^{1/4}$ for $t \ge 3$, we have in both subcases

$$\frac{\log\log v(d)}{v(d)^{1-\eta}} \le \frac{1}{v(d)^{3/4-\eta}}, \qquad \frac{\log\log v(d)}{v(d)} \le \frac{1}{v(d)^{3/4}}.$$

Collecting the estimates from both subcases, we arrive at

$$S \ll \frac{(\log x)^{1-\varepsilon\eta}}{\log 2}\sum_{d\ge 2}\frac{\mu^2(d)}{d\,v(d)^{3/4-\eta}} + \frac{\log x}{(1-\varepsilon)\log\log x}\sum_{d\ge 2}\frac{\mu^2(d)}{d\,v(d)^{3/4}}.$$

By Lemma 2.3, both series converge (taking $\alpha = \frac{3}{4} - \eta > 0$ and $\alpha = \frac{3}{4}$, respectively). Therefore

$$S \ll (\log x)^{1-\varepsilon\eta} + \frac{\log x}{\log\log x}.$$

Substituting back into (8) and recalling $L(x) < 4\log x$, we conclude

$$T \ll \log x \cdot \log\log x \cdot \Big(\log\log x + (\log x)^{1-\varepsilon\eta} + \frac{\log x}{\log\log x}\Big) \ll \log^2 x,$$

which establishes (3).

Returning to (1), the total contribution of Case III is $O\big(\frac{x}{\log^2 x}\cdot\log^2 x\big) = O(x)$.

Combining all three cases, we obtain $\sum_{n\le x} r(n)^2 \ll x$. □

Proof of Theorem 1.1.

By Lemma 3.2 we have $\sum_{n\le x} r(n) \gg x$, and by Lemma 3.3 we have $\sum_{n\le x} r(n)^2 \ll x$. Applying the Cauchy–Schwarz inequality,

$$\sum_{n\le x} r(n) \le \Big(\sum_{\substack{n\le x\\ r(n)\ge 1}} 1\Big)^{1/2}\Big(\sum_{n\le x} r(n)^2\Big)^{1/2},$$

the two estimates above force

$$\#\{n \le x : r(n) \ge 1\} \gg x,$$

i.e., the set of integers representable as $p + F_{2^k} + F_q$ has a positive lower asymptotic density. □

4. Proof of Theorem 1.2

4.1. The construction

Let

$$\mathcal{P}_0 = \{3, 5, 7, 11, 23, 31, 41, 47, 61, 241, 769, 1103, 1601, 2161, 2521, 3041, 20641, 2207, 1087, 4481\}.$$

The primes in $\mathcal{P}_0$ arise from the factorizations of the Fibonacci numbers

$$F_4, F_5, F_8, F_{10}, F_{15}, F_{16}, F_{20}, F_{24}, F_{30}, F_{32}, F_{40}, F_{48}, F_{60}, F_{64}, F_{80}, F_{96}, F_{120},$$

whose indices are all of the form $2^s3^t5^w$, where s, t, w are nonnegative integers.

Let

$$M = 2\prod_{\ell\in\mathcal{P}_0}\ell = 232918976932328041044109389142371078168219931230.$$

Let

$$n_0 = 155398989363933186474726425105344056488153156387.$$

A direct verification shows that, for every $\ell \in \mathcal{P}_0$,

$$n_0 \equiv c_\ell \pmod{\ell},$$

where

$$\begin{aligned} &c_3 = 2, \quad c_5 = 2, \quad c_7 = 6, \quad c_{11} = 5, \quad c_{23} = 22, \quad c_{31} = 6, \quad c_{41} = 4,\\ &c_{47} = 46, \quad c_{61} = 8, \quad c_{241} = 12, \quad c_{769} = 502, \quad c_{1103} = 493, \quad c_{1601} = 1369,\\ &c_{2161} = 3, \quad c_{2521} = 76, \quad c_{3041} = 301, \quad c_{20641} = 8, \quad c_{2207} = 2206,\\ &c_{1087} = 1, \quad c_{4481} = 4480. \end{aligned}$$

Since M is even and n_0 is odd, every integer in the residue class

$$n \equiv n_0 \pmod{M}$$

is odd.

We shall show in Subsections 4.2–4.10 that this arithmetic progression has no term that can be represented as

$$p + F_{2^k} + F_q,$$

where p and q are primes and k is a nonnegative integer.

The general strategy, used in Subsections 4.3–4.10, is as follows. Fix k and classify q according to its residue modulo, a suitable period T. For each class of q, choose a prime $\ell \in \mathcal{P}_0$ such that the equation, read modulo ℓ, gives

$$p \equiv n_0 - F_{2^k} - F_q \equiv 0 \pmod{\ell},$$

so that $p = \ell$. Then choose another *witness prime* m and verify that

$$F_q \not\equiv n_0 - F_{2^k} - \ell \pmod{m},$$

which excludes the possibility that $p = \ell$, yielding a contradiction.

In addition, we need the following four specific strategies in the proof.

(i) We choose primes so that, for some q and k, both $F_q \bmod \ell$ and $F_{2^k} \bmod \ell$ are fixed.

(ii) We first try small primes and primes already used, in order to rule out the possibility that the equation holds.

(iii) If certain residue classes of q modulo T_1 must be excluded but only information modulo T_2 is available, we work modulo $\mathrm{lcm}(T_1, T_2)$.

(iv) After the first three strategies, only a few specific residue classes of q remain; we then use large, previously unused primes to eliminate the remaining possibilities. Large primes typically appear only to fill these gaps.

4.2. A parity reduction

Lemma 4.1. *(parity reduction) Let* $n \equiv n_0 \pmod{M}$ *and suppose that*

$$n = p + F_{2^k} + F_q$$

with p, q *prime and* $k \in \mathbb{N}_0$. *Then* p *is odd and* $q \neq 3$.

Proof. Since $3 \nmid 2^k$, the number F_{2^k} is odd. Recall that F_m is even if and only if $3 \mid m$; in particular, F_q is odd for every prime $q \neq 3$.

Suppose first that $q = 3$. Then $F_q = 2$, so $p = n - F_{2^k} - 2$ is even, and being prime it must equal 2; hence $n = F_{2^k} + 4$. If $k \geq 2$, then $z(3) = 4 \mid 2^k$, so $F_{2^k} \equiv 0 \pmod 3$ and $n \equiv 1 \pmod 3$, contradicting $n_0 \equiv 2 \pmod 3$. If $k = 0$ or $k = 1$, then $F_{2^k} = 1$ and $n = 5 \not\equiv 2 \equiv n_0 \pmod 5$. Thus $q = 3$ is impossible.

Hence $q \neq 3$, so F_q is odd, and therefore $p = n - F_{2^k} - F_q$ is odd as well. □

By the above observations, it remains to prove that

$$n = p + F_{2^k} + F_q$$

has no solution when $n \equiv n_0 \pmod{M}$, p is an odd prime, $k \geq 0$, and q is a prime with $q \neq 3$. We split this into two cases: $q \notin \{2, 3, 5\}$ (Subsections 4.3–4.8) and $q \in \{2, 5\}$ (Subsections 4.9 and 4.10). These two cases cover all primes $q \neq 3$.

4.3. The case $k \geq 6$

In this subsection, we use $(\mathbb{Z}/128\mathbb{Z})^\times$ to classify q by cases.

$$T = 128, \qquad \varphi(128) = 64.$$

Since

$$z(3) = 4, \quad z(7) = 8, \quad z(47) = 16, \quad z(2207) = 32, \quad z(1087) = 64, \quad z(4481) = 64$$

all divide 2^k, the number F_{2^k} vanishes modulo each of these primes.

Step 1. Cover

$$q \equiv 3, 5 \pmod 8.$$

Take $\ell = 3$. Since $n_0 \equiv 2 \pmod 3$, $F_{2^k} \equiv 0 \pmod 3$, and $F_q \equiv 2 \pmod 3$, we obtain

$$p \equiv 2 - 0 - 2 \equiv 0 \pmod 3,$$

so $p = 3$. To exclude $p = 3$, take the witness prime $m = 7$. From $q \equiv 3, 5 \pmod 8$, we get $q \equiv 3, 5, 11, 13 \pmod{16}$, and hence

$$F_q \in \{2, 5\} \pmod 7.$$

But the target value

$$n_0 - F_{2^k} - 3 \equiv 6 - 0 - 3 \equiv 3 \pmod 7$$

does not lie in $\{2, 5\}$, a contradiction. This step removes 32 classes; the remaining classes satisfy

$$q \equiv \pm 1 \pmod 8.$$

Step 2. Cover

$$q \equiv 7, 9 \pmod{16}.$$

Take $\ell = 7$. Since $n_0 \equiv 6 \pmod 7$, $F_{2^k} \equiv 0 \pmod 7$, and $F_q \equiv 6 \pmod 7$, we obtain

$$p \equiv 6 - 0 - 6 \equiv 0 \pmod 7,$$

so $p = 7$. To exclude $p = 7$, take the witness prime $m = 47$. From $q \equiv 7, 9 \pmod{16}$, we get $q \equiv 7, 9, 23, 25 \pmod{32}$, and hence

$$F_q \in \{13, 34\} \pmod{47}.$$

But the target value

$$n_0 - F_{2^k} - 7 \equiv 46 - 0 - 7 \equiv 39 \pmod{47}$$

does not lie in $\{13, 34\}$, a contradiction. This step removes 16 classes; the remaining classes satisfy

$$q \equiv \pm 1 \pmod{16}.$$

Step 3. Cover

$$q \equiv 15, 17 \pmod{32}.$$

Take $\ell = 47$. Since $n_0 \equiv -1 \pmod{47}$, $F_{2^k} \equiv 0 \pmod{47}$, and $F_q \equiv -1 \pmod{47}$, we obtain

$$p \equiv -1 - 0 - (-1) \equiv 0 \pmod{47},$$

so $p = 47$. To exclude $p = 47$, take the witness prime $m = 3$. From $q \equiv 15, 17 \pmod{32}$, we get $q \equiv \pm 1 \pmod 8$, and hence

$$F_q \equiv 1 \pmod 3.$$

But

$$n_0 - F_{2^k} - 47 \equiv 2 - 0 - 2 \equiv 0 \pmod 3,$$

a contradiction. This step removes 8 classes; the remaining classes satisfy

$$q \equiv \pm 1 \pmod{32}.$$

Step 4. Cover

$$q \equiv 31, 33 \pmod{64}.$$

Take $\ell = 2207$. Since $n_0 \equiv -1 \pmod{2207}$, $F_{2^k} \equiv 0 \pmod{2207}$, and $F_q \equiv -1 \pmod{2207}$, we obtain

$$p \equiv -1 - 0 - (-1) \equiv 0 \pmod{2207},$$

so $p = 2207$. To exclude $p = 2207$, take the witness prime $m = 3$. From $q \equiv 31, 33 \pmod{64}$, we get $q \equiv \pm 1 \pmod{8}$, and hence

$$F_q \equiv 1 \pmod{3}.$$

But

$$n_0 - F_{2^k} - 2207 \equiv 2 - 0 - 2 \equiv 0 \pmod{3},$$

a contradiction. This step removes 4 classes; the remaining classes satisfy

$$q \equiv \pm 1 \pmod{64},$$

that is, the four classes

$$\{1, 63, 65, 127\}$$

modulo 128.

Step 5. Cover

$$q \equiv 1, 127 \pmod{128}.$$

Take $\ell = 1087$. Since $n_0 \equiv 1 \pmod{1087}$, $F_{2^k} \equiv 0 \pmod{1087}$, and $F_q \equiv 1 \pmod{1087}$, we obtain

$$p \equiv 1 - 0 - 1 \equiv 0 \pmod{1087},$$

so $p = 1087$. To exclude $p = 1087$, take the witness prime $m = 7$. From $q \equiv 1, 127 \pmod{128}$, we get $q \equiv 1, 15 \pmod{16}$, and hence

$$F_q \equiv 1 \pmod{7}.$$

But

$$n_0 - 1087 \equiv 6 - 2 \equiv 4 \pmod{7},$$

a contradiction.

Step 6. Cover

$$q \equiv 63, 65 \pmod{128}.$$

Take $\ell = 4481$. Since $n_0 \equiv -1 \pmod{4481}$, $F_{2^k} \equiv 0 \pmod{4481}$, and $F_q \equiv -1 \pmod{4481}$, we obtain

$$p \equiv -1 - 0 - (-1) \equiv 0 \pmod{4481},$$

so $p = 4481$. To exclude $p = 4481$, take the witness prime $m = 3$. From $q \equiv 63, 65 \pmod{128}$, we get $q \equiv 7, 1 \pmod{8}$, and hence

$$F_q \equiv 1 \pmod{3}.$$

But

$$n_0 - 4481 \equiv 2 - 2 \equiv 0 \pmod 3,$$

a contradiction.

The six steps remove

$$32,\ 16,\ 8,\ 4,\ 2,\ 2$$

classes respectively, 64 in total, and hence cover all odd classes modulo 128.

4.4. The cases $k = 0, 1$.

In this subsection, we use $(\mathbb{Z}/160\mathbb{Z})^\times$ to classify q by cases.

$$T = 160, \qquad F_{2^k} = 1, \qquad \varphi(160) = 64.$$

There are 64 reduced classes modulo 160. We define the classes $C_1, \ldots, C_7$ in turn; at the end,

$$C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5 \cup C_6 \cup C_7 = (\mathbb{Z}/160\mathbb{Z})^\times,$$

and the C_i are pairwise disjoint.

Step 1. Let

$$C_1 = \{a \in (\mathbb{Z}/160\mathbb{Z})^\times : a \equiv \pm 1 \pmod 8\},$$

which consists of 32 classes. Take $\ell = 3$. Since

$$n_0 \equiv 2 \pmod 3, \quad F_{2^k} = 1 \equiv 1 \pmod 3, \quad F_q \equiv 1 \pmod 3,$$

we obtain

$$p \equiv 2 - 1 - 1 \equiv 0 \pmod 3,$$

so $p = 3$. To exclude $p = 3$, take the witness prime $m = 7$. We have

$$F_q \in \{1, 6\} \pmod 7,$$

while

$$n_0 - 1 - 3 \equiv 6 - 4 \equiv 2 \pmod 7,$$

a contradiction.

Step 2. Let

$$B_2 = \{a \in (\mathbb{Z}/160\mathbb{Z})^\times : a \equiv 19, 21 \pmod{40}\}.$$

Its elements are congruent to 3 or 5 modulo 8, hence it is disjoint from C_1; it consists of eight classes, which we denote by C_2. Take $\ell = 5$. Since

$$n_0 \equiv 2 \pmod 5, \quad F_{2^k} = 1 \equiv 1 \pmod 5, \quad F_q \equiv 1 \pmod 5,$$

we obtain

$$p \equiv 2 - 1 - 1 \equiv 0 \pmod 5,$$

so $p = 5$. To exclude $p = 5$, take the witness prime $m = 7$. From $q \equiv 19, 21 \pmod{40}$, we get $q \equiv 3, 5, 11, 13 \pmod{16}$, and hence

$$F_q \in \{2, 5\} \pmod{7}.$$

But

$$n_0 - 1 - 5 \equiv 6 - 1 - 5 \equiv 0 \pmod{7},$$

a contradiction.

Step 3. Let

$$B_3 = \{a \in (\mathbb{Z}/160\mathbb{Z})^\times : a \equiv 5, 11 \pmod{16}\}.$$

After removing the classes already in C_2, this covers 12 new classes, which we denote by C_3. Take $\ell = 7$. Since

$$n_0 \equiv 6 \pmod{7}, \quad F_{2^k} = 1 \equiv 1 \pmod{7}, \quad F_q \equiv 5 \pmod{7},$$

we obtain

$$p \equiv 6 - 1 - 5 \equiv 0 \pmod{7},$$

so $p = 7$. To exclude $p = 7$, take the witness prime $m = 3$. From $q \equiv 5, 11 \pmod{16}$, we get $q \equiv 3, 5 \pmod{8}$, and hence

$$F_q \equiv 2 \pmod{3}.$$

But

$$n_0 - 1 - 7 \equiv 2 - 1 - 1 \equiv 0 \pmod{3},$$

a contradiction.

Step 4. Let

$$B_4 = \{a \in (\mathbb{Z}/160\mathbb{Z})^\times : a \equiv 13, 19 \pmod{32}\}.$$

After removing the classes already covered, this covers six new classes, which we denote by C_4. Take $\ell = 47$. Since

$$n_0 \equiv -1 \pmod{47}, \quad F_{2^k} = 1 \equiv 1 \pmod{47}, \quad F_q \equiv -2 \pmod{47},$$

we obtain

$$p \equiv -1 - 1 - (-2) \equiv 0 \pmod{47},$$

so $p = 47$. To exclude $p = 47$, take the witness prime $m = 7$. From $q \equiv 13, 19 \pmod{32}$, we get $q \equiv 3, 13 \pmod{16}$, and hence

$$F_q \equiv 2 \pmod{7}.$$

But

$$n_0 - 1 - 47 \equiv 6 - 1 - 5 \equiv 0 \pmod{7},$$

a contradiction.

Step 5. Let

$$B_5 = \{a \in (\mathbb{Z}/160\mathbb{Z})^\times : a \equiv 3, 157 \pmod{160}\},$$

which consists of two classes, denoted by C_5. Take $\ell = 2161$. Since

$$n_0 \equiv 3 \pmod{2161}, \quad F_{2^k} = 1 \equiv 1 \pmod{2161}, \quad F_q \equiv 2 \pmod{2161},$$

we obtain

$$p \equiv 3 - 1 - 2 \equiv 0 \pmod{2161},$$

so $p = 2161$. To exclude $p = 2161$, take the witness prime $m = 3$. From $q \equiv 3, 157 \pmod{160}$, we get $q \equiv 3, 5 \pmod{8}$, and hence

$$F_q \equiv 2 \pmod{3}.$$

But

$$n_0 - 1 - 2161 \equiv 2 - 1 - 1 \equiv 0 \pmod{3},$$

a contradiction.

Step 6. Let

$$B_6 = \{a \in (\mathbb{Z}/160\mathbb{Z})^\times : a \equiv 67, 93 \pmod{160}\},$$

which consists of two classes, denoted by C_6. Take $\ell = 1601$. Since

$$n_0 \equiv 1369 \pmod{1601}, \quad F_{2^k} = 1 \equiv 1 \pmod{1601}, \quad F_q \equiv 1368 \pmod{1601},$$

we obtain

$$p \equiv 1369 - 1 - 1368 \equiv 0 \pmod{1601},$$

so $p = 1601$. To exclude $p = 1601$, take the witness prime $m = 5$. From $q \equiv 67, 93 \pmod{160}$, we get $q \equiv 7, 13 \pmod{20}$, and hence,

$$F_q \equiv 3 \pmod{5}.$$

But

$$n_0 - 1 - 1601 \equiv 2 - 1 - 1 \equiv 0 \pmod{5},$$

a contradiction.

Step 7. Let

$$B_7 = \{a \in (\mathbb{Z}/160\mathbb{Z})^\times : a \equiv 29, 131 \pmod{160}\},$$

which consists of two classes, denoted by C_7. Take $\ell = 3041$. Since

$$n_0 \equiv 301 \pmod{3041}, \quad F_{2^k} = 1 \equiv 1 \pmod{3041}, \quad F_q \equiv 300 \pmod{3041},$$

we obtain

$$p \equiv 301 - 1 - 300 \equiv 0 \pmod{3041},$$

so $p = 3041$. To exclude $p = 3041$, take the witness prime $m = 5$. From $q \equiv 29, 131 \pmod{160}$, we get $q \equiv 9, 11 \pmod{20}$, and hence

$$F_q \equiv 4 \pmod{5}.$$

But

$$n_0 - 1 - 3041 \equiv 2 - 1 - 1 \equiv 0 \pmod{5},$$

a contradiction.

All 64 reduced classes modulo 160 are now covered.

4.5. The case $k = 2$

In this layer, we use $(\mathbb{Z}/120\mathbb{Z})^\times$ to classify q by cases.

$$T = 120, \qquad F_{2^k} = F_4 = 3, \qquad \varphi(120) = 32.$$

The argument is identical in form to that of Subsections 4.3 and 4.4, so we record all the data in Table 3. Each row of the table is read as follows. The first column describes a family of reduced residue classes for q modulo T; the second column counts the *new* classes, namely those not covered by the previous rows. The third and fourth columns give a prime $\ell \in \mathcal{P}_0$ such that

$$p \equiv n_0 - F_{2^k} - F_q \equiv 0 \pmod{\ell},$$

so $p = \ell$. In the last three columns, m is the witness prime, the set S consists of the values taken by F_q mod m on the classes of the row, and $t \equiv n_0 - F_{2^k} - \ell \pmod{m}$. Since $t \notin S$ in every row, the possibility $p = \ell$ is excluded, a contradiction.

Table 3. The case $k = 2$: Covering data ($T = 120,\ F_{2^k} = 3$).

classes of q	#new	ℓ	$F_q \pmod{\ell}$	m	$F_q \pmod{m}$	$t \pmod{m}$
$q \equiv 3, 5 \pmod 8$	16	3	2	7	$\{2, 5\}$	0
$q \equiv 3, 7 \pmod{10}$	8	11	2	3	$\{1\}$	0
$q \equiv 9, 11 \pmod{20}$	4	5	4	3	$\{1\}$	0
$q \equiv \pm 1 \pmod{40}$	4	41	1	3	$\{1\}$	0

The entries in the second column sum to $16 + 8 + 4 + 4 = 32 = \varphi(120)$, so all 32 reduced classes modulo 120 are covered.

4.6. The case $k = 3$

In this layer, we use $(\mathbb{Z}/240\mathbb{Z})^\times$ to classify q by cases.

$$T = 240, \qquad F_{2^k} = F_8 = 21, \qquad \varphi(240) = 64.$$

The data are recorded in Table 4, which is read exactly as Table 3.

Table 4. The case $k = 3$: Covering data ($T = 240,\ F_{2^k} = 21$).

classes of q	#new	ℓ	$F_q \pmod{\ell}$	m	$F_q \pmod{m}$	$t \pmod{m}$
$q \equiv 3, 5 \pmod 8$	32	3	2	7	$\{2, 5\}$	3
$q \equiv 7, 9 \pmod{16}$	16	7	6	47	$\{13, 34\}$	18
$q \equiv \pm 1 \pmod{20}$	4	5	1	3	$\{1\}$	0
$q \equiv 13, 17 \pmod{30}$	4	31	16	5	$\{2, 3\}$	0
$q \equiv 23, 37 \pmod{60}$	2	61	-13	5	$\{2\}$	0
$q \equiv 31, 89 \pmod{120}$	2	2521	55	5	$\{4\}$	0
$q \equiv 49, 191 \pmod{240}$	2	241	-9	5	$\{4\}$	0
$q \equiv 113, 127 \pmod{240}$	2	20641	-13	5	$\{3\}$	0

For example, the new classes in the third row are $1, 79, 161, 239 \pmod{240}$.

The entries in the second column sum to $32 + 16 + 4 + 4 + 2 + 2 + 2 + 2 = 64 = \varphi(240)$, so all 64 reduced classes modulo 240 are covered.

4.7. The case $k = 4$

In this layer, we use $(\mathbb{Z}/96\mathbb{Z})^\times$ to classify q by cases.

$$T = 96, \qquad F_{2^k} = F_{16} = 987, \qquad \varphi(96) = 32.$$

The data are recorded in Table 5, which is read exactly as Table 3.

Table 5. The case $k = 4$: Covering data ($T = 96,\ F_{2^k} = 987$).

classes of q	#new	ℓ	$F_q \pmod{\ell}$	m	$F_q \pmod{m}$	$t \pmod{m}$
$q \equiv 3, 5 \pmod 8$	16	3	2	7	$\{2, 5\}$	3
$q \equiv 7, 9 \pmod{16}$	8	7	6	47	$\{13, 34\}$	39
$q \equiv 15, 17 \pmod{32}$	4	47	-1	3	$\{1\}$	0
$q \equiv \pm 1 \pmod{48}$	2	23	1	3	$\{1\}$	0
$q \equiv 31, 65 \pmod{96}$	2	1103	609	3	$\{1\}$	0

For example, the new classes in the fourth row are $1, 95 \pmod{96}$.

The entries in the second column sum to $16 + 8 + 4 + 2 + 2 = 32 = \varphi(96)$, so all 32 reduced classes modulo 96 are covered.

4.8. The case $k = 5$

In this layer, we use $(\mathbb{Z}/192\mathbb{Z})^\times$ to classify q by cases.

$$T = 192, \qquad F_{2^k} = F_{32} = 2178309, \qquad \varphi(192) = 64.$$

The data are recorded in Table 6, which is read exactly as Table 3.

Table 6. The case $k = 5$: Covering data ($T = 192,\ F_{2^k} = 2178309$).

classes of q	#new	ℓ	$F_q \pmod{\ell}$	m	$F_q \pmod{m}$	$t \pmod{m}$
$q \equiv 3, 5 \pmod 8$	32	3	2	7	$\{2, 5\}$	3
$q \equiv 7, 9 \pmod{16}$	16	7	6	47	$\{13, 34\}$	39
$q \equiv 15, 17 \pmod{32}$	8	47	-1	3	$\{1\}$	0
$q \equiv 31, 33 \pmod{64}$	4	2207	-1	3	$\{1\}$	0
$q \equiv 31, 65 \pmod{96}$	2	1103	609	3	$\{1\}$	0
$q \equiv 1, 191 \pmod{192}$	2	769	1	7	$\{1\}$	0

For example, the new classes in the fifth row are $65, 127 \pmod{192}$.

The entries in the second column sum to $32 + 16 + 8 + 4 + 2 + 2 = 64 = \varphi(192)$, so all 64 reduced classes modulo 192 are covered.

4.9. A supplementary check for $q = 5$

In the cases with $5 \nmid T$, the prime $q = 5$ is itself a reduced class modulo T and is already included in the covering tables above. In the three cases with $5 \mid T$, a separate verification is needed; the data are recorded in Table 7, which is read exactly as Table 3, with $F_q = F_5 = 5$.

Table 7. The supplementary check for $q = 5$.

k	ℓ	$F_5 \pmod{\ell}$	m	$F_5 \pmod{m}$	$t \pmod{m}$
0, 1	7	5	3	2	0
2	3	2	5	0	1
3	3	2	5	0	3

In every row, $t \not\equiv F_5 \pmod{m}$, so $p = \ell$ is excluded. Hence $q = 5$ is excluded as well.

4.10. The case $q = 2$

When $q = 2$, we have $F_q = F_2 = 1$. The data for the six layers are recorded in Table 8, which is read exactly as Table 3.

Table 8. The case $q = 2$.

k	ℓ	$F_2 \pmod{\ell}$	m	$F_2 \pmod{m}$	$t \pmod{m}$
≥ 6	1087	1	7	1	4
0, 1	3	1	7	1	2
2	41	1	3	1	0
3	5	1	3	1	0
4	23	1	3	1	0
5	769	1	7	1	0

In every row, $t \not\equiv 1 \equiv F_2 \pmod{m}$, so $p = \ell$ is excluded. Hence the equation

$$n = p + F_{2^k} + F_q$$

has no solution with $q = 2$.

4.11. Completion of the proof of Theorem 1.2

Suppose, for contradiction, that for some $t \geq 0$, the integer

$$n = n_0 + Mt$$

admits a representation

$$n = p + F_{2^k} + F_q,$$

where $p, q \in \mathcal{P}$ and $k \in \mathbb{N}_0$.

By Lemma 4.1, the case $p = 2$, $q = 3$ is impossible, and the case $q = 2$ is excluded by Subsection 4.10. Hence we may assume that p, q are odd primes with $q \neq 3$. The case $q = 5$ has been treated in Subsection 4.9. In Subsections 4.3–4.8, one can always obtain a contradiction for $q \notin \{2, 3, 5\}$. Therefore no such representation exists. This completes the proof of Theorem 1.2.

As a consequence, we can obtain the density assertion made in the abstract.

Corollary 4.2. *The set of positive integers that cannot be represented as $p + F_{2^k} + F_q$, with p, q prime and $k \in \mathbb{N}_0$, has lower asymptotic density at least*

$$\frac{1}{2} + \frac{1}{M},$$

where M is as in Subsection 4.1. In particular, this lower asymptotic density exceeds $\frac{1}{2}$.

Proof. From the discussion on page 3, we know that the even integers not admitting such a representation form a set of lower density $\frac{1}{2}$. On the other hand, by Theorem 1.2, the arithmetic progression $n \equiv n_0 \pmod{M}$ consists entirely of non-representable odd integers and has density $1/M$. This completes the proof.

5. Conclusions

Using the first- and second-moment estimates developed by Romanoff, we prove that the set of integers of the form $p + F_{2^k} + F_q$ has a positive lower asymptotic density. In the opposite direction, inspired by Erdős's construction and applications of covering congruences, we prove that there exists an infinite arithmetic progression consisting of positive odd integers that avoids all integers of the form $p + F_{2^k} + F_q$.

Use of Generative-AI tools declaration

The author declares that he has no Artificial Intelligence (AI) tools were used in the creation of this article.

Acknowledgements

We sincerely thank the reviewers for their careful reading of the manuscript, and for their valuable suggestions on strengthening the content and streamlining the proofs, especially for simplifying the proof of Lemma 3.1 and providing directional guidance on Theorem 1.2.

Conflict of interest

The author declares no conflicts of interest in this paper.

References

1. Y. G. Chen, X. G. Sun, On Romanoff's constant, *J. Number Theory*, **106** (2004), 275–284. https://doi.org/10.1016/j.jnt.2003.11.009
2. Y. G. Chen, R. J. Wang, The sum of a prime and a term of exponential sequences, *Acta Math. Sin. (Chinese Ser.)*, **67** (2024), 259–272. https://doi.org/10.12386/A20220173
3. Y. G. Chen, J. Z. Xu, On integers of the form $p + 2^{k_1^{r_1}} + \cdots + 2^{k_t^{r_t}}$, *J. Number Theory*, **258** (2024), 66–93. https://doi.org/10.1016/j.jnt.2023.10.018

4. Y. G. Chen, On integers of the form $p + 2^{k_1^{r_1}} + \cdots + 2^{k_t^{r_t}}$, II, *Front. Math.*, 2026. https://doi.org/10.1007/s11464-026-0057-0
5. R. Crocker, On the sum of a prime and two powers of two, *Pac. J. Math.*, **36** (1971), 103–107.
6. Y. Ding, On a problem of Romanoff type, *Acta Arith.*, **205** (2022), 53–62. https://doi.org/10.4064/aa220207-14-6
7. A. Dubickas, Sums of primes and quadratic linear recurrence sequences, *Acta Math. Sin.*, **29** (2013), 2251–2260. https://doi.org/10.1007/s10114-013-1059-x
8. C. Elsholtz, F. Luca, S. Planitzer, Romanov type problems, *Ramanujan J.*, **47** (2018), 267–289. https://doi.org/10.1007/s11139-017-9972-8
9. C. Elsholtz, J. C. Schlage-Puchta, On Romanov's constant, *Math. Z.*, **288** (2018), 713–724. https://doi.org/10.1007/s00209-017-1908-x
10. P. Erdős, On integers of the form $2^k + p$ and some related problems, *Summa Brasil. Math.*, **2** (1950), 113–123.
11. G. H. Hardy, E. M. Wright, *An introduction to the theory of numbers*, 5 Eds., Oxford: Oxford University Press, 1979.
12. K. S. E. Lee, On the sum of a prime and a Fibonacci number, *Int. J. Number Theory*, **6** (2010), 1669–1676. https://doi.org/10.1142/S1793042110003708
13. Z. Liu, M. Xue, The sum of a prime and a Fibonacci number, *Int. J. Number Theory*, **17** (2021), 1815–1823. https://doi.org/10.1142/S1793042121500640
14. H. L. Montgomery, R. C. Vaughan, The large sieve, *Mathematika*, **20** (1973), 119–134. https://doi.org/10.1112/S0025579300004708
15. M. B. Nathanson, *Additive number theory: The classical bases*, In: Graduate Texts in Mathematics, New York: Springer, **164** (1996). https://doi.org/10.1007/978-1-4757-3845-2
16. H. Pan, On the integers not of the form $p + 2^a + 2^b$, *Acta Arith.*, **148** (2011), 55–61.
17. N. P. Romanoff, Über einige Sätze der additiven Zahlentheorie, *Math. Ann.*, **109** (1934), 668–678. https://doi.org/10.1007/BF01449161
18. J. Šiurys, On integers not of the form $F_n \pm p^a$, *Int. J. Number Theory*, **12** (2016), 505–512. https://doi.org/10.1142/S1793042116500317
19. L. Somer, *Distribution of residues of certain second-order linear recurrences modulo p*, In: Applications of Fibonacci Numbers, Dordrecht: Springer, **3** (1990), 311–324. https://doi.org/10.1007/978-94-009-1910-5_34
20. Z. W. Sun, M. H. Le, On integers not of the form $c(2^a + 2^b) + p^\alpha$, *Acta Arith.*, **99** (2001), 183–190.
21. R. J. Wang, On the sum of a Lucas number and a prime, *Period. Math. Hung.*, **90** (2025), 434–439. https://doi.org/10.1007/s10998-025-00630-8
22. R. J. Wang, On arithmetic progressions of positive integers avoiding $p + F_m$ and $q + L_n$, *Front. Math.*, 2026. https://doi.org/10.1007/s11464-025-0187-9
23. R. J. Wang, Y. G. Chen, On the sum of a Fibonacci number and a prime, *Int. J. Number Theory*, **19** (2023), 873–889. https://doi.org/10.1142/S1793042123500446

24. J. Z. Xu, Y. G. Chen, On integers of the form $p + 2^{a^2} + 2^{b^2}$, *Int. J. Number Theory*, **21** (2025), 153–175. https://doi.org/10.1142/S1793042125500083

25. J. Z. Xu, Y. G. Chen, On the sum of a prime and two Fibonacci numbers, *arXiv Preprint*, 2025. https://doi.org/10.48550/arXiv.2506.03631

26. P. Z. Yuan, On integers not of the form $c(2^a + 2^b) + p^\alpha$, *Acta Math. Sin. (Chinese Ser.)*, **47** (2004), 139–144.



© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0)