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Research article

Lie intersection graph frameworks associated with group actions in topological groups theory

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Abstract: In this article, we define the Lie intersection graph $Lie(\mathbb{GA})$ for continuous Lie group actions on topological semigroups. The vertices of this graph represent the subacts of $P_c(\mathcal{A})$, and two distinct vertices are adjacent if the corresponding subacts intersect in a simple subact. We analyze to what extent algebraic and topological properties of the original actions such as simple and semisimple or Artinian, can be detected by graph invariants, namely connectedness, diameter, girth, bipartiteness and clique number. Also, chordless paths are used to describe some properties of $Lie(\mathbb{GA})$. A chordless topology is defined on the set of vertices, and we prove that it is Alexandroff. These results show that the chordless and monophonic theories capture finer information than classical connectivity, further enriching the relationship between Lie group actions, graph theory, and topology. We also explore how the orbit structure and invariant subact factorization affect the behavior $Lie(\mathbb{GA})$ as a whole. Among our results, we relate invariant filtrations with graph-theoretic decompositions into blocks and articulation points. Our results illustrate how graph invariants may be used to identify intricacy and hierarchy in structural continuous representation of Lie groups.

Keywords: Lie group action; topological semigroup; intersection graph; simple subact; connectivity; clique number; chordless topology
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1. Introduction

Graphs associated with algebraic structures are the most natural way to capture the combinatorial aspects of an algebraic phenomenon. In these constructions vertices represent algebraic substructures

and edges correspond to significant algebraic interactions. One of the most successful examples of this methodology is intersection graphs, where adjacency is given by a certain type of intersection between two algebraic elements. These graphs enable structural attributes like simplicity, decomposition, and chain conditions to be examined by graph invariants such as connectivity, diameter, girth, and clique number. Many studies have indicated that graphs arising from intersections encode fine information about the internal structure of the algebra from which they are obtained. The concept of intersection graphs was generalized beyond group theory. Mukherjee and Sen [1] defined intersection graphs of ideals of rings, transferring intersection graph techniques to ring theory.

More refinements were also given for the commutative ring case, Abdioglu et al. [2] investigated graph invariants of co-intersection graphs of ideals in commutative rings, highlighting connections between algebraic structures and graph properties. Recent studies have also emphasized the interaction between graph theory and topology in different contexts. For example, Damag et al. [3] investigated directed topological structures arising from graph-based models in complex systems, while Damag et al. [4] studied monophonic sets and rough directed topological spaces with applications to directed networks. These works further highlight the growing connections between graph theory techniques and topological frameworks. Mohd and Ahmed [5] introduced the idea of simple intersection graphs in which adjacency is determined by certain minimum intersection requirements, thereby creating graphs with predictable combinatorial properties. The development of intersection graph constructions in more general contexts is based on semigroup theory and the theory of actions. Seminal classical texts by Howie [6], Lallement [7], Grillet [8] and Kilp et al. [9] cover the background necessary for this exposition in regards to semigroups, monoids, and actions, respectively. In this setting, chain conditions have been considered in a number of works, in particular for commutative monoids by Davvaz and Nazemian [10] and for Rees congruences of $\mathbb{D}$-acts by Khosravi and Roueentan [11]. Intersection graph methods were in particular specialized to $\mathbb{D}$-acts by Rasouli and Tehranian [12], and were generalized to semigroup actions by Derrfien et al. [13]. In recent years, Liang et al. [14] generalized the simple intersection framework to $\mathbb{D}$-acts and bridged concepts from ring theory and action theory. Historically the attention was mostly on groups and being early work in this direction. Csakány and Pollák [15] defined the subgroup intersection graph of a finite group by considering the proper non-trivial subgroups as vertices, and joining two vertices if the corresponding subgroups have a non-trivial intersection. This notion was then tailored to the finite abelian groups framework by Zelinka [16] considering the effect of commutativity on the structure of the resulting graph. Later research generalized these results to wider classes of finite groups. Shen [17] studied some graph theoretic invariants like diameter and girth while Rajkumar and Devi [18] studied intersection graphs of cyclic subgroups and characterized several graph structures arising from finite groups. Connectivity became very quickly the focus of attention for the intersection graph of subgroups. Kayacan [19] investigated connectivity properties of intersection graphs of finite groups. More recently, more precise results have been obtained by Beheshtipour and Jafarian Amiri [20] that they examined clique numbers of intersection graphs of finite groups and established connections between clique structure and group solvability. More recently, Alzubaidi [21] investigated P-intersection graphs of acts over semigroups, focusing on connectivity, clique structure, and monophonic convexity. Ahmed and Moh'd [22] introduced the simple-intersection graph of modules and investigated its connectivity, diameter, girth, clique number, and domination number. Spanier [23] provided fundamental concepts of algebraic topology, which are used as a reference for the preliminaries of this work. The work also investigated

controlling sets and monophonic topology, further extending the connection between semigroup acts, graph theory, and topology. Nevertheless, the literature is sparse in studying intersection graphs from topological groups and their actions, where algebraic structure meets topology and continuity. Our paper seeks to make a contribution in this direction through the analysis of intersection graphs derived from topological group actions, notably path spaces.

The present work is closely related to the intersection and simple-intersection graphs of acts studied in [12–14]. We also note the recent work on Ψ-intersection graphs of acts over topological semigroups, where several graph invariants and monophonic structures were investigated. The novelty of the present work is therefore not the intersection-based adjacency rule itself. Our focus is on continuous Lie group actions and on the structure of their invariant subacts. In this setting, we study the corresponding Lie intersection graph and develop its graph-theoretic and topological properties. The results in Sections 4–7 are presented in this specific Lie-action framework, and their relation to the earlier act-based results is clarified where appropriate.

We develop a graph-theoretic approach to continuous Lie group actions on topological semigroups via intersection-based constructions. We define the Lie intersection graph, denoted by $Lie(\mathbb{G}\mathcal{A})$, with vertices as non-trivial invariant subacts and adjacency derived by intersections that yield simple subacts. In Section 2, we bring back the preliminaries requisite from group actions, topological semigroups, Lie groups, and graph theory. In Section 3, we define the Lie intersection graph $Lie(\mathbb{G}\mathcal{A})$, give some examples and show some elementary structural properties. Section 4 is concerned with cycles, girth, bipartiteness and related local properties of $Lie(\mathbb{G}\mathcal{A})$. Section 5 deals with several global graph invariants: the connectivity, the diameter and the clique structure and relates them to algebraic conditions, in particular to the Artinian property of the act underlying the $\mathbb{G}$-act. In Section 6, we turn to chordless paths, define the chordless topology on the vertices of $Lie(\mathbb{G}\mathcal{A})$ and prove that it is Alexandroff.

2. Preliminaries

Let $\mathcal{A}$ be a non-empty set and $\mathbb{G}$ be any group. The set $\mathcal{A}$ is said to be a $\mathbb{G}$-action ($\mathbb{G}$-act) if there exists a mapping $\mathbb{G} \times \mathcal{A} \to \mathcal{A} : (s, a) \mapsto sa$ such that $(st)a = s(ta)$ and $1a = a$ for all $s, t \in \mathbb{G}$ and $a \in \mathcal{A}$. A non-empty subset $\mathcal{B}$ of $\mathcal{A}$ is said to be a subact of $\mathcal{A}$ if $s\mathcal{B} \subseteq \mathcal{B}$ for every $s \in \mathbb{G}$ and is called non-trivial if it is a proper, non-empty subact of $\mathcal{A}$. Observe that $\mathbb{G}$ itself is a $\mathbb{G}$-act under its own multiplication. In this case, the subacts of $\mathbb{G}$ are precisely the left subacts of $\mathbb{G}$, namely the non-empty subsets $\mathcal{Z}$ of $\mathbb{G}$ satisfying $\mathbb{G}\mathcal{Z} \subseteq \mathcal{Z}$. An element $x \in \mathbb{G}$ is called a left zero element if $xs = x$ for all $s \in \mathbb{G}$. A non-empty $\mathbb{G}$-act is said to be simple if it admits no non-trivial subact, and $\mathbb{D}(\mathcal{A})$ denotes the number of simple subacts of $\mathcal{A}$. A non-trivial subact M of $\mathcal{A}$ is called a minimal subact if it contains no proper subact of $\mathcal{A}$. Dually, a maximal subact of $\mathcal{A}$ is a non-trivial subact M such that there exists no subact of $\mathcal{A}$ properly contained between M and $\mathcal{A}$. Given a family $\{\mathcal{A}_k \mid k \in \Delta\}$ of $\mathbb{G}$-acts, the coproduct $\coprod_{k\in\Delta} \mathcal{A}_k$ of this family is defined as the disjoint union $\bigcup_{k\in\Delta}(\mathcal{A}_k \times \{k\})$ equipped with the action $s(a, k) = (sa, k)$ for all $s \in \mathbb{G}$, $a \in \mathcal{A}_k$, and $k \in \Delta$. A $\mathbb{G}$-act is called completely reducible if it can be expressed as a coproduct of simple subacts. A $\mathbb{G}$-act $\mathcal{A}$ is said to be semisimple if it is the coproduct of its simple subacts. Each simple subact occurring in such a decomposition is called a component of $\mathcal{A}$. The socle of $\mathcal{A}$, denoted by $\mathrm{Soc}(\mathcal{A})$, is defined as the coproduct of all simple subacts of $\mathcal{A}$. Consequently, $\mathcal{A} = \mathrm{Soc}(\mathcal{A})$ if and only if $\mathcal{A}$ is a semisimple $\mathbb{G}$-act. A $\mathbb{G}$-act $\mathcal{A}$ is called Artinian

if every descending chain of subacts of $\mathcal{A}$ is eventually stationary. Equivalently, $\mathcal{A}$ is Artinian if and only if every non-empty collection of subacts of $\mathcal{A}$ contains a minimal element. An element $a \in \mathcal{A}$ is called a fixed element if $sa = a$ for all $s \in \mathbb{G}$.

Here we recall some backgrounds concern the graph theory. Let $\mathcal{G}$ be a graph with vertex set $\mathcal{V}(\mathcal{G})$ and edge set $\mathcal{E}(\mathcal{G})$. A path is a sequence of vertices in which each consecutive pair of vertices is joined by an edge, and no vertex is repeated. A graph is connected if there is a path between every pair of vertices in the graph. An Euler path is a path in a graph that uses every edge exactly once. A circuit is a path that starts and ends at the same vertex, with no repeated edges. For distinct vertices v_1 and v_2 in $\mathcal{V}(\mathcal{G})$, a $v_1 - v_2$ path is a path whose starting vertex is v_1 and whose ending vertex is v_2. The length of the shortest $v_1 - v_2$ path is denoted by $\partial(v_1, v_2)$. If no such path exists in $\mathcal{G}$, then $\partial(v_1, v_2) = \infty$. The order of $\mathcal{G}$, denoted by $O(\mathcal{G})$, is the number of vertices of $\mathcal{G}$. The diameter of $\mathcal{G}$, written as diam($\mathcal{G}$), is the supremum of the set $\{\partial(v_1, v_2) : v_1, v_2 \in \mathcal{V}(\mathcal{G})\}$. The number of vertices adjacent to a vertex v is called the degree of v and is denoted by ∂_v. A graph is called simple if it has no loops and no multiple edges. The null graph is a graph with no edges. The simple graph C_n, with $n > 2$, consists of a single closed path and is called a cycle graph. A complete bipartite graph $\mathcal{K}_{n,m}$ is a bipartite graph whose vertex set can be partitioned into two disjoint subsets such that every vertex in one subset is adjacent to every vertex in the other. A simple graph $\mathcal{K}_n$ is called a complete graph. A wheel graph $\mathcal{W}_n$ is obtained by connecting a single vertex to all vertices of a cycle. The girth of a graph $\mathcal{G}$ is the length of its shortest cycle and is denoted by $\mathcal{G}_g$. A clique of $\mathcal{G}$ is a complete subgraph, and the clique number of $\mathcal{G}$ is denoted by $\xi(\mathcal{G})$. A path component of a graph is a maximal connected subgraph. A controlling set $\mathbb{C}$ of a graph $\mathcal{G}$ is a non-empty subset of $\mathcal{V}(\mathcal{G})$ such that every vertex not in $\mathbb{C}$ is adjacent to at least one vertex in $\mathbb{C}$. Let $Con(\mathcal{G})$ be the family of all controlling sets of $\mathcal{G}$. The control number of $\mathcal{G}$ is denoted by $\Delta(\mathcal{G})$ and defined by $\Delta(\mathcal{G}) = \inf\{O(\mathbb{C}) : \mathbb{C} \in Con(\mathcal{G})\}$. An edge connecting two non-adjacent points in a path is called a chord edge. A monophonic path is a chordless path.

Theorem 2.1. [22] A graph contains an Euler circuit if and only if every vertex in the graph has an even degree. Likewise, a graph contains an Euler path if and only if exactly two vertices have odd degrees, while all remaining vertices have even degrees.

3. The Lie intersection graph $Lie(\mathbb{G}\mathcal{A})$

For further background on topological spaces and continuous functions, the reader may consult [23]. A topological semigroup is a semigroup $\mathbb{G}$ endowed with a topology for which the multiplication map $\mathbb{G} \times \mathbb{G} \to \mathbb{G}$ is continuous. If, in addition, the semigroup $\mathbb{G}$ is a group and the inversion map $\mathbb{G} \to \mathbb{G}$ is continuous, then $\mathbb{G}$ is called a topological group. Let $I = [0, 1]$ be the closed interval endowed with the usual topology on $\mathbb{R}$. The path group $\mathbb{P}(\mathbb{G})$ associated with a topological group $\mathbb{G}$ is defined as the set of all continuous functions $\alpha : I \to \mathbb{G}$. The group operation on $\mathbb{P}(\mathbb{G})$ is defined pointwise by $(\alpha\beta)(t) = \alpha(t)\beta(t)$ for all $\alpha, \beta \in \mathbb{P}(\mathbb{G})$ and $t \in I$. When equipped with the compact–open topology, $\mathbb{P}(\mathbb{G})$ becomes a topological group; see [23]. For $g \in \mathbb{G}$, define the map $\widehat{\widehat{g}} : I \to \mathbb{G}$ by $\widehat{\widehat{g}}(t) = g$ for all $t \in I$. Such maps are called constant paths. The set $\mathbb{P}_c(\mathbb{G}) := \{\widehat{\widehat{g}} \in \mathbb{P}(\mathbb{G}) : g \in \mathbb{G}\}$ forms a subgroup of $\mathbb{P}(\mathbb{G})$ and is itself a topological group. In the setting of topological groups, action mappings are assumed to be continuous. Let $\mathbb{G}$ and $\mathcal{A}$ be topological groups, and suppose that $\mathcal{A}$ is a $\mathbb{G}$-act with a continuous action $\mathbb{G} \times \mathcal{A} \to \mathcal{A} : (g, a) \mapsto ga$. Then the path group $\mathbb{P}(\mathcal{A})$ becomes a $\mathbb{G}$-act under the continuous action $\mathbb{G} \times \mathbb{P}(\mathcal{A}) \to \mathbb{P}(\mathcal{A}) : (g, \alpha) \mapsto g\alpha$, where $(g\alpha)(t) = g\alpha(t)$ for all $t \in I$. A smooth manifold

of dimension n is a Hausdorff, second countable topological space M together with the atlas family $\{(U_i, \varphi_i)\}_{i\in I}$, $\varphi_i : U_i \to \mathbb{R}^n$, such that the transition maps $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ are smooth (all partial derivatives of every order exist and are continuous). A Lie group is a topological group $\mathbb{G}$ equipped with a smooth manifold structure.

Definition 3.1. Let $\mathbb{G}$ be a Lie group, $\mathcal{A}$ be a topological semigroup, and $\mathbb{P}_c(\mathcal{A})$ be a $\mathbb{G}$-act. The Lie intersection graph $Lie(\mathbb{G}\mathcal{A})$ associated with the $\mathbb{G}$-act $\mathbb{P}_c(\mathcal{A})$ is defined to be a simple undirected graph whose vertex set consists of all subacts of $\mathbb{P}_c(\mathcal{A})$. Two distinct vertices in $Lie(\mathbb{G}\mathcal{A})$ are adjacent if and only if their intersection is a simple subact of $\mathbb{P}_c(\mathcal{A})$.

Example 3.2. Let $\mathbb{G} = \{1\}$ be a trivial Lie group and $\mathbb{Z}_3 = \{0, 1, 2\}$ with the discrete topology. Note that any mapping from discrete space into any space is continuous, and so $\mathbb{P}_c(\mathbb{Z}_3)$ is a $\mathbb{G}$-act, where $1 \cdot \widehat{\widehat{x}} = \widehat{\widehat{x}}$ for all $\widehat{\widehat{x}} \in \mathbb{P}_c(\mathbb{Z}_3)$. Since $\mathbb{G}$ is a trivial Lie group, every non-empty subset of $\mathbb{P}_c(\mathbb{Z}_3)$ is a subact. Hence

$$\mathcal{V}[Lie(\mathbb{G}\mathbb{Z}_3)] = \{\{\widehat{\widehat{0}}\}, \{\widehat{\widehat{1}}\}, \{\widehat{\widehat{2}}\}, \{\widehat{\widehat{0}}, \widehat{\widehat{1}}\}, \{\widehat{\widehat{0}}, \widehat{\widehat{2}}\}, \{\widehat{\widehat{1}}, \widehat{\widehat{2}}\}, \mathbb{P}_c(\mathbb{Z}_3)\}.$$

The Lie intersection graph $Lie(\mathbb{Z}_3\mathbb{G})$ is illustrated in Figure 1.

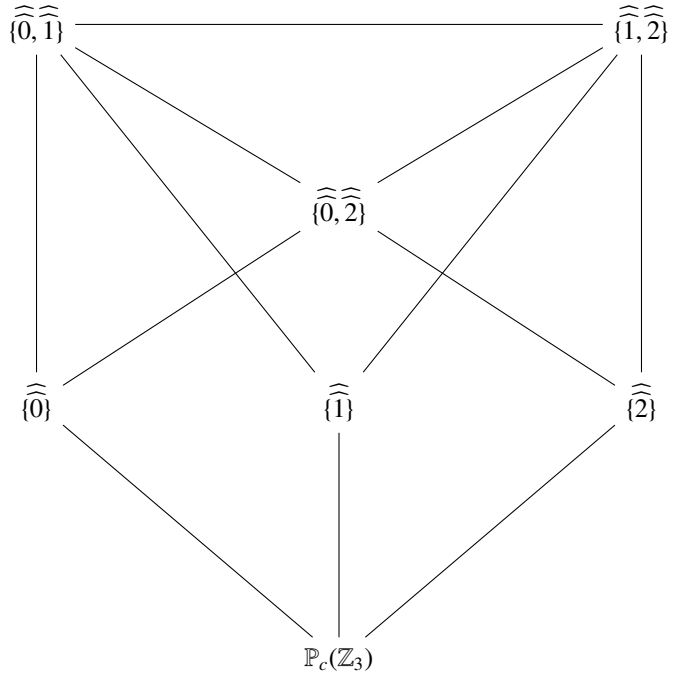


Figure 1. Lie intersection graph $Lie(\mathbb{G}\mathcal{A})$ I.

Example 3.3. Let $\mathbb{G}$ be any Lie group and $\mathcal{A} = \{1, 2, 3, 4\}$ with any topology under the operation $xy = x$. It is clear that $\mathbb{P}_c(\mathcal{A})$ is a $\mathbb{G}-$act under the action $\mathbb{G} \times \mathbb{P}_c(\mathcal{A}) \to \mathbb{P}_c(\mathcal{A}) : (g, \widehat{\widehat{k}}) \mapsto \widehat{\widehat{k}}$ for all $k \in \mathcal{A}$ and the subacts of $\mathbb{P}_c(\mathcal{A})$ are all non-empty subsets of $\mathbb{P}_c(\mathcal{A})$. That is,

$$\mathcal{V}[Lie(\mathbb{G}\mathcal{A})] = P(\mathbb{P}_c(\mathcal{A})) - \{\emptyset\}.$$

The Lie intersection graph $Lie(\mathbb{G}\mathbb{G})$ is illustrated in Figure 2.

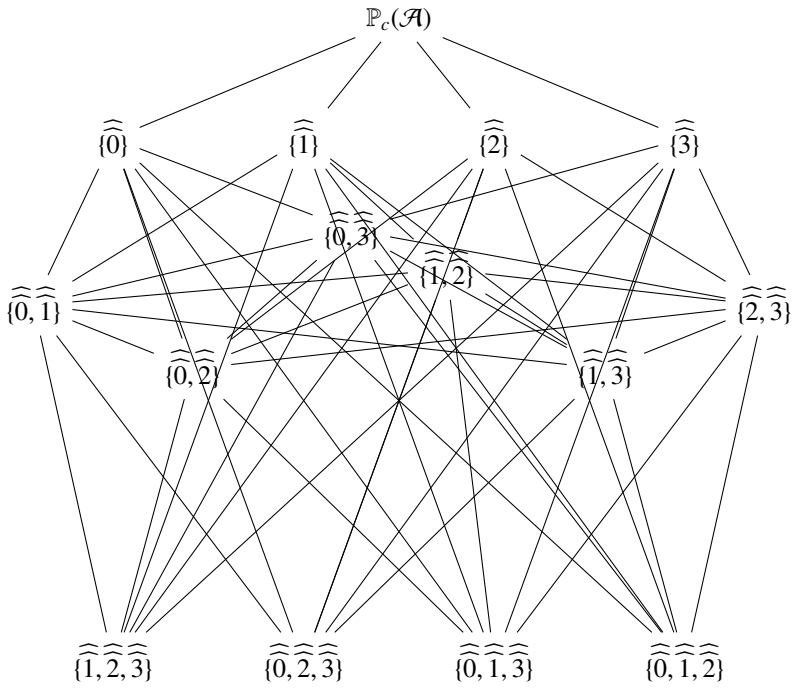


Figure 2. The Lie intersection graph $Lie(\mathbb{G}\mathcal{A})$ II.

Henceforth, we assume that $\mathcal{A}$ is a topological semigroup, unless stated otherwise.

Theorem 3.4. Let $\mathbb{G}$ be a Lie group and $\mathbb{P}_c(\mathcal{A})$ be a $\mathbb{G}$-act. Then $\mathbb{P}_c(\mathcal{A})$ is either a simple $\mathbb{G}$-act or contains no simple subacts if and only if $Lie(\mathbb{G}\mathcal{A})$ is a null graph.

Proof. Suppose first that $Lie(\mathbb{G}\mathcal{A})$ is a null graph, that is, it contains no edges. Assume, toward a contradiction, that $\mathbb{P}_c(\mathcal{A})$ is not simple and that it contains a simple subact $\mathcal{B}$. Since $\mathcal{B} \subseteq \mathbb{P}_c(\mathcal{A})$, we have

$$\mathbb{P}_c(\mathcal{A}) \cap \mathcal{B} = \mathcal{B}.$$

Because $\mathcal{B}$ is a simple subact, it follows from the definition of $Lie(\mathbb{G}\mathcal{A})$ that $\mathbb{P}_c(\mathcal{A})$ and $\mathcal{B}$ are adjacent vertices. This contradicts the assumption that $Lie(\mathbb{G}\mathcal{A})$ has no edges. Therefore, $\mathbb{P}_c(\mathcal{A})$ must either be simple or contain no simple subacts.

Conversely, suppose that $\mathbb{P}_c(\mathcal{A})$ is simple or contains no simple subacts. If $\mathbb{P}_c(\mathcal{A})$ is simple, then it has no proper non-empty subacts, and consequently the graph $Lie(\mathbb{G}\mathcal{A})$ has at most one vertex and therefore no edges. If $\mathbb{P}_c(\mathcal{A})$ contains no simple subacts, then the intersection of any two distinct subacts of $\mathbb{P}_c(\mathcal{A})$ cannot be a simple subact. Hence no two vertices of $Lie(\mathbb{G}\mathcal{A})$ are adjacent. In either case, $Lie(\mathbb{G}\mathcal{A})$ is a null graph. □

For instance, if $\mathcal{A}$ consists of a single element, then $Lie(\mathbb{G}\mathcal{A})$ is a null graph for every Lie group $\mathbb{G}$.

Theorem 3.5. Let $\mathbb{G}$ be a Lie group and $\mathbb{P}_c(\mathcal{A})$ be a semisimple $\mathbb{G}$-act with completely reducible decomposition

$$\mathbb{P}_c(\mathcal{A}) = \bigsqcup_{i=1}^{n} \mathcal{A}_i.$$

Then

$$\mathbb{D}(\mathbb{P}_c(\mathcal{A})) = n,$$

$$\mathbb{D}(\mathcal{A}_i) = 2^{n-1} - 1 \qquad \text{for } i = 1, 2, \ldots, n,$$

and

$$\mathbb{D}(\mathcal{B}) = m\, 2^{n-m}$$

whenever $\mathcal{B}$ is a subact with m components, where $2 \leq m \leq n$.

Proof. Since $\mathbb{P}_c(\mathcal{A})$ is semisimple, its simple subacts are exactly the components

$$\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n.$$

Hence the number of simple subacts of $\mathbb{P}_c(\mathcal{A})$ is n, and so

$$\mathbb{D}(\mathbb{P}_c(\mathcal{A})) = n.$$

Now fix $i \in \{1, 2, \ldots, n\}$. Since $\mathbb{P}_c(\mathcal{A})$ is semisimple, every subact of $\mathbb{P}_c(\mathcal{A})$ is a coproduct of some of the simple components $\mathcal{A}_1, \ldots, \mathcal{A}_n$. A subact is adjacent to $\mathcal{A}_i$ in $Lie(\mathbb{G}\mathcal{A})$ if and only if it contains $\mathcal{A}_i$ together with at least one additional component. Thus every neighbor of $\mathcal{A}_i$ has the form

$$\mathcal{A}_i \bigsqcup \left(\bigsqcup_{j \in J} \mathcal{A}_j \right),$$

where J is a subset of $\{1, 2, \ldots, n\} \setminus \{i\}$. Therefore the number of such neighbors is exactly the number of nonempty subsets of the remaining $n - 1$ components, namely $2^{n-1} - 1$. Hence

$$\mathbb{D}(\mathcal{A}_i) = 2^{n-1} - 1.$$

Finally, let $\mathcal{B}$ be a subact with exactly m components, where $2 \leq m \leq n$. Write

$$\mathcal{B} = \bigsqcup_{r=1}^{m} \mathcal{A}_{i_r}.$$

A vertex C is adjacent to $\mathcal{B}$ if and only if $\mathcal{B} \cap C$ is a simple subact. Since $\mathbb{P}_c(\mathcal{A})$ is semisimple, this happens precisely when C contains exactly one of the m components of $\mathcal{B}$ and may contain any subset of the remaining $n - m$ components not belonging to $\mathcal{B}$. Choose one fixed component of $\mathcal{B}$, say $\mathcal{A}_{i_r}$. The number of subacts C whose intersection with $\mathcal{B}$ is exactly $\mathcal{A}_{i_r}$ equals the number of subsets of the $n - m$ components outside $\mathcal{B}$, namely

$$\sum_{k=0}^{n-m} \binom{n-m}{k} = 2^{n-m}.$$

Since this can be done independently for each of the m components of $\mathcal{B}$, we obtain

$$\mathbb{D}(\mathcal{B}) = m\, 2^{n-m}.$$

This completes the proof. □

Theorem 3.6. Let $\mathbb{G}$ be a Lie group and $\mathbb{P}_c(\mathcal{A})$ be a $\mathbb{G}$-act. Then $Lie(\mathbb{G}\mathcal{A})$ is isomorphic to the wheel graph $\mathcal{W}_{n+1} = \mathcal{K}_1 + C_n$ for some $n \geq 3$ if and only if there exist distinct subacts

$$Q, C_1, C_2, \ldots, C_n \subseteq \mathbb{P}_c(\mathcal{A})$$

such that

(1) $V(Lie(\mathbb{G}\mathcal{A})) = \{Q, C_1, \ldots, C_n\}$;
(2) $Q \cap C_i$ is a simple subact for each i;
(3) $C_i \cap C_{i+1}$ is a simple subact for each i;
(4) if $j \not\equiv i \pm 1 \pmod{n}$, then $C_i \cap C_j$ is not a simple subact.

Proof. Suppose that $Lie(\mathbb{G}\mathcal{A})$ is isomorphic to the wheel graph $\mathcal{W}_{n+1}$. Recall that $\mathcal{W}_{n+1}$ contains a unique central vertex of degree n. Let Q denote the vertex of $Lie(\mathbb{G}\mathcal{A})$ corresponding to this central vertex.

The remaining vertices lie on the rim cycle C_n. Denote their corresponding subacts by

$$C_1, C_2, \ldots, C_n$$

in cyclic order. Since Q is adjacent to every C_i, the definition of $Lie(\mathbb{G}\mathcal{A})$ implies that $Q \cap C_i$ is a simple subact for each i. Moreover, each vertex C_i of the rim is adjacent exactly to C_{i-1} and C_{i+1}. Hence $C_i \cap C_{i+1}$ is a simple subact, while $C_i \cap C_j$ is not simple whenever $j \not\equiv i \pm 1 \pmod{n}$. Therefore the stated conditions follow.

Conversely, assume that subacts $Q, C_1, \ldots, C_n$ satisfy the above conditions. Then Q is adjacent to each C_i, so it plays the role of the central vertex. Furthermore the vertices $C_1, \ldots, C_n$ form a cycle C_n since each C_i is adjacent only to its two neighbors C_{i-1} and C_{i+1}. Thus the resulting graph consists of a cycle together with one vertex adjacent to all vertices of the cycle. Therefore

$$Lie(\mathbb{G}\mathcal{A}) \cong \mathcal{W}_{n+1}.$$

□

4. Intersection patterns in $Lie(\mathbb{G}\mathcal{A})$

In this section, we study key properties of the Lie intersection graph $Lie(\mathbb{G}\mathcal{A})$, with emphasis on cycles, girth, and bipartiteness. We establish precise conditions for the existence of short cycles and derive strong restrictions on the possible girth values, highlighting the role of simple and non-simple subacts. These results prepare the ground for the connectivity and global invariants examined in the next section.

Proposition 4.1. Let $\mathbb{G}$ be any Lie group and $\mathbb{P}_c(\mathcal{A})$ be a $\mathbb{G}$-act. Then the graph $Lie(\mathbb{G}\mathcal{A})$ contains a cycle of length 3 if and only if $Lie(\mathbb{G}\mathcal{A})$ has at least two adjacent non-simple subacts.

Proof. Assume first that $Lie(\mathbb{G}\mathcal{A})$ contains a cycle of length 3, say $\mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3 - \mathcal{A}_1$. By the definition of $Lie(\mathbb{G}\mathcal{A})$, the intersections $\mathcal{A}_1 \cap \mathcal{A}_2$, $\mathcal{A}_2 \cap \mathcal{A}_3$, and $\mathcal{A}_3 \cap \mathcal{A}_1$ are simple subacts of $\mathbb{P}_c(\mathcal{A})$. If two vertices of the cycle were simple subacts, then their intersection would be equal to one of them, which is impossible since distinct simple subacts intersect trivially. Hence, at most one vertex in the cycle can

be simple. Consequently, at least two vertices must be non-simple subacts, and since they are joined by an edge in the cycle, they are adjacent non-simple subacts.

Conversely, suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are two distinct non-simple subacts of $\mathbb{P}_c(\mathcal{A})$ that are adjacent in $Lie(\mathbb{G}\mathcal{A})$. Then $\mathcal{A}_1 \cap \mathcal{A}_2$ is a simple subact of $\mathbb{P}_c(\mathcal{A})$. Since $\mathcal{A}_1 \cap \mathcal{A}_2 \subseteq \mathcal{A}_1$ and $\mathcal{A}_1 \cap \mathcal{A}_2 \subseteq \mathcal{A}_2$, it follows that $\mathcal{A}_1 \cap \mathcal{A}_2$ is adjacent to both $\mathcal{A}_1$ and $\mathcal{A}_2$. Therefore, $\mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_1 \cap \mathcal{A}_2 - \mathcal{A}_1$ forms a cycle of length 3 in $Lie(\mathbb{G}\mathcal{A})$. □

Proposition 4.2. Let $\mathbb{G}$ be any Lie group and $\mathbb{P}_c(\mathcal{A})$ be a semisimple $\mathbb{G}$-act. If $\mathbb{P}_c(\mathcal{A})$ has more than two components, then $Lie(\mathbb{G}\mathcal{A})_g = 3$. Otherwise, $Lie(\mathbb{G}\mathcal{A})_g = \infty$.

Proof. If $\mathbb{P}_c(\mathcal{A})$ has at most two components, then $Lie(\mathbb{G}\mathcal{A})$ is either a null graph or acyclic. In both cases, the graph contains no cycles, and hence $Lie(\mathbb{G}\mathcal{A})_g = \infty$. Now suppose that $\mathbb{P}_c(\mathcal{A})$ has at least three components, say $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$. Then the subacts $\mathcal{A}_1 \sqcup \mathcal{A}_2$ and $\mathcal{A}_2 \sqcup \mathcal{A}_3$ are distinct non-simple subacts whose intersections are the simple subact $\mathcal{A}_2$. By Proposition 4.1, the graph $Lie(\mathbb{G}\mathcal{A})$ contains a cycle of length 3, and therefore $Lie(\mathbb{G}\mathcal{A})_g = 3$. □

Theorem 4.3. Let $\mathbb{P}_c(\mathcal{A})$ be a $\mathbb{G}$-act. If the graph $Lie(\mathbb{G}\mathcal{A})$ contains cycles, then either $Lie(\mathbb{G}\mathcal{A})_g$ is even or $Lie(\mathbb{G}\mathcal{A})_g = 3$.

Proof. Assume that $Lie(\mathbb{G}\mathcal{A})$ contains a cycle and $\mathcal{A}_1 - \mathcal{A}_2 - \cdots - \mathcal{A}_n - \mathcal{A}_1$ is a cycle of minimum length $n > 3$. By minimality, no two adjacent vertices in this cycle can both be non-simple subacts; otherwise, Proposition 4.1 would imply the existence of a 3-cycle. Moreover, two distinct simple subacts cannot be adjacent, since their intersection is trivial. Hence, adjacent vertices must alternate between simple and non-simple subacts. Without loss of generality, assume that $\mathcal{A}_1$ is simple. Then $\mathcal{A}_2$ is non-simple, $\mathcal{A}_3$ is simple, and so on. Thus, $\mathcal{A}_k$ is simple if k is odd and non-simple if k is even. If n were odd, then $\mathcal{A}_n$ would be simple and adjacent to $\mathcal{A}_1$, contradicting the fact that two distinct simple subacts cannot be adjacent. Therefore, n is even, and since n is the length of a shortest cycle, $Lie(\mathbb{G}\mathcal{A})_g$ is even. □

Theorem 4.4. Let $\mathbb{G}$ be any Lie group and $\mathbb{P}_c(\mathcal{A})$ be a $\mathbb{G}$-act. Then $Lie(\mathbb{G}\mathcal{A})_g \in \{3, 4, \infty\}$.

Proof. If $\mathbb{P}_c(\mathcal{A})$ has no simple subacts, then $Lie(\mathbb{G}\mathcal{A})$ contains no cycles, and hence $Lie(\mathbb{G}\mathcal{A})_g = \infty$. Suppose that $\mathbb{P}_c(\mathcal{A})$ contains exactly one simple subact $\mathcal{A}_1$. If $\mathcal{A}_1$ is maximal, then the only edge in $Lie(\mathbb{G}\mathcal{A})$ is $\mathbb{P}_c(\mathcal{A}) - \mathcal{A}_1$, so the graph is acyclic and $Lie(\mathbb{G}\mathcal{A})_g = \infty$. If $\mathcal{A}_1$ is not maximal and is contained in a unique proper non-simple subact, then again no cycle exists. If $\mathcal{A}_1$ is contained in more than one proper non-simple subact, then either two such subacts intersect in $\mathcal{A}_1$, yielding a 3-cycle by Proposition 4.1, or the graph is a star with center $\mathcal{A}_1$ and has infinite girth. Finally, suppose that $\mathbb{P}_c(\mathcal{A})$ contains at least two distinct simple subacts $\mathcal{A}_1$ and $\mathcal{A}_2$. If $\mathbb{P}_c(\mathcal{A}) = \mathcal{A}_1 \sqcup \mathcal{A}_2$, then $Lie(\mathbb{G}\mathcal{A})_g = \infty$ by Proposition 4.2. Otherwise, the sequence $\mathbb{P}_c(\mathcal{A}) - \mathcal{A}_1 - \mathcal{A}_1 \sqcup \mathcal{A}_2 - \mathcal{A}_2 - \mathbb{P}_c(\mathcal{A})$ forms a cycle of length 4. This cycle is minimal unless a 3-cycle exists, which again occurs precisely under the condition of Proposition 4.1. □

Let $\mathbb{G}$ and $\mathcal{A}$ be any two Lie groups. Let $\mathbb{P}_c(\mathcal{A})$ be a $\mathbb{G}$-act. For any subact $\mathcal{B}$ of $\mathbb{P}_c(\mathcal{A})$ and for $k \in \{1, 2, 3, \cdots\}$, the copy $\mathbb{G}$-act $\mathcal{B}^{(k)}$ of $\mathcal{B}$ is $\mathcal{B}^{(k)} = \mathcal{B} \times \{k\}$ with the continuous action $g \cdot (\widehat{x}, k) = (g \cdot \widehat{x}, k)$ for all $g \in \mathbb{G}$ and $\widehat{x} \in \mathcal{B}$.

Example 4.5. Let $\mathbb{G} = \mathbb{R}$ be the additive Lie group with its usual topology and $\mathbb{Z}_3$ be the cyclic group of order 3 equipped with the discrete topology. Assume that $\mathbb{G}$ acts trivially on each copy of $\mathbb{Z}_3$ (that is, $g \cdot x = x$ for all $g \in \mathbb{G}$ and $x \in \mathbb{Z}_3$). Consider

$$\mathcal{A} = \mathbb{Z}_3^{(1)} \sqcup \mathbb{Z}_3^{(2)} \sqcup \mathbb{Z}_3^{(3)} \sqcup \mathbb{Z}_3^{(4)},$$

where each copy $\mathbb{Z}_3^{(k)}$ of $\mathbb{Z}_3$ is a simple $\mathbb{G}$-act. Then

$$\mathbb{P}_c(\mathcal{A}) = \mathbb{P}_c(\mathbb{Z}_3)^{(1)} \sqcup \mathbb{P}_c(\mathbb{Z}_3)^{(2)} \sqcup \mathbb{P}_c(\mathbb{Z}_3)^{(3)} \sqcup \mathbb{P}_c(\mathbb{Z}_3)^{(4)}$$

is a semisimple $\mathbb{G}$-act with four components. Consequently, $Lie(\mathbb{G}\mathcal{A})$ contains the following two distinct cycles of length 4:

$$\mathcal{D}: \quad \mathbb{P}_c(\mathbb{Z}_3)^{(1)} - (\mathbb{P}_c(\mathbb{Z}_3)^{(1)} \sqcup \mathbb{P}_c(\mathbb{Z}_3)^{(2)}) - \mathbb{P}_c(\mathbb{Z}_3)^{(2)} - (\mathbb{P}_c(\mathbb{Z}_3)^{(1)} \sqcup \mathbb{P}_c(\mathbb{Z}_3)^{(2)} \sqcup \mathbb{P}_c(\mathbb{Z}_3)^{(3)}) - \mathbb{P}_c(\mathbb{Z}_3)^{(1)}$$

and

$$\mathcal{D}': \quad (\mathbb{P}_c(\mathbb{Z}_3)^{(1)} \sqcup \mathbb{P}_c(\mathbb{Z}_3)^{(4)}) - (\mathbb{P}_c(\mathbb{Z}_3)^{(1)} \sqcup \mathbb{P}_c(\mathbb{Z}_3)^{(2)}) - (\mathbb{P}_c(\mathbb{Z}_3)^{(2)} \sqcup \mathbb{P}_c(\mathbb{Z}_3)^{(3)}) - (\mathbb{P}_c(\mathbb{Z}_3)^{(3)} \sqcup \mathbb{P}_c(\mathbb{Z}_3)^{(4)}) - (\mathbb{P}_c(\mathbb{Z}_3)^{(1)} \sqcup \mathbb{P}_c(\mathbb{Z}_3)^{(4)}).$$

Note that the cycle $\mathcal{D}$ follows the pattern used in the proof of Theorem 4.3, so its vertices alternate between a simple subact and a non-simple subact. In contrast, the cycle $\mathcal{D}'$ consists entirely of non-simple subacts. Moreover, by Proposition 4.2, the cycle $\mathcal{D}$ is not of minimum length, and by Theorem 4.3 the cycle $\mathcal{D}'$ cannot be of minimum length.

Example 4.6. Let $\mathbb{G} = \{1, a, b\}$ be a nilpotent Lie group endowed with the discrete topology, where 1 is a zero element and the multiplication is defined by $a^2 = b^2 = ab = ba = 1$. Then $\mathbb{G}$ has exactly two proper subacts $J_1 = \{1, a\}$ and $J_2 = \{1, b\}$. Moreover, J_1 and J_2 are simple $\mathbb{G}$-acts. Consider $\mathcal{A} = \mathbb{P}_c(\mathbb{G})^{(1)} \sqcup \mathbb{P}_c(J_1)^{(2)} \sqcup \mathbb{P}_c(J_2)^{(3)}$. Then we obtain the $\mathbb{G}$-act $\mathbb{P}_c(\mathcal{A}) = \mathbb{P}_c(\mathbb{G})^{(1)} \sqcup \mathbb{P}_c(J_1)^{(2)} \sqcup \mathbb{P}_c(J_2)^{(3)}$, where the superscripts indicate distinct copies. In this case, $\mathbb{P}_c(\mathcal{A})$ has exactly three proper non-trivial subacts: $\mathbb{P}_c(\mathbb{G})^{(1)} \sqcup \mathbb{P}_c(J_1)^{(2)}$, $\mathbb{P}_c(\mathbb{G})^{(1)} \sqcup \mathbb{P}_c(J_2)^{(3)}$, and $\mathbb{P}_c(J_1)^{(2)} \sqcup \mathbb{P}_c(J_2)^{(3)}$. Hence, in the Lie intersection graph $Lie(\mathbb{G}\mathcal{A})$ we obtain the cycle

$$\mathbb{P}_c(\mathbb{G})^{(1)} \sqcup \mathbb{P}_c(J_1)^{(2)} - \mathbb{P}_c(J_1)^{(2)} - \mathbb{P}_c(J_1)^{(2)} \sqcup \mathbb{P}_c(J_2)^{(3)} - \mathbb{P}_c(J_2)^{(3)} - \mathbb{P}_c(\mathbb{G})^{(1)} \sqcup \mathbb{P}_c(J_1)^{(2)},$$

which is a cycle of length 4. Therefore, $Lie(\mathbb{G}\mathcal{A})_g = 4$.

Theorem 4.7. Let $\mathbb{P}_c(\mathcal{A})$ be a $\mathbb{G}$-act. If $3 < Lie(\mathbb{G}\mathcal{A})_g < \infty$, then the graph $Lie(\mathbb{G}\mathcal{A})$ is bipartite.

Proof. Let $3 < Lie(\mathbb{G}\mathcal{A})_g < \infty$. Then $Lie(\mathbb{G}\mathcal{A})$ contains no cycles of length 3. Consequently, no two adjacent vertices of $Lie(\mathbb{G}\mathcal{A})$ can both be non-simple subacts. Indeed, if there existed two adjacent non-simple subacts, then by Proposition 4.1, $Lie(\mathbb{G}\mathcal{A})$ would contain a cycle of length 3, which contradicts the assumption $Lie(\mathbb{G}\mathcal{A})_g > 3$. Moreover, two distinct simple subacts cannot be adjacent in $Lie(\mathbb{G}\mathcal{A})$, since the intersection of two simple subacts is either empty or equal to one of them, and therefore cannot be simple. Hence, no two simple subacts are adjacent in $Lie(\mathbb{G}\mathcal{A})$. It follows that every edge of $Lie(\mathbb{G}\mathcal{A})$ joins a simple subact to a non-simple subact. Thus, the vertex set of $Lie(\mathbb{G}\mathcal{A})$ can be partitioned into two disjoint sets, where $\mathcal{V}$ is the set of all simple subacts of $\mathbb{P}_c(\mathcal{A})$ and $\mathcal{V}'$ is the set of all non-simple subacts of $\mathbb{P}_c(\mathcal{A})$, such that no two vertices within the same set are adjacent. Therefore, $Lie(\mathbb{G}\mathcal{A})$ is bipartite with bipartition $\mathcal{V}$ and $\mathcal{V}'$. □

Theorem 4.8. Let $\mathbb{G}$ be any Lie group and $\mathbb{P}_c(\mathcal{A})$ be a $\mathbb{G}$-act. Then $Lie(\mathbb{G}\mathcal{A})$ is bipartite if and only if for every simple subact $\mathcal{B}$ of $\mathbb{P}_c(\mathcal{A})$, any two vertices of $Lie(\mathbb{G}\mathcal{A})$ that are adjacent to $\mathcal{B}$ are not adjacent to each other.

Proof. Suppose first that $Lie(\mathbb{G}\mathcal{A})$ is bipartite. Let $\mathcal{V}_n$ and $\mathcal{V}_m$ be a bipartition of its vertex set and $\mathcal{B}$ be a simple subact of $\mathbb{P}_c(\mathcal{A})$. Without loss of generality, assume that $\mathcal{B} \in \mathcal{V}_n$. Then every vertex adjacent to $\mathcal{B}$ belongs to $\mathcal{V}_m$. Since no two vertices of $\mathcal{V}_m$ are adjacent, it follows that no two neighbors of $\mathcal{B}$ are adjacent. Thus, for every simple subact $\mathcal{B}$ of $\mathbb{P}_c(\mathcal{A})$, any two vertices of $Lie(\mathbb{G}\mathcal{A})$ that are adjacent to $\mathcal{B}$ are not adjacent to each other.

Conversely, assume that for every simple subact $\mathcal{B}$ of $\mathbb{P}_c(\mathcal{A})$, no two vertices adjacent to $\mathcal{B}$ are adjacent to each other. Let $\mathcal{V}_1$ denote the set of all simple subacts of $\mathbb{P}_c(\mathcal{A})$ and $\mathcal{V}_2$ denote the set of all non-simple subacts of $\mathbb{P}_c(\mathcal{A})$. By the definition of the simple intersection graph, two simple subacts cannot be adjacent, and hence there are no edges among vertices of $\mathcal{V}_1$. Now suppose, toward a contradiction, that there exist two vertices $\mathcal{C}, \mathcal{D} \in \mathcal{V}_2$ that are adjacent. Then $\mathcal{C} \cap \mathcal{D}$ is a simple subact of $\mathbb{P}_c(\mathcal{A})$, say $\mathcal{B}$. Thus, both $\mathcal{C}$ and $\mathcal{D}$ are adjacent to $\mathcal{B}$, contradicting the assumption that no two vertices adjacent to a simple subact are adjacent to each other. Therefore, there are no edges among vertices of $\mathcal{V}_2$. Consequently, every edge of $Lie(\mathbb{G}\mathcal{A})$ joins a vertex in $\mathcal{V}_1$ to a vertex in $\mathcal{V}_2$, and hence $Lie(\mathbb{G}\mathcal{A})$ is bipartite. □

In general, a bipartite graph may admit more than one bipartition. However, for the graph $Lie(\mathbb{G}\mathcal{A})$, the existence of the $(\mathcal{V}, \mathcal{V}')$-bipartition where $\mathcal{V}$ denotes the set of all simple subacts of $\mathbb{P}_c(\mathcal{A})$ and $\mathcal{V}'$ denotes the set of all non-simple subacts of $\mathbb{P}_c(\mathcal{A})$—is both necessary and sufficient for $Lie(\mathbb{G}\mathcal{A})$ to be bipartite. As will be shown below, this specific bipartition plays a central role in determining the overall structure of $Lie(\mathbb{G}\mathcal{A})$. It is also worth noting that any subact which does not contain a simple subact corresponds to an isolated vertex in $Lie(\mathbb{G}\mathcal{A})$. Such isolated vertices do not affect the bipartiteness of the graph and may be assigned arbitrarily to either part of a bipartition. Consequently, when studying bipartitions of $Lie(\mathbb{G}\mathcal{A})$, it suffices to focus on those vertices corresponding to subacts that contain simple subacts, since these vertices alone determine the bipartite structure of the graph.

Theorem 4.9. Let $\mathbb{G}$ be any Lie group and let $\mathbb{P}_c(\mathcal{A})$ be a $\mathbb{G}$-act. Then $Lie(\mathbb{G}\mathcal{A})$ is bipartite if and only if the $(\mathcal{V}, \mathcal{V}')$-bipartition of $Lie(\mathbb{G}\mathcal{A})$ exists, where $\mathcal{V}$ is the set of all simple subacts of $\mathbb{P}_c(\mathcal{A})$ and $\mathcal{V}'$ is the set of all non-simple subacts of $\mathbb{P}_c(\mathcal{A})$.

Proof. Assume first that $Lie(\mathbb{G}\mathcal{A})$ is bipartite. Let

$$\mathcal{V} = \{\text{all simple subacts of } \mathbb{P}_c(\mathcal{A})\}$$

and

$$\mathcal{V}' = \{\text{all non-simple subacts of } \mathbb{P}_c(\mathcal{A})\}.$$

We show that $(\mathcal{V}, \mathcal{V}')$ is a bipartition of $Lie(\mathbb{G}\mathcal{A})$.

First, no two vertices of $\mathcal{V}$ are adjacent. Indeed, if $\mathcal{B}, \mathcal{C} \in \mathcal{V}$ were adjacent, then $\mathcal{B} \cap \mathcal{C}$ would be a simple subact. Since both $\mathcal{B}$ and $\mathcal{C}$ are simple, this would force $\mathcal{B} = \mathcal{C}$, contradicting the fact that the vertices are distinct.

Next, no two vertices of $\mathcal{V}'$ are adjacent. Suppose, to the contrary, that $\mathcal{M}, \mathcal{N} \in \mathcal{V}'$ are adjacent. Then $\mathcal{M} \cap \mathcal{N}$ is a simple subact, say $\mathcal{B}$. Hence $\mathcal{B}$ is adjacent to both $\mathcal{M}$ and $\mathcal{N}$, and therefore

$$\mathcal{M} - \mathcal{B} - \mathcal{N} - \mathcal{M}$$

is a cycle of length 3, which is impossible in a bipartite graph. Thus no two vertices of $\mathcal{V}'$ are adjacent. Therefore every edge of $Lie(\mathbb{G}\mathcal{A})$ joins a vertex of $\mathcal{V}$ to a vertex of $\mathcal{V}'$, and so the $(\mathcal{V}, \mathcal{V}')$-bipartition exists.

Conversely, assume that the $(\mathcal{V}, \mathcal{V}')$-bipartition exists. Then, by definition, no two vertices in $\mathcal{V}$ are adjacent and no two vertices in $\mathcal{V}'$ are adjacent. Hence every edge joins a vertex in $\mathcal{V}$ to a vertex in $\mathcal{V}'$, and therefore $Lie(\mathbb{G}\mathcal{A})$ is bipartite. □

A subact $\mathcal{B}$ of $\mathbb{P}_c(\mathcal{A})$ is called essential in $\mathbb{P}_c(\mathcal{A})$ if every non-empty subact of $\mathbb{P}_c(\mathcal{A})$ has a non-empty intersection with $\mathcal{B}$.

Theorem 4.10. Let $\mathbb{P}_c(\mathcal{A})$ be a $\mathbb{G}$-act and let $Lie(\mathbb{G}\mathcal{A})$ be bipartite. Then $Lie(\mathbb{G}\mathcal{A})$ is complete bipartite if and only if $\mathrm{Soc}(\mathbb{P}_c(\mathcal{A}))$ is an essential subact of $\mathbb{P}_c(\mathcal{A})$.

Proof. Assume first that $Lie(\mathbb{G}\mathcal{A})$ is complete bipartite. Let $\mathcal{C}$ be any non-empty subact of $\mathbb{P}_c(\mathcal{A})$. If $\mathcal{C}$ is simple, then $\mathcal{C} \subseteq \mathrm{Soc}(\mathbb{P}_c(\mathcal{A}))$, and hence $\mathrm{Soc}(\mathbb{P}_c(\mathcal{A})) \cap \mathcal{C} = \mathcal{C} \neq \emptyset$. Now suppose that $\mathcal{C}$ is non-simple. Let $\mathcal{B}$ be any simple subact of $\mathbb{P}_c(\mathcal{A})$. Since $Lie(\mathbb{G}\mathcal{A})$ is complete bipartite, every simple vertex is adjacent to every non-simple vertex. In particular, $\mathcal{B}$ is adjacent to $\mathcal{C}$. Therefore $\mathcal{B} \cap \mathcal{C}$ is a simple subact. Since $\mathcal{B}$ is simple, $\mathcal{B} \cap \mathcal{C}$ is a non-empty subact of $\mathcal{B}$, and thus $\mathcal{B} \cap \mathcal{C} = \mathcal{B}$. Hence $\mathcal{B} \subseteq \mathcal{C}$. Since $\mathcal{B}$ was arbitrary, every simple subact of $\mathbb{P}_c(\mathcal{A})$ is contained in $\mathcal{C}$. Therefore $\mathrm{Soc}(\mathbb{P}_c(\mathcal{A})) \subseteq \mathcal{C}$, and so $\mathrm{Soc}(\mathbb{P}_c(\mathcal{A})) \cap \mathcal{C} = \mathrm{Soc}(\mathbb{P}_c(\mathcal{A})) \neq \emptyset$. Thus $\mathrm{Soc}(\mathbb{P}_c(\mathcal{A}))$ is essential in $\mathbb{P}_c(\mathcal{A})$. Conversely, suppose that $\mathrm{Soc}(\mathbb{P}_c(\mathcal{A}))$ is essential in $\mathbb{P}_c(\mathcal{A})$. Let $\mathcal{V}$ be the set of all simple subacts of $\mathbb{P}_c(\mathcal{A})$, and let $\mathcal{V}'$ be the set of all non-simple subacts of $\mathbb{P}_c(\mathcal{A})$. Since $Lie(\mathbb{G}\mathcal{A})$ is bipartite, there are no edges among vertices of $\mathcal{V}$ and no edges among vertices of $\mathcal{V}'$. Thus it suffices to show that every vertex of $\mathcal{V}$ is adjacent to every vertex of $\mathcal{V}'$. Let $\mathcal{B} \in \mathcal{V}$ and $\mathcal{C} \in \mathcal{V}'$. Since $\mathrm{Soc}(\mathbb{P}_c(\mathcal{A}))$ is essential, we have $\mathrm{Soc}(\mathbb{P}_c(\mathcal{A})) \cap \mathcal{C} \neq \emptyset$. Hence $\mathcal{C}$ contains a simple subact; denote it by $\mathcal{D}$. If $\mathcal{D} \neq \mathcal{B}$, then $\mathcal{B} \cap \mathcal{D} = \emptyset$. Consider the subact $\mathcal{B} \sqcup \mathcal{D}$. It is non-simple, since it contains two distinct simple subacts. Moreover $(\mathcal{B} \sqcup \mathcal{D}) \cap \mathcal{C} = \mathcal{D}$, which is simple. Hence $\mathcal{B} \sqcup \mathcal{D}$ is adjacent to $\mathcal{C}$. But both $\mathcal{B} \sqcup \mathcal{D}$ and $\mathcal{C}$ are non-simple vertices, contradicting the bipartiteness of $Lie(\mathbb{G}\mathcal{A})$. Therefore $\mathcal{D} = \mathcal{B}$, and so $\mathcal{B} \subseteq \mathcal{C}$. Hence $\mathcal{B} \cap \mathcal{C} = \mathcal{B}$, which is simple. Thus $\mathcal{B}$ is adjacent to $\mathcal{C}$. Since $\mathcal{B} \in \mathcal{V}$ and $\mathcal{C} \in \mathcal{V}'$ were arbitrary, every simple vertex is adjacent to every non-simple vertex. Therefore $Lie(\mathbb{G}\mathcal{A})$ is complete bipartite. □

Theorem 4.11. Let $\mathbb{P}_c(\mathcal{A})$ be a $\mathbb{G}$-act and let $Lie(\mathbb{G}\mathcal{A})$ be bipartite. Then $Lie(\mathbb{G}\mathcal{A})$ is complete bipartite if and only if every non-empty subact of $\mathbb{P}_c(\mathcal{A})$ contains a simple subact and $\mathrm{diam}(Lie(\mathbb{G}\mathcal{A})) \leq 2$.

Proof. Assume first that $Lie(\mathbb{G}\mathcal{A})$ is complete bipartite. Let $\mathcal{C}$ be any non-empty subact of $\mathbb{P}_c(\mathcal{A})$. If $\mathcal{C}$ is simple, then it already contains a simple subact, namely itself. If $\mathcal{C}$ is non-simple, then every non-simple vertex is adjacent to every simple vertex. Hence $\mathcal{C}$ is adjacent to some simple subact $\mathcal{B}$, and therefore $\mathcal{B} \cap \mathcal{C}$ is a simple subact. Thus $\mathcal{C}$ contains a simple subact. Moreover, in a complete bipartite graph any two vertices in different parts are adjacent, so their distance is 1, while any two distinct vertices in the same part have a common neighbor, so their distance is 2. Hence $\mathrm{diam}(Lie(\mathbb{G}\mathcal{A})) \leq 2$. Conversely, suppose that every non-empty subact of $\mathbb{P}_c(\mathcal{A})$ contains a simple subact and that $\mathrm{diam}(Lie(\mathbb{G}\mathcal{A})) \leq 2$. Let $\mathcal{B}$ be a simple subact and let $\mathcal{M}$ be a non-simple subact of $\mathbb{P}_c(\mathcal{A})$. By assumption, $\mathcal{M}$ contains a simple subact, and therefore $\mathcal{M}$ is not an isolated vertex. Also, $\mathcal{B}$ is not isolated, since it is adjacent to $\mathbb{P}_c(\mathcal{A})$. Hence both $\mathcal{B}$ and $\mathcal{M}$ belong to the non-trivial component of $Lie(\mathbb{G}\mathcal{A})$, and $d(\mathcal{B}, \mathcal{M}) \leq 2$. By Theorem 4.9, since $Lie(\mathbb{G}\mathcal{A})$ is bipartite, the $(\mathcal{V}, \mathcal{V}')$-bipartition exists, where $\mathcal{V}$ is the set of all simple subacts and $\mathcal{V}'$ is the set of all non-simple subacts. Thus $\mathcal{B} \in \mathcal{V}$

and $\mathcal{M} \in \mathcal{V}'$ lie in opposite parts. In a bipartite graph, any path of length 2 starts and ends in the same part. Therefore $d(\mathcal{B}, \mathcal{M}) \neq 2$. Since $\mathcal{B} \neq \mathcal{M}$, we must have $d(\mathcal{B}, \mathcal{M}) = 1$. Hence $\mathcal{B}$ is adjacent to $\mathcal{M}$. Since $\mathcal{B}$ and $\mathcal{M}$ were arbitrary, every simple vertex is adjacent to every non-simple vertex. Together with the bipartiteness of the graph, this shows that $Lie(\mathbb{G}\mathcal{A})$ is complete bipartite. □

5. Global structure of $Lie(\mathbb{G}\mathcal{A})$

In this section, we study global properties of the Lie intersection graph $Lie(\mathbb{G}\mathcal{A})$, focusing on connectivity, diameter, and clique structure. We relate these graph-theoretic invariants to algebraic conditions on the underlying $\mathbb{G}$-act, particularly the Artinian property, and show how simple subacts govern the formation and size of cliques. These results clarify how global graph structure reflects the internal organization of $\mathbb{P}_c(\mathcal{A})$.

Theorem 5.1. Let $\mathbb{G}$ be any Lie group and let $\mathbb{P}_c(\mathcal{A})$ be a $\mathbb{G}$-act. If every non-empty subact of $\mathbb{P}_c(\mathcal{A})$ contains a simple subact, then $Lie(\mathbb{G}\mathcal{A})$ is connected.

Proof. Let $\mathcal{B}$ and C be two arbitrary non-empty subacts of $\mathbb{P}_c(\mathcal{A})$. By assumption, there exist simple subacts $\mathcal{B}_1 \subseteq \mathcal{B}$ and $C_1 \subseteq C$. If $\mathcal{B}_1 = C_1$, then $\mathcal{B} - \mathcal{B}_1 - C$ is a path in $Lie(\mathbb{G}\mathcal{A})$. Assume now that $\mathcal{B}_1 \neq C_1$. Then the subact $\mathcal{B}_1 \sqcup C_1$ is non-simple, and $\mathcal{B}_1 \cap (\mathcal{B}_1 \sqcup C_1) = \mathcal{B}_1, \quad C_1 \cap (\mathcal{B}_1 \sqcup C_1) = C_1$, both of which are simple subacts. Hence $\mathcal{B} - \mathcal{B}_1 - (\mathcal{B}_1 \sqcup C_1) - C_1 - C$ is a path in $Lie(\mathbb{G}\mathcal{A})$. Therefore any two vertices of $Lie(\mathbb{G}\mathcal{A})$ are joined by a path, and so $Lie(\mathbb{G}\mathcal{A})$ is connected. □

Corollary 5.2. Let $\mathbb{G}$ be any Lie group and let $\mathbb{P}_c(\mathcal{A})$ be an Artinian $\mathbb{G}$-act. Then $Lie(\mathbb{G}\mathcal{A})$ is connected.

Proof. Since $\mathbb{P}_c(\mathcal{A})$ is Artinian, every non-empty subact contains a minimal subact, and every minimal subact is simple. Hence the result follows from Theorem 5.1. □

Proposition 5.3. Let $\mathbb{G}$ be any Lie group and let $\mathbb{P}_c(\mathcal{A})$ be an Artinian $\mathbb{G}$-act. Then, for any two distinct non-empty subacts $\mathcal{B}$ and C of $\mathbb{P}_c(\mathcal{A})$, one has $1 \leq \partial(\mathcal{B}, C) \leq 4$.

Proof. Let $\mathcal{B}$ and C be two distinct non-empty subacts of $\mathbb{P}_c(\mathcal{A})$. Since $\mathbb{P}_c(\mathcal{A})$ is Artinian, every non-empty subact contains a simple subact. Thus there exist simple subacts $\mathcal{B}_1 \subseteq \mathcal{B}$ and $C_1 \subseteq C$. Hence $\partial(\mathcal{B}, C)$ is finite, so $\partial(\mathcal{B}, C) \geq 1$. We consider the following cases. If both $\mathcal{B}$ and C are simple, then $\mathcal{B} - \mathbb{P}_c(\mathcal{A}) - C$ is a path of length 2. Hence $\partial(\mathcal{B}, C) \leq 2$. Assume that exactly one of $\mathcal{B}$ and C is simple; without loss of generality, let $\mathcal{B}$ be simple. If $\mathcal{B} \subseteq C$, then $\mathcal{B}$ is adjacent to C, and so $\partial(\mathcal{B}, C) = 1$. If $\mathcal{B} \cap C = \emptyset$, let K be a simple subact contained in C. Then $\mathcal{B} - (\mathcal{B} \sqcup K) - C$ is a path of length 2, and therefore $\partial(\mathcal{B}, C) \leq 2$. Finally, assume that neither $\mathcal{B}$ nor C is simple. If $\mathcal{B} \cap C$ is a simple subact, then $\mathcal{B}$ and C are adjacent, so $\partial(\mathcal{B}, C) = 1$. If $\mathcal{B} \cap C$ is non-empty but non-simple, then it contains a simple subact K, and hence $\mathcal{B} - K - C$ is a path of length 2. Therefore $\partial(\mathcal{B}, C) \leq 2$. If $\mathcal{B} \cap C = \emptyset$, choose simple subacts $\mathcal{A}_1 \subseteq \mathcal{B}$ and $\mathcal{A}_2 \subseteq C$. Then $\mathcal{B} - \mathcal{A}_1 - (\mathcal{A}_1 \sqcup \mathcal{A}_2) - \mathcal{A}_2 - C$ is a path of length 4. Hence $\partial(\mathcal{B}, C) \leq 4$. Thus in all cases $1 \leq \partial(\mathcal{B}, C) \leq 4$. □

Theorem 5.4. Let $\mathbb{G}$ be any Lie group and let $\mathbb{P}_c(\mathcal{A})$ be a $\mathbb{G}$-act. Every vertex of $Lie(\mathbb{G}\mathcal{A})$ having positive degree lies in the same path component as the vertex $\mathbb{P}_c(\mathcal{A})$, and all remaining components of $Lie(\mathbb{G}\mathcal{A})$ are isolated vertices.

Proof. Let $\mathcal{B}$ be a non-empty subact of $\mathbb{P}_c(\mathcal{A})$ such that $\deg(\mathcal{B}) > 0$ in $Lie(\mathbb{G}\mathcal{A})$. We show that $\mathcal{B}$ is connected to $\mathbb{P}_c(\mathcal{A})$ by a path. If $\mathcal{B}$ is simple, then $\mathcal{B} \cap \mathbb{P}_c(\mathcal{A}) = \mathcal{B}$ is simple, and therefore $\mathcal{B}$ is adjacent to $\mathbb{P}_c(\mathcal{A})$. Now assume that $\mathcal{B}$ is not simple. Since $\deg(\mathcal{B}) > 0$, there exists a non-empty subact C such that $\mathcal{B}$ is adjacent to C. By the definition of $Lie(\mathbb{G}\mathcal{A})$, this adjacency implies that $\mathcal{B} \cap C$ is a simple subact of $\mathbb{P}_c(\mathcal{A})$. If C is simple, then $\mathcal{B} - C - \mathbb{P}_c(\mathcal{A})$ is a path. If C is non-simple, then $\mathcal{B} \cap C$ is a simple subact, and hence $\mathcal{B} - (\mathcal{B} \cap C) - \mathbb{P}_c(\mathcal{A})$ is a path. Thus every vertex of positive degree belongs to the path component containing $\mathbb{P}_c(\mathcal{A})$. Consequently, every other component consists entirely of vertices of degree zero, that is, isolated vertices. □

The path component of $Lie(\mathbb{G}\mathcal{A})$ that contains the vertex $\mathbb{P}_c(\mathcal{A})$ is called the $\mathbb{P}_c(\mathcal{A})$-component of $Lie(\mathbb{G}\mathcal{A})$. By Theorem 5.4, in many situations it is sufficient to restrict attention to the $\mathbb{P}_c(\mathcal{A})$-component of $Lie(\mathbb{G}\mathcal{A})$. Indeed, when $Lie(\mathbb{G}\mathcal{A})$ is not connected, any vertex outside the $\mathbb{P}_c(\mathcal{A})$-component is necessarily isolated and therefore contributes no additional information to path-related properties of the graph. Consequently, the study of disconnected simple-intersection graphs may be reduced to their $\mathbb{P}_c(\mathcal{A})$-components. Furthermore, in view of Theorem 5.4, if $\mathbb{P}_c(\mathcal{A})$ is a semisimple but non-simple $\mathbb{G}$-act, then $Lie(\mathbb{G}\mathcal{A})$ contains no isolated vertices and is therefore connected. Nevertheless, even in this case, the graph $Lie(\mathbb{G}\mathcal{A})$ need not be complete.

Theorem 5.5. Let $\mathbb{P}_c(\mathcal{A})$ be a $\mathbb{G}$-act. Then $Lie(\mathbb{G}\mathcal{A})$ is complete if and only if $\mathbb{P}_c(\mathcal{A})$ has exactly one non-empty proper subact. In this case, $Lie(\mathbb{G}\mathcal{A})$ is isomorphic to the complete graph $\mathcal{K}_2$.

Proof. Assume that $Lie(\mathbb{G}\mathcal{A})$ is complete. Suppose, toward a contradiction, that $\mathbb{P}_c(\mathcal{A})$ has two distinct non-empty proper subacts $\mathcal{B}$ and C. If both $\mathcal{B}$ and C are simple, then they cannot be adjacent, since two distinct simple subacts do not intersect in a simple subact. If at least one of them is non-simple, say $\mathcal{B}$, then $\mathbb{P}_c(\mathcal{A}) \cap \mathcal{B} = \mathcal{B}$, which is not simple. Hence $\mathbb{P}_c(\mathcal{A})$ is not adjacent to $\mathcal{B}$, contradicting completeness. Therefore $\mathbb{P}_c(\mathcal{A})$ has exactly one non-empty proper subact.

Conversely, assume that $\mathbb{P}_c(\mathcal{A})$ has a unique non-empty proper subact $\mathcal{B}$. Then the vertex set of $Lie(\mathbb{G}\mathcal{A})$ is $\{\mathbb{P}_c(\mathcal{A}), \mathcal{B}\}$. Since $\mathcal{B}$ is the unique proper non-empty subact, it must be simple. Hence $\mathbb{P}_c(\mathcal{A}) \cap \mathcal{B} = \mathcal{B}$ is simple, so the two vertices are adjacent. Therefore $Lie(\mathbb{G}\mathcal{A}) \cong \mathcal{K}_2$. □

Theorem 5.6. Let $\mathbb{P}_c(\mathcal{A})$ be a $\mathbb{G}$-act. Then the distance between any two non-isolated vertices of $Lie(\mathbb{G}\mathcal{A})$ is at most 2.

Proof. Let $\mathcal{B}$ and C be two non-isolated vertices of $Lie(\mathbb{G}\mathcal{A})$. If $\mathcal{B}$ is adjacent to C, then $\partial(\mathcal{B}, C) = 1$. Assume that $\mathcal{B}$ and C are not adjacent. First, suppose that both $\mathcal{B}$ and C are simple. Then each of them is adjacent to $\mathbb{P}_c(\mathcal{A})$, and hence $\mathcal{B} - \mathbb{P}_c(\mathcal{A}) - C$ is a path of length 2. Next, suppose that exactly one of $\mathcal{B}$ and C is simple; without loss of generality, let $\mathcal{B}$ be simple. Since C is non-isolated, there exists a vertex $\mathcal{D}$ adjacent to C. Then $C \cap \mathcal{D}$ is a simple subact. If $\mathcal{B} \cap (C \cap \mathcal{D}) \neq \emptyset$, then by simplicity we get $C \cap \mathcal{D} \subseteq \mathcal{B}$, and therefore $\mathcal{B} - (C \cap \mathcal{D}) - C$ is a path of length 2. If instead $\mathcal{B} \cap (C \cap \mathcal{D}) = \emptyset$, then $\mathcal{B} \sqcup (C \cap \mathcal{D})$ is a subact adjacent to both $\mathcal{B}$ and C, so again $\partial(\mathcal{B}, C) \leq 2$. Finally, suppose that neither $\mathcal{B}$ nor C is simple. Since both are non-isolated, there exist vertices $\mathcal{D}$ and $\mathcal{E}$ such that $\mathcal{B}$ is adjacent to $\mathcal{D}$ and $\mathcal{E}$ is adjacent to C. Then $\mathcal{B} \cap \mathcal{D}$ and $C \cap \mathcal{E}$ are simple subacts. If $\mathcal{B} \cap (C \cap \mathcal{E}) \neq \emptyset$, then by simplicity $C \cap \mathcal{E} \subseteq \mathcal{B}$, and hence $\mathcal{B} - (C \cap \mathcal{E}) - C$ is a path of length 2. Similarly, if $(\mathcal{B} \cap \mathcal{D}) \cap C \neq \emptyset$, then $\mathcal{B} - (\mathcal{B} \cap \mathcal{D}) - C$ is a path of length 2. If both intersections are empty, then the subact $(\mathcal{B} \cap \mathcal{D}) \sqcup (C \cap \mathcal{E})$ is adjacent to both $\mathcal{B}$ and C. Thus again $\partial(\mathcal{B}, C) \leq 2$. Therefore the distance between any two non-isolated vertices of $Lie(\mathbb{G}\mathcal{A})$ is at most 2. □

Let $\mathcal{B}$ be a simple subact of $\mathbb{P}_c(\mathcal{A})$. We say that two nonempty subacts C and D are adjacent through $\mathcal{B}$ provided that $C \cap D = \mathcal{B}$.

Theorem 5.7. Let $\mathbb{P}_c(\mathcal{A})$ be a $\mathbb{G}$-act. Every clique in $Lie(\mathbb{G}\mathcal{A})$ contains at most one simple subact.

Proof. Assume, toward a contradiction, that there exists a clique Θ in $Lie(\mathbb{G}\mathcal{A})$ containing two distinct simple subacts $\mathcal{B}$ and C. Since Θ is a clique, the vertices $\mathcal{B}$ and C must be adjacent in $Lie(\mathbb{G}\mathcal{A})$. By the definition of $Lie(\mathbb{G}\mathcal{A})$, this implies that $\mathcal{B} \cap C$ is a simple subact. However, $\mathcal{B} \cap C$ is a subact of the simple $\mathbb{G}$-act $\mathcal{B}$. Hence, $\mathcal{B} \cap C$ is either $\varnothing$ or $\mathcal{B}$. Similarly, since C is simple, the intersection $\mathcal{B} \cap C$ is either $\varnothing$ or C. Because $\mathcal{B} \cap C$ is assumed to be simple, it must be non-empty, and therefore we obtain $\mathcal{B} \cap C = \mathcal{B}$ and $\mathcal{B} \cap C = C$. This forces $\mathcal{B} = C$, which contradicts the assumption that $\mathcal{B}$ and C are distinct. Therefore, no clique in $Lie(\mathbb{G}\mathcal{A})$ can contain two different simple subacts, and the claim follows. □

The previous theorem shows that cliques in $Lie(\mathbb{G}\mathcal{A})$ fall into exactly two categories: (i) cliques whose vertices are all non-simple subacts, (ii(1) cliques that contain a unique simple subact. The next example illustrates both possibilities.

Example 5.8. Let $\mathbb{P}_c(\mathcal{A}) = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \mathcal{A}_3 \sqcup \mathcal{A}_4 \sqcup \mathcal{A}_5$ be a semisimple $\mathbb{G}$-act with five simple components.
(1) A clique with no simple vertices. Consider the four non-simple subacts

$$Y_2 = \mathcal{A}_1 \sqcup \mathcal{A}_2, \qquad Y_3 = \mathcal{A}_1 \sqcup \mathcal{A}_3, \qquad Y_4 = \mathcal{A}_1 \sqcup \mathcal{A}_4, \qquad Y_5 = \mathcal{A}_1 \sqcup \mathcal{A}_5.$$

Then for all distinct $i, j \in \{2, 3, 4, 5\}$, we have $Y_i \cap Y_j = \mathcal{A}_1$, and since $\mathcal{A}_1$ is simple, every pair among Y_2, Y_3, Y_4, Y_5 is adjacent in $Lie(\mathbb{G}\mathcal{A})$. Hence $\{Y_2, Y_3, Y_4, Y_5\}$ induces a clique in $Lie(\mathbb{G}\mathcal{A})$ whose vertices are all non-simple.
(2) A clique containing exactly one simple vertex. Now include the simple subact $\mathcal{A}_1$ and set

$$\Theta = \{\mathcal{A}_1,\ Y_2,\ Y_3,\ Y_4,\ Y_5\}.$$

We have $\mathcal{A}_1 \cap Y_i = \mathcal{A}_1$ for $i \in \{2, 3, 4, 5\}$, so $\mathcal{A}_1$ is adjacent to each Y_i. Moreover, the vertices Y_2, Y_3, Y_4, Y_5 are pairwise adjacent by part (1). Therefore, Θ is a clique of $Lie(\mathbb{G}\mathcal{A})$ containing exactly one simple subact.

Theorem 5.9. Let $\mathbb{P}_c(\mathcal{A})$ be a $\mathbb{G}$-act and let Θ be a clique in $Lie(\mathbb{G}\mathcal{A})$. Then Θ contains exactly one simple subact $\mathcal{B}$ if and only if $\Theta = \{\mathcal{B}\} \cup \mathcal{P}$, where $\mathcal{P} \subseteq N(\mathcal{B})$ and every pair of distinct vertices in $\mathcal{P}$ is adjacent through $\mathcal{B}$.

Proof. Assume first that Θ is a clique containing a unique simple subact $\mathcal{B}$. Let $C \in \mathcal{V}(\Theta) \setminus \{\mathcal{B}\}$. Since Θ is a clique, C is adjacent to $\mathcal{B}$, and therefore $C \in N(\mathcal{B})$. Hence $\mathcal{V}(\Theta) \setminus \{\mathcal{B}\} \subseteq N(\mathcal{B})$. Set $\mathcal{P} = \mathcal{V}(\Theta) \setminus \{\mathcal{B}\}$. Then $\Theta = \{\mathcal{B}\} \cup \mathcal{P}$ and $\mathcal{P} \subseteq N(\mathcal{B})$.

Now let $C, \mathcal{D} \in \mathcal{P}$ with $C \neq \mathcal{D}$. Since Θ is a clique, C and $\mathcal{D}$ are adjacent, so $C \cap \mathcal{D}$ is a simple subact. Because $\mathcal{B}$ is the unique simple subact belonging to Θ, and since $\mathcal{B} \subseteq C$ and $\mathcal{B} \subseteq \mathcal{D}$ need not be assumed a priori, we argue as follows. Both C and $\mathcal{D}$ are adjacent to $\mathcal{B}$, so $\mathcal{B} \cap C = \mathcal{B}$, and $\mathcal{B} \cap \mathcal{D} = \mathcal{B}$. Thus $\mathcal{B} \subseteq C$ and $\mathcal{B} \subseteq \mathcal{D}$. Since $C \cap \mathcal{D}$ is simple and contains $\mathcal{B}$, we must have $C \cap \mathcal{D} = \mathcal{B}$. Hence every pair of distinct vertices in $\mathcal{P}$ is adjacent through $\mathcal{B}$.

Conversely, suppose that $\Theta = \{\mathcal{B}\} \cup \mathcal{P}$, where $\mathcal{P} \subseteq N(\mathcal{B})$ and every pair of distinct vertices in $\mathcal{P}$ is adjacent through $\mathcal{B}$. Then every vertex of $\mathcal{P}$ is adjacent to $\mathcal{B}$, and any two distinct vertices of $\mathcal{P}$ are adjacent through $\mathcal{B}$. Therefore every pair of distinct vertices in Θ is adjacent, so Θ is a clique.

Finally, since every vertex in $\mathcal{P}$ is adjacent to $\mathcal{B}$, we have $\mathcal{B} \subseteq C$ for every $C \in \mathcal{P}$. If some $C \in \mathcal{P}$ were simple, then from $\mathcal{B} \subseteq C$ and the simplicity of both $\mathcal{B}$ and C, it would follow that $C = \mathcal{B}$, contradicting $C \in \mathcal{P}$. Hence $\mathcal{B}$ is the unique simple subact in Θ. □

Theorem 5.10. Let $\mathbb{P}_c(\mathcal{A})$ be a $\mathbb{G}$-act and let Θ be a clique in $Lie(\mathbb{G}\mathcal{A})$ containing a unique simple subact $\mathcal{B}$. Then $O(\mathcal{V}(\Theta)) > 2$ if and only if $\mathbb{P}_c(\mathcal{A}) \notin \mathcal{V}(\Theta)$ and $\mathcal{V}(\Theta)$ contains at least two proper non-simple subacts that are pairwise adjacent through $\mathcal{B}$.

Proof. Assume that $O(\mathcal{V}(\Theta)) > 2$. Since Θ contains a unique simple subact $\mathcal{B}$, Theorem 5.9 implies that every vertex of $\mathcal{V}(\Theta) \setminus \{\mathcal{B}\}$ is a non-simple subact adjacent to $\mathcal{B}$, and any two such vertices are adjacent through $\mathcal{B}$. Because $O(\mathcal{V}(\Theta)) > 2$, there are at least two such vertices.

Moreover, $\mathbb{P}_c(\mathcal{A}) \notin \mathcal{V}(\Theta)$, because $\mathbb{P}_c(\mathcal{A})$ is adjacent only to simple subacts: Indeed, for any subact $C \subseteq \mathbb{P}_c(\mathcal{A})$, one has $\mathbb{P}_c(\mathcal{A}) \cap C = C$, so adjacency with $\mathbb{P}_c(\mathcal{A})$ occurs only when C is simple. Hence every vertex of $\mathcal{V}(\Theta) \setminus \{\mathcal{B}\}$ is a proper non-simple subact. Therefore $\mathcal{V}(\Theta)$ contains at least two proper non-simple subacts that are pairwise adjacent through $\mathcal{B}$.

Conversely, suppose that $\mathbb{P}_c(\mathcal{A}) \notin \mathcal{V}(\Theta)$ and that $\mathcal{V}(\Theta)$ contains at least two proper non-simple subacts that are pairwise adjacent through $\mathcal{B}$. Then these two vertices, together with $\mathcal{B}$, form three distinct vertices of the clique Θ. Hence $O(\mathcal{V}(\Theta)) > 2$. □

Let $\mathcal{B}$ be a simple subact of $\mathbb{P}_c(\mathcal{A})$. A maximal clique induced by $\mathcal{B}$ is a clique in $Lie(\mathbb{G}\mathcal{A})$ that contains $\mathcal{B}$ and is not properly contained in any larger clique of $Lie(\mathbb{G}\mathcal{A})$. For every simple subact $\mathcal{B}$ of $\mathbb{P}_c(\mathcal{A})$, the edge $\mathcal{B} - \mathbb{P}_c(\mathcal{A})$ always determines a maximal clique induced by $\mathcal{B}$. We refer to this clique as the trivial maximal clique induced by $\mathcal{B}$.

If $O(N(\mathcal{B})) = 1$, then this trivial clique is the only maximal clique induced by $\mathcal{B}$. However, when $O(N(\mathcal{B})) > 1$, there may exist another maximal clique induced by $\mathcal{B}$, consisting of $\mathcal{B}$ together with proper non-simple subacts in $N(\mathcal{B})$ that are pairwise adjacent through $\mathcal{B}$. We denote such a non-trivial maximal clique by $\Theta(\mathcal{B})$ when it exists. Note that $O(\mathcal{V}(\Theta(\mathcal{B}))) > 2$.

Example 5.11. Let $\mathbb{P}_c(\mathcal{A}) = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \mathcal{A}_3$ be a semisimple $\mathbb{G}$-act with exactly three simple components. Consider the simple subact $\mathcal{A}_1$. The only subacts of $\mathbb{P}_c(\mathcal{A})$ that contain $\mathcal{A}_1$ are $\mathcal{A}_1$ itself, $\mathcal{A}_1 \sqcup \mathcal{A}_2$, $\mathcal{A}_1 \sqcup \mathcal{A}_3$, and $\mathbb{P}_c(\mathcal{A})$. Hence $N(\mathcal{A}_1) = \{\mathcal{A}_1 \sqcup \mathcal{A}_2,\ \mathcal{A}_1 \sqcup \mathcal{A}_3,\ \mathbb{P}_c(\mathcal{A})\}$. Moreover, $(\mathcal{A}_1 \sqcup \mathcal{A}_2) \cap (\mathcal{A}_1 \sqcup \mathcal{A}_3) = \mathcal{A}_1$ is simple, so $\mathcal{A}_1 \sqcup \mathcal{A}_2$ and $\mathcal{A}_1 \sqcup \mathcal{A}_3$ are adjacent through $\mathcal{A}_1$. Therefore, the set $\{\mathcal{A}_1,\ \mathcal{A}_1 \sqcup \mathcal{A}_2,\ \mathcal{A}_1 \sqcup \mathcal{A}_3\}$ is a clique in $Lie(\mathbb{G}\mathcal{A})$ induced by $\mathcal{A}_1$, and it properly contains the trivial clique $\{\mathcal{A}_1, \mathbb{P}_c(\mathcal{A})\}$. In particular, when $O(N(\mathcal{A}_1)) > 1$, the trivial maximal clique induced by $\mathcal{A}_1$ need not be the only maximal clique induced by $\mathcal{A}_1$.

Example 5.12. Let $\mathbb{P}_c(\mathcal{A}) = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \mathcal{A}_3$ be a semisimple $\mathbb{G}$-act with exactly three simple components. Consider the simple subact $\mathcal{A}_1$. The subacts of $\mathbb{P}_c(\mathcal{A})$ that contain $\mathcal{A}_1$ are $\mathcal{A}_1$, $\mathcal{A}_1 \sqcup \mathcal{A}_2$, $\mathcal{A}_1 \sqcup \mathcal{A}_3$, and $\mathbb{P}_c(\mathcal{A})$. Hence $N(\mathcal{A}_1) = \{\mathcal{A}_1 \sqcup \mathcal{A}_2,\ \mathcal{A}_1 \sqcup \mathcal{A}_3,\ \mathbb{P}_c(\mathcal{A})\}$. Moreover, $(\mathcal{A}_1 \sqcup \mathcal{A}_2) \cap (\mathcal{A}_1 \sqcup \mathcal{A}_3) = \mathcal{A}_1$, which is a simple subact. Therefore $\mathcal{A}_1 \sqcup \mathcal{A}_2$ and $\mathcal{A}_1 \sqcup \mathcal{A}_3$ are adjacent through $\mathcal{A}_1$. Consequently, $\{\mathcal{A}_1, \mathcal{A}_1 \sqcup \mathcal{A}_2, \mathcal{A}_1 \sqcup \mathcal{A}_3\}$ forms a clique in $Lie(\mathbb{G}\mathcal{A})$ induced by $\mathcal{A}_1$. This clique properly contains the trivial clique $\{\mathcal{A}_1, \mathbb{P}_c(\mathcal{A})\}$. Thus, when $O(N(\mathcal{A}_1)) > 1$, the trivial maximal clique induced by $\mathcal{A}_1$ need not be the only maximal clique induced by $\mathcal{A}_1$.

Let X be a non-empty collection of simple subacts of $\mathbb{P}_c(\mathcal{A})$. A clique in $Lie(\mathbb{G}\mathcal{A})$ is said to be induced by X if it satisfies the following two conditions: First, for any two distinct vertices of the clique, their adjacency occurs through some element of X; second, for each $\mathcal{B} \in X$, there exist two vertices of the clique whose intersection is exactly $\mathcal{B}$.

We now show that whenever a non-empty set X of simple subacts gives rise to cliques in $Lie(\mathbb{G}\mathcal{A})$, there exists a maximal such clique.

Theorem 5.13. Let $\mathbb{P}_c(\mathcal{A})$ be a $\mathbb{G}$-act and let X be a non-empty collection of simple subacts of $\mathbb{P}_c(\mathcal{A})$ such that there exists at least one clique in $Lie(\mathbb{G}\mathcal{A})$ induced by X. Then there exists a maximal clique in $Lie(\mathbb{G}\mathcal{A})$ induced by X.

Proof. Let Σ denote the family of all cliques in $Lie(\mathbb{G}\mathcal{A})$ that are induced by X. By assumption, Σ is non-empty. We partially order Σ by inclusion.

Let C be a chain in Σ, and set $\Theta^* = \bigcup_{\Theta \in C} \Theta$. We claim that $\Theta^* \in \Sigma$.

Let $\mathcal{U}, \mathcal{V} \in \Theta^*$ with $\mathcal{U} \neq \mathcal{V}$. Then there exist $\Theta_1, \Theta_2 \in C$ such that $\mathcal{U} \in \Theta_1$ and $\mathcal{V} \in \Theta_2$. Since C is a chain, either $\Theta_1 \subseteq \Theta_2$ or $\Theta_2 \subseteq \Theta_1$. Hence $\mathcal{U}, \mathcal{V}$ belong to a common clique in C, and therefore they are adjacent. Thus Θ^* is a clique.

Moreover, if $\mathcal{U}, \mathcal{V} \in \Theta^*$ are distinct, then their adjacency occurs through some element of X, because this already holds in the member of C that contains both of them. Also, for each $\mathcal{B} \in X$, there exists some $\Theta_{\mathcal{B}} \in C$ and two vertices of $\Theta_{\mathcal{B}}$ whose intersection is exactly $\mathcal{B}$. These two vertices also belong to Θ^*. Therefore Θ^* is induced by X, so $\Theta^* \in \Sigma$.

Hence every chain in Σ has an upper bound in Σ. By Zorn's lemma, Σ contains a maximal element, which is a maximal clique induced by X. □

Note that if $X = \{\mathcal{B}\}$ consists of a single simple subact, then a maximal clique induced by X is either the trivial clique $\mathcal{B} - \mathbb{P}_c(\mathcal{A})$ or a non-trivial maximal clique of the form $\Theta(\mathcal{B})$, when the latter exists. In general, maximal cliques induced by X need not be unique.

Theorem 5.14. Let $\mathbb{P}_c(\mathcal{A})$ be a $\mathbb{G}$-act. If Θ is a clique of $Lie(\mathbb{G}\mathcal{A})$, then there exists a unique non-empty set X of simple subacts of $\mathbb{P}_c(\mathcal{A})$ such that Θ is induced by X.

Proof. Let Θ be a clique in $Lie(\mathbb{G}\mathcal{A})$, and define $X = \{\mathcal{B} : \mathcal{B} = \mathcal{U} \cap \mathcal{V}$ for some distinct $\mathcal{U}, \mathcal{V} \in \mathcal{V}(\Theta)\}$. Since Θ is a clique, every such intersection is a simple subact, and thus X is a set of simple subacts.

By construction, whenever $\mathcal{U}$ and $\mathcal{V}$ are distinct vertices of Θ, their adjacency occurs through some element of X, namely $\mathcal{U} \cap \mathcal{V}$. Also, for every $\mathcal{B} \in X$, there exist two vertices of Θ whose intersection is exactly $\mathcal{B}$. Hence Θ is induced by X.

To prove uniqueness, let Y be another non-empty set of simple subacts such that Θ is induced by Y. If $\mathcal{B} \in X$, then by definition there exist distinct vertices $\mathcal{U}, \mathcal{V} \in \Theta$ such that $\mathcal{U} \cap \mathcal{V} = \mathcal{B}$. Since Θ is induced by Y, this intersection must belong to Y. Thus $X \subseteq Y$. Similarly, $Y \subseteq X$. Therefore $X = Y$. □

For the graph $Lie(\mathbb{G}\mathcal{A})$, the following observations hold. First, if $Lie(\mathbb{G}\mathcal{A})$ has finite order, then $2 \leq \xi(Lie(\mathbb{G}\mathcal{A})) \leq O(\mathcal{V}(Lie(\mathbb{G}\mathcal{A})))$. Second, the clique number satisfies $\xi(Lie(\mathbb{G}\mathcal{A})) = \sup\{O(\mathcal{V}(\Theta(X))) : X \neq \emptyset$ is a set of simple subacts of $\mathbb{P}_c(\mathcal{A})\}$.

6. The chordless property in $Lie(\mathbb{G}\mathcal{A})$

This section is devoted to the study of chordless paths and the corresponding monophonic notions in $Lie(\mathbb{G}\mathcal{A})$. Monophonic paths represent a stricter notion of connectivity than ordinary paths, as they exclude the presence of chords and therefore encode finer structural information. In particular, they allow us to examine how simple intersections arise without introducing shortcuts, yielding a more delicate understanding of both local and global connectivity properties. We investigate how the action-theoretic decomposition of the act $\mathcal{A}$ under $\mathbb{G}$, together with the distribution of simple subacts, influences the existence and behavior of monophonic paths. These considerations lead to consequences for path-based invariants in $Lie(\mathbb{G}\mathcal{A})$, especially those sensitive to chordless configurations.

Let $\mathbb{G}$ be a Lie group, and let $\mathbb{P}_c(\mathcal{A})$ be a $\mathbb{G}$-act. Let $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{V}(Lie(\mathbb{G}\mathcal{A}))$. We say that $\mathcal{A}_1$ is a chordless subact of $\mathcal{A}_2$ if there exists a monophonic path of length 2 between $\mathcal{A}_1$ and $\mathcal{A}_2$. Thus, the definition does not require the existence of a longest monophonic path and remains valid when $Lie(\mathbb{G}\mathcal{A})$ is infinite. For any $\mathcal{B} \in \mathcal{V}(Lie(\mathbb{G}\mathcal{A}))$, the chordless open neighborhood of $\mathcal{B}$, denoted by $\mathbb{L}^c(\mathcal{B})$, is defined by

$$\mathbb{L}^c(\mathcal{B}) = \{C \in \mathcal{V}(Lie(\mathbb{G}\mathcal{A})) : \text{there exists a monophonic path of length 2 between } \mathcal{B} \text{ and } C\}.$$

The corresponding chordless closed neighborhood of $\mathcal{B}$ is given by

$$\mathbb{L}^c[\mathcal{B}] = \mathbb{L}^c(\mathcal{B}) \cup \{\mathcal{B}\}.$$

We define the collection $\mathbb{L}^c(Lie(\mathbb{G}\mathcal{A}))$ by

$$\mathbb{L}^c(Lie(\mathbb{G}\mathcal{A})) = \{\mathbb{L}^c[\mathcal{B}] : \mathcal{B} \in \mathcal{V}(Lie(\mathbb{G}\mathcal{A}))\}.$$

The chordless topological space (or simply, the chordless space) associated with $Lie(\mathbb{G}\mathcal{A})$ is the pair

$$(\mathcal{V}(Lie(\mathbb{G}\mathcal{A})), T_{\mathbb{L}^c(Lie(\mathbb{G}\mathcal{A}))}),$$

where $T_{\mathbb{L}^c(Lie(\mathbb{G}\mathcal{A}))}$ denotes the topology generated by the subbasis $\mathbb{L}^c(Lie(\mathbb{G}\mathcal{A}))$.

Example 6.1. In Example 3.2, the subacts are all non-empty proper subsets of $\mathbb{P}_c(\mathbb{Z}_4)$, that is, $\mathcal{V}[Lie(\mathbb{G}\mathbb{Z}_3)] = P(\mathbb{P}_c(\mathbb{Z}_3)) \setminus \{\emptyset\}$. The graph $Lie(\mathbb{Z}_3\mathbb{G}))$ in Figure 2 that has the subacts set $\mathcal{V}(Lie(\mathbb{Z}_3\mathbb{G}))$ has the subbasis $\mathbb{L}^c(Lie(\mathbb{Z}_3\mathbb{G})) = P(\mathbb{P}_c(\mathbb{Z}_3)) \setminus \{\emptyset\}$. That is, $T_{\mathbb{L}^c(Lie(\mathbb{Z}_3\mathbb{G}))}$ is discrete.

Example 6.2. In Example 3.3, the graph $Lie(\mathbb{Z}_3\mathbb{G})$ in Figure 1 has the subacts set $\mathcal{V}[Lie(\mathbb{G}\mathcal{A})] = P(\mathbb{P}_c(\mathcal{A})) - \{\emptyset\}$ and the subbasis $\mathbb{L}^c(Lie(\mathbb{G}\mathcal{A})) = P(\mathbb{P}_c(\mathcal{A})) - \{\emptyset\}$. That is, $T_{\mathbb{L}^c(Lie(\mathbb{G}\mathcal{A}))}$ is discrete.

Let $Lie(\mathbb{G}\mathcal{A})$ be any graph, and let $\mathcal{B} \in \mathcal{V}(Lie(\mathbb{G}\mathcal{A}))$ be an isolated subact. Then the singleton set $\{\mathcal{B}\}$ is open in the topological space $(\mathcal{V}(Lie(\mathbb{G}\mathcal{A})), T_{\mathbb{L}^c(Lie(\mathbb{G}\mathcal{A}))})$. Moreover, for any isolated edge $\mathcal{B}\mathcal{B}' \in \mathcal{E}(Lie(\mathbb{G}\mathcal{A}))$, the singleton sets $\{\mathcal{B}\}$ and $\{\mathcal{B}'\}$ are both open. A topology in which the intersection of any family of open sets is again an open set is called an Alexandroff topology. In the following results, we assume that $Lie(\mathbb{G}\mathcal{A})$ has locally finite property, that is, every vertex has finite degree.

Theorem 6.3. Let $Lie(\mathbb{G}\mathcal{A})$ be locally finite. Then the chordless space $(\mathcal{V}(Lie(\mathbb{G}\mathcal{A})), T_{\mathbb{L}^c(Lie(\mathbb{G}\mathcal{A}))})$ has the Alexandroff property.

Proof. Let $\mathcal{F} = \{\mathbb{L}^c[\mathcal{B}] : \mathcal{B} \in F \subseteq \mathcal{V}(Lie(\mathbb{G}\mathcal{A}))\}$ be any family in $\mathbb{L}^c(Lie(\mathbb{G}\mathcal{A}))$. We show that $\cap_{\mathcal{B}\in F}\mathbb{L}^c[\mathcal{B}]$ is an open set. If $\cap_{\mathcal{B}\in F}\mathbb{L}^c[\mathcal{B}] = \emptyset$, then it is open. Suppose that $\cap_{\mathcal{B}\in F}\mathbb{L}^c[\mathcal{B}] \neq \emptyset$, and let $\mathcal{B}' \in \cap_{\mathcal{B}\in F}\mathbb{L}^c[\mathcal{B}]$. Then $\mathcal{B}' \in \mathbb{L}^c[\mathcal{B}]$ for all $\mathcal{B} \in F$. Since the chordless relation is symmetric, we have $\mathcal{B} \in \mathbb{L}^c[\mathcal{B}']$ for all $\mathcal{B} \in F$. Thus, $F \subseteq \mathbb{L}^c[\mathcal{B}']$. Since $Lie(\mathbb{G}\mathcal{A})$ is locally finite, $\mathbb{L}^c[\mathcal{B}']$ is finite, and hence F is finite. Therefore, $\cap_{\mathcal{B}\in F}\mathbb{L}^c[\mathcal{B}]$ is a finite intersection of members of the subbasis $\mathbb{L}^c(Lie(\mathbb{G}\mathcal{A}))$, and hence it is open. Consequently, arbitrary intersections of open sets are open, and the chordless topology is Alexandroff. □

By using the theorem above, we can put $\mathbb{O}(\mathcal{B})$ to denote the smallest open set w here $\mathcal{B} \in \mathbb{O}(\mathcal{B})$. Note that if the subact $\mathcal{B} \in \mathcal{V}(Lie(\mathbb{G}\mathcal{A}))$ is an isolated subact in a graph $Lie(\mathbb{G}\mathcal{A}) = (\mathcal{V}(Lie(\mathbb{G}\mathcal{A})), \mathcal{E}(Lie(\mathbb{G}\mathcal{A})))$, then $\mathbb{O}(\mathcal{B}) = \{\mathcal{B}\}$. Also $\mathbb{O}(\mathcal{B}) = \{\mathcal{B}\}$ and $\mathbb{O}(\mathcal{B}') = \{\mathcal{B}'\}$ if the edge γ is an isolated edge with $J_\gamma = \{\mathcal{B}, \mathcal{B}'\}$.

Theorem 6.4. $\mathbb{O}(\mathcal{B}) = \cap_{\mathcal{B}'\in\mathbb{L}^c[\mathcal{B}]}\mathbb{L}^c[\mathcal{B}']$ for all $\mathcal{B} \in \mathcal{V}(Lie(\mathbb{G}\mathcal{A}))$.

Proof. By the concept of $\mathbb{L}^c(Lie(\mathbb{G}\mathcal{A}))$, $\mathbb{L}^c[\mathcal{B}]$ is an open set in $(\mathcal{V}(Lie(\mathbb{G}\mathcal{A})), T_{\mathbb{L}^c(Lie(\mathbb{G}\mathcal{A}))})$ for all $\mathcal{B} \in \mathcal{V}(Lie(\mathbb{G}\mathcal{A}))$. From Theorem 6.3, $\cap_{\mathcal{B}'\in\mathbb{L}^c[\mathcal{B}]}\mathbb{L}^c[\mathcal{B}']$ is an open set. If $\mathcal{B}' \in \mathbb{L}^c[\mathcal{B}]$, then it is clear that $\mathcal{B} \in \mathbb{L}^c[\mathcal{B}']$. Hence $\cap_{\mathcal{B}'\in\mathbb{L}^c[\mathcal{B}]}\mathbb{L}^c[\mathcal{B}']$ is an open set involving $\mathcal{B}$. By the concept of $\mathbb{O}(\mathcal{B})$, we get $\mathbb{O}(\mathcal{B}) \subseteq \cap_{\mathcal{B}'\in\mathbb{L}^c[\mathcal{B}]}\mathbb{L}^c[\mathcal{B}']$. Let $\mathbb{O}(\mathcal{B}) = \cap_{\mathcal{B}'\in F}\mathbb{L}^c[\mathcal{B}']$ for some subsets F of $\mathcal{V}(Lie(\mathbb{G}\mathcal{A}))$. Then $\mathcal{B} \in \mathbb{L}^c[\mathcal{B}']$ for all $\mathcal{B}' \in F$. This implies $\mathcal{B}' \in \mathbb{L}^c[\mathcal{B}]$ for all $\mathcal{B}' \in F$. Then $F \subseteq \mathbb{L}^c[\mathcal{B}]$. Therefore

$$\cap_{\mathcal{B}'\in\mathbb{L}^c[\mathcal{B}]}\mathbb{L}^c[\mathcal{B}'] \subseteq \cap_{\mathcal{B}'\in F}\mathbb{L}^c[\mathcal{B}'] = \mathbb{O}(\mathcal{B}).$$

So $\mathbb{O}(\mathcal{B}) = \cap_{\mathcal{B}'\in\mathbb{L}^c[\mathcal{B}]}\mathbb{L}^c[\mathcal{B}']$. □

Theorem 6.5. Let $\mathcal{B}, \mathcal{B}' \in \mathcal{V}(Lie(\mathbb{G}\mathcal{A}))$. Then $\mathcal{B} \in \mathbb{O}(\mathcal{B}')$ if and only if $\mathbb{L}^c[\mathcal{B}'] \subseteq \mathbb{L}^c[\mathcal{B}]$.

Proof. Let $\mathcal{B} \in \mathbb{O}(\mathcal{B}')$. From the theorem above, $\mathbb{L}^c[\mathcal{B}'] = \cap_{\mathcal{B}''\in\mathbb{L}^c[\mathcal{B}']}\mathbb{L}^c[\mathcal{B}'']$. Then

$$\mathcal{B} \in \cap_{\mathcal{B}''\in\mathbb{L}^c[\mathcal{B}']}\mathbb{L}^c[\mathcal{B}''].$$

That is, $\mathcal{B} \in \mathbb{L}^c[\mathcal{B}'']$ for all $\mathcal{B}'' \in \mathbb{L}^c[\mathcal{B}']$. Hence $\mathcal{B}'' \in \mathbb{L}^c[\mathcal{B}]$ for all $\mathcal{B}'' \in \mathbb{L}^c[\mathcal{B}']$. Then $\mathbb{L}^c[\mathcal{B}'] \subseteq \mathbb{L}^c[\mathcal{B}]$.

Conversely, let $\mathbb{L}^c[\mathcal{B}'] \subseteq \mathbb{L}^c[\mathcal{B}]$. Hence

$$\mathcal{B} \in \cap_{\mathcal{B}''\in\mathbb{L}^c[\mathcal{B}]}\mathbb{L}^c[\mathcal{B}''] \subseteq \cap_{\mathcal{B}''\in\mathbb{L}^c[\mathcal{B}']}\mathbb{L}^c[\mathcal{B}''] = \mathbb{O}(\mathcal{B}').$$

That is, $\mathcal{B} \in \mathbb{O}(\mathcal{B}')$. □

Theorem 6.6. In the graph $Lie(\mathbb{G}\mathcal{A})$, $T_{\mathbb{L}^c(Lie(\mathbb{G}\mathcal{A}))}$ is a discrete topology if and only if $\mathbb{L}^c[\mathcal{B}'] \nsubseteq \mathbb{L}^c[\mathcal{B}]$ and $\mathbb{L}^c[\mathcal{B}] \nsubseteq \mathbb{L}^c[\mathcal{B}']$ for all $\mathcal{B}, \mathcal{B}' \in \mathcal{V}(Lie(\mathbb{G}\mathcal{A}))$.

Proof. Let $(\mathcal{V}(Lie(\mathbb{G}\mathcal{A})), T_{\mathbb{L}^c(Lie(\mathbb{G}\mathcal{A}))})$ be a discrete space. Hence $\mathbb{L}^c[\mathcal{B}] = \{\mathcal{B}\}$ for all $\mathcal{B} \in \mathcal{V}(Lie(\mathbb{G}\mathcal{A}))$. Then for $\mathcal{B} \neq \mathcal{B}' \in \mathcal{V}(Lie(\mathbb{G}\mathcal{A}))$, $\mathcal{B}' \notin \mathbb{L}^c[\mathcal{B}]$ and $\mathcal{B} \notin \mathbb{L}^c[\mathcal{B}']$. Hence $\mathbb{L}^c[\mathcal{B}'] \nsubseteq \mathbb{L}^c[\mathcal{B}]$ and $\mathbb{L}^c[\mathcal{B}] \nsubseteq \mathbb{L}^c[\mathcal{B}']$.

Conversely, let $\mathcal{B}$ be any subact in $\mathcal{V}(Lie(\mathbb{G}\mathcal{A}))$. Note that $\mathcal{B} \in \mathbb{L}^c[\mathcal{B}]$. If $\mathcal{B}' \neq \mathcal{B} \in \mathcal{V}(Lie(\mathbb{G}\mathcal{A}))$ and $\mathcal{B}' \in \mathbb{L}^c[\mathcal{B}]$, then by Theorem 6.5, $\mathbb{L}^c[\mathcal{B}] \subseteq \mathbb{L}^c[\mathcal{B}']$,and this contradicts the above hypothesis. Then $\mathbb{O}(\mathcal{B}) = \{\mathcal{B}\}$ for all $\mathcal{B} \in \mathcal{V}(Lie(\mathbb{G}\mathcal{A}))$. That is, $(\mathcal{V}(Lie(\mathbb{G}\mathcal{A})), T_{\mathbb{L}^c(Lie(\mathbb{G}\mathcal{A}))})$ is a discrete space. □

Let $F \subseteq \mathcal{V}(Lie(\mathbb{G}\mathcal{A}))$. In $(\mathcal{V}(Lie(\mathbb{G}\mathcal{A})), T_{\mathbb{L}^c(Lie(\mathbb{G}\mathcal{A}))})$, $\overline{F}$ denotes the closure set of F, the smallest closed sets involving F. Note that $\mathcal{B} \in \overline{F}$ if and only if $H \cap F \neq \emptyset$ for any open H involving $\mathcal{B}$. A set F is said to be dense in A if $\overline{F} = A$, that is, for all open sets O, $F \cap O \neq \emptyset$.

Theorem 6.7. Let $\mathcal{B} \in \mathcal{V}(Lie(\mathbb{G}\mathcal{A}))$. Then $\overline{\mathbb{L}^c[\mathcal{B}]} \subseteq \overline{\mathbb{L}^c[\mathcal{B}']}$ for all $\mathcal{B}' \in \mathbb{L}^c[\mathcal{B}]$.

Proof. Let $\mathcal{B}'' \in \overline{\mathbb{L}^c[\mathcal{B}]}$. Then for every open set F containing $\mathcal{B}''$, $F \cap \mathbb{L}^c[\mathcal{B}] \neq \emptyset$. Since $\mathbb{L}^c[\mathcal{B}] \subseteq \mathbb{L}^c[\mathcal{B}']$ for all $\mathcal{B}' \in \mathbb{L}^c[\mathcal{B}]$, then $F \cap \mathbb{L}^c[\mathcal{B}'] \neq \emptyset$ for all $\mathcal{B}' \in \mathbb{L}^c[\mathcal{B}]$. Then $\mathcal{B}'' \in \overline{\mathbb{L}^c[\mathcal{B}']}$ for all $\mathcal{B}' \in \mathbb{L}^c[\mathcal{B}]$. Hence $\overline{\mathbb{L}^c[\mathcal{B}]} \subseteq \overline{\mathbb{L}^c[\mathcal{B}']}$ for all $\mathcal{B}' \in \mathbb{L}^c[\mathcal{B}]$. □

Corollary 6.8. Let $\mathcal{B} \in \mathcal{V}(Lie(\mathbb{G}\mathcal{A}))$. Then $\overline{\{\mathcal{B}\}} \subseteq \overline{\mathbb{L}^c[\mathcal{B}']}$ for all $\mathcal{B}' \in \mathbb{L}^c[\mathcal{B}]$.

Corollary 6.9. Let $\mathcal{B}, \mathcal{B}' \in \mathcal{V}(Lie(\mathbb{G}\mathcal{A}))$. Then $\mathcal{B} \in \overline{\{\mathcal{B}'\}}$ if and only if $\mathbb{L}^c[\mathcal{B}] \subseteq \mathbb{L}^c[\mathcal{B}']$.

7. On controlling cliques

In this section, we introduce and study controlling sets of simple subacts and their role in determining the clique structure of the Lie intersection graph $Lie(\mathbb{G}\mathcal{A})$. We show how families of simple subacts control adjacency patterns among non-simple subacts and lead to the existence and description of maximal cliques. These results provide a unified framework for understanding clique formation in terms of simple subact interactions. By $\mathcal{Z}(\mathbb{P}_c(\mathcal{A}))$, we denote the set of all simple subacts of $\mathbb{P}_c(\mathcal{A})$ together with all non-simple subacts that contain no simple subacts.

Theorem 7.1. Let $\mathbb{G}$ be any Lie group and let $\mathbb{P}_c(\mathcal{A})$ be a $\mathbb{G}$-act. Then $\mathcal{Z}(\mathbb{P}_c(\mathcal{A}))$ is an indecomposable controlling set of $Lie(\mathbb{G}\mathcal{A})$.

Proof. Let $\mathcal{B}$ be an arbitrary vertex of $Lie(\mathbb{G}\mathcal{A})$ such that $\mathcal{B} \notin \mathcal{Z}(\mathbb{P}_c(\mathcal{A}))$. By the definition of $\mathcal{Z}(\mathbb{P}_c(\mathcal{A}))$, the subact $\mathcal{B}$ must contain a simple subact, say $\mathcal{B}_1$. Consequently, $\mathcal{B}$ is adjacent to $\mathcal{B}_1$ in $Lie(\mathbb{G}\mathcal{A})$. Thus, every vertex outside $\mathcal{Z}(\mathbb{P}_c(\mathcal{A}))$ is adjacent to at least one vertex in $\mathcal{Z}(\mathbb{P}_c(\mathcal{A}))$, which shows that $\mathcal{Z}(\mathbb{P}_c(\mathcal{A}))$ is a controlling set. Moreover, vertices of $\mathcal{Z}(\mathbb{P}_c(\mathcal{A}))$ are pairwise non-adjacent, and each vertex in $\mathcal{Z}(\mathbb{P}_c(\mathcal{A}))$ is required to control itself. Hence removing any vertex from $\mathcal{Z}(\mathbb{P}_c(\mathcal{A}))$ destroys the controlling property. Therefore, $\mathcal{Z}(\mathbb{P}_c(\mathcal{A}))$ is indecomposable. □

Proposition 7.2. Let $\mathbb{G}$ be any Lie group and let $\mathbb{P}_c(\mathcal{A})$ be a $\mathbb{G}$-act. Then $\Delta(Lie(\mathbb{G}\mathcal{A})) \leq O(\mathcal{Z}(\mathbb{P}_c(\mathcal{A})))$. Moreover, if $Lie(\mathbb{G}\mathcal{A})$ is connected, then $\mathcal{Z}(\mathbb{P}_c(\mathcal{A}))$ consists only of simple subacts of $\mathbb{P}_c(\mathcal{A})$.

Proof. By Theorem 7.1, $\mathcal{Z}(\mathbb{P}_c(\mathcal{A}))$ is a controlling set of $Lie(\mathbb{G}\mathcal{A})$. By definition of the control number, $\Delta(Lie(\mathbb{G}\mathcal{A})) = \inf\{O(\mathbb{C}) : \mathbb{C} \in Con(Lie(\mathbb{G}\mathcal{A}))\}$, and hence $\Delta(Lie(\mathbb{G}\mathcal{A})) \leq O(\mathcal{Z}(\mathbb{P}_c(\mathcal{A})))$. If a non-simple subact of $\mathbb{P}_c(\mathcal{A})$ contains no simple subact, then it has no neighbors in $Lie(\mathbb{G}\mathcal{A})$ and is therefore an isolated vertex. If $Lie(\mathbb{G}\mathcal{A})$ is connected, it admits no isolated vertices, so no such non-simple subacts can exist. Consequently, $\mathcal{Z}(\mathbb{P}_c(\mathcal{A}))$ consists entirely of simple subacts. □

Note that $\mathcal{Z}(\mathbb{P}_c(\mathcal{A}))$ is not the only possible controlling set of $Lie(\mathbb{G}\mathcal{A})$. However, we shall show in subsequent work that every controlling set can be reconstructed as an indecomposable controlling set whose vertices consist of the isolated vertices of $Lie(\mathbb{G}\mathcal{A})$ together with certain semisimple subacts. Let $\mathbb{C}$ be a controlling set of $Lie(\mathbb{G}\mathcal{A})$. We denote by $\overline{\mathbb{C}}$ the set consisting of the following vertices: all non-simple subacts of $\mathbb{P}_c(\mathcal{A})$ that contain no simple subacts; all vertices $\mathcal{B} \in \mathbb{C}$ such that $\mathcal{B} = \mathrm{Soc}(\mathcal{B})$; all vertices $\mathcal{B} \in \mathbb{C}$ with $\mathcal{B} \neq \mathrm{Soc}(\mathcal{B})$ and $\mathrm{Soc}(\mathcal{B}) \in \mathbb{C}$; and all vertices $\mathrm{Soc}(\mathcal{B})$ such that $\mathcal{B} \neq \mathrm{Soc}(\mathcal{B})$ and $\mathrm{Soc}(\mathcal{B}) \notin \mathbb{C}$.

Proposition 7.3. Let $\mathbb{P}_c(\mathcal{A})$ be a $\mathbb{G}$-act. If $\mathbb{C}$ is a controlling set of $Lie(\mathbb{G}\mathcal{A})$, then $\overline{\mathbb{C}}$ is also a controlling set and $O(\overline{\mathbb{C}}) \leq O(\mathbb{C})$.

Proof. Assume that $\mathbb{C}$ is a controlling set of $Lie(\mathbb{G}\mathcal{A})$, and let $\mathcal{B}$ be an arbitrary vertex. If $\mathcal{B}$ is non-simple and contains a simple subact, then either $\mathcal{B} \notin \mathbb{C}$, in which case there exists $C \in \mathbb{C}$ with $\mathcal{B} - C$ and hence $\mathcal{B} - \mathrm{Soc}(C)$, or $\mathcal{B} \in \mathbb{C}$, in which case one of the defining conditions of $\overline{\mathbb{C}}$ applies. In all cases, $\mathcal{B}$ is adjacent to some vertex of $\overline{\mathbb{C}}$, and thus $\overline{\mathbb{C}}$ is controlling.

Define a map $h : \mathbb{C} \to \overline{\mathbb{C}}$ by $h(\mathcal{D}) = \mathcal{D}$ if $\mathcal{D} = \mathrm{Soc}(\mathcal{D})$ or $\mathrm{Soc}(\mathcal{D}) \in \mathbb{C}$, and $h(\mathcal{D}) = \mathrm{Soc}(\mathcal{D})$ otherwise. The map h is surjective, which implies $O(\overline{\mathbb{C}}) \leq O(\mathbb{C})$. □

Proposition 7.4. Let $\mathbb{P}_c(\mathcal{A})$ be a non-simple $\mathbb{G}$-act, and let $\mathbb{C}$ be an indecomposable controlling set of $Lie(\mathbb{G}\mathcal{A})$. Then $\mathbb{P}_c(\mathcal{A})$ and $\mathrm{Soc}(\mathcal{A})$ cannot both belong to $\mathbb{C}$. Moreover, if $\mathbb{P}_c(\mathcal{A}) \notin \mathbb{C}$, then $\mathbb{C}$ must contain at least one simple subact of $\mathbb{P}_c(\mathcal{A})$.

Proof. If $\mathbb{P}_c(\mathcal{A}) \neq \mathrm{Soc}(\mathcal{A})$ and both belong to $\mathbb{C}$, then $\mathbb{C}$ is decomposable, contradicting the assumption. The final assertion follows from the fact that $\mathbb{P}_c(\mathcal{A})$ is adjacent only to simple subacts. □

Example 7.5. Let $\mathbb{P}_c(\mathcal{A}) = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \mathcal{A}_3 \sqcup \mathcal{A}_4$ be a semisimple $\mathbb{G}$-act, where $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ are simple subacts. Consider the set $\mathbb{C} = \{\mathcal{A}_1 \sqcup \mathcal{A}_2,\ \mathcal{A}_3 \sqcup \mathcal{A}_4\}$. Then $\mathbb{C}$ is a controlling set of $Lie(\mathbb{G}\mathcal{A})$. Indeed, every vertex containing $\mathcal{A}_1$ or $\mathcal{A}_2$ is adjacent to $\mathcal{A}_1 \sqcup \mathcal{A}_2$ whenever their intersection is one of these simple subacts, and similarly every vertex containing $\mathcal{A}_3$ or $\mathcal{A}_4$ is controlled by $\mathcal{A}_3 \sqcup \mathcal{A}_4$ whenever their intersection is simple. Moreover, $\mathbb{C}$ is indecomposable. If $\mathcal{A}_1 \sqcup \mathcal{A}_2$ is removed, then the simple subacts $\mathcal{A}_1$ and $\mathcal{A}_2$ are not controlled. If $\mathcal{A}_3 \sqcup \mathcal{A}_4$ is removed, then the simple subacts $\mathcal{A}_3$ and $\mathcal{A}_4$ are not controlled. Hence no proper subset of $\mathbb{C}$ is a controlling set, and therefore $\mathbb{C}$ is an indecomposable controlling set.

In $Lie(\mathbb{G}\mathcal{A})$, if $\mathcal{B} - C$, then adjacency is preserved under passing to socles. In particular, $\mathcal{B} - \mathrm{Soc}(C)$, $\mathrm{Soc}(\mathcal{B}) - C$, and $\mathrm{Soc}(\mathcal{B}) - \mathrm{Soc}(C)$ all hold whenever the socles are defined. Thus, adjacency relations remain valid after replacing one or both vertices by their socles. Conversely, if $\mathrm{Soc}(\mathcal{B}) - \mathrm{Soc}(C)$, then necessarily $\mathcal{B} - \mathrm{Soc}(C)$ and $\mathrm{Soc}(\mathcal{B}) - C$ hold. However, in this situation $\mathcal{B}$ and C themselves need not be adjacent in $Lie(\mathbb{G}\mathcal{A})$. Motivated by these observations, we show in the next proposition that any controlling set of $Lie(\mathbb{G}\mathcal{A})$ can be replaced by an indecomposable controlling set whose vertices consist of semisimple subacts of $\mathbb{P}_c(\mathcal{A})$, together with the isolated vertices of the graph.

Proposition 7.6. Let $\mathbb{P}_c(\mathcal{A})$ be a $\mathbb{G}$-act, and let $\mathbb{C}$ be a controlling set of $Lie(\mathbb{G}\mathcal{A})$. Then there exists an indecomposable controlling set $\mathcal{P}$ such that $O(\mathcal{P}) \leq O(\mathbb{C})$, where $\mathcal{P}$ consists of semisimple subacts of $\mathbb{P}_c(\mathcal{A})$ together with the isolated vertices of $Lie(\mathbb{G}\mathcal{A})$. Moreover, if $\mathbb{P}_c(\mathcal{A}) \notin \mathcal{P}$, then $\mathcal{P}$ contains at least one simple subact of $\mathbb{P}_c(\mathcal{A})$.

Proof. By Proposition 7.3, we may replace $\mathbb{C}$ by $\overline{\mathbb{C}}$ so that $O(\overline{\mathbb{C}}) \leq O(\mathbb{C})$, and every vertex of $\overline{\mathbb{C}}$ is semisimple, except possibly vertices $\mathcal{B} \neq \mathrm{Soc}(\mathcal{B})$ with $\mathrm{Soc}(\mathcal{B}) \in \overline{\mathbb{C}}$. For each such vertex $\mathcal{B}$, choose a simple subact $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{B}$ and replace $\mathcal{B}$ by $\mathcal{T}_{\mathcal{B}}$. Denote the resulting set by $\mathbb{C}_0$. Then $\mathbb{C}_0$ consists entirely of semisimple subacts and satisfies $O(\mathbb{C}_0) \leq O(\overline{\mathbb{C}})$. Let C be a vertex of $Lie(\mathbb{G}\mathcal{A})$ not contained in $\mathbb{C}_0$. If $C \notin \overline{\mathbb{C}}$, then C is adjacent to some vertex of $\overline{\mathbb{C}}$, and hence to a vertex of $\mathbb{C}_0$ by construction. If $C \in \overline{\mathbb{C}}$, then it was replaced by $\mathcal{T}_C$, and we have $C - \mathcal{T}_C$. Thus every vertex of $Lie(\mathbb{G}\mathcal{A})$ is adjacent to some vertex of $\mathbb{C}_0$, showing that $\mathbb{C}_0$ is a controlling set. Since $\mathbb{C}_0$ may be decomposable, it contains an indecomposable controlling subset $\mathcal{P}$. The final assertion follows from Proposition 7.4. □

Example 7.7. Let $\mathbb{P}_c(\mathcal{A}) = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \mathcal{A}_3 \sqcup \mathcal{A}_4 \sqcup \mathcal{A}_5$ be a semisimple $\mathbb{G}$-act with five simple components. Then the set $\mathcal{P} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4 \sqcup \mathcal{A}_5\}$ is an indecomposable controlling set of $Lie(\mathbb{G}\mathcal{A})$ of least possible cardinality. Hence $\Delta(Lie(\mathbb{G}\mathcal{A})) = 4$.

Proposition 7.8. Let $\mathbb{P}_c(\mathcal{A})$ be a semisimple $\mathbb{G}$-act with n components, where $n > 1$. Then $\Delta(Lie(\mathbb{G}\mathcal{A})) \leq n - 1$.

Proof. Write $\mathbb{P}_c(\mathcal{A}) = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \cdots \sqcup \mathcal{A}_n$. If $n = 2$, then $\{\mathbb{P}_c(\mathcal{A})\}$ is an indecomposable controlling set of $Lie(\mathbb{G}\mathcal{A})$ of minimum cardinality, and hence $\Delta(Lie(\mathbb{G}\mathcal{A})) = 1$. Assume now that $n > 2$, and define $\mathbb{C} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{n-2}, \mathcal{A}_{n-1} \sqcup \mathcal{A}_n\}$. Let $\mathcal{B}$ be a vertex of $Lie(\mathbb{G}\mathcal{A})$ not belonging to $\mathbb{C}$. If $\mathcal{B} = \mathbb{P}_c(\mathcal{A})$ or $\mathcal{B}$ contains one of the components $\mathcal{A}_j$ with $1 \leq j \leq n-2$, then $\mathcal{B}$ is adjacent to $\mathcal{A}_j \in \mathbb{C}$. Otherwise, $\mathcal{B}$ must be equal to either $\mathcal{A}_{n-1}$ or $\mathcal{A}_n$, and in this case $\mathcal{B}$ is adjacent to $\mathcal{A}_{n-1} \sqcup \mathcal{A}_n \in \mathbb{C}$. Thus every vertex outside $\mathbb{C}$ is adjacent to some vertex of $\mathbb{C}$, so $\mathbb{C}$ is an indecomposable controlling set of cardinality $n - 1$. Consequently, $\Delta(Lie(\mathbb{G}\mathcal{A})) \leq n - 1$. □

Theorem 7.9. Let $\mathbb{P}_c(\mathcal{A})$ be a semisimple $\mathbb{G}$-act with n components, where $n > 1$. Then $\Delta(Lie(\mathbb{G}\mathcal{A})) = n - 1$.

Proof. For $n = 2$, the result follows immediately, since $\{\mathbb{P}_c(\mathcal{A})\}$ is an indecomposable controlling set of minimum size, and hence $\Delta(Lie(\mathbb{G}\mathcal{A})) = 1$. Assume that the statement holds for all semisimple $\mathbb{G}$-acts with n components, and let $\mathbb{P}_c(\mathcal{A})$ be semisimple with $n + 1$ components. By Proposition 7.8, we have $\Delta(Lie(\mathbb{G}\mathcal{A})) \leq n$. Suppose, toward a contradiction, that $\Delta(Lie(\mathbb{G}\mathcal{A})) < n$. Then there exists an indecomposable controlling set $\mathbb{C}$ of $Lie(\mathbb{G}\mathcal{A})$ such that $O(\mathbb{C}) < n$. By Proposition 7.3, we may assume that $\mathbb{C}$ contains a simple subact $\mathcal{A}_1$ of $\mathbb{P}_c(\mathcal{A})$. Choose a vertex $\mathcal{B} \notin \mathbb{C}$ such that $\mathcal{A}_1 \nsubseteq \mathcal{B}$. Then $\mathcal{B}$ is not adjacent to $\mathcal{A}_1$. Since $\mathbb{C}$ is indecomposable, there exists $\mathcal{C} \in \mathbb{C}$ such that $\mathcal{B}$ is adjacent to $\mathcal{C}$ through a simple subact $\mathcal{A}_2 \neq \mathcal{A}_1$. Since $\mathbb{P}_c(\mathcal{A})$ is semisimple, every vertex $\mathcal{D} \in \mathbb{C}$ is a coproduct of simple components. For each $\mathcal{D} \in \mathbb{C}$, define

$$\mathcal{D}_0 = \coprod_{\substack{\mathcal{A}_j \subseteq \mathcal{D} \\ j \neq 1}} \mathcal{A}_j.$$

After omitting the empty subact, define

$$\mathbb{C}_0 = \{\mathcal{D}_0 : \mathcal{D} \in \mathbb{C},\ \mathcal{D}_0 \neq \emptyset\}.$$

Thus, $\mathbb{C}_0$ is obtained by deleting the component $\mathcal{A}_1$ from every vertex of $\mathbb{C}$ that contains it. Then $O(\mathbb{C}_0) < O(\mathbb{C}) < n$, and $\mathbb{C}_0$ is a controlling set of $\mathcal{G}_{\mathbb{G}}(\mathcal{A}_0)$, where $\mathcal{A}_0$ is the semisimple $\mathbb{G}$-act obtained from $\mathbb{P}_c(\mathcal{A})$ by deleting the component $\mathcal{A}_1$. The act $\mathcal{A}_0$ has exactly n components. Hence $\Delta(\mathcal{G}_{\mathbb{G}}(\mathcal{A}_0)) < n - 1$, contradicting the induction hypothesis. Therefore, $\Delta(Lie(\mathbb{G}\mathcal{A})) = n - 1$. □

8. Conclusions

In this paper, we introduced the Lie intersection graph $Lie(\mathbb{G}\mathcal{A})$ for continuous actions of Lie groups on topological semigroups and studied how its algebraic and topological structure reflects properties of the underlying $\mathbb{G}$-act. By defining adjacency through intersections that are simple subacts, we investigated classical graph invariants such as connectivity, diameter, girth, bipartiteness, and clique number in terms of simplicity, semisimplicity, and the Artinian property of $\mathbb{P}_c(\mathcal{A})$. In particular, we

showed that the Artinian property of $\mathbb{P}_c(\mathcal{A})$ ensures the connectedness of $Lie(\mathbb{G}\mathcal{A})$. This establishes a relation between the chain condition on subacts and the connectivity of the associated graph. We also characterized bipartite and complete bipartite Lie intersection graphs in terms of essential socles and minimal subacts, and showed that the formation and size of cliques are related to the distribution of simple subacts. In addition to the usual notion of connectivity, we defined chordless paths and the resulting chordless topology on the vertex set of $Lie(\mathbb{G}\mathcal{A})$. We established that this topology is Alexandroff, revealing a natural order-theoretic structure associated with the graph. Thus, the Lie intersection graph provides a combinatorial and topological framework for studying continuous Lie group actions on topological semigroups.

Author contributions

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Use of Generative-AI tools declaration

The authors declare that they have not used artificial intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that there is no conflict of interests.

References

1. I. Chakrabarty, S. Ghosh, T. K. Mukherjee, M. K Sen, Intersection graphs of ideals of rings, *Discrete Math.*, **309** (2009), 5381–5392. https://doi.org/10.1016/j.disc.2008.11.034
2. C. Abdioğlu, N. ur Rehman, S. A. Mir, S. Rawade, Some graph invariants of Co-Intersection graph of ideals of a commutative ring, *Discret. Math. Algorithms Appl.*, 2026, 2650002. https://doi.org/10.1142/S1793830926500023

3. F. H. Damag, A. Saif, A. Kiliçman, E. E. Ali, M. B. Mesmouli, On m-negative sets and out mondirected topologies in the human nervous system, *Mathematics*, **12** (2024), 3763. https://doi.org/10.3390/math12233763
4. F. H. Damag, A. Saif, A. Kiliçman, F. Alhubairah, K. M. Saad, E. E. Ali, et al., Monophonic sets and rough directed topological spaces: Applications with some directed networks, *AIMS Mathematics*, **10** (2025), 17623–17641. https://doi.org/10.3934/math.2025787
5. F. Moh'd, M. Ahmed, Simple-intersection graphs of rings, *AIMS Mathematics*, **8** (2023), 1040–1054. https://doi.org/10.3934/math.2023051
6. J. M. Howie, *Fundamentals of semigroup theory*, Oxford University Press, 1995. https://doi.org/10.1093/oso/9780198511946.001.0001
7. G. Lallement, *Semigroups and combinatorial applications*, New York: Wiley, 1979.
8. P. A. Grillet, *Semigroups: An introduction to the structure theory*, New York: Routledge, 2017. https://doi.org/10.4324/9780203739938
9. M. Kilp, U. Knauer, A. V. Mikhalev, *Monoids, acts and categories*, Walter de Gruyter, 2011.
10. B. Davvaz, Z. Nazemian, Chain conditions on commutative monoids, *Semigroup Forum*, **100** (2020), 732–742. https://doi.org/10.1007/s00233-019-10032-1
11. R. Khosravi, M. Roueentan, Chain conditions on (Rees) congruences of $\mathbb{S}$-acts, *J. Algebra Appl.*, **23** (2024), 2450085. http://doi.org/10.1142/S0219498824500853
12. H. Rasouli, A. Tehranian, Intersection graphs of $\mathbb{S}$-acts, *Bull. Malays. Math. Sci. Soc.*, **38** (2015), 1575–1587. https://doi.org/10.1007/s40840-014-0098-5
13. A. Delfan, H. Rasouli, A. Tehranian, Intersection graphs associated with semigroup acts, *Categ. Gen. Algebr. Struct. Appl.*, **11** (2019), 131–148. https://doi.org/10.29252/cgasa.11.1.131
14. X. Liang, Y. Wang, X. Chen, Simple-intersection graphs of $\mathbb{S}$-acts, *Commun. Comb. Optim.*, 2025, 1–22. https://doi.org/10.22049/cco.2025.30632.2558
15. B. Csakany, G. Pollak, The graph of subgroups of a finite group, *Czech. Math. J.*, **19** (1969), 241–247. https://doi.org/10.21136/cmj.1969.100891
16. B. Zelinka, Intersection graphs of finite abelian groups, *Czech. Math. J.*, **25** (1975), 171–174. https://doi.org/10.21136/CMJ.1975.101307
17. R. Shen, Intersection graphs of subgroups of finite groups, *Czech. Math. J.*, **60** (2010), 945–950. https://doi.org/10.1007/s10587-010-0085-4
18. R. Rajkumar, P. Devi, Intersection graphs of cyclic subgroups of groups, *Electron. Notes Discrete Math.*, **53** (2016), 15–24. https://doi.org/10.1016/j.endm.2016.05.003
19. S. Kayacan, Connectivity of intersection graphs of finite groups, *Commun. Algebra*, **46** (2018), 1492–1505. https://doi.org/10.1080/00927872.2017.1347662
20. A. Beheshtipour, S. M. J. Amiri, The clique number of the intersection graph of a finite group, *Bull. Iran. Math. Soc.*, **49** (2023), 74. https://doi.org/10.1007/s41980-023-00804-5

21. H. Alzubaidi, On the *P*-intersection graphs of acts over semigroups: Applications to connectivity and monophonic convexity with clique structure, *AIMS Mathematics*, **11** (2026), 20619–20644. https://doi.org/10.3934/math.2026839

22. M. Ahmed, F. Moh'd, A new intersection-graph type for modules, *Commun. Algebra*, **52** (2024), 2065–2078. https://doi.org/10.1080/00927872.2023.2280716

23. E. H. Spanier, *Algebraic topology*, New York: Springer, 1966. https://doi.org/10.1007/978-1-4684-9322-1



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