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Research article

Regular linear forms of a monomial ideal

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Abstract: Let I be a monomial ideal of a polynomial ring S over a field k. We study the regular elements and zero divisors of S/I, along with the associated primes of S/I and its quotients by a regular linear form. Using the concept of a vertex cover of a hypergraph, we provide a method that produces a regular linear form on S/I from an arbitrary regular element of S/I in the maximal ideal of S. For each associated prime P of S/I, we exhibit an explicit monomial $\mathbf{m}$, expressed in terms of a maximal cover of the exponent vectors of the minimal generators of I, that annihilates P (that is, $P\mathbf{m} \subset I$). Then, we determine which sets of powers of variables can be added to I to force depth zero in the quotient, leading to a notion of stability that generalizes the classical notion of a stable set in a graph and that is governed entirely by the maximal covers of the exponent vectors of I. Finally, when I is the edge ideal of a simple graph G and r is a regular linear form on S/I, for each minimal vertex cover C of G, we characterize when the product of the variables outside C is a nonzero element of $S/\langle I, r\rangle$ whose annihilator is the associated prime $\langle C, r\rangle$.

Keywords: regular element; zero divisor; linear form; associated prime; maximal cover; stable vector; depth; monomial ideal
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1. Introduction

Let $S := k[\mathbf{x}]$ be a polynomial ring over a field k in the variables $\mathbf{x} := \{x_1, \ldots, x_n\}$, and let I be a monomial ideal of S. We study regular elements and zero divisors in the quotient S-module $M := S/I$ and how the associated primes of S/I_G are affected when a regular linear form r is adjoined to the edge ideal I_G of a graph G (Definition 2.17). Many algebraic properties of this quotient module stem from the combinatorial structure of I, an interplay between commutative algebra and combinatorics

that traces back to the Stanley–Reisner correspondence; see [1–3] and the references therein.

An element r in the maximal ideal $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$ of S is *regular* on M if, whenever $q \in S$ and $rq \in I$, it follows that $q \in I$. The element r is regular on M if and only if

$$I = (I \colon r),$$

where $(I \colon r) := \{g \in S \mid gr \in I\}$ is a *colon ideal*. If r is not regular on M, then it is called a *zero divisor* of M. Note that verifying whether r is regular on M is an instance of the ideal membership problem.

For convenience, we use the following multi-index notation to denote the monomials of S:

$$\mathbf{x}^{\mathbf{g}} := x_1^{g_1} \cdots x_n^{g_n} \text{ for } \mathbf{g} = (g_1, \ldots, g_n) \in \mathbb{N}^n.$$

By Dickson's lemma [3, Lemma 3.3.3], any monomial ideal I is minimally generated by a set of monomials $\{\mathbf{x}^{\mathbf{g}} : \mathbf{g} \in \mathbf{G}\}$, where $\mathbf{G}$ is a (finite) antichain in the poset $(\mathbb{N}^n, \leq)$, with the usual componentwise order on $\mathbb{N}^n$. The monomial ideal $I_{\mathbf{G}}$, given by an antichain $\mathbf{G} \subset \mathbb{N}^n$, is the ideal of S generated by all monomials $\mathbf{x}^{\mathbf{g}}$ with $\mathbf{g} \in \mathbf{G}$. This ideal is also called the *antichain ideal* of $\mathbf{G}$ [4]. This gives a one-to-one correspondence between the antichains of $\mathbb{N}^n$ and the monomial ideals of S, which allows us to interpret algebraic properties of $S/I_{\mathbf{G}}$ and $I_{\mathbf{G}}$ using the combinatorics of $\mathbf{G}$. For simplicity, we assume that every variable in S appears in at least one minimal generator of $I_{\mathbf{G}}$.

A prime ideal P of S is an *associated prime* of S/I if $P = (I \colon \mathbf{x}^{\mathbf{a}})$ for some monomial $\mathbf{x}^{\mathbf{a}} \in S$. We denote the set of associated primes of S/I by $\operatorname{Ass}(S/I)$. By Lemma 2.1, an element $r \in \mathfrak{m}$ is regular on M if and only if $r \notin P$ for every $P \in \operatorname{Ass}(S/I)$. An ideal of S is called *irreducible* if it cannot be written as an intersection of two ideals of S that properly contain the ideal. A monomial ideal I has a *unique irreducible decomposition*,

$$I = I_1 \cap \cdots \cap I_m,$$

such that each I_i is an irreducible monomial ideal of the form $I_i = \langle \{x_k^{b_k} | \, b_k \geq 1\} \rangle$ for some $\mathbf{b} = (b_1, \ldots, b_n)$ in $\mathbb{N}^n \setminus \{0\}$ and $I \neq \bigcap_{i \neq j} I_i$ for $j = 1, \ldots, m$ [3, Proposition 6.1.25]. The ideals $I_1, \ldots, I_m$ are the irreducible components of I, and their radicals are the associated primes of S/I. A monomial $\mathbf{x}^{\mathbf{c}}$ *annihilates* an associated prime ideal P of S/I if $P \subset (I \colon \mathbf{x}^{\mathbf{c}})$ and $\mathbf{x}^{\mathbf{c}} \notin I$. If S/I has no embedded associated primes and $\mathbf{x}^{\mathbf{c}}$ annihilates P, then $P = (I \colon \mathbf{x}^{\mathbf{c}})$.

The central combinatorial object of this paper is the *maximal ∞-cover* of an antichain (*maximal cover* for short) [4, Subsection 2.2] (see Definition 2.5). A *cover* of an antichain $\mathbf{G}$ is a vector $\mathbf{c} \in \mathbb{N}_\infty^n$, where $\mathbb{N}_\infty = (\mathbb{N} \setminus \{0\}) \cup \{\infty\}$, such that every $\mathbf{g} \in \mathbf{G}$ satisfies $c_j \leq g_j$ for some index j, and it is *maximal* if it is maximal for the componentwise order. In our previous work [4, Theorem 2.12], we showed that the one-to-one correspondence:

$$\begin{aligned} &\mathbb{N}^n \setminus \{0\} \to \{\text{irreducible monomial ideals of } S\} \\ &\mathbf{b} = (b_1, \ldots, b_n) \to \mathfrak{m}^{\mathbf{b}} := \langle \{x_i^{b_i} | \, b_i \geq 1\} \rangle, \end{aligned}$$

induces, after writing ∞ instead of 0 for an absent exponent, a bijection $\mathbf{c} \mapsto \mathfrak{m}^{\mathbf{c}} = \langle x_i^{c_i} : c_i < \infty \rangle$ between the maximal covers of $\mathbf{G}$ and the irreducible components of $I_{\mathbf{G}}$, and therefore one recovers the associated primes of $S/I_{\mathbf{G}}$. Another way to describe the irreducible components of $I_{\mathbf{G}}$ appears in [5, Proposition 1], using the poset of monomial ideals under inclusion. If $I_{\mathcal{H}}$ is the edge ideal generated by the monomials that correspond to the edges of a hypergraph $\mathcal{H}$, then the minimal vertex

covers of $\mathcal{H}$ in the classical sense of the hypergraph theory are precisely the minimal generating sets of the irreducible components of $I_{\mathcal{H}}$ [3, Section 6.3], that is, the notion of a maximal cover of an antichain generalizes the concept of a minimal vertex cover of a hypergraph.

The present paper shows that maximal covers carry more information than the bijection above records; additionally, they produce explicit annihilators for the associated primes of $S/I_{\mathbf{G}}$, and they decide which powers of variables force depth zero in the quotient. The following are our main results.

(i) If I is a monomial ideal of S and $\operatorname{depth}(S/I) > 0$, then by Lemma 2.1 and Corollary 3.1, the maximal ideal $\mathfrak{m}$ is not an associated prime of S/I and $x_1+\cdots+x_n$ is regular on S/I. For monomial ideals, we obtain more than existence: associating a hypergraph to an arbitrary regular element q on S/I, every vertex cover of that hypergraph yields a regular linear form on S/I (Theorem 2.6).

(ii) Each associated prime of $S/I_{\mathbf{G}}$ is annihilated by a monomial explicitly constructed from the corresponding maximal cover (Theorem 2.15), which upgrades the correspondence above to an explicit list of annihilators. The proof rests on a property of maximal covers given in Lemma 2.14.

(iii) Then, we address a natural question: which sets of powers of variables $P^{\mathbf{a}} = \{x_i^{a_i}\}$ must be added to $I_{\mathbf{G}}$ to force depth zero in the quotient? The answer leads to a notion of stability for vectors of $\mathbb{N}_\infty^n$ relative to an antichain of $\mathbb{N}^n$ (Definition 3.5) which only depends on the support of the vector and generalizes the classical notion of a stable set in a graph. Every maximal cover $\mathbf{c}$ produces a canonical stable vector $\overline{\mathbf{c}}$ such that $S/(I_{\mathbf{G}} + \mathfrak{m}^{\mathbf{d}})$ has depth zero whenever $\mathbf{d} \geq \overline{\mathbf{c}} + \mathbf{1}$ and $\operatorname{supp}(\mathbf{d}) = \operatorname{supp}(\overline{\mathbf{c}})$ (Theorem 3.6), and conversely, a vector $\mathbf{a} \geq \operatorname{lcm}(\mathbf{G})$ is stable for $\mathbf{G}$ if and only if $\mathbf{a} \geq \overline{\mathbf{c}}$ for some maximal cover $\mathbf{c}$ (Theorem 3.9). Therefore, maximal covers account for all stable vectors.

(iv) Finally, we ask how the associated primes of S/I_G are affected when a regular linear form r is adjoined to the edge ideal I_G of a simple graph G. Let C be a minimal vertex cover of G, let $P = \langle C \rangle$, and let $\mathbf{m} = \prod_{x_i \notin C} x_i$. A standard depth argument shows that $\langle P, r \rangle$ is again an associated prime of $S/\langle I_G, r \rangle$ (Remark 3.14); we look for an explicit witness. We give a combinatorial criterion in terms of the neighborhoods of the maximal stable set $V(G) \setminus C$ for $\mathbf{m}$ to remain nonzero in $S/\langle I_G, r \rangle$, and in that case, $\langle P, r \rangle$ is the annihilator of $\mathbf{m}$ (Theorem 3.13). Example 3.15 shows that $\mathbf{m}$ may vanish and that not every associated prime of $S/\langle I_G, r \rangle$ arises in this way.

Regular elements and regular sequences on quotients by monomial ideals have been approached from several angles. Initially regular sequences, that is, sequences that are regular on the quotient by an initial monomial ideal, were introduced in [6], and the analysis was extended to squares of monomial ideals in [7]. Systems of parameters and the Cohen–Macaulay property were studied in [8], while [9] constructed regular sequences of linear binomials on edge ideals using regular edges. More recently, and after this paper was submitted, it was shown in [10] that pairwise separated stars of degree one form a regular sequence on the quotient by a monomial ideal over an algebraically closed field; that work also described the minimal primes of the ideal obtained by adjoining linear forms with pairwise disjoint supports to a monomial ideal, and studied the associated primes when the linear forms are stars on an edge ideal. The relation with item (iv) is described at the end of Subsection 3.2. For the general theory of regular sequences and depth, we refer to [11], [12, Part III], and [3, Chapter 3]. The point of view adopted here is complementary: we neither fix an initial ideal nor restrict to edge ideals, but read regular linear forms and annihilators of associated primes directly off the maximal covers of the

antichain $\mathbf{G}$, which also underlie the signature invariants studied in [4, 13]. The associated primes of $S/I_{\mathbf{G}}$ can also be recovered from those of $S/\operatorname{sgn}(I_{\mathbf{G}})$, where $\operatorname{sgn}(I_{\mathbf{G}})$ denotes the signature of $I_{\mathbf{G}}$ [13], a fine invariant that retains essential algebraic information about $I_{\mathbf{G}}$.

For the reader's convenience, we list the notations used throughout the paper.

Symbol	Meaning
$S = k[\mathbf{x}]$	polynomial ring over k in $\mathbf{x} = \{x_1, \dots, x_n\}$
$\mathfrak{m}$	the maximal ideal $\langle x_1, \dots, x_n\rangle$ of S
$\mathbf{x}^{\mathbf{g}}$	the monomial $x_1^{g_1} \cdots x_n^{g_n}$, for $\mathbf{g} \in \mathbb{N}^n$
$\mathbf{m}$	a monomial of S
p, $\mathfrak{p}$	a polynomial $p = \sum_{i=1}^{s} a_i p_i$ of S and its coset $p + I$ in M
$\mathsf{s}(C)$	the sum of the variables in a nonempty set $C \subseteq \mathbf{x}$
$\mathbf{G}$	a finite antichain of $(\mathbb{N}^n, \leq)$
$I_{\mathbf{G}}$	the antichain ideal $\langle \mathbf{x}^{\mathbf{g}} \mid \mathbf{g} \in \mathbf{G}\rangle$
M, M'	the quotient modules S/I and $S/\langle I, r\rangle$
$\mathbb{N}_\infty$	the set $(\mathbb{N} \setminus \{0\}) \cup \{\infty\}$, ordered by $a \leq \infty$
$\mathbf{c}$	a cover of $\mathbf{G}$, that is, $\mathbf{c} \in \mathbb{N}_\infty^n$ such that every $\mathbf{g} \in \mathbf{G}$ admits an index j with $c_j \leq g_j$
$\operatorname{supp}(\mathbf{c})$	the set of indices i with $c_i \neq \infty$
A^c	the complement $\{1, \dots, n\} \setminus A$ of a set of indices A
$\mathfrak{m}^{\mathbf{c}}$	the irreducible monomial ideal $\langle \{x_i^{c_i} \mid c_i < \infty\}\rangle$
$\sqrt{\mathbf{c}}$	the prime ideal generated by the variables indexed by $\operatorname{supp}(\mathbf{c})$
$\mathbf{b}_{\infty\to 0}$	the vector of $\mathbb{N}^n$ obtained from $\mathbf{b}$ by replacing each ∞ by 0
G, I_G	a graph with vertex set $\mathbf{x}$ and its edge ideal (Subsection 2.4)
$N_G(W)$	the set of vertices adjacent to some vertex of $W \subseteq \mathbf{x}$

2. Regular elements and zero divisors modulo a monomial ideal

To avoid repetition, we continue to use the notation and terminology introduced in Section 1. Let $I = I_{\mathbf{G}}$ be a monomial ideal of S, and let $M = S/I$.

2.1. Zero divisors and maximal covers of antichains

Recall that a prime ideal P of a ring S is an associated prime of an S-module M if $P = \operatorname{Ann}_S(\mathfrak{p}) = \{r \in S : r\mathfrak{p} = 0_M\}$ for some nonzero $\mathfrak{p} \in M$. The following algebraic characterization of the zero divisors of a module in terms of its associated primes is well known; for instance, see [3, Lemma 2.1.19].

Lemma 2.1. *If M is an S-module and $Z(M)$ is the set of zero divisors of M, then*

$$Z(M) = \bigcup_{P \in \operatorname{Ass}_S(M)} P\,,$$

where $\operatorname{Ass}_S(M)$ is the set of associated primes of M.

Any associated prime P of M is generated by a subset of the variables $\mathbf{x}$, and there is a monomial $\mathbf{m}$ in S such that $P = (I\colon \mathbf{m})$. Moreover, when I is a square-free monomial ideal, its associated primes correspond to the minimal vertex covers of the associated hypergraph; for instance, see [14, page 279].

For convenience, we recall how we represent the cosets of M. Let $\mathrm{Mon}(S)$ and $\mathrm{Mon}(I)$ be the sets of monomials in S and I, respectively, and let $\overline{\mathrm{Mon}}(M)$ be the cosets in M of $\mathrm{Mon}(M) := \mathrm{Mon}(S)\backslash\mathrm{Mon}(I)$. We call $\overline{\mathrm{Mon}}(M)$ the set of monomials of M. To simplify the notation, we denote the coset $p + I \in M$ of an element p in S by the same letter, but in Fraktur (German Gothic) type, that is, $\mathfrak{p} = p + I$. Then, each nonzero element $\mathfrak{p}$ in M is the coset of a polynomial p of the following form:

$$p = \sum_{i=1}^{s} a_i p_i,$$

where $a_i \in k \setminus \{0\}$ and $p_1, \ldots, p_s$ are distinct elements of $\mathrm{Mon}(M)$. We use this representation for cosets of M.

Remark 2.2. If $p = \sum_{i=1}^{s} a_i p_i$ is a polynomial in S with $a_i \in k \setminus \{0\}$ for all $1 \leq i \leq s$ and distinct monomials $p_1, \ldots, p_s$, then $\mathfrak{p} = 0_M$ if and only if $p_i \in I$ for all $1 \leq i \leq s$.

Proposition 2.3. *An element p in S that is nonzero in M is a zero divisor of M if and only if there exists $\mathfrak{q} \in \overline{\mathrm{Mon}}(M)$ such that $p\mathfrak{q} = 0_M$.*

Proof. This follows from Lemma 2.1, since the associated primes of M have the form $(I\colon \mathbf{x}^{\mathbf{a}})$ for some $\mathbf{x}^{\mathbf{a}} \in S$. □

Example 2.4. When I is not a monomial ideal, Proposition 2.3 does not hold. For instance, if $I = \langle x^2 - y^2 \rangle$ is an ideal in $S = k[x, y, z]$ and $M = S/I$ is its quotient S-module, then $p = x + y$ and $q = x - y$ are zero divisors of M. However, $p\mathfrak{e} \neq 0_M$ and $q\mathfrak{e} \neq 0_M$ hold for any monomial $\mathfrak{e}$ of M.

2.1.1. Maximal covers of antichains

We begin by recalling the concept of a *maximal cover* of an antichain. Every explicit construction in the following sections, annihilating monomials and depth-zero criteria alike, is built from a maximal cover.

Here, $\mathbb{N}_\infty = (\mathbb{N} \setminus \{0\}) \cup \{\infty\}$ denotes the extended positive natural numbers, ordered by the usual order on $\mathbb{N}$, with the convention that $a \leq \infty$ for every $a \in \mathbb{N} \setminus \{0\}$. Similarly, let $\mathbb{N}_\infty^n$ be equipped with the componentwise order induced by that on $\mathbb{N}_\infty$.

Definition 2.5. A vector $\mathbf{c} \in \mathbb{N}_\infty^n$ is a cover of an antichain $\mathbf{G}$ whenever, for each $\mathbf{g} \in \mathbf{G}$, there exists $1 \leq j \leq n$ such that $c_j \leq g_j$. A cover $\mathbf{c}$ is a maximal cover if it is maximal with respect to this order. We denote the set of maximal covers of $\mathbf{G}$ by $\mathcal{T}(\mathbf{G})$.

As usual, the *support* of a vector $\mathbf{a} \in \mathbb{N}^n$, denoted $\mathrm{supp}(\mathbf{a})$, is the set of indices i such that $a_i > 0$, and the *support* of a monomial $\mathbf{x}^{\mathbf{a}}$, denoted $\mathrm{supp}(\mathbf{x}^{\mathbf{a}})$, is the set of variables whose index is in the support of $\mathbf{a}$. For convenience, we identify the support of the vector $\mathbf{a}$ with that of the monomial $\mathbf{x}^{\mathbf{a}}$. Analogously, the support of a vector $\mathbf{c} \in \mathbb{N}_\infty^n$ is the set of indices i such that $c_i \neq \infty$. A *vertex cover* of a hypergraph $\mathcal{H}$ is a set of vertices whose intersection with every edge of $\mathcal{H}$ is nonempty. If $I_{\mathbf{H}} = I_{\mathcal{H}}$ is the edge ideal of $\mathcal{H}$ whose corresponding antichain is $\mathbf{H}$, then a minimal vertex cover C of $\mathcal{H}$ corresponds to

the maximal cover $\mathbf{c} \in \{1, \infty\}^n$ of $\mathbf{H}$ whose support is C, that is, we set the entry c_i of $\mathbf{c}$ equal to ∞ whenever $x_i \notin C$, and $c_i = 1$ otherwise.

With this combinatorial vocabulary in place, we turn to its algebraic payoff: we associate to each element of M a hypergraph on the variables, whose covers directly translate into linear regular elements and zero divisors of M. Every explicit construction that follows is read off a maximal cover: the annihilators of Theorem 2.15, the depth-zero criteria of Theorems 3.6 and 3.9, and the associated primes of Theorem 3.13.

2.2. Vertex covers, zero divisors and regular elements of M

Given $\mathfrak{q} = \sum_{i=1}^{s} a_i q_i$ in M, let $\mathcal{H}_{\mathfrak{q}}$ be the hypergraph with the set of variables $\mathbf{x}$ as its vertex set and the supports of the monomials q_i in $\mathfrak{q}$ as its edge set. Given a nonempty set of variables C, let $\mathsf{s}(C)$ be the sum of the variables in C. As the following result shows, when q is regular on M, each cover of $\mathcal{H}_{\mathfrak{q}}$ defines a linear regular element of M.

Theorem 2.6. *Let I be a monomial ideal of S and $M = S/I$. If $q \in \mathfrak{m}$ is a regular element of M and C is a cover of $\mathcal{H}_{\mathfrak{q}}$, then $\mathsf{s}(C)$ is a regular element of M.*

Proof. Let $\mathfrak{q} = \sum_{i=1}^{s} a_i q_i$, $C = \{x_{j_1}, \ldots, x_{j_t}\}$, and $\mathsf{s}(C) = x_{j_1} + \cdots + x_{j_t}$. We argue by contradiction, assuming that $\mathsf{s}(C)$ is not regular in M. In other words, $\mathsf{s}(C)$ is a zero divisor of M. Then, by Lemma 2.1, there exists an associated prime P of M such that $\mathsf{s}(C) \in P$. Hence, there is a monomial $\mathbf{m} \in S \setminus I$ such that $(I\colon \mathbf{m}) = P$. As P is generated by a subset of variables in $\mathbf{x}$, one has that $x_{j_1}, \ldots, x_{j_t} \in P$, and consequently $x_{j_1}\mathbf{m}, \ldots, x_{j_t}\mathbf{m} \in I$. Since C is a vertex cover of the hypergraph $\mathcal{H}_{\mathfrak{q}}$, it follows that $q_1, \ldots, q_s \in \langle x_{j_1}, \ldots, x_{j_t}\rangle$. Hence, $\mathbf{m}q_i \in I$ for $i = 1, \ldots, s$. Therefore,

$$q\mathbf{m} = \Big(\sum_{i=1}^{s} a_i q_i\Big)\mathbf{m} = \sum_{i=1}^{s} a_i \mathbf{m} q_i \in I,$$

that is, $\mathfrak{q}\mathbf{m} = 0_M$, which is a contradiction since q is regular on M. □

A dual version of Theorem 2.6 for zero divisors is not valid, that is, if q is a zero divisor of M and C is a cover of $\mathcal{H}_{\mathfrak{q}}$, then $\mathsf{s}(C)$ is not necessarily a zero divisor, as shown by the following example.

Example 2.7. Let $I = \langle xy, yz\rangle$ be a monomial ideal in the polynomial ring $S = k[x, y, z]$, and let $q = x^2z + xz^2$. Since $qy = x^2zy + xz^2y = (xy)xz + (xy)z^2 = 0_M$, q is a zero divisor of M, but $C = \{x, y\}$ is a cover of $\mathcal{H}_{\mathfrak{q}}$, while $\mathsf{s}(C) = x + y$ is not a zero divisor in M.

Nevertheless, every zero divisor q of M yields a linear zero divisor of the form $\mathsf{s}(C)$, with C a cover of $\mathcal{H}_{\mathfrak{q}}$.

Proposition 2.8. *If q is a zero divisor of M, then there exists a cover C of $\mathcal{H}_{\mathfrak{q}}$ such that $\mathsf{s}(C)$ is a zero divisor of M.*

Proof. By Lemma 2.1, there is an associated prime P of S/I and a monomial $\mathbf{m} \in S$ such that $P = (I\colon \mathbf{m})$ and $q \in P$. Since P is generated by variables, one has $P = \langle x_{j_1}, \ldots, x_{j_t}\rangle$. Writing $q = \sum_{i=1}^{s} a_i q_i$, $a_i \in k \setminus \{0\}$, and $q = \sum_{i=1}^{t} x_{j_i} b_i$, $b_i \in S$, it follows that $C := \{x_{j_1}, \ldots, x_{j_t}\}$ is a cover of $\mathcal{H}_{\mathfrak{q}}$ and $\mathsf{s}(C)$ is a zero divisor of M. □

Remark 2.9. (a) $p = \sum_{i=1}^{s} x_i$ is a zero divisor (regular element) of M if and only if $p = \sum_{i=1}^{s} a_i x_i$ is a zero divisor (regular element) of M for any $a_i \in k \setminus \{0\}$; see Remark 2.2.

(b) The hypothesis $a_i \neq 0$ in (a) cannot be dropped, and regular sequences of linear forms with integer coefficients, including those of length one, may depend on the field characteristic because some coefficients may vanish (Example 2.10).

Example 2.10. Let ℓ be a prime number, $S = k[x, y]$ be a polynomial ring over a field k, and let $I = \langle xy \rangle$. Then, $x - \ell y$ is regular on S/I if $\mathrm{char}(k) \neq \ell$ and is a zero divisor of S/I if $\mathrm{char}(k) = \ell$.

For this reason, in what follows, we mean a *linear form* to be one whose coefficients are all 1 and do not vanish in any characteristic, that is, each linear form p is equal to $\mathsf{s}(C)$ for some $C \subset \mathbf{x}$.

Corollary 2.11. *Let $C \subset D$ be subsets of the set of variables $\mathbf{x}$. If $\mathsf{s}(D)$ is a zero divisor of M, then $\mathsf{s}(C)$ is also a zero divisor of M, that is,*

$$\Delta(M) = \{C \subset \mathbf{x} : \mathsf{s}(C) \textit{ is a zero divisor of } M\} \cup \{\emptyset\}$$

is a simplicial complex on $\mathbf{x}$.

Proof. By Proposition 2.3, $\mathsf{s}(D)$ is a zero divisor of M if and only if there is a monomial $\mathfrak{q} \in \overline{\mathrm{Mon}}(M)$ with $\mathsf{s}(D)\mathfrak{q} = 0_M$, that is, $x_i\mathfrak{q} = 0_M$ for every $x_i \in D$. Since $C \subseteq D$, the same $\mathfrak{q}$ satisfies $x_i\mathfrak{q} = 0_M$ for every $x_i \in C$, so $\mathsf{s}(C)\mathfrak{q} = 0_M$ and $\mathsf{s}(C)$ is a zero divisor of M. Hence, $\Delta(M)$ is closed under taking subsets, and it is therefore a simplicial complex. □

Corollary 2.12. *Let C and D be subsets of the variables $\mathbf{x}$. If $C \subset D$ and $\mathsf{s}(C)$ is a regular element of M, then $\mathsf{s}(D)$ is a regular element of M.*

Proof. This is the contrapositive of Corollary 2.11: if $\mathsf{s}(D)$ were a zero divisor of M, then $\mathsf{s}(C)$ would be one too, contrary to the hypothesis. □

2.3. Maximal covers annihilate associated primes

In this subsection, we exhibit an explicit monomial, expressed in terms of a maximal cover of the exponent vectors of the minimal generators of I, that annihilates each associated prime ideal; see Theorem 2.15.

Definition 2.13. Given a maximal cover $\mathbf{c}$ of $\mathbf{G}$, let $\sqrt{\mathbf{c}}$ be the prime ideal generated by the variables in the support of $\mathbf{c}$. In other words, $\sqrt{\mathbf{c}}$ is the radical of $\mathfrak{m}^{\mathbf{c}} := \langle \{x_i^{c_i} : c_i < \infty\} \rangle$. Given a set of vectors $\mathbf{G}$ in $\mathbb{N}^n$, let $\mathrm{lcm}(\mathbf{G})_i = \max\{g_i : \mathbf{g} \in \mathbf{G}\} \in \mathbb{N}$.

Before presenting one of our main results, we need the following property.

Lemma 2.14. *Let $\mathbf{c}$ be a maximal cover of an antichain $\mathbf{G}$. If $c_i \neq \infty$, then there exists $\mathbf{g} \in \mathbf{G}$ such that $g_i = c_i$ and $g_j < c_j$ for all $j \neq i$.*

Proof. Let $1 \leq i \leq n$ be such that $c_i \neq \infty$. We proceed by contradiction, assuming that $c_i \neq h_i$ for all $\mathbf{h} \in \mathbf{G}$. We consider two cases: either $c_i > h_i$ for all $\mathbf{h} \in \mathbf{G}$, or there exists $\mathbf{f} \in \mathbf{G}$ such that $c_i < f_i$. First, if $c_i > h_i$ for all $\mathbf{h} \in \mathbf{G}$, consider the vector $\mathbf{d}$ defined by $d_i = \infty$ and $d_j = c_j$ for all $j \neq i$. Since $c_i > h_i$ for every $\mathbf{h} \in \mathbf{G}$, no generator is covered at the index i, so $\mathbf{d}$ is again a cover of $\mathbf{G}$; moreover $\mathbf{d} > \mathbf{c}$, because $d_i = \infty > c_i$. This contradicts the maximality of $\mathbf{c}$.

Consequently, there must exist $\mathbf{f} \in \mathbf{G}$ such that $c_i < f_i$. Let f_i be the smallest such value. Then, no generator $\mathbf{h} \in \mathbf{G}$ has an i-th coordinate in the interval $[c_i, f_i - 1]$. Let $\mathbf{e}_i$ be the canonical vector with 1 in the i-th entry and 0 elsewhere. Since no generator falls in the gap created, $\mathbf{c} + \mathbf{e}_i$ is a cover, which again contradicts the maximality of $\mathbf{c}$. Thus, there exists some $\mathbf{g} \in \mathbf{G}$ such that $c_i = g_i$.

Now, we prove the second condition. Note that if $c_j = \infty$, then $c_j > g_j$, because $g_j \in \mathbb{N}$. Therefore, we focus on the indices where $c_j \neq \infty$. Suppose that for every $\mathbf{g} \in \mathbf{G}$ with $g_i = c_i$, there exists some $j \neq i$ such that $g_j \geq c_j$. Then, $\mathbf{c} + \mathbf{e}_i$ would still be a cover of $\mathbf{G}$: a generator $\mathbf{g}$ with $g_i = c_i$ is covered at an index $j \neq i$ by assumption; one with $g_i > c_i$ is covered at i, since $c_i + 1 \leq g_i$; and one with $g_i < c_i$ is covered by $\mathbf{c}$ at an index $j \neq i$, where $\mathbf{c}$ and $\mathbf{c} + \mathbf{e}_i$ agree. This is impossible by the maximality of $\mathbf{c}$. Therefore, there must exist a $\mathbf{g} \in \mathbf{G}$ such that $g_i = c_i$ and $g_j < c_j$ for all $j \neq i$. □

The next result will be used in the next section to produce quotients of S by monomial ideals that have depth zero by adding powers of variables to a monomial ideal I. The key idea is that, by Lemma 2.14, every coordinate c_i in the support of $\mathbf{c}$ is attained by some generator $\mathbf{g} \in \mathbf{G}$, that is, $g_i = c_i$. Consequently, multiplying $\mathbf{m}$ by the corresponding variable always lands in $I_{\mathbf{G}}$, while multiplying by any variable outside the support of $\mathbf{c}$ never does.

Theorem 2.15. *Let $I_{\mathbf{G}}$ be the antichain ideal of $\mathbf{G}$, $\mathbf{c}$ be a maximal cover of $\mathbf{G}$, and*

$$\mathbf{m} = \left(\prod_{i \in \operatorname{supp}(\mathbf{c})} x_i^{c_i - 1} \right) \cdot \mathbf{m}',$$

where $\operatorname{supp}(\mathbf{m}') = \operatorname{supp}(\mathbf{c})^c$ is the complement of $\operatorname{supp}(\mathbf{c})$ in $\{1, \ldots, n\}$, and $\prod_{i \notin \operatorname{supp}(\mathbf{c})} x_i^{\operatorname{lcm}(\mathbf{G})_i} \mid \mathbf{m}'$. Then, $\mathbf{m}$ is a nonzero monomial in $S/I_{\mathbf{G}}$ that annihilates the associated prime ideal $\sqrt{\mathbf{c}}$ of $S/I_{\mathbf{G}}$. In particular, the annihilator is explicitly given and can be read off directly from the maximal cover $\mathbf{c}$ and the exponents of the generators of $I_{\mathbf{G}}$.

Proof. Let $\mathbf{d} \in \mathbb{N}^n$ be such that $\mathbf{m} = \mathbf{x}^{\mathbf{d}}$. First, $\mathbf{m}$ is nonzero in $S/I_{\mathbf{G}}$; otherwise, there would exist $\mathbf{g} \in \mathbf{G}$ such that $\mathbf{x}^{\mathbf{g}} \mid \mathbf{m}$, which contradicts the fact that $\mathbf{c}$ is a cover of $\mathbf{G}$ because $\mathbf{d} < \mathbf{c}$.

Moreover, by the maximality of $\mathbf{c}$, for any $1 \leq i \leq n$ such that $c_i \neq \infty$, there exists $\mathbf{g} \in \mathbf{G}$ such that $g_i = c_i$ and $g_j < c_j$ for any $j \neq i$, that is, for all $1 \leq i \leq n$ such that $c_i \neq \infty$, $\mathbf{x}^{\mathbf{g}} \mid \mathbf{x}^{\mathbf{d}} \cdot x_i$, and therefore $x_i \mathbf{m} = 0_{S/I_{\mathbf{G}}}$ for all $1 \leq i \leq n$ such that $c_i < \infty$. On the other hand, let i be such that $c_i = \infty$. If $\mathbf{x}^{\mathbf{g}}$ divided $x_i \mathbf{m}$ for some $\mathbf{g} \in \mathbf{G}$, then, as $\mathbf{c}$ is a cover, $c_j \leq g_j$ for some j; this j lies in $\operatorname{supp}(\mathbf{c})$, so $j \neq i$ and $g_j \leq d_j = c_j - 1$, which is a contradiction. Hence, $x_i \mathbf{m} \neq 0_{S/I_{\mathbf{G}}}$ for all $1 \leq i \leq n$ such that $c_i = \infty$. □

Note that any divisor of $\mathbf{m}$ is nonzero in $S/I_{\mathbf{G}}$.

Example 2.16. Let $I_{\mathbf{G}} = \langle x_1^2 x_3, x_2^2, x_1 x_2 x_3^2 \rangle$ be the antichain ideal of

$$\mathbf{G} = \{(2, 0, 1), (0, 2, 0), (1, 1, 2)\}.$$

A direct calculation shows that $(1, 2, \infty)$, $(2, 1, \infty)$, $(\infty, 2, 1)$, and $(2, 2, 2)$ are the maximal covers of $\mathbf{G}$. Taking $\mathbf{c} = (2, 2, 2)$, Theorem 2.15 says that $\mathbf{x}^{\mathbf{c}-\mathbf{1}} = x_1 x_2 x_3$ annihilates the maximal ideal $\langle x_1, x_2, x_3 \rangle$.

2.4. Edge ideals of graphs

So far, the monomial ideal has been an arbitrary antichain ideal $I_{\mathbf{G}}$, described through its antichain $\mathbf{G} \subset \mathbb{N}^n$ and the maximal covers of $\mathbf{G}$. Several results below concern the edge ideal of a graph, for which these notions take a graph-theoretic form; this subsection fixes the terminology and the translation between the two settings. We keep the boldface letter $\mathbf{G}$ for antichains and the italic letter G for graphs.

Definition 2.17. Let G be a graph, possibly with loops, with vertex set $V(G) = \{x_1, \dots, x_n\}$ and edge set $E(G)$. The *edge ideal* I_G of G is the ideal of S generated by all monomials $x_e := x_i x_j$ such that $e = \{x_i, x_j\} \in E(G)$ is an edge of G. If $e = \{x_i, x_i\}$ is a loop, then $x_e = x_i^2$.

A set W of vertices of a graph G is *stable* if no edge of G, including loops, has both endpoints in W, and it is a *vertex cover* of G if it meets every edge of G. Thus, W is a stable set of G if and only if $V(G) \setminus W$ is a vertex cover of G, and the minimal vertex covers of G are the complements of the maximal stable sets of G. The *neighborhood* $N_G(W)$ of a set W of vertices of G is the set of vertices x_i such that $\{x_i, x_j\} \in E(G)$ for some $x_j \in W$.

If G is a simple graph, that is, a graph without loops, then $I_G = I_{\mathbf{G}}$ for the antichain

$$\mathbf{G} = \{\mathbf{e}_i + \mathbf{e}_j : \{x_i, x_j\} \in E(G)\}$$

of $\{0, 1\}^n$, where $\mathbf{e}_i$ denotes the i-th canonical vector. Since, by our standing assumption, every variable appears in some edge, $\operatorname{lcm}(\mathbf{G}) = \mathbf{1}$, and the notions of the previous subsections translate as follows.

(i) The maximal covers of $\mathbf{G}$ are the vectors $\mathbf{c} \in \{1, \infty\}^n$ whose support is a minimal vertex cover C of G (Subsection 2.1.1), and then $\sqrt{\mathbf{c}} = \langle C \rangle$. Hence, the associated primes of S/I_G are the ideals $\langle C \rangle$, where C is a minimal vertex cover of G.

(ii) For such a cover $\mathbf{c}$, taking $\mathbf{m}' = \prod_{x_i \notin C} x_i$ in Theorem 2.15 gives the annihilator $\mathbf{m} = \prod_{x_i \notin C} x_i$ of $\langle C \rangle$, the product of the variables of the maximal stable set $V(G) \setminus C$.

The graph-theoretic form of the stable vectors in Section 3 appears after Definition 3.5.

Remark 2.18. Let G be a graph, and let W be a stable set of G with $N_G(W) \neq \emptyset$. Then, $\prod_{x_j \in W} x_j$ is nonzero in S/I_G and $\langle N_G(W) \rangle \prod_{x_j \in W} x_j \subseteq I_G$; in particular, the linear form $\mathsf{s}(N_G(W))$ is a zero divisor of S/I_G. Indeed, $\prod_{x_j \in W} x_j$ is square-free and no edge has both endpoints in W, so it is not in I_G; if $x_i \in N_G(W)$, with $\{x_i, x_k\} \in E(G)$ and $x_k \in W$, then $x_i x_k$ divides $x_i \prod_{x_j \in W} x_j$.

Example 2.19. Let $I_{C_4} = \langle x_1x_2, x_2x_3, x_3x_4, x_1x_4 \rangle$ be the edge ideal of the four-cycle. Its two maximal covers are $\mathbf{c}_1 = (1, \infty, 1, \infty)$ and $\mathbf{c}_2 = (\infty, 1, \infty, 1)$, corresponding to the minimal vertex covers $\{x_1, x_3\}$ and $\{x_2, x_4\}$. By Theorem 2.15, the monomial x_2x_4 annihilates $\langle x_1, x_3 \rangle$, x_1x_3 annihilates $\langle x_2, x_4 \rangle$, and one has $\operatorname{Ass}(S/I_{C_4}) = \{\langle x_1, x_3 \rangle, \langle x_2, x_4 \rangle\}$.

3. Depth zero and associated primes

In this section, we answer the question: which powers of variables can be added to a monomial ideal to force its quotient S-module to have depth zero? Then, we introduce the notion of stability for vectors in $\mathbb{N}_\infty^n$ relative to an antichain in $\mathbb{N}^n$. In the last subsection, we study how the associated primes of S/I_G change when a regular linear form is adjoined to the edge ideal of a graph G.

Corollary 3.1. *Let I be a monomial ideal of S and $M = S/I$. The quotient S-module M has depth zero if and only if $\mathsf{s}(\mathbf{x})$ is a zero divisor of M.*

Proof. The module M has depth zero if and only if every element of $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$ is a zero divisor of M, that is, by Lemma 2.1, if and only if $\mathfrak{m}$ is contained in the union of the associated primes of M; by prime avoidance, this happens if and only if $\mathfrak{m}$ is an associated prime of M. Since the associated primes of M are generated by variables, $\mathfrak{m}$ is the only one that can contain $\mathsf{s}(\mathbf{x}) = x_1 + \cdots + x_n$; hence, by Lemma 2.1 again, $\mathfrak{m}$ is an associated prime of M if and only if $\mathsf{s}(\mathbf{x})$ is a zero divisor of M. □

Corollary 3.2. *Let G be a graph, possibly with loops, let $\{x_1, \ldots, x_n\}$ be its vertex set, and let I_G be its edge ideal. If G has a vertex x_k with a loop adjacent to every other vertex of G, then $\mathsf{s}(\mathbf{x})$ is a zero divisor in $M = S/I_G$, and M has depth zero.*

Proof. Let x_k be such a vertex. The loop at x_k gives $x_k^2 \in I_G$, and the other edges at x_k give $x_i x_k \in I_G$ for every $i \neq k$, that is, $x_1 x_k, \ldots, x_n x_k \in I_G$. Thus,

$$\mathsf{s}(\mathbf{x}) x_k = (x_1 + \cdots + x_n) x_k = x_1 x_k + \cdots + x_n x_k \in I_G,$$

and the class of $\mathsf{s}(\mathbf{x}) x_k$ in M is 0_M, while the class of x_k is nonzero, because I_G is generated in degree two. Therefore, $\mathsf{s}(\mathbf{x})$ is a zero divisor of M and, by Corollary 3.1, M has depth equal to zero. □

The loop at x_k cannot be omitted. For instance, if $n \geq 2$ and G is the star with center x_1, then $I_G = \langle x_1 \rangle \cap \langle x_2, \ldots, x_n \rangle$, so $\mathfrak{m}$ is not an associated prime of S/I_G and $\operatorname{depth}(S/I_G) > 0$.

For general antichain ideals, a characterization of when S/I has depth zero in terms of its generators was given in [15, Theorem 5.4.4]. We include a new, short combinatorial proof of this fact that only relies on maximal covers, the correspondence of [4, Theorem 2.12] recalled in Section 1, and the results developed so far.

Proposition 3.3 ([15, Theorem 5.4.4])**.** *The quotient $S/I_{\mathbf{G}}$ has depth zero if and only if there is a subset $\{\mathbf{g}^{(1)}, \ldots, \mathbf{g}^{(n)}\} \subseteq \mathbf{G}$ that satisfies the following conditions:*

(i) *$g_i^{(i)} > g_i^{(j)}$ for all $i \neq j$; and*

(ii) *no element of $\mathbf{G}$ is strictly less than $(g_1^{(1)}, g_2^{(2)}, \ldots, g_n^{(n)})$ in every coordinate, that is, there is no $\mathbf{h} \in \mathbf{G}$ with $h_i < g_i^{(i)}$ for all $1 \leq i \leq n$.*

Proof. By Corollary 3.1, $S/I_{\mathbf{G}}$ has depth zero if and only if $\mathsf{s}(\mathbf{x})$ is a zero divisor of $S/I_{\mathbf{G}}$, if and only if the maximal ideal $\langle x_1, \ldots, x_n \rangle$ is an associated prime of $S/I_{\mathbf{G}}$, if and only if there exists a maximal cover $\mathbf{c}$ of $\mathbf{G}$ such that $\sqrt{\mathfrak{m}^{\mathbf{c}}} = \langle x_1, \ldots, x_n \rangle$.

($\Rightarrow$) By the maximality of $\mathbf{c}$ and Lemma 2.14, for every i, choose a generator $\mathbf{g}^{(i)} \in \mathbf{G}$ such that $c_i = g_i^{(i)}$ and $c_j > g_j^{(i)}$ whenever $i \neq j$. Then, for $i \neq j$, $c_i = g_i^{(i)} > g_i^{(j)}$, which is (i). By the definition of cover, no element of $\mathbf{G}$ has coordinates all strictly smaller than the corresponding coordinates of $\mathbf{c}$.

($\Leftarrow$) Consider $\mathbf{a} = (g_1^{(1)}, \ldots, g_n^{(n)})$. We claim that $\mathbf{a}$ is a maximal cover of $\mathbf{G}$. By (*ii*), no $\mathbf{h} \in \mathbf{G}$ satisfies $h_i < a_i$ for every $1 \leq i \leq n$; equivalently, every $\mathbf{h} \in \mathbf{G}$ admits an index j with $a_j \leq h_j$. This is precisely the statement that $\mathbf{a}$ is a cover of $\mathbf{G}$.

To prove maximality, suppose not, and let $\mathbf{b} > \mathbf{a}$ be a cover of $\mathbf{G}$. Choose j with $a_j < b_j$. Then, $b_j > a_j = g_j^{(j)}$, while for every $k \neq j$, condition (i) gives $b_k \geq a_k = g_k^{(k)} > g_k^{(j)}$. Hence, $g_k^{(j)} < b_k$ for every k, so $\mathbf{b}$ does not cover $\mathbf{g}^{(j)}$, which is a contradiction. Hence, $\mathbf{a}$ is a maximal cover, and thus $\sqrt{\mathfrak{m}^{\mathbf{a}}} = \langle x_1, \ldots, x_n \rangle$. □

3.1. Adding powers of variables to a monomial ideal

Definition 3.4. Given $\mathbf{a} \in \mathbb{N}_\infty^n$, its set of powers is the set of monomials $P^{\mathbf{a}}$ such that $\mathfrak{m}^{\mathbf{a}} = \langle P^{\mathbf{a}} \rangle$. In other words, $P^{\mathbf{a}} = \{x_i^{a_i} : 1 \leq i \leq n\}$, where we use the convention that $x^\infty = 0$.

In this subsection, we determine which sets of powers $P^{\mathbf{a}}$ can be added to the antichain ideal $I_{\mathbf{G}}$ so that the resulting S-module has depth zero. We may assume that $\mathbf{a} \geq \operatorname{lcm}(\mathbf{G}) + \mathbf{1}$: if $\mathbf{G}'$ is the set of $\mathbf{g} \in \mathbf{G}$ with $g_i < a_i$ for every i, then $I_{\mathbf{G}} + \mathfrak{m}^{\mathbf{a}} = I_{\mathbf{G}'} + \mathfrak{m}^{\mathbf{a}}$, because every $\mathbf{x}^{\mathbf{g}}$ with $\mathbf{g} \notin \mathbf{G}'$ is a multiple of some $x_i^{a_i}$, and $\mathbf{a} \geq \operatorname{lcm}(\mathbf{G}') + \mathbf{1}$.

Definition 3.5 (Stable vectors)**.** For any $\mathbf{b} \in \mathbb{N}_\infty^n$, let $\mathbf{b}_{\infty\to 0}$ be the vector in $\mathbb{N}^n$ obtained by replacing the ∞ entries of $\mathbf{b}$ with zero. A vector $\mathbf{a} \in \mathbb{N}_\infty^n$ is *stable* for $\mathbf{G}$ whenever $\mathbf{x}^{\mathbf{d}_{\infty\to 0}}$ is not in $I_{\mathbf{G}}$ for all $\mathbf{d} \geq \mathbf{a}$. In other words, $\mathbf{x}^{\mathbf{d}_{\infty\to 0}}$ is nonzero in the quotient ring $S/I_{\mathbf{G}}$.

Since $I_{\mathbf{G}}$ is a monomial ideal, $\mathbf{a}$ is stable for $\mathbf{G}$ if and only if no $\mathbf{g} \in \mathbf{G}$ satisfies $\operatorname{supp}(\mathbf{g}) \subseteq \operatorname{supp}(\mathbf{a})$. Indeed, every $\mathbf{d} \geq \mathbf{a}$ satisfies $\operatorname{supp}(\mathbf{d}_{\infty\to 0}) \subseteq \operatorname{supp}(\mathbf{a})$, so $\mathbf{x}^{\mathbf{g}} \mid \mathbf{x}^{\mathbf{d}_{\infty\to 0}}$ forces $\operatorname{supp}(\mathbf{g}) \subseteq \operatorname{supp}(\mathbf{a})$; conversely, if $\operatorname{supp}(\mathbf{g}) \subseteq \operatorname{supp}(\mathbf{a})$, then the vector $\mathbf{d}$ with $d_i = \max(a_i, g_i)$ for $i \in \operatorname{supp}(\mathbf{a})$ and $d_i = \infty$ otherwise satisfies $\mathbf{d} \geq \mathbf{a}$ and $\mathbf{x}^{\mathbf{g}} \mid \mathbf{x}^{\mathbf{d}_{\infty\to 0}}$. In particular, stability only depends on $\operatorname{supp}(\mathbf{a})$, and it generalizes the concept of a stable set in a graph: if G is a simple graph and $\mathbf{G}$ is the antichain of I_G (Subsection 2.4), then $\mathbf{a}$ is stable for $\mathbf{G}$ if and only if the variables indexed by $\operatorname{supp}(\mathbf{a})$ form a stable set of G.

Theorem 3.6. *Let* $\mathbf{c}$ *be a maximal cover of* $\mathbf{G}$*, and let*

$$\overline{\mathbf{c}}_i = \begin{cases} \operatorname{lcm}(\mathbf{G})_i & \text{if } c_i = \infty, \\ \infty & \text{otherwise}, \end{cases}$$

where $\operatorname{lcm}(\mathbf{G})_i = \max\{g_i : \mathbf{g} \in \mathbf{G}\}$*. Then,* $\overline{\mathbf{c}}$ *is stable for* $\mathbf{G}$ *and the quotient* S*-module*

$$M'_{\mathbf{d}} = S/(I_{\mathbf{G}} + \mathfrak{m}^{\mathbf{d}})$$

has depth equal to zero for all $\mathbf{d} \geq \overline{\mathbf{c}} + \mathbf{1}$ *with* $\operatorname{supp}(\mathbf{d}) = \operatorname{supp}(\overline{\mathbf{c}})$ *(recall that* $\infty + a = \infty$ *for all* $a \in \mathbb{N}$*).*

Proof. The idea of the proof rests on Theorem 2.15: the annihilating monomial $\mathbf{m}$ built there from $\mathbf{c}$ survives the addition of $\mathfrak{m}^{\mathbf{d}}$ once $\mathbf{d}$ is large enough, and it is annihilated by every variable. Consequently, the maximal ideal itself becomes an associated prime.

First, we show that $\overline{\mathbf{c}}$ is stable for $\mathbf{G}$. Let $\mathbf{d} \geq \overline{\mathbf{c}}$, and let $\mathbf{g} \in \mathbf{G}$. Since $\mathbf{c}$ is a cover of $\mathbf{G}$, there is an index j with $c_j \leq g_j$, and such a j lies in $\operatorname{supp}(\mathbf{c})$, so that $\overline{\mathbf{c}}_j = \infty$ and hence $d_j = \infty$. Then, $(\mathbf{d}_{\infty\to 0})_j = 0 < c_j \leq g_j$, so $\mathbf{x}^{\mathbf{g}}$ does not divide $\mathbf{x}^{\mathbf{d}_{\infty\to 0}}$, and $\overline{\mathbf{c}}$ is stable. Now, we prove that $S/(I_{\mathbf{G}} + \mathfrak{m}^{\mathbf{d}})$ has depth zero for all $\mathbf{d} \geq \overline{\mathbf{c}} + \mathbf{1}$ with $\operatorname{supp}(\mathbf{d}) = \operatorname{supp}(\overline{\mathbf{c}})$.

Again, by Theorem 2.15, $\mathbf{m} = \mathbf{x}^{(\mathbf{c}-\mathbf{1})_{\infty\to 0}}\mathbf{x}^{(\mathbf{d}-\mathbf{1})_{\infty\to 0}}$ is nonzero in $S/I_{\mathbf{G}}$. Since $\mathbf{m}$ is not divisible by any power in $P^{\mathbf{d}}$, $\mathbf{m}$ is also nonzero in $M'_{\mathbf{d}}$.

Since $\mathbf{c}$ is a maximal cover for $\mathbf{G}$, by Lemma 2.14, for all $i \in \operatorname{supp}(\mathbf{c})$, there exists $\mathbf{g} \in \mathbf{G}$ such that $g_i = c_i$ and $g_j < c_j$ for every $j \neq i$. Thus, $\mathbf{x}^{\mathbf{g}} \mid \mathbf{m} \cdot x_i$, and therefore $x_i\mathbf{m} = 0_{M'_{\mathbf{d}}}$. On the other hand, if $i \notin \operatorname{supp}(\mathbf{c})$, then $x_i^{d_i} \mid \mathbf{m} \cdot x_i$, and hence $x_i\mathbf{m} = 0_{M'_{\mathbf{d}}}$. Consequently, $\operatorname{Ann}_{M'_{\mathbf{d}}}(\mathbf{m}) = \langle x_1, \ldots, x_n \rangle$ and therefore $M'_{\mathbf{d}}$ has depth zero. □

Remark 3.7. The hypothesis $\operatorname{supp}(\mathbf{d}) = \operatorname{supp}(\overline{\mathbf{c}})$ in Theorem 3.6 cannot be omitted. Let $S = k[x_1, x_2]$ and $\mathbf{G} = \{(1, 1)\}$, so that $I_{\mathbf{G}} = \langle x_1x_2 \rangle$ and $\operatorname{lcm}(\mathbf{G}) = (1, 1)$. The vector $\mathbf{c} = (1, \infty)$ is a maximal

cover of $\mathbf{G}$, whence $\overline{\mathbf{c}} = (\infty, 1)$ and $\overline{\mathbf{c}} + \mathbf{1} = (\infty, 2)$. Taking $\mathbf{d} = (\infty, \infty) \geq \overline{\mathbf{c}} + \mathbf{1}$ gives $\mathfrak{m}^{\mathbf{d}} = \langle 0 \rangle$, so that $M'_{\mathbf{d}} = k[x_1, x_2]/\langle x_1 x_2 \rangle$. Since $\mathrm{Ass}(M'_{\mathbf{d}}) = \{\langle x_1 \rangle, \langle x_2 \rangle\}$, the linear form $\mathsf{s}(\mathbf{x})$ is regular on $M'_{\mathbf{d}}$ and $\mathrm{depth}(M'_{\mathbf{d}}) = 1$.

Example 3.8. Let $\mathbf{G} = \{(2, 0, 1), (0, 2, 0), (1, 1, 2)\}$. Then, $\mathrm{lcm}(\mathbf{G}) = (2, 2, 2)$. The vector $\mathbf{a} = (2, \infty, \infty)$ is stable for $\mathbf{G}$: for every $\mathbf{d} \geq \mathbf{a}$, the monomial $\mathbf{x}^{\mathbf{d}_{\infty \to 0}} = x_1^{d_1}$ is a power of a single variable and hence lies outside $I_{\mathbf{G}}$. Now, consider the maximal cover $\mathbf{c} = (\infty, 2, 1)$, whose support is $\{2, 3\}$. Since $\mathrm{lcm}(\mathbf{G}) = (2, 2, 2)$, we obtain $\overline{\mathbf{c}} = (2, \infty, \infty)$, so Theorem 3.6 applies to every $\mathbf{d} = (d_1, \infty, \infty)$ with $3 \leq d_1 < \infty$; the smallest such choice adds the single power x_1^3 to $I_{\mathbf{G}}$. The associated annihilator is $\mathbf{m} = x_1^2 x_2$, which is nonzero modulo $I_{\mathbf{G}}$ and satisfies $x_1\mathbf{m}, x_2\mathbf{m}, x_3\mathbf{m} \in \langle I_{\mathbf{G}}, x_1^3 \rangle$, so $\langle x_1, x_2, x_3 \rangle = \mathrm{Ann}(\mathbf{m})$ and $S/\langle I_{\mathbf{G}}, x_1^3 \rangle$ has depth zero.

The next result shows that the stable vectors above $\mathrm{lcm}(\mathbf{G})$ are exactly those dominated by some $\overline{\mathbf{c}}$, so that maximal covers account for all of them.

Theorem 3.9. *Let* $\mathbf{G}$ *be a nonempty antichain, and let* $\mathbf{a} \in \mathbb{N}_\infty^n$ *with* $\mathbf{a} \geq \mathrm{lcm}(\mathbf{G})$*. Then,* $\mathbf{a}$ *is stable for* $\mathbf{G}$ *if and only if* $\mathbf{a} \geq \overline{\mathbf{c}}$ *for some maximal cover* $\mathbf{c}$ *of* $\mathbf{G}$*.*

Proof. First, assume that $\mathbf{a}$ is stable and set $A = \mathrm{supp}(\mathbf{a})$. Let $\mathbf{b} \in \mathbb{N}_\infty^n$ be given by $b_i = \infty$ for $i \in A$ and $b_i = 1$ for $i \notin A$. Every $\mathbf{g} \in \mathbf{G}$ has an index $j \in \mathrm{supp}(\mathbf{g}) \setminus A$, and for such a j, one has $b_j = 1 \leq g_j$; hence, $\mathbf{b}$ is a cover of $\mathbf{G}$.

We claim that $\mathbf{b}$ lies below a maximal cover. If $\mathbf{e}$ is a cover with $e_i > \mathrm{lcm}(\mathbf{G})_i$ for some i with $e_i \neq \infty$, then no $\mathbf{g} \in \mathbf{G}$ satisfies $e_i \leq g_i$, so replacing e_i by ∞ again yields a cover, and it is strictly larger. Consequently, every cover is dominated by a cover all of whose finite entries e_i satisfy $e_i \leq \mathrm{lcm}(\mathbf{G})_i$, and there are only finitely many such vectors. Among the finitely many such covers that dominate $\mathbf{b}$ (as guaranteed by the previous sentence), choose a maximal one, $\mathbf{c}$. Then, $\mathbf{c}$ is a maximal cover: a strictly larger cover would be dominated by one of these vectors, which would also dominate $\mathbf{b}$ and be strictly larger than $\mathbf{c}$.

Since $c_i \geq b_i = \infty$ for every $i \in A$, we get $\mathrm{supp}(\mathbf{c}) \subseteq A^c$. Now, let $1 \leq i \leq n$. If $i \in \mathrm{supp}(\mathbf{c})$, then $i \notin A$, so $\overline{\mathbf{c}}_i = \infty = a_i$. If $i \notin \mathrm{supp}(\mathbf{c})$, then $\overline{\mathbf{c}}_i = \mathrm{lcm}(\mathbf{G})_i \leq a_i$ by hypothesis. Hence, $\mathbf{a} \geq \overline{\mathbf{c}}$.

Conversely, assume $\mathbf{a} \geq \overline{\mathbf{c}}$ for some maximal cover $\mathbf{c}$ of $\mathbf{G}$. Then, $\overline{\mathbf{c}}_i = \infty$ for every $i \in \mathrm{supp}(\mathbf{c})$, so $\mathrm{supp}(\mathbf{a}) \subseteq \mathrm{supp}(\overline{\mathbf{c}}) = \mathrm{supp}(\mathbf{c})^c$. Given $\mathbf{g} \in \mathbf{G}$, there is j with $c_j \leq g_j$, and necessarily $j \in \mathrm{supp}(\mathbf{c})$; then, $g_j \geq c_j \geq 1$, so $j \in \mathrm{supp}(\mathbf{g})$ while $j \notin \mathrm{supp}(\mathbf{a})$ and $\mathbf{a}$ is stable for $\mathbf{G}$. □

Corollary 3.10. *If* W *is a maximal stable set of a simple graph* G*, then*

$$\mathrm{depth}(S/\langle I_G, x_i^2 : x_i \in W \rangle) = 0.$$

Proof. Let $\mathbf{G}$ be the antichain of I_G, and let $\mathbf{c} \in \{1, \infty\}^n$ be the maximal cover of $\mathbf{G}$ whose support is the minimal vertex cover $V(G) \setminus W$ (Subsection 2.4). Since $\mathrm{lcm}(\mathbf{G}) = \mathbf{1}$, the vector $\overline{\mathbf{c}}$ has entry 1 at the indices of the variables of W and ∞ elsewhere. For $\mathbf{d} = \overline{\mathbf{c}} + \mathbf{1}$, one has $\mathrm{supp}(\mathbf{d}) = \mathrm{supp}(\overline{\mathbf{c}})$ and $\mathfrak{m}^{\mathbf{d}} = \langle x_i^2 : x_i \in W \rangle$, so the claim follows from Theorem 3.6. □

Adding the set of powers $P^{\mathbf{a}} = \{x_i^{a_i} : 1 \leq i \leq n\}$ to a monomial ideal forces its maximal covers to include the indices i in their support, and therefore the variables x_i in their associated primes. Observe that the set of covers of an antichain is closed under passing to smaller vectors, so a cover $\mathbf{c}$ is maximal if and only if $\mathbf{c} + \mathbf{e}_i$ fails to be a cover for every $i \in \mathrm{supp}(\mathbf{c})$.

Theorem 3.11. *If* $\mathbf{G}$ *is an antichain and* $\mathbf{a} \geq \operatorname{lcm}(\mathbf{G}) + \mathbf{1}$*, then*

$$\mathcal{T}(\mathbf{G} \sqcup \mathbf{G}^{\mathbf{a}}) = \{\min(\mathbf{c}, \mathbf{a}) : \mathbf{c} \in \mathcal{T}(\mathbf{G})\},$$

where $\mathbf{G}^{\mathbf{a}}$ *is such that* $\mathfrak{m}^{\mathbf{a}} = I_{\mathbf{G}^{\mathbf{a}}}$*. Here,* $\mathbf{G} \sqcup \mathbf{G}^{\mathbf{a}}$ *need not be an antichain; its covers are defined as in Definition 2.5, and they coincide with those of the antichain of its minimal elements, which generates* $I_{\mathbf{G}} + \mathfrak{m}^{\mathbf{a}}$*.*

Proof. ($\supseteq$) Let $\mathbf{c} \in \mathcal{T}(\mathbf{G})$ and $\mathbf{c}' = \min(\mathbf{c}, \mathbf{a})$. Since $\mathbf{a} \geq \operatorname{lcm}(\mathbf{G}) + \mathbf{1}$ and $\operatorname{lcm}(\mathbf{G})_i \geq c_i$ for all $i \in \operatorname{supp}(\mathbf{c})$ (by Lemma 2.14, $c_i = g_i$ for some $\mathbf{g} \in \mathbf{G}$),

$$c'_i = \begin{cases} c_i & \text{if } i \in \operatorname{supp}(\mathbf{c}), \\ a_i & \text{otherwise.} \end{cases}$$

First, $\mathbf{c}'$ is a cover of $\mathbf{G} \sqcup \mathbf{G}^{\mathbf{a}}$. Indeed, given $\mathbf{g} \in \mathbf{G}$, there is j with $c_j \leq g_j$; such a j lies in $\operatorname{supp}(\mathbf{c})$, so $c'_j = c_j \leq g_j$. Given $a_i\mathbf{e}_i \in \mathbf{G}^{\mathbf{a}}$, the only index that can serve is i itself, and $c'_i = \min(c_i, a_i) \leq a_i$. It remains to show that $\mathbf{c}'$ is maximal, that is, that $\mathbf{c}' + \mathbf{e}_i$ is not a cover of $\mathbf{G} \sqcup \mathbf{G}^{\mathbf{a}}$ for every $i \in \operatorname{supp}(\mathbf{c}')$. If $i \in \operatorname{supp}(\mathbf{c})$, then, by the maximality of $\mathbf{c}$, some $\mathbf{g} \in \mathbf{G}$ is not covered by $\mathbf{c} + \mathbf{e}_i$, that is, $g_j < (\mathbf{c} + \mathbf{e}_i)_j$ for every j; since $g_j \leq \operatorname{lcm}(\mathbf{G})_j < a_j = c'_j$ for every $j \notin \operatorname{supp}(\mathbf{c})$, the vector $\mathbf{c}' + \mathbf{e}_i$ does not cover $\mathbf{g}$ either. If $i \notin \operatorname{supp}(\mathbf{c})$, then $c'_i = a_i$, and $\mathbf{c}' + \mathbf{e}_i$ does not cover $a_i\mathbf{e}_i$, since its i-th entry is $a_i + 1$ and i is the only index that can serve.

($\subseteq$) Let $\mathbf{c}'$ be a maximal cover of $\mathbf{G} \sqcup \mathbf{G}^{\mathbf{a}}$ and

$$c_i = \begin{cases} c'_i & \text{if } c'_i \leq \operatorname{lcm}(\mathbf{G})_i, \\ \infty & \text{otherwise.} \end{cases}$$

Since $\mathbf{c}'$ is a cover of $\mathbf{G} \sqcup \mathbf{G}^{\mathbf{a}}$, $\mathbf{c}$ is a cover of $\mathbf{G}$. Moreover, $\mathbf{c}$ is maximal. Otherwise, $\mathbf{c} + \mathbf{e}_i$ is a cover of $\mathbf{G}$ for some $i \in \operatorname{supp}(\mathbf{c})$, and then $\mathbf{c}' + \mathbf{e}_i$ is a cover of $\mathbf{G} \sqcup \mathbf{G}^{\mathbf{a}}$: it covers each $\mathbf{g} \in \mathbf{G}$ at an index of $\operatorname{supp}(\mathbf{c})$, where $\mathbf{c}'$ and $\mathbf{c}$ agree, and it covers each $a_l\mathbf{e}_l \in \mathbf{G}^{\mathbf{a}}$ at the index l, because $c'_i + 1 = c_i + 1 \leq \operatorname{lcm}(\mathbf{G})_i + 1 \leq a_i$. This contradicts the maximality of $\mathbf{c}'$.

Finally, we show the equality $\mathbf{c}' = \min(\mathbf{c}, \mathbf{a})$. For $i \in \operatorname{supp}(\mathbf{c})$, $c'_i = c_i \leq \operatorname{lcm}(\mathbf{G})_i < a_i$. For $i \notin \operatorname{supp}(\mathbf{c})$, we have $c'_i > \operatorname{lcm}(\mathbf{G})_i$ or $c'_i = \infty$, so raising the i-th entry of $\mathbf{c}'$ preserves the covering of $\mathbf{G}$; by the maximality of $\mathbf{c}'$, it must fail to cover $\mathbf{G}^{\mathbf{a}}$, which forces $c'_i = a_i = \min(c_i, a_i)$. □

As a consequence, we can express the associated primes of $S/I_{\mathbf{G} \sqcup \mathbf{G}^{\mathbf{a}}}$ in terms of the associated primes of $S/I_{\mathbf{G}}$ and $\mathbf{a}$.

Corollary 3.12. *If* $\mathbf{G}$ *is an antichain and* $\mathbf{a} \geq \operatorname{lcm}(\mathbf{G}) + \mathbf{1}$*, then*

$$\operatorname{Ass}(S/I_{\mathbf{G} \sqcup \mathbf{G}^{\mathbf{a}}}) = \{P + \sqrt{\mathfrak{m}^{\mathbf{a}}} : P \in \operatorname{Ass}(S/I_{\mathbf{G}})\}.$$

Proof. Since any associated prime of $S/I_{\mathbf{H}}$ for an antichain $\mathbf{H}$ is the radical of one of the factors in the irredundant irreducible decomposition of $I_{\mathbf{H}}$, the result follows from Theorem 3.11 and [4, Theorem 2.12]. □

3.2. Adding a linear form to an edge ideal

Now, we leave general antichain ideals and return to edge ideals: we study how the associated primes of S/I_G change when a linear form is added.

Throughout this subsection, G is a simple graph, that is, a graph without loops. By Subsection 2.4, the associated primes of S/I_G are the ideals $P = \langle C \rangle$, where C is a minimal vertex cover of G, and P is annihilated by the product $\mathbf{m} = \prod_{x_i \notin C} x_i$ of the variables of the maximal stable set $D = V(G) \setminus C$.

Theorem 3.13. *Let C be a minimal vertex cover of a simple graph G, let $D = V(G) \setminus C$, let $P = \langle C \rangle$, and let $\mathbf{m} = \prod_{x_i \in D} x_i$. Let $r = \mathsf{s}(U)$ be a linear regular element of S/I_G, and let $M' = S/\langle I_G, r \rangle$. Then, the following hold:*

(a) *$\mathbf{m} \neq 0_{M'}$ if and only if $U \setminus \{x_j\} \not\subseteq N_G(D \setminus \{x_j\})$ for every $x_j \in D$; and*

(b) *if $\mathbf{m} \neq 0_{M'}$, then $\langle P, r \rangle = \operatorname{Ann}_{M'}(\mathbf{m})$, and it is an associated prime of M'.*

Proof. First, C is a minimal vertex cover if and only if D is a maximal stable set of G. Thus, $N_G(D \setminus \{x_j\}) \subseteq N_G(D) \subseteq C$ for every $x_j \in D$. Since P is an associated prime of S/I_G, we have $I_G \subseteq P$. Moreover, since r is regular on S/I_G, Lemma 2.1 gives $r \notin P$; as P is generated by the variables of C, this means that $U \not\subseteq C$, that is, $U \cap D \neq \emptyset$. Hence, the condition in (a) can fail at $x_j \in D$ only if $U \cap D = \{x_j\}$; otherwise, $U \setminus \{x_j\}$ contains a vertex of D, which does not lie in C. The ideal $\langle P, r \rangle = \langle C \cup \{\mathsf{s}(U \cap D)\} \rangle$ is generated by linear forms, so it is a prime ideal.

Under the isomorphism $S/P \cong k[D]$, the ideal $\langle P, r \rangle$ corresponds to the ideal generated by $\mathsf{s}(U \cap D)$, and $\mathbf{m}$ corresponds to itself. If $|U \cap D| \geq 2$, then $\mathbf{m} \notin \langle P, r \rangle$, since every divisor of a monomial in $k[D]$ is a scalar multiple of a monomial, so the linear form $\mathsf{s}(U \cap D)$, which has at least two terms, does not divide $\mathbf{m}$.

When $U \cap D = \{x_v\}$, we use the following notation. Let $D' = D \setminus \{x_v\}$, let $\mathbf{m}_v = \mathbf{m}/x_v$, and let $\rho = \sum_{x_w \in U \cap C} x_w$, with $\rho = 0$ if $U \cap C = \emptyset$, so that $r = x_v + \rho$. For $x_w \in C$, let K_w be the ideal of $k[D']$ generated by the variables of $N_G(\{x_w\}) \cap D'$.

Claim. If $U \cap D = \{x_v\}$ and $f \in k[D']$, then $f\mathbf{m} \in \langle I_G, r \rangle$ if and only if $f\mathbf{m}_v \in K_w$ for every $x_w \in U \cap C$.

Let $S' = k[\mathbf{x} \setminus \{x_v\}]$, and let $\sigma \colon S \to S'$ be the k-algebra map with $\sigma(x_v) = -\rho$ that fixes the other variables. It induces an isomorphism $S/\langle r \rangle \cong S'$, which carries $\langle I_G, r \rangle$ onto the ideal $\overline{I}$ of S' generated by the monomials $x_a x_b$ with $\{x_a, x_b\} \in E(G)$ and $x_v \notin \{x_a, x_b\}$, and by the forms $\rho\, x_b$ with $x_b \in N_G(\{x_v\})$. Grade S' by the total degree in the variables of C. These generators are homogeneous: since C is a vertex cover and D is stable, each monomial $x_a x_b$ has degree 1 or 2, and since $N_G(\{x_v\}) \subseteq C$, each form $\rho\, x_b$ has degree 2. Hence, $\overline{I}$ is homogeneous, its component of degree 0 is zero, and its component of degree 1 is $\bigoplus_{x_w \in C} x_w K_w$. Now, $\sigma(f\mathbf{m}) = -f\rho\, \mathbf{m}_v = -\sum_{x_w \in U \cap C} x_w\, f\mathbf{m}_v$ is homogeneous of degree 1, so $f\mathbf{m} \in \langle I_G, r \rangle$ if and only if $f\mathbf{m}_v \in K_w$ for every $x_w \in U \cap C$. This proves the claim.

(a) Since $U \cap D \neq \emptyset$, there are two cases.

Case $|U \cap D| \geq 2$. The condition in (a) holds, and $\mathbf{m} \neq 0_{M'}$: by the first paragraph, $\mathbf{m} \notin \langle P, r \rangle$ and $\langle I_G, r \rangle \subseteq \langle P, r \rangle$.

Case $U \cap D = \{x_v\}$. By the first paragraph, the condition in (a) reduces to $U \cap C \not\subseteq N_G(D')$. By the claim with $f = 1$, $\mathbf{m} \in \langle I_G, r \rangle$ if and only if $\mathbf{m}_v \in K_w$ for every $x_w \in U \cap C$. Since $\mathbf{m}_v$ is the product of the variables of D' and K_w is generated by some of them, $\mathbf{m}_v \in K_w$ if and only if x_w has a neighbor in D'. Hence, $\mathbf{m} = 0_{M'}$ if and only if $U \cap C \subseteq N_G(D')$, that is, if and only if the condition in (a) fails.

(b) Assume that $\mathbf{m} \neq 0_{M'}$; by (a), the condition in (a) holds. First, $\langle P, r\rangle \subseteq \operatorname{Ann}_{M'}(\mathbf{m})$. Indeed, $r\mathbf{m} \in \langle I_G, r\rangle$, and every $x_j \in C$ has a neighbor $x_i \in D$: by the maximality of D, the set $D \cup \{x_j\}$ is not stable, and since G has no loops, some edge $x_i x_j$ has $x_i \in D$. Then, $x_i x_j$ divides $x_j\mathbf{m}$, so $x_j\mathbf{m} \in I_G$.

For the reverse inclusion, we consider the same two cases as in (a).

Case $|U \cap D| \geq 2$. By the first paragraph, $\mathbf{m} \notin \langle P, r\rangle$; since $\langle P, r\rangle$ is prime, $(\langle P, r\rangle \colon \mathbf{m}) = \langle P, r\rangle$. Hence,

$$\operatorname{Ann}_{M'}(\mathbf{m}) = (\langle I_G, r\rangle \colon \mathbf{m}) \subseteq (\langle P, r\rangle \colon \mathbf{m}) = \langle P, r\rangle.$$

Case $U \cap D = \{x_v\}$. Since condition (a) holds, we can choose $x_c \in U \cap C$ with no neighbor in D', so $K_c = 0$. Let $u \in S$ with $u\mathbf{m} \in \langle I_G, r\rangle$, and let u_0 be the sum of the terms of u that involve no variable of $C \cup \{x_v\}$, so that $u_0 \in k[D']$ and $u - u_0 \in \langle C \cup \{x_v\}\rangle = \langle P, r\rangle$. By the inclusion $\langle P, r\rangle \subseteq \operatorname{Ann}_{M'}(\mathbf{m})$, also $u_0\mathbf{m} \in \langle I_G, r\rangle$, and the claim with $f = u_0$ gives $u_0\mathbf{m}_v \in K_c = 0$. Hence, $u_0 = 0$ and $u \in \langle P, r\rangle$.

In both cases, $\operatorname{Ann}_{M'}(\mathbf{m}) = \langle P, r\rangle$, which is a prime ideal; being the annihilator of a nonzero element of M', it is an associated prime of M'.

Note that we used the regularity of r only to ensure $r \notin P$. □

Since rescaling the variables fixes I_G and P and multiplies $\mathbf{m}$ by a nonzero scalar, Theorem 3.13 also holds for every linear form $\sum_{x_i \in U} a_i x_i$ with $a_i \in k \setminus \{0\}$ that is regular on S/I_G.

Remark 3.14. That $\langle P, r\rangle$ is an associated prime of $S/\langle I_G, r\rangle$ does not depend on the condition in Theorem 3.13 (a), and it holds for every monomial ideal: if I is a monomial ideal of S, $P \in \operatorname{Ass}(S/I)$, and r is a linear form that is regular on $M = S/I$, then $Q = \langle P, r\rangle$ is an associated prime of $S/\langle I, r\rangle$. Indeed, $r \notin P$ by Lemma 2.1, and since P is generated by variables, S/P is a polynomial ring in which the image of r is a nonzero linear form; hence, Q is a prime ideal, and Q/P is a nonzero principal prime ideal of the domain S/P, so that $\dim S_Q/PS_Q = 1$. Since PS_Q is an associated prime of M_Q, [11, Proposition 1.2.13] gives $\operatorname{depth} M_Q \leq 1$; since r is regular on M_Q, it follows that $\operatorname{depth} M_Q = 1$ and, by [11, Proposition 1.2.10(d)], that $\operatorname{depth}(M/rM)_Q = 0$. Thus, every element of QS_Q is a zero divisor of $(M/rM)_Q$, so, by Lemma 2.1 and prime avoidance, QS_Q is contained in, and hence equal to, an associated prime of $(M/rM)_Q$; therefore, $Q \in \operatorname{Ass}(S/\langle I, r\rangle)$. What Theorem 3.13 adds, for edge ideals, is an explicit witness: it characterizes when the monomial $\mathbf{m}$ is nonzero modulo $\langle I_G, r\rangle$, and in that case, it identifies $\langle P, r\rangle$ as the annihilator of $\mathbf{m}$.

In general, not every associated prime of $M' = S/\langle I_G, r\rangle$ is of the form $\langle P, r\rangle$ with $P \in \operatorname{Ass}(S/I_G)$. Neither is it always the case that $\mathbf{m} \neq 0_{M'}$, as shown in the following example, where the condition in Theorem 3.13 (a) fails for both minimal vertex covers of G, while both ideals $\langle P, r\rangle$ are still associated primes of M' (Remark 3.14).

Example 3.15. Let $I = \langle x_1x_2, x_2x_3, x_3x_4, x_1x_4\rangle$ be the edge ideal of the cycle with four vertices C_4 and $M = k[x_1, x_2, x_3, x_4]/I$. By Example 2.19,

$$\operatorname{Ass}(S/I) = \{\langle x_1, x_3\rangle, \langle x_2, x_4\rangle\},$$

$\langle x_1, x_3\rangle = \operatorname{Ann}_M(x_2x_4)$, $\langle x_2, x_4\rangle = \operatorname{Ann}_M(x_1x_3)$, and $r = x_1 + x_2$ is a regular linear form of M, because it lies in neither of these two primes (Lemma 2.1). Thus, the ideals of the form $\langle P, r\rangle$ with $P \in \operatorname{Ass}(S/I)$ are the ideals $\langle x_1, x_2, x_3\rangle$ and $\langle x_1, x_2, x_4\rangle$. However, using Macaulay2 [16],

$$\operatorname{Ass}(S/\langle I, r\rangle) = \{\langle x_1, x_2, x_3\rangle, \langle x_1, x_2, x_4\rangle, \langle x_1, x_2, x_3, x_4\rangle\},$$

where

$$\langle x_1, x_2, x_3\rangle = \operatorname{Ann}_{M'}(x_4), \quad \langle x_1, x_2, x_4\rangle = \operatorname{Ann}_{M'}(x_3), \quad \langle x_1, x_2, x_3, x_4\rangle = \operatorname{Ann}_{M'}(x_1) = \operatorname{Ann}_{M'}(x_2).$$

Therefore $\{\langle P, r\rangle\}_{P\in \operatorname{Ass}(S/I)} \subsetneq \operatorname{Ass}(S/\langle I, r\rangle)$. Moreover, $\mathbf{m} = 0_{M'}$ for both minimal vertex covers: modulo $\langle I, r\rangle$ one has $x_2 \equiv -x_1$, so $x_2x_4 \equiv -x_1x_4 \in I$ and $x_1x_3 \equiv -x_2x_3 \in I$.

A *regular sequence* on M is an ordered sequence $\{r_1, \ldots, r_s\}$ of forms in $\mathfrak{m}$ such that r_i is regular on $M/\langle r_1, \ldots, r_{i-1}\rangle M$ for every $1 \leq i \leq s$ and $\langle r_1, \ldots, r_s\rangle M \neq M$. One might expect that linear forms with disjoint support that are regular elements form a regular sequence. However, this is not true in general, as we will see below.

Example 3.16. Let $I_{C_4} = \langle x_1x_2, x_2x_3, x_3x_4, x_1x_4\rangle$ and $M = k[x_1, x_2, x_3, x_4]/I_{C_4}$. The linear forms $r_1 = x_1 + x_2$ and $r_2 = x_3 + x_4$ have disjoint support, and they are regular elements in M, because neither lies in $\langle x_1, x_3\rangle$ or in $\langle x_2, x_4\rangle$ (Example 2.19). However, the set $\{r_1, r_2\}$ is not a regular sequence, since $\operatorname{depth}(M) = 1$, so no regular sequence of length two exists.

As the following example shows, a set $\{r_1, \ldots, r_s\}$ of s regular linear forms on S/I need not form a regular sequence, even when $s \leq \operatorname{depth}(S/I)$.

Example 3.17. Let

$$I = \langle x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1, y_1y_2, y_2y_3, y_3y_4, y_4y_5, y_5y_1, x_1y_1, x_2y_2, x_3y_3\rangle$$

be the edge ideal of the graph given in Figure 1 in the polynomial ring $S = k[x_1, \ldots, x_5, y_1, \ldots, y_5]$ and $M = S/I$.

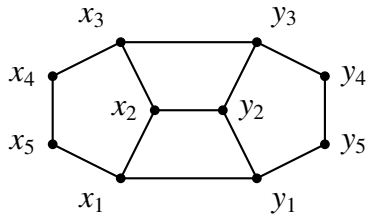


Figure 1. Graph G such that S/I_G has depth three, and which admits two regular linear forms r_1 and r_2 with disjoint support such that r_2 is not regular in $S/\langle I, r_1\rangle$.

Using Macaulay2 [16], one checks that $r_1 = x_1+x_2+x_3$ and $r_2 = y_1+y_2+y_3$ are regular linear forms in M with disjoint support. Since $\operatorname{depth}(M) = 3$, depth alone does not obstruct $\{r_1, r_2\}$ from being a regular sequence of M.

However, r_2 is not regular in $M' = M/\langle r_1\rangle M$ because $\mathbf{m} = x_3x_5y_5$ is nonzero in M' and

$$r_2\mathbf{m} = (y_1 + y_2 + y_3)x_3x_5y_5 = y_1x_3x_5y_5 + y_2x_3x_5y_5 + y_3x_3x_5y_5$$

is equal to zero. To check this, since $y_1y_5 \in I$ and $x_3y_3 \in I$, the first and third terms

$$y_1x_3x_5y_5 = x_3x_5(y_1y_5), \quad y_3x_3x_5y_5 = x_5y_5(x_3y_3),$$

are equal to zero in M'. On the other hand, since $r_1 = x_1 + x_2 + x_3 = 0_{M'}$, $x_3 = -x_1 - x_2$, and therefore

$$y_2x_3x_5y_5 = y_2(-x_1 - x_2)x_5y_5 = -y_2x_1x_5y_5 - y_2x_2x_5y_5,$$

which is equal to zero because x_2y_2 and x_1x_5 are in I. Therefore, $\{r_1, r_2\}$ is not a regular sequence of M.

These examples show that regularity alone, even paired with disjoint supports, does not guarantee a regular sequence; some additional combinatorial structure is needed. The results in this subsection relate to the preprint [10], which appeared after we submitted this paper; we obtained our results independently. For an edge ideal, a star in the sense of [10] is a sum of variables whose support contains the closed neighborhood of one of them. Given pairwise disjoint stars $f_1, \ldots, f_q$, [10, Theorem 3.10] gives a combinatorial condition under which $\langle P, f_1, \ldots, f_q \rangle$ is an associated prime of $S/\langle I_G, f_1, \ldots, f_q \rangle$. When $q = 1$, the hypothesis of [10, Theorem 3.10] is equivalent to the condition in Theorem 3.13 (a), so that in this case, the monomial $\mathbf{m}$ is a witness of the associated prime given there; since a star on I_G is a regular element [6, Lemma 3.9], the conclusion of [10, Theorem 3.10] for $q = 1$ also follows from Remark 3.14. For a single linear form, Theorem 3.13 applies to any regular linear form r, not necessarily a star, characterizes when $\mathbf{m} \neq 0$ in $S/\langle I_G, r \rangle$, and identifies $\langle P, r \rangle$ as the annihilator of $\mathbf{m}$, whereas [10] treats several stars at once. In [10, Example 3.6], a prime of the form $\langle P, r \rangle$ is an embedded associated prime; in Example 3.15, the embedded associated prime is not of this form. Finally, one instance of the additional structure mentioned above is given by [10, Theorem 2.9]: over an algebraically closed field, pairwise separated stars of degree one form a regular sequence, and for edge ideals, separated stars are simply stars with pairwise disjoint supports; the linear forms in Examples 3.16 and 3.17 are not stars.

Author contributions

J. Ibarguen, D. S. Moran, C. E. Valencia and R. H. Villarreal contributed equally to this work regarding Conceptualization, Methodology, Formal analysis, Investigation, Software, Writing–original draft preparation, and Writing–review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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