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Research article

Critical fractional orders and quadratic loss of sensitivity in Mittag–Leffler dynamics

Areen Al-khateeb*

Department of Mathematics, Faculty of Science, Jadara University, Irbid 21110, Jordan

* **Correspondence:** Email: Areen.k@jadara.edu.jo.

Abstract: The dependence of fractional dynamical systems on the differentiation order is commonly studied through sensitivity, monotonicity, or inverse order determination. In this paper, we investigated a complementary phenomenon: critical fractional orders at which the first-order sensitivity of a Mittag–Leffler response with respect to its order vanishes. We first considered the map

$$\alpha \longmapsto E_\alpha(\lambda), \qquad 0 < \alpha < 1,$$

for sufficiently small nonzero λ. A uniform C^1 perturbation argument shows that this map possesses a unique critical order $\alpha_*(\lambda)$ in $(0, 1)$, emerging from the distinguished value α_0 determined by $\psi(\alpha_0 + 1) = 0$. The critical point is proved to be nondegenerate; it is a strict minimum for $\lambda < 0$ and a strict maximum for $\lambda > 0$. Consequently, the usual linear order sensitivity is lost at the critical point and is replaced by the quadratic local law

$$\left|E_{\alpha_*(\lambda)+h}(\lambda) - E_{\alpha_*(\lambda)}(\lambda)\right| \asymp |h|^2.$$

We also derived an explicit first-order asymptotic expansion of the critical branch. The analysis was then extended to the dynamical map $E_\alpha(\lambda t^\alpha)$ arising from a scalar Caputo evolution equation. An explicit unperturbed critical-order curve was obtained and shown to persist, on compact time intervals, as a unique nondegenerate critical surface for small $|\lambda|$. High-precision numerical computations confirmed the critical-branch displacement, the minimum–maximum classification, and the predicted quadratic scaling. The results characterize a local loss of first-order order sensitivity that is distinct from the monotonicity regimes usually employed in fractional-order identification.

Keywords: fractional order; Mittag–Leffler function; order sensitivity; critical order; Caputo fractional equation
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1. Introduction

Fractional differential equations provide a natural framework for evolution processes in which the present state depends nonlocally on the past. Their analytical foundations, including the Riemann–Liouville and Caputo operators and the corresponding initial-value problems, are now well - established; see, for example, [1–3]. In the scalar Caputo equation, λ is the growth/relaxation coefficient: negative values represent relaxation and positive values growth. For a dimensional time variable, its units depend on the order, $[\lambda] = T^{-\alpha}$. Thus comparisons across orders require a fixed reference time and a consistent nondimensionalization. In our analysis, "sufficiently small" means that $|\lambda|$ lies below the positive perturbation threshold furnished by the uniform estimates in Section 3 (and, on a time interval J, by Lemma 5.1). It is a sufficient local hypothesis, not a claimed necessary threshold. The small-argument approximation additionally requires $|\lambda|t^{\alpha} \ll 1$; the two conditions should not be conflated.

A central role in this theory is played by the Mittag–Leffler function, which replaces the exponential function in many linear fractional evolution equations and governs characteristic relaxation and subdiffusive responses. Its analytic structure, generalizations, and applications to fractional differential equations have been developed extensively in [4–6].

For a fractional model, the differential order is not merely a formal parameter. It controls the temporal memory encoded by the operator and can substantially change the observable response of the corresponding dynamical system. This has motivated investigations of continuous dependence and sensitivity with respect to parameters in fractional differential equations. Guo, Ma, and Wu [7] introduced fractional sensitivity functions and sensitivity equations for Caputo differential equations, while Kharazmi and Zayernouri [8] developed a fractional sensitivity equation method for model construction and parameter estimation. Continuous dependence of solutions on a fractional order has also been considered for generalized fractional Cauchy-type problems [9]. These studies demonstrate that variation of the fractional order can be treated quantitatively, but they do not address the local geometry created when sensitivity with respect to the order itself vanishes.

At the level of the special function governing the scalar dynamics, parameter differentiation has been studied more directly. Apelblat [10] derived derivatives of one- and two-parameter Mittag–Leffler functions with respect to their parameters using Laplace-transform, integral-representation, and direct-series approaches. In particular, direct differentiation leads naturally to series whose coefficients contain quotients of digamma and gamma functions. Parameter differentiation has subsequently been developed for integral Mittag–Leffler and related Wright functions by Apelblat and González-Santander [11]. Consequently, differentiation of a Mittag–Leffler function with respect to its order is already an established analytical operation. The question considered here is different: we investigate the zeros of that order derivative, their uniqueness and nondegeneracy, and the local change of sensitivity that occurs at such zeros.

A closely related line of research arises in inverse fractional-order problems. Determination of an unknown fractional order from additional observations has been studied for several classes of subdiffusion and diffusion-wave equations. Ashurov and Umarov [12] proved unique determination of the order of a subdiffusion equation from data at a fixed time, while Alimov and Ashurov [13] obtained an order-identification result for a Caputo subdiffusion equation with a general positive self-adjoint spatial operator. Yamamoto [14] established uniqueness results for simultaneously determining

fractional orders and initial values from several types of observations. These results illustrate the close connection between the behavior of the order-to-observation map and the solvability of fractional inverse problems.

For single-time estimation of α from a response $y = F_{\lambda,t}(\alpha)$, a Newton step divides by $\partial_\alpha F_{\lambda,t}$; near a critical order, this step becomes ill-conditioned. The scalar Gauss–Newton normal equation similarly contains $(\partial_\alpha F_{\lambda,t})^2$. Under independent Gaussian measurement noise of variance σ^2, the single-observation Fisher information is $(\partial_\alpha F_{\lambda,t})^2/\sigma^2$, which vanishes at the critical order. These statements concern local sensitivity, not global structural nonidentifiability. Measurements at several suitably chosen times can avoid a simultaneously vanishing sensitivity vector.

The scope of order recovery has subsequently been broadened in several directions. Jin and Kian [15] considered the recovery of multiple fractional orders and their weights in an unknown medium, and later established recovery of the differentiation order from boundary flux measurements even when several properties of the medium are unknown [16]. Jing and Yamamoto [17] studied simultaneous identification of the fractional order together with other coefficients and data in fractional diffusion-wave equations. Related investigations have addressed variable fractional orders [18], simultaneous recovery of an order and a source term [19], and inverse determination in mixed-type fractional equations [20]. Thus, identifiability of fractional orders is a substantial and active subject, but its usual formulation asks whether the unknown order can be reconstructed from prescribed data.

Monotonicity of the order-to-observation map provides one particularly effective mechanism behind such uniqueness results. Li, Wang, Jia, and Zhang [21] investigated the determination of the fractional order in a time-fractional diffusion equation from a measurement at one space–time point, with monotonicity of the corresponding Mittag–Leffler response playing a central role. More recently, Ashurov and Sulaymonov [22] simultaneously considered temporal and spatial fractional orders, while Sulaymonov and Ashurov [23] established strict monotonicity in the fractional parameter for specific Mittag–Leffler profiles and applied it to order determination in subdiffusion equations. These recent developments reinforce the importance of understanding the fractional order as an active dynamical parameter rather than only as a fixed index of differentiation.

The monotone regime, however, represents only one possible geometry of the order-to-observation map. If the derivative with respect to the fractional order vanishes at an interior point, first-order sensitivity is lost and monotonicity necessarily changes locally unless the zero has higher multiplicity. Parameter-derivative formulas such as those in [10, 11] provide the analytical ingredients for detecting such points, while the sensitivity literature [7, 8] explains their relevance to parameter perturbations. Nevertheless, these theories do not by themselves characterize the number, multiplicity, perturbative persistence, or local scaling of zeros of the order derivative.

This motivates the forward problem studied in the present paper. We begin with

$$F_\lambda(\alpha) = E_\alpha(\lambda), \qquad 0 < \alpha < 1, \tag{1.1}$$

where λ is regarded as a perturbation parameter. At $\lambda = 0$, one has $E_\alpha(0) \equiv 1$, so that the entire order-to-observation map is degenerate. For nonzero but sufficiently small $|\lambda|$, the leading-order structure of the order derivative is governed by

$$q(\alpha) = \frac{\psi(\alpha + 1)}{\Gamma(\alpha + 1)}.$$

Since the digamma function has a unique zero between 1 and 2, this identifies a distinguished reference

order

$$\alpha_0 \in (0,1), \qquad \psi(\alpha_0 + 1) = 0,$$

with $\alpha_0 \approx 0.461632145$.

The condition $\psi(1+\alpha_0) = 0$ arises because $\psi(x) = d(\log\Gamma(x))/dx$: the leading order derivative of $1/\Gamma(1+\alpha)$ vanishes precisely at this value. At a general observation time, the competing factor t^α shifts the condition to $\psi(1+\alpha) = \log t$. Thus criticality describes cancellation between the order dependence of the time factor and of the gamma normalization.

The first contribution of this paper is to prove that, for every sufficiently small nonzero $|\lambda|$, the map $\alpha \mapsto E_\alpha(\lambda)$ possesses a *unique* critical order $\alpha_*(\lambda) \in (0,1)$. The proof combines a uniform C^1 perturbation estimate with a localization argument, so that uniqueness holds on the full order interval rather than only in a small neighborhood supplied by the implicit-function theorem. We further prove that this critical point is nondegenerate and classify its geometry: it is a strict minimum for $\lambda < 0$ and a strict maximum for $\lambda > 0$.

The second contribution concerns the local sensitivity law at the critical order. Since

$$\partial_\alpha F_\lambda(\alpha_*(\lambda)) = 0$$

while the second derivative is nonzero, the usual linear response to an order perturbation disappears. We establish instead that

$$|F_\lambda(\alpha_*(\lambda) + h) - F_\lambda(\alpha_*(\lambda))| \asymp |h|^2, \qquad h \to 0. \tag{1.2}$$

Thus a nondegenerate critical order produces a quadratic loss of local order sensitivity. In addition, the critical branch is shown to be smooth near $\lambda = 0$, and its first-order displacement

$$\alpha_*(\lambda) = \alpha_0 + \alpha_*'(0)\lambda + O(\lambda^2)$$

is obtained explicitly in terms of gamma, digamma, and trigamma functions.

Finally, we pass from the static Mittag–Leffler profile to the dynamical observation map

$$F_{\lambda,t}(\alpha) = E_\alpha(\lambda t^\alpha), \tag{1.3}$$

which arises from the scalar Caputo equation ${}^C D_t^\alpha x = \lambda x$ with normalized initial data. The unperturbed critical orders form the explicit curve

$$\alpha_0(t) = \psi^{-1}(\log t) - 1, \qquad e^{-\gamma} < t < e^{1-\gamma}.$$

On every compact subinterval of this time range, we prove that this curve persists, for sufficiently small $|\lambda|$, as a unique critical surface $\alpha_*(\lambda, t)$. We determine its first-order displacement and show that nondegeneracy and quadratic sensitivity persist throughout the perturbative dynamical regime.

To the best of our knowledge, existing works on Mittag–Leffler parameter differentiation, fractional sensitivity, monotonicity, and fractional-order recovery do not provide this combination of *global uniqueness of a perturbative critical order, nondegeneracy and extremum classification, quadratic local sensitivity, explicit critical-branch asymptotics, and persistence of the critical structure in the dynamical map* $E_\alpha(\lambda t^\alpha)$. The distinction is important: the present problem is not an inverse

reconstruction problem and does not assume monotonicity. Instead, it studies the forward geometry precisely at the point where first-order order sensitivity and local monotonicity break down. The numerical experiments reported later are used to verify the theoretical branch expansion, the extremum classification, and the quadratic exponent.

The scalar scope is deliberate. For a diagonal system ${}^{C}D_t^{\alpha}x = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)x$, each observed component with a nonzero initial coefficient has a scalar Mittag–Leffler factor and can be analyzed componentwise under the corresponding small-parameter assumptions. A general matrix-valued or coupled nonlinear system need not share one critical order, and no such general theorem is asserted here. Related fractional-order synchronization studies in higher-dimensional neural systems illustrate this broader setting [24, 25]; their synchronization criteria are distinct from the order-sensitivity question addressed here.

The remainder of the paper is organized as follows. Section 2 introduces the required properties of the Mittag–Leffler, gamma, digamma, and trigamma functions, and derives the order-derivative structure. Section 3 establishes the uniform perturbation estimates. Section 4 develops the critical-order theory, including uniqueness and nondegeneracy. Section 5 extends the analysis to the dynamical map $E_{\alpha}(\lambda t^{\alpha})$. Section 6 provides numerical verification, followed by a comparison with short- and long-time approximations. The final section contains the conclusions and directions for further work.

2. Mathematical setting and order-derivative structure

This section introduces the fractional evolution problem and the parameter-dependent Mittag–Leffler functions used throughout the paper. Only the analytical background required for the subsequent perturbation analysis is collected here.

2.1. The scalar Caputo evolution problem

Let $0 < \alpha < 1$ and consider the scalar initial-value problem

$$ {}^{C}D_t^{\alpha}x(t) = \lambda x(t), \qquad t > 0, \qquad x(0) = 1, \tag{2.1}$$

where $\lambda \in \mathbb{R}$. The Caputo derivative of order $\alpha \in (0, 1)$ is defined by

$$ {}^{C}D_t^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} x'(\tau)\, d\tau, \tag{2.2}$$

whenever the right-hand side is well - defined; see, for example, [1, 2].

The solution of (2.1) is represented by the Mittag–Leffler function as

$$ x_{\alpha}(t) = E_{\alpha}(\lambda t^{\alpha}), \tag{2.3}$$

where

$$ E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \tag{2.4}$$

Standard properties are given in [5].

For fixed λ and $t > 0$, define the order-to-observation map

$$ F_{\lambda,t}(\alpha) := E_{\alpha}(\lambda t^{\alpha}), \qquad 0 < \alpha < 1. \tag{2.5}$$

At the reference observation time $t = 1$, this reduces to

$$F_\lambda(\alpha) := F_{\lambda,1}(\alpha) = E_\alpha(\lambda). \tag{2.6}$$

A critical fractional order for the observation map (2.5) is a value $\alpha_* \in (0, 1)$ satisfying

$$\partial_\alpha F_{\lambda,t}(\alpha_*) = 0. \tag{2.7}$$

If, in addition,

$$\partial_{\alpha\alpha} F_{\lambda,t}(\alpha_*) \neq 0, \tag{2.8}$$

we call α_* a nondegenerate critical fractional order.

2.2. Differentiation with respect to the fractional order

Parameter differentiation of Mittag–Leffler functions has been studied previously, and formulas involving the gamma, digamma, and polygamma functions are known; see, for example, [10]. Here we record the particular representation needed for the order map (2.5).

Recall that the digamma and trigamma functions are defined by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \qquad \psi_1(x) = \psi'(x). \tag{2.9}$$

For $x > 0$,

$$\psi_1(x) = \psi'(x) = \sum_{n=0}^{\infty} \frac{1}{(n + x)^2} > 0.$$

and hence ψ is strictly increasing on $(0, \infty)$.

Differentiating the kth term of (2.5) with respect to α gives

$$\frac{\partial}{\partial\alpha}\left(\frac{\lambda^k t^{\alpha k}}{\Gamma(\alpha k + 1)}\right) = \frac{k\lambda^k t^{\alpha k}}{\Gamma(\alpha k + 1)}\left[\log t - \psi(\alpha k + 1)\right]. \tag{2.10}$$

Consequently,

$$\partial_\alpha F_{\lambda,t}(\alpha) = \sum_{k=1}^{\infty} \frac{k\lambda^k t^{\alpha k}}{\Gamma(\alpha k + 1)}\left[\log t - \psi(\alpha k + 1)\right]. \tag{2.11}$$

For fixed t in a compact subset of $(0, \infty)$ and λ in a bounded set, the series and the differentiated series converge uniformly on compact subintervals of $(0, 1)$. The stronger uniform estimates up to $\alpha = 0$ that are required for the perturbation argument will be proved separately in Section 3.

At $t = 1$, formula (2.11) becomes

$$F'_\lambda(\alpha) = -\sum_{k=1}^{\infty} \frac{k\lambda^k \psi(\alpha k + 1)}{\Gamma(\alpha k + 1)}. \tag{2.12}$$

Since $F_0(\alpha) = 1$, we normalize the derivative for $\lambda \neq 0$ by

$$H(\alpha, \lambda) := -\frac{F'_\lambda(\alpha)}{\lambda} = \sum_{k=1}^{\infty} \frac{k\lambda^{k-1} \psi(\alpha k + 1)}{\Gamma(\alpha k + 1)}. \tag{2.13}$$

The natural limiting profile at $\lambda = 0$ is

$$q(\alpha) := \frac{\psi(\alpha + 1)}{\Gamma(\alpha + 1)}. \tag{2.14}$$

The precise statement that $H(\cdot, \lambda)$ converges to q in the C^1 topology as $\lambda \to 0$ is one of the key perturbation results of the next section.

For the time-dependent observation map, it is convenient to introduce the corresponding normalized quantity

$$G(\alpha, \lambda, t) := -\frac{1}{\lambda}\partial_\alpha F_{\lambda,t}(\alpha), \qquad \lambda \neq 0. \tag{2.15}$$

Using (2.11), we obtain

$$G(\alpha, \lambda, t) = \sum_{k=1}^{\infty} k\lambda^{k-1} t^{\alpha k} \frac{\psi(\alpha k + 1) - \log t}{\Gamma(\alpha k + 1)}. \tag{2.16}$$

Its continuous extension to $\lambda = 0$ is therefore

$$G(\alpha, 0, t) = \frac{t^\alpha[\psi(\alpha + 1) - \log t]}{\Gamma(\alpha + 1)}. \tag{2.17}$$

At $t = 1$, $G(\alpha, \lambda, 1) = H(\alpha, \lambda)$.

2.3. The distinguished digamma zero

The limiting profile (2.14) has a particularly simple zero structure. Since $\Gamma(\alpha + 1) > 0$, its zeros coincide with those of $\psi(\alpha + 1)$. The classical identities

$$\psi(1) = -\gamma, \qquad \psi(2) = 1 - \gamma, \tag{2.18}$$

where γ denotes the Euler–Mascheroni constant, together with the strict positivity of ψ_1, imply that there exists a unique $\alpha_0 \in (0, 1)$ satisfying

$$\psi(\alpha_0 + 1) = 0. \tag{2.19}$$

Equivalently,

$$\alpha_0 = \psi^{-1}(0) - 1.$$

Numerically,

$$\alpha_0 \approx 0.461632145.$$

Moreover, differentiating (2.14) gives

$$q'(\alpha) = \frac{\psi_1(\alpha + 1) - \psi(\alpha + 1)^2}{\Gamma(\alpha + 1)}. \tag{2.20}$$

At $\alpha = \alpha_0$, relation (2.19) yields

$$q'(\alpha_0) = \frac{\psi_1(\alpha_0 + 1)}{\Gamma(\alpha_0 + 1)} > 0. \tag{2.21}$$

Thus α_0 is a simple zero of the limiting profile q.

For the time-dependent limiting profile (2.17), the zero condition takes the form

$$\psi(\alpha + 1) = \log t. \tag{2.22}$$

Since ψ maps $(1, 2)$ strictly increasingly onto $(-\gamma, 1-\gamma)$, equation (2.22) has a unique solution in $(0, 1)$ precisely when

$$e^{-\gamma} < t < e^{1-\gamma}. \tag{2.23}$$

For such t, we denote this limiting critical-order curve by

$$\alpha_0(t) := \psi^{-1}(\log t) - 1. \tag{2.24}$$

In particular,

$$\alpha_0(1) = \alpha_0.$$

3. Critical-order perturbation theory

We now study the emergence of critical fractional orders from the degenerate state $\lambda = 0$. At this value,

$$F_0(\alpha) = E_\alpha(0) \equiv 1,$$

and hence the unnormalized order derivative vanishes identically. The normalization introduced in Section 2 removes this degeneracy and reveals a distinguished limiting profile. The main purpose of this section is to prove that, for sufficiently small nonzero λ, this profile generates exactly one critical fractional order in the full interval $(0, 1)$.

Throughout this section,

$$f(x) := \frac{\psi(x)}{\Gamma(x)}, \qquad x \geq 1, \tag{3.1}$$

so that

$$H(\alpha, \lambda) = \sum_{k=1}^{\infty} k\lambda^{k-1} f(\alpha k + 1) \tag{3.2}$$

and

$$q(\alpha) = f(\alpha + 1) = \frac{\psi(\alpha + 1)}{\Gamma(\alpha + 1)}. \tag{3.3}$$

3.1. Uniform perturbation of the limiting profile

We first establish the estimate that underlies the entire critical-order analysis.

lemma 3.1 (Uniform C^1 perturbation)**.** *Let $0 < r < 1$. There exists a constant $C_r > 0$ such that*

$$\|H(\cdot, \lambda) - q\|_{C^1([0,1])} \leq C_r|\lambda|, \qquad |\lambda| \leq r. \tag{3.4}$$

Consequently,

$$H(\cdot, \lambda) \longrightarrow q \quad \text{in } C^1([0, 1]) \quad \text{as } \lambda \to 0. \tag{3.5}$$

Proof. We first observe that both f and f' are bounded on $[1,\infty)$. Indeed,

$$f'(x) = \frac{\psi_1(x) - \psi(x)^2}{\Gamma(x)}. \tag{3.6}$$

The standard large-x asymptotic relations

$$\psi(x) = \log x + O(x^{-1}), \qquad \psi_1(x) = x^{-1} + O(x^{-2}),$$

together with the growth of $\Gamma(x)$, imply

$$f(x) \to 0, \qquad f'(x) \to 0 \qquad (x \to \infty).$$

Since f and f' are continuous on $[1,\infty)$, the quantities

$$M_0 := \sup_{x\geq 1} |f(x)|, \qquad M_1 := \sup_{x\geq 1} |f'(x)| \tag{3.7}$$

are finite.

Separating the first term of (3.2), we have

$$H(\alpha,\lambda) - q(\alpha) = \sum_{k=2}^{\infty} k\lambda^{k-1} f(\alpha k + 1). \tag{3.8}$$

Hence, uniformly for $\alpha \in [0,1]$,

$$|H(\alpha,\lambda) - q(\alpha)| \leq M_0 \sum_{k=2}^{\infty} k|\lambda|^{k-1}. \tag{3.9}$$

For $0 \leq x < 1$,

$$\sum_{k=2}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2} - 1 = \frac{x(2-x)}{(1-x)^2}. \tag{3.10}$$

Therefore, if $|\lambda| \leq r < 1$,

$$\|H(\cdot,\lambda) - q\|_\infty \leq \frac{2M_0}{(1-r)^2}|\lambda|. \tag{3.11}$$

We next differentiate (3.2) with respect to α. The uniform convergence required for termwise differentiation follows from the estimates below. We obtain

$$\partial_\alpha H(\alpha,\lambda) = \sum_{k=1}^{\infty} k^2\lambda^{k-1} f'(\alpha k + 1). \tag{3.12}$$

Since

$$q'(\alpha) = f'(\alpha + 1),$$

it follows that

$$\partial_\alpha H(\alpha,\lambda) - q'(\alpha) = \sum_{k=2}^{\infty} k^2\lambda^{k-1} f'(\alpha k + 1). \tag{3.13}$$

Thus

$$|\partial_\alpha H(\alpha,\lambda) - q'(\alpha)| \le M_1 \sum_{k=2}^{\infty} k^2 |\lambda|^{k-1}. \tag{3.14}$$

For $0 \le x < 1$,

$$\sum_{k=1}^{\infty} k^2 x^{k-1} = \frac{1+x}{(1-x)^3}, \tag{3.15}$$

and hence

$$\sum_{k=2}^{\infty} k^2 x^{k-1} = \frac{1+x}{(1-x)^3} - 1 = \frac{x(4-3x+x^2)}{(1-x)^3}. \tag{3.16}$$

Therefore, for $|\lambda| \le r$,

$$\|\partial_\alpha H(\cdot,\lambda) - q'\|_\infty \le \frac{4M_1}{(1-r)^3}|\lambda|. \tag{3.17}$$

Combining (3.11) and (3.17) gives (3.4), after enlarging the constant C_r if necessary. □

Remark 3.2. *The uniformity in Lemma 3.1 includes the endpoint $\alpha = 0$. In particular, the proof does not rely on a lower bound of the form $\Gamma(\alpha k + 1) \to \infty$ uniformly in α, which would fail at $\alpha = 0$. Instead, the bounded functions f and f' together with the geometric factor $|\lambda|^{k-1}$ provide the required uniform control.*

3.2. Existence and global uniqueness of the critical order

We now use Lemma 3.1 to show that the unique zero of the limiting profile persists as a unique zero of the perturbed problem on the entire interval $(0, 1)$.

Recall from Section 2 that there exists a unique $\alpha_0 \in (0, 1)$ satisfying

$$\psi(\alpha_0 + 1) = 0, \tag{3.18}$$

and that

$$q'(\alpha_0) = \frac{\psi_1(\alpha_0+1)}{\Gamma(\alpha_0+1)} > 0. \tag{3.19}$$

Theorem 3.3 (Unique critical-order branch)**.** *There exists $\lambda_0 > 0$ such that, for every*

$$0 < |\lambda| < \lambda_0,$$

the map

$$F_\lambda(\alpha) = E_\alpha(\lambda), \qquad 0 < \alpha < 1,$$

possesses exactly one critical fractional order

$$\alpha_*(\lambda) \in (0, 1).$$

Equivalently,

$$H(\alpha_*(\lambda), \lambda) = 0. \tag{3.20}$$

Moreover,

$$\alpha_*(\lambda) \longrightarrow \alpha_0 \qquad \text{as } \lambda \to 0. \tag{3.21}$$

Proof. Since $q'(\alpha_0) > 0$, the continuity of q' allows us to choose $\delta > 0$ such that

$$I := [\alpha_0 - \delta, \alpha_0 + \delta] \subset (0, 1)$$

and

$$q'(\alpha) \geq m_1 > 0, \qquad \alpha \in I, \tag{3.22}$$

for some constant m_1.

By Lemma 3.1,

$$\|\partial_\alpha H(\cdot, \lambda) - q'\|_\infty \longrightarrow 0 \qquad (\lambda \to 0).$$

Hence there exists $\lambda_1 > 0$ such that, whenever $|\lambda| < \lambda_1$,

$$\partial_\alpha H(\alpha, \lambda) \geq \frac{m_1}{2} > 0, \qquad \alpha \in I. \tag{3.23}$$

Thus $H(\cdot, \lambda)$ is strictly increasing on I and can have at most one zero there.

Choose δ smaller, if necessary, so that

$$q(\alpha_0 - \delta) < 0, \qquad q(\alpha_0 + \delta) > 0.$$

The uniform C^0 convergence in Lemma 3.1 implies that, for all sufficiently small $|\lambda|$,

$$H(\alpha_0 - \delta, \lambda) < 0, \qquad H(\alpha_0 + \delta, \lambda) > 0.$$

The intermediate value theorem therefore gives a zero $\alpha_*(\lambda) \in I$. By (3.23), this zero is unique in I.

It remains to exclude zeros outside I. Let

$$K := [0, 1] \setminus (\alpha_0 - \delta, \alpha_0 + \delta).$$

Since q has the unique zero α_0, the compactness of K yields

$$m_0 := \min_{\alpha \in K} |q(\alpha)| > 0. \tag{3.24}$$

Again by uniform convergence, there exists $\lambda_2 > 0$ such that

$$\|H(\cdot, \lambda) - q\|_\infty < \frac{m_0}{2}$$

whenever $|\lambda| < \lambda_2$. Consequently, for $\alpha \in K$,

$$|H(\alpha, \lambda)| \geq |q(\alpha)| - |H(\alpha, \lambda) - q(\alpha)| > \frac{m_0}{2},$$

so $H(\cdot, \lambda)$ has no zeros in K.

Taking

$$\lambda_0 := \min\{\lambda_1, \lambda_2\},$$

we conclude that $H(\cdot, \lambda)$ has exactly one zero in $(0, 1)$ for every $0 < |\lambda| < \lambda_0$.

Finally, let $\lambda_j \to 0$. Since $\alpha_*(\lambda_j) \in I$, every convergent subsequence has a limit $\bar{\alpha} \in I$. From

$$H(\alpha_*(\lambda_j), \lambda_j) = 0$$

and the uniform convergence $H(\cdot, \lambda_j) \to q$, we obtain $q(\bar{\alpha}) = 0$. The uniqueness of the zero of q implies $\bar{\alpha} = \alpha_0$. Hence the whole family satisfies (3.21). □

Remark 3.4. *Theorem 3.3 is stronger than a direct application of the implicit function theorem. The latter would yield only a locally unique branch near* $(\alpha_0, 0)$. *The uniform perturbation estimate excludes additional critical orders away from* α_0 *and therefore establishes uniqueness on the full interval* $(0, 1)$.

3.3. Regularity of the critical-order branch

The preceding theorem provides global uniqueness in the order variable. We next record the local regularity of the resulting branch.

Proposition 3.5. *After possibly decreasing λ_0, the function*

$$\lambda \longmapsto \alpha_*(\lambda)$$

extends continuously to $\lambda = 0$ by setting

$$\alpha_*(0) := \alpha_0,$$

and is continuously differentiable in a neighborhood of $\lambda = 0$. Moreover,

$$\alpha_*'(\lambda) = -\frac{\partial_\lambda H(\alpha_*(\lambda), \lambda)}{\partial_\alpha H(\alpha_*(\lambda), \lambda)}. \tag{3.25}$$

Proof. The series defining H and its first derivatives with respect to α and λ converge locally uniformly for $|\lambda| < 1$. Thus H is continuously differentiable in a neighborhood of $(\alpha_0, 0)$. Furthermore,

$$H(\alpha_0, 0) = q(\alpha_0) = 0$$

and, by (3.19),

$$\partial_\alpha H(\alpha_0, 0) = q'(\alpha_0) > 0.$$

The implicit function theorem therefore yields a unique C^1 function $\widetilde{\alpha}(\lambda)$ near $\lambda = 0$ satisfying

$$H(\widetilde{\alpha}(\lambda), \lambda) = 0, \qquad \widetilde{\alpha}(0) = \alpha_0.$$

By the global uniqueness established in Theorem 3.3, this local branch coincides with $\alpha_*(\lambda)$ for sufficiently small nonzero λ. Differentiating the identity

$$H(\alpha_*(\lambda), \lambda) = 0$$

gives (3.25). □

Theorem 3.3 shows that the degeneracy of $F_0(\alpha) \equiv 1$ does not lead to an arbitrary collection of critical orders once λ is perturbed away from zero. Instead, a single order is selected from the continuum, and this selected order converges to the distinguished digamma zero α_0. In the next section, we investigate the geometry of this critical point, prove its nondegeneracy, classify it according to the sign of λ, and derive the associated quadratic loss of order sensitivity.

4. Nondegeneracy and quadratic loss of order sensitivity

The previous section established that, for sufficiently small nonzero λ, the map

$$F_\lambda(\alpha) = E_\alpha(\lambda)$$

has exactly one critical order $\alpha_*(\lambda) \in (0, 1)$, and that $\alpha_*(\lambda) \to \alpha_0$ as $\lambda \to 0$, where α_0 is determined by

$$\psi(\alpha_0 + 1) = 0.$$

We now investigate the local geometry of this critical point. The main conclusions are that the critical order is nondegenerate, its type is determined by the sign of λ, and the usual first-order sensitivity with respect to the fractional order is replaced by a quadratic law at the critical branch.

Throughout this section, $\lambda_0 > 0$ is chosen sufficiently small so that the conclusions of Theorem 3.3 and Proposition 3.5 hold for $0 < |\lambda| < \lambda_0$.

4.1. Nondegeneracy of the critical order

Recall that

$$F'_\lambda(\alpha) = -\lambda H(\alpha, \lambda), \tag{4.1}$$

where

$$H(\alpha, \lambda) = \sum_{k=1}^{\infty} k\lambda^{k-1} \frac{\psi(\alpha k + 1)}{\Gamma(\alpha k + 1)}.$$

Differentiating (4.1) with respect to α gives

$$F''_\lambda(\alpha) = -\lambda\, \partial_\alpha H(\alpha, \lambda). \tag{4.2}$$

The C^1 perturbation established in Lemma 3.1 immediately yields the following nondegeneracy result.

Theorem 4.1 (Nondegeneracy and classification)**.** *There exists $\lambda_1 \in (0, \lambda_0]$ such that, for every $0 < |\lambda| < \lambda_1$, the unique critical order $\alpha_*(\lambda)$ is nondegenerate:*

$$F''_\lambda(\alpha_*(\lambda)) \neq 0. \tag{4.3}$$

More precisely,

$$\operatorname{sgn} F''_\lambda(\alpha_*(\lambda)) = -\operatorname{sgn}(\lambda). \tag{4.4}$$

Consequently,

$$\lambda < 0 \quad \Longrightarrow \quad \alpha_*(\lambda) \text{ is a strict local minimum},$$

whereas

$$\lambda > 0 \quad \Longrightarrow \quad \alpha_*(\lambda) \text{ is a strict local maximum}.$$

Proof. From Lemma 3.1,

$$\partial_\alpha H(\cdot, \lambda) \longrightarrow q' \quad \text{uniformly on } [0, 1] \quad \text{as } \lambda \to 0.$$

Furthermore,

$$\alpha_*(\lambda) \longrightarrow \alpha_0.$$

Hence

$$\partial_\alpha H(\alpha_*(\lambda), \lambda) \longrightarrow q'(\alpha_0) \qquad (\lambda \to 0). \tag{4.5}$$

Since

$$q'(\alpha_0) = \frac{\psi_1(\alpha_0 + 1)}{\Gamma(\alpha_0 + 1)} > 0,$$

there exists $\lambda_1 > 0$ such that

$$\partial_\alpha H(\alpha_*(\lambda), \lambda) > 0 \tag{4.6}$$

whenever $0 < |\lambda| < \lambda_1$.

Using (4.2),

$$F''_\lambda(\alpha_*(\lambda)) = -\lambda \partial_\alpha H(\alpha_*(\lambda), \lambda).$$

The second factor is positive by (4.6). Therefore $F''_\lambda(\alpha_*(\lambda))$ is nonzero and has a sign opposite to that of λ. The classification follows from the standard second-derivative criterion. □

Remark 4.2. *Theorem 4.1 identifies a qualitative change in the order profile across $\lambda = 0$. At $\lambda = 0$, $F_0(\alpha) \equiv 1$, so every order is formally critical. For $\lambda \neq 0$ sufficiently small, this complete degeneracy collapses to a single nondegenerate critical order. The sign of the perturbation then determines whether the selected critical point is a minimum or a maximum.*

4.2. Quadratic loss of first-order sensitivity

Away from a critical order, a perturbation of α produces, to the leading order, a linear variation of F_λ. At $\alpha = \alpha_*(\lambda)$, however,

$$F'_\lambda(\alpha_*(\lambda)) = 0,$$

and the linear contribution disappears. The nondegeneracy established above therefore changes the local scaling from first to second order.

Theorem 4.3 (Quadratic critical scaling)**.** *Let $0 < |\lambda| < \lambda_1$ be fixed and write*

$$\alpha_* = \alpha_*(\lambda).$$

Then

$$F_\lambda(\alpha_* + h) - F_\lambda(\alpha_*) = \frac{1}{2} F''_\lambda(\alpha_*) h^2 + o(h^2) \qquad (h \to 0). \tag{4.7}$$

In particular,

$$\lim_{h \to 0} \frac{F_\lambda(\alpha_* + h) - F_\lambda(\alpha_*)}{h^2} = \frac{1}{2} F''_\lambda(\alpha_*) \neq 0. \tag{4.8}$$

Consequently, there exist constants $c_\lambda, C_\lambda > 0$ and $\delta_\lambda > 0$ such that

$$c_\lambda |h|^2 \leq |F_\lambda(\alpha_* + h) - F_\lambda(\alpha_*)| \leq C_\lambda |h|^2 \tag{4.9}$$

whenever $|h| < \delta_\lambda$ and $\alpha_* + h \in (0, 1)$.

Proof. For fixed nonzero λ with $|\lambda| < \lambda_1$, the function $\alpha \mapsto F_\lambda(\alpha)$ is twice continuously differentiable in a neighborhood of α_*. Taylor's formula gives

$$F_\lambda(\alpha_* + h) = F_\lambda(\alpha_*) + F'_\lambda(\alpha_*) h + \frac{1}{2} F''_\lambda(\alpha_*) h^2 + o(h^2).$$

Since α_* is critical,

$$F'_\lambda(\alpha_*) = 0,$$

and (4.7) follows.

By Theorem 4.1, $F''_\lambda(\alpha_*) \neq 0$, which proves (4.8). Hence

$$\frac{|F_\lambda(\alpha_* + h) - F_\lambda(\alpha_*)|}{|h|^2} \longrightarrow \frac{1}{2} |F''_\lambda(\alpha_*)| > 0.$$

The two-sided estimate (4.9) follows by choosing $\delta_\lambda > 0$ sufficiently small. □

Remark 4.4 (Loss of first-order order sensitivity)**.** *At a regular order $\bar{\alpha}$ for which $F'_\lambda(\bar{\alpha}) \neq 0$,*

$$F_\lambda(\bar{\alpha} + h) - F_\lambda(\bar{\alpha}) = F'_\lambda(\bar{\alpha})h + o(h),$$

so the local response is of order $|h|$. In contrast, Theorem 4.3 shows that, at the critical order, the response is of order $|h|^2$. Thus the vanishing of the first-order order sensitivity produces a genuine quadratic degeneracy of the order-to-response map.

The preceding result also has an immediate local inverse interpretation. Near a regular order, an output perturbation of size ε corresponds locally to an order displacement of size $O(\varepsilon)$. At a nondegenerate critical order, the corresponding scale becomes $O(\varepsilon^{1/2})$.

Corollary 4.5 (Square-root inverse scaling)**.** *Under the assumptions of Theorem 4.3, set*

$$\Delta F = |F_\lambda(\alpha_* + h) - F_\lambda(\alpha_*)|.$$

Then, for sufficiently small h,

$$\frac{1}{\sqrt{C_\lambda}}(\Delta F)^{1/2} \leq |h| \leq \frac{1}{\sqrt{c_\lambda}}(\Delta F)^{1/2}. \tag{4.10}$$

Proof. This follows immediately by taking square roots in (4.9). □

Remark 4.6. *The term "loss of sensitivity" is used here in a precise local sense: the first derivative with respect to the fractional order vanishes at $\alpha_*(\lambda)$. The result does not, by itself, constitute a global inverse-problem or structural-identifiability theorem. Rather, it describes the local geometry of the forward order-to-response map near its unique critical order.*

4.3. Asymptotic location of the critical branch

We finally determine the first-order displacement of $\alpha_*(\lambda)$ from the limiting digamma zero α_0.

Theorem 4.7 (First-order branch asymptotics)**.** *As $\lambda \to 0$,*

$$\alpha_*(\lambda) = \alpha_0 + \kappa_0\lambda + O(\lambda^2), \tag{4.11}$$

where

$$\kappa_0 = -\frac{2\,\psi(2\alpha_0 + 1)/\Gamma(2\alpha_0 + 1)}{\psi_1(\alpha_0 + 1)/\Gamma(\alpha_0 + 1)}. \tag{4.12}$$

Equivalently,

$$\kappa_0 = -\frac{2\,\Gamma(\alpha_0 + 1)\psi(2\alpha_0 + 1)}{\Gamma(2\alpha_0 + 1)\psi_1(\alpha_0 + 1)}. \tag{4.13}$$

Proof. By Proposition 3.5,

$$H(\alpha_*(\lambda), \lambda) = 0$$

and the branch is differentiable at $\lambda = 0$. Therefore,

$$\alpha'_*(0) = -\frac{\partial_\lambda H(\alpha_0, 0)}{\partial_\alpha H(\alpha_0, 0)}. \tag{4.14}$$

From

$$H(\alpha,\lambda)=\sum_{k=1}^{\infty}k\lambda^{k-1}f(\alpha k+1),$$

we obtain

$$\partial_\lambda H(\alpha,\lambda)=\sum_{k=2}^{\infty}k(k-1)\lambda^{k-2}f(\alpha k+1).$$

At $\lambda=0$, only the term $k=2$ survives, giving

$$\partial_\lambda H(\alpha_0,0)=2f(2\alpha_0+1)=\frac{2\psi(2\alpha_0+1)}{\Gamma(2\alpha_0+1)}. \tag{4.15}$$

On the other hand,

$$\partial_\alpha H(\alpha_0,0)=q'(\alpha_0)=\frac{\psi_1(\alpha_0+1)}{\Gamma(\alpha_0+1)}, \tag{4.16}$$

where we used $\psi(\alpha_0+1)=0$.

Substitution of (4.15) and (4.16) into (4.14) gives (4.12)–(4.13).

Finally, the defining series is smooth with respect to (α,λ) in a neighborhood of $(\alpha_0,0)$, so the implicit branch is sufficiently smooth there. Taylor expansion in λ yields

$$\alpha_*(\lambda)=\alpha_0+\alpha_*'(0)\lambda+O(\lambda^2),$$

which proves (4.11). □

The coefficient κ_0 has a definite sign. Since α_0+1 is the unique positive zero of ψ and ψ is strictly increasing,

$$2\alpha_0+1>\alpha_0+1 \quad\Longrightarrow\quad \psi(2\alpha_0+1)>0.$$

Moreover,

$$\Gamma(\alpha_0+1)>0, \qquad \Gamma(2\alpha_0+1)>0, \qquad \psi_1(\alpha_0+1)>0.$$

Thus (4.13) implies

$$\kappa_0<0. \tag{4.17}$$

We therefore obtain the following directional information.

Corollary 4.8. *For sufficiently small $|\lambda|$,*

$$\lambda>0 \quad\Longrightarrow\quad \alpha_*(\lambda)<\alpha_0,$$

whereas

$$\lambda<0 \quad\Longrightarrow\quad \alpha_*(\lambda)>\alpha_0,$$

to first order in λ.

Proof. The assertion follows from

$$\alpha_*(\lambda)-\alpha_0=\kappa_0\lambda+O(\lambda^2)$$

and $\kappa_0<0$. □

4.4. A normalized curvature law

The preceding results also identify the leading curvature of the critical profile as $\lambda \to 0$.

Proposition 4.9. *The curvature at the critical branch satisfies*

$$F''_\lambda(\alpha_*(\lambda)) = -\lambda q'(\alpha_0) + o(\lambda) \qquad (\lambda \to 0). \tag{4.18}$$

Equivalently,

$$\lim_{\lambda\to 0,\ \lambda\neq 0} \frac{F''_\lambda(\alpha_*(\lambda))}{\lambda} = -\frac{\psi_1(\alpha_0+1)}{\Gamma(\alpha_0+1)} < 0. \tag{4.19}$$

Proof. From (4.2),

$$\frac{F''_\lambda(\alpha_*(\lambda))}{\lambda} = -\partial_\alpha H(\alpha_*(\lambda), \lambda).$$

Since

$$\alpha_*(\lambda) \to \alpha_0$$

and

$$\partial_\alpha H(\cdot, \lambda) \to q'$$

uniformly as $\lambda \to 0$, we obtain

$$\partial_\alpha H(\alpha_*(\lambda), \lambda) \to q'(\alpha_0).$$

Using

$$q'(\alpha_0) = \frac{\psi_1(\alpha_0+1)}{\Gamma(\alpha_0+1)} > 0$$

proves both assertions. □

Proposition 4.9 shows that the critical point becomes progressively flatter as $\lambda \to 0$, with curvature of exact order $|\lambda|$. Thus two distinct degeneracies coexist near the unperturbed state: the first-order sensitivity vanishes at $\alpha_*(\lambda)$, while the second-order sensitivity itself tends to zero linearly as $\lambda \to 0$. Nevertheless, for every fixed sufficiently small $\lambda \neq 0$, the critical point remains nondegenerate and the local order response is genuinely quadratic.

5. Dynamical critical-order curves

We now return to the complete order-to-observation map

$$F_{\lambda,t}(\alpha) = E_\alpha(\lambda t^\alpha), \qquad 0 < \alpha < 1, \quad t > 0. \tag{5.1}$$

The analysis of Sections 3 and 4 corresponds to the reference time $t = 1$. Here we show that the critical-order mechanism persists for a nontrivial interval of observation times and generates a smooth two-parameter critical surface.

Recall that

$$G(\alpha, \lambda, t) = -\frac{1}{\lambda}\partial_\alpha F_{\lambda,t}(\alpha), \qquad \lambda \neq 0, \tag{5.2}$$

with

$$G(\alpha, \lambda, t) = \sum_{k=1}^{\infty} k\lambda^{k-1} t^{\alpha k} \frac{\psi(\alpha k + 1) - \log t}{\Gamma(\alpha k + 1)}. \tag{5.3}$$

Its continuous extension to $\lambda = 0$ is

$$G_0(\alpha, t) = \frac{t^\alpha[\psi(\alpha+1) - \log t]}{\Gamma(\alpha+1)}. \tag{5.4}$$

5.1. Unperturbed critical curve and uniform perturbation

Since

$$\frac{t^\alpha}{\Gamma(\alpha+1)} > 0,$$

the equation

$$G_0(\alpha, t) = 0$$

is equivalent to

$$\psi(\alpha+1) = \log t.$$

Because ψ is strictly increasing and

$$\psi(1) = -\gamma, \qquad \psi(2) = 1 - \gamma,$$

there exists a unique solution $\alpha_0(t) \in (0, 1)$ if and only if

$$e^{-\gamma} < t < e^{1-\gamma}. \tag{5.5}$$

For such t,

$$\alpha_0(t) = \psi^{-1}(\log t) - 1. \tag{5.6}$$

Differentiation gives

$$\alpha_0'(t) = \frac{1}{t\,\psi_1(\alpha_0(t)+1)} > 0. \tag{5.7}$$

Hence α_0 is strictly increasing, with

$$\alpha_0(t) \to 0 \quad \text{as } t \downarrow e^{-\gamma}$$

and

$$\alpha_0(t) \to 1 \quad \text{as } t \uparrow e^{1-\gamma}.$$

We next extend the perturbation argument of Lemma 3.1 uniformly with respect to t.

lemma 5.1 (Uniform dynamical perturbation)**.** *Let*

$$J = [t_-, t_+] \Subset \left(e^{-\gamma}, e^{1-\gamma}\right).$$

Then there exist constants $\lambda_J > 0$ and $C_J > 0$ such that

$$\sup_{t\in J} \|G(\cdot, \lambda, t) - G_0(\cdot, t)\|_{C^1([0,1])} \le C_J|\lambda| \tag{5.8}$$

whenever $|\lambda| < \lambda_J$.

Proof. For $t \in J$, define

$$b_t(x) = \frac{\psi(x) - \log t}{\Gamma(x)}, \qquad c_t(x) = \frac{\psi_1(x) - [\psi(x) - \log t]^2}{\Gamma(x)}.$$

Since J is compact and $\Gamma(x)$ dominates the logarithmic growth of ψ and ψ_1, there exist finite constants

$$M_{0,J} = \sup_{\substack{x \geq 1 \\ t \in J}} |b_t(x)|, \qquad M_{1,J} = \sup_{\substack{x \geq 1 \\ t \in J}} |c_t(x)|.$$

Let

$$T_J = \max\{1, t_+\}.$$

Separating the $k = 1$ term in (5.3),

$$G - G_0 = \sum_{k=2}^{\infty} k\lambda^{k-1} t^{\alpha k} b_t(\alpha k + 1).$$

Since

$$t^{\alpha k} \leq T_J^k,$$

we obtain

$$|G - G_0| \leq M_{0,J} T_J \sum_{k=2}^{\infty} k(|\lambda| T_J)^{k-1}.$$

Choose λ_J so that $|\lambda| T_J \leq r < 1$. The geometric-series estimate used in Section 3 then gives

$$\sup_{(\alpha,t) \in [0,1] \times J} |G - G_0| \leq C_J^{(0)} |\lambda|.$$

Differentiating with respect to α yields

$$\partial_\alpha G = \sum_{k=1}^{\infty} k^2 \lambda^{k-1} t^{\alpha k} c_t(\alpha k + 1).$$

Again separating the first term,

$$\partial_\alpha G - \partial_\alpha G_0 = \sum_{k=2}^{\infty} k^2 \lambda^{k-1} t^{\alpha k} c_t(\alpha k + 1),$$

and therefore

$$|\partial_\alpha G - \partial_\alpha G_0| \leq M_{1,J} T_J \sum_{k=2}^{\infty} k^2 (|\lambda| T_J)^{k-1}.$$

The estimate from (3.16) gives

$$\sup_{(\alpha,t) \in [0,1] \times J} |\partial_\alpha G - \partial_\alpha G_0| \leq C_J^{(1)} |\lambda|.$$

Combining the two bounds proves (5.8). □

5.2. Existence and uniqueness of the dynamical critical surface

Theorem 5.2 (Unique critical-order surface)**.** *Let*

$$J \Subset \left(e^{-\gamma}, e^{1-\gamma}\right).$$

Then there exists $\lambda_J > 0$ such that, for every

$$0 < |\lambda| < \lambda_J$$

and every $t \in J$, the map

$$\alpha \mapsto F_{\lambda,t}(\alpha)$$

has exactly one critical order

$$\alpha_*(\lambda, t) \in (0, 1).$$

Moreover,

$$\sup_{t\in J} |\alpha_*(\lambda, t) - \alpha_0(t)| \longrightarrow 0 \qquad (\lambda \to 0), \tag{5.9}$$

and, after defining

$$\alpha_*(0, t) := \alpha_0(t),$$

the map

$$(\lambda, t) \mapsto \alpha_*(\lambda, t)$$

is C^1 near $\{0\} \times J$.

Proof. At the unperturbed zero curve,

$$\psi(\alpha_0(t) + 1) = \log t.$$

Differentiating (5.4) with respect to α gives

$$\partial_\alpha G_0(\alpha_0(t), t) = \frac{t^{\alpha_0(t)}\psi_1(\alpha_0(t) + 1)}{\Gamma(\alpha_0(t) + 1)} > 0. \tag{5.10}$$

Since J is compact, there exist $\delta > 0$ and $m_1 > 0$ such that

$$\partial_\alpha G_0(\alpha, t) \geq m_1 > 0$$

whenever

$$|\alpha - \alpha_0(t)| \leq \delta, \qquad t \in J.$$

By Lemma 5.1, for sufficiently small $|\lambda|$,

$$\partial_\alpha G(\alpha, \lambda, t) \geq \frac{m_1}{2} > 0$$

in the same tubular neighborhood. Thus $G(\cdot, \lambda, t)$ is strictly increasing there.

The signs of G_0 at $\alpha_0(t) - \delta$ and $\alpha_0(t) + \delta$ are negative and positive, respectively, uniformly for $t \in J$. Uniform convergence therefore preserves these signs for small $|\lambda|$, and the intermediate value theorem yields exactly one zero in the tube.

To exclude additional zeros, set

$$K_\delta = \{(\alpha, t) \in [0,1] \times J : |\alpha - \alpha_0(t)| \geq \delta\}.$$

Since G_0 has no zeros on this compact set,

$$m_0 = \min_{K_\delta} |G_0| > 0.$$

For sufficiently small $|\lambda|$,

$$|G - G_0| < \frac{m_0}{2},$$

and therefore

$$|G| > \frac{m_0}{2}$$

on K_δ. Hence no other critical orders exist.

The uniform convergence (5.9) follows from the same compactness argument. Finally, (5.10) and the implicit function theorem give a local C^1 surface; global uniqueness makes the local representations agree. □

5.3. Branch displacement and persistence of quadratic degeneracy

The first-order displacement of the critical surface can be computed explicitly.

Theorem 5.3. *For each fixed*

$$t \in \left(e^{-\gamma}, e^{1-\gamma}\right),$$

the critical branch satisfies

$$\alpha_*(\lambda, t) = \alpha_0(t) + \kappa(t)\lambda + O(\lambda^2), \qquad \lambda \to 0, \tag{5.11}$$

where

$$\kappa(t) = -\frac{2t^{\alpha_0(t)}\Gamma(\alpha_0(t)+1)\left[\psi(2\alpha_0(t)+1) - \psi(\alpha_0(t)+1)\right]}{\Gamma(2\alpha_0(t)+1)\psi_1(\alpha_0(t)+1)}. \tag{5.12}$$

In particular,

$$\kappa(t) < 0.$$

Furthermore, for $t \in J$ and sufficiently small nonzero λ, the critical order is nondegenerate and

$$\operatorname{sgn}\left[\partial_{\alpha\alpha} F_{\lambda,t}(\alpha_*(\lambda, t))\right] = -\operatorname{sgn}(\lambda). \tag{5.13}$$

Hence it is a strict minimum for $\lambda < 0$ and a strict maximum for $\lambda > 0$.

Finally, for each fixed (λ, t) in this regime,

$$F_{\lambda,t}(\alpha_* + h) - F_{\lambda,t}(\alpha_*) = \frac{1}{2}\partial_{\alpha\alpha} F_{\lambda,t}(\alpha_*)h^2 + o(h^2), \tag{5.14}$$

and therefore

$$\left|F_{\lambda,t}(\alpha_* + h) - F_{\lambda,t}(\alpha_*)\right| \asymp |h|^2. \tag{5.15}$$

Proof. Differentiating

$$G(\alpha_*(\lambda, t), \lambda, t) = 0$$

with respect to λ at $\lambda = 0$ gives

$$\partial_\lambda \alpha_*(0, t) = -\frac{G_\lambda(\alpha_0(t), 0, t)}{G_\alpha(\alpha_0(t), 0, t)}.$$

From (5.3),

$$G_\lambda(\alpha, 0, t) = 2t^{2\alpha} \frac{\psi(2\alpha + 1) - \log t}{\Gamma(2\alpha + 1)}.$$

Using

$$\log t = \psi(\alpha_0(t) + 1)$$

and (5.10) yields (5.12). Since ψ is strictly increasing,

$$\psi(2\alpha_0(t) + 1) > \psi(\alpha_0(t) + 1),$$

so $\kappa(t) < 0$. Taylor expansion of the implicit branch gives (5.11).

Next,

$$\partial_\alpha F_{\lambda,t}(\alpha) = -\lambda G(\alpha, \lambda, t),$$

and therefore

$$\partial_{\alpha\alpha} F_{\lambda,t}(\alpha) = -\lambda \partial_\alpha G(\alpha, \lambda, t).$$

By Lemma 5.1 and (5.9),

$$\partial_\alpha G(\alpha_*(\lambda, t), \lambda, t) \longrightarrow \partial_\alpha G_0(\alpha_0(t), t) > 0$$

uniformly on compact J. Hence, (5.13) follows.

Finally,

$$\partial_\alpha F_{\lambda,t}(\alpha_*) = 0$$

and

$$\partial_{\alpha\alpha} F_{\lambda,t}(\alpha_*) \neq 0.$$

Taylor's theorem gives (5.14), and the two-sided estimate (5.15) follows immediately. □

Remark 5.4. *The observation time determines the unperturbed critical curve $\alpha_0(t)$, while the dynamical coefficient λ shifts this curve to $\alpha_*(\lambda, t)$. The critical response remains nondegenerate for nonzero sufficiently small λ, but its first-order sensitivity with respect to the fractional order vanishes, leaving a quadratic response in the order perturbation.*

The next section provides numerical computations for the critical curve, the first-order branch displacement, and the quadratic sensitivity law.

6. Numerical verification

This section provides numerical verification of the principal analytical predictions obtained in Sections 3–5. The computations are performed with high-precision arithmetic. The Mittag–Leffler function and the series defining G are evaluated to high numerical precision, while the critical order $\alpha_*(\lambda, t)$ is obtained by solving

$$G(\alpha, \lambda, t) = 0$$

inside $(0, 1)$. No finite-difference approximation of the order derivative is required in locating the critical points.

6.1. The unperturbed critical curve

We first verify the explicit curve

$$\alpha_0(t) = \psi^{-1}(\log t) - 1, \qquad e^{-\gamma} < t < e^{1-\gamma}. \tag{6.1}$$

The admissible observation-time interval is numerically

$$0.5614594836 < t < 1.5262051116.$$

Representative values are reported in Table 1.

Table 1. Unperturbed critical order and first-order branch coefficients.

t	$\alpha_0(t)$	$\kappa(t)$
0.70	0.148256674	– 0.268416302
0.85	0.305900569	– 0.505138449
1.00	0.461632145	– 0.702122682
1.20	0.667349966	– 0.918068172
1.40	0.871619408	– 1.091734762

The values confirm the strict increase predicted by (5.7). In particular,

$$\alpha_0(1) = 0.461632144968\ldots,$$

which is the distinguished order determined by the unique zero of $\psi(\alpha + 1)$ in $(0, 1)$.

Figure 1 displays the complete unperturbed critical curve over the admissible time interval.

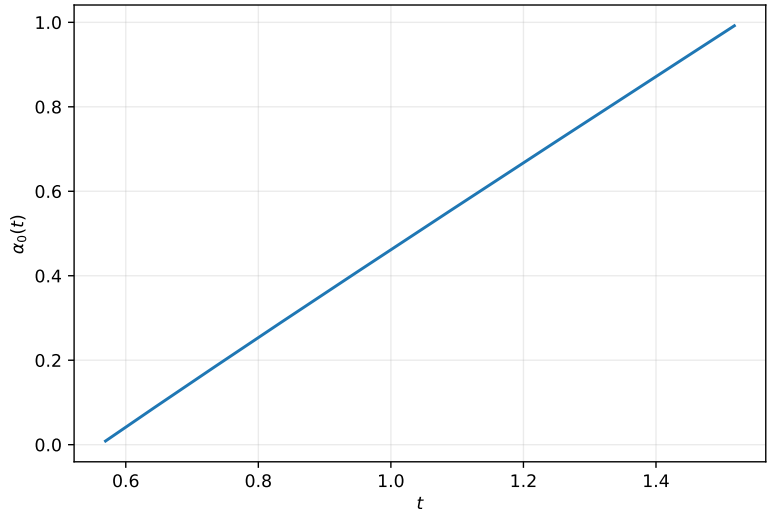


Figure 1. The unperturbed critical-order curve $\alpha_0(t) = \psi^{-1}(\log t) - 1$.

6.2. *Accuracy of the first-order branch approximation*

We next compare the numerically computed critical order with the first-order prediction

$$\alpha_*(\lambda, t) = \alpha_0(t) + \kappa(t)\lambda + O(\lambda^2). \tag{6.2}$$

Table 2 gives representative results for three observation times.

Table 2. Exact numerical critical orders and first-order approximations.

t	λ	$\alpha_*(\lambda, t)$	$\alpha_0(t) + \kappa(t)\lambda$	absolute error
0.85	−0.10	0.358892373	0.356414414	2.4780×10^{-3}
0.85	−0.05	0.331780360	0.331157492	6.2287×10^{-4}
0.85	−0.02	0.316103180	0.316003338	9.9842×10^{-5}
0.85	0.02	0.295897734	0.295797800	9.9934×10^{-5}
1.00	−0.10	0.533795994	0.531844413	1.9516×10^{-3}
1.00	−0.05	0.497251071	0.496738279	5.1279×10^{-4}
1.00	0.05	0.427081069	0.426526011	5.5506×10^{-4}
1.00	0.10	0.393709304	0.391419877	2.2894×10^{-3}
1.20	−0.05	0.713404958	0.713253375	1.5158×10^{-4}
1.20	−0.02	0.685740174	0.685711330	2.8844×10^{-5}
1.20	0.02	0.649023454	0.648988603	3.4851×10^{-5}
1.20	0.05	0.621691827	0.621446558	2.4527×10^{-4}

The decrease in the approximation error as $|\lambda|$ decreases is consistent with the $O(\lambda^2)$ remainder in (6.2). The sign of the displacement is also correctly predicted by $\kappa(t) < 0$: positive λ moves the critical order to the left of $\alpha_0(t)$, whereas negative λ moves it to the right.

6.3. *Classification of the critical point*

We now examine the local geometry of

$$\alpha \longmapsto F_{\lambda,1}(\alpha)$$

for the symmetric parameter choices $\lambda = \pm 0.05$. The corresponding critical orders are

$$\alpha_*(-0.05, 1) = 0.497251071402\ldots$$

and

$$\alpha_*(0.05, 1) = 0.427081069396\ldots.$$

Figure 2 shows the local profiles. For $\lambda < 0$, the critical point is a strict minimum, whereas for $\lambda > 0$, it is a strict maximum, in agreement with Theorem 5.3.

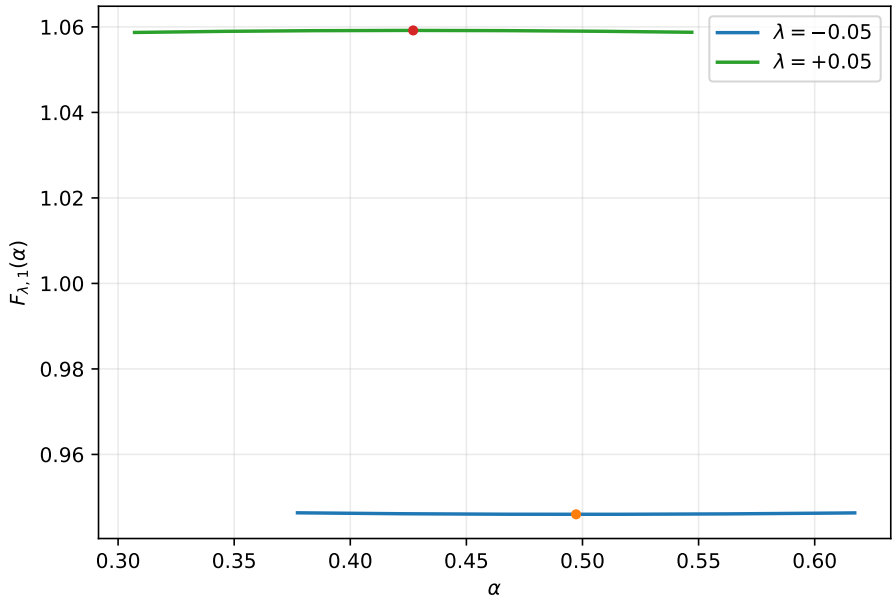


Figure 2. Local profiles of $F_{\lambda,1}(\alpha)$ near their critical orders for $\lambda = -0.05$ and $\lambda = 0.05$.

6.4. *Quadratic loss of order sensitivity*

Finally, we test the local scaling law

$$\left|F_{\lambda,t}(\alpha_* + h) - F_{\lambda,t}(\alpha_*)\right| \asymp |h|^2. \tag{6.3}$$

We again take $t = 1$ and $\lambda = \pm 0.05$. Define

$$\Delta(h) = \left|F_{\lambda,1}(\alpha_* + h) - F_{\lambda,1}(\alpha_*)\right|.$$

The observed convergence orders are calculated from

$$p(h_j, h_{j+1}) = \frac{\log[\Delta(h_{j+1})/\Delta(h_j)]}{\log(h_{j+1}/h_j)}. \tag{6.4}$$

For $\lambda = -0.05$, the computed values are

$$p = 1.999136, \quad 1.999914, \quad 1.999991, \quad 1.999999,$$

as h decreases from 10^{-2} to 10^{-6}. For $\lambda = 0.05$, the corresponding values are

$$p = 1.998443, \quad 1.999845, \quad 1.999984, \quad 1.999998.$$

Thus the observed exponent converges rapidly to the theoretical value 2.

Figure 3 gives the corresponding log–log representation. The numerical curves become parallel to the reference slope-2 line as $h \to 0$.

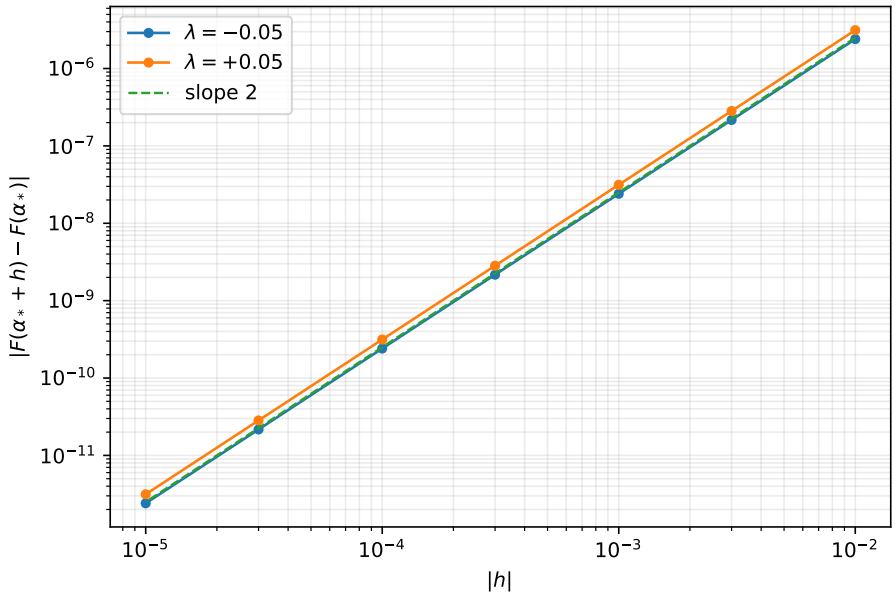


Figure 3. Log–log verification of the quadratic critical-order scaling for $\lambda = \pm 0.05$ at $t = 1$.

The computations therefore support three independent analytical predictions: the monotone motion of the unperturbed critical order with observation time, the first-order displacement of the perturbed critical surface, and the quadratic loss of order sensitivity at a nondegenerate critical order. In particular, the numerical results distinguish clearly between the location of the critical order and the local geometry of the order-to-observation map around it.

7. Critical orders in short- and long-time approximations

The reviewer-requested comparison must distinguish the domain of validity of an approximation from the stationary points of its truncated expression. We write $a = -\lambda > 0$ in the relaxation regime.

7.1. Small-argument approximation

The first nonconstant truncation is

$$S_{\lambda,t}(\alpha) = 1 + \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}, \qquad \lambda \neq 0. \tag{7.1}$$

Its order derivative is exactly

$$\partial_\alpha S_{\lambda,t}(\alpha) = \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}[\log t - \psi(1+\alpha)]. \tag{7.2}$$

Since ψ is strictly increasing, this truncation has precisely one stationary order in $(0,1)$ if and only if $e^{-\gamma} < t < e^{1-\gamma}$, namely, $\alpha_{\text{short}}(t) = \psi^{-1}(\log t) - 1 = \alpha_0(t)$. At that point,

$$\partial_{\alpha\alpha} S_{\lambda,t}(\alpha_{\text{short}}) = -\frac{\lambda t^{\alpha_{\text{short}}}}{\Gamma(1+\alpha_{\text{short}})}\psi_1(1+\alpha_{\text{short}}) \neq 0. \tag{7.3}$$

Thus it is a minimum for $\lambda < 0$ and a maximum for $\lambda > 0$. Theorem 5.3 yields, for each fixed admissible t,

$$\alpha_*(\lambda, t) - \alpha_{\text{short}}(t) = \kappa(t)\lambda + O(\lambda^2), \qquad \lambda \to 0, \tag{7.4}$$

with $\kappa(t)$ defined in that theorem. This comparison is uniform on compact time intervals as allowed by the perturbation estimates, but the small-argument truncation itself should be used only when $|\lambda| t^\alpha$ is small.

7.2. Large-argument relaxation approximation

For $\lambda = -a < 0$ and fixed $0 < \alpha < 1$, the leading large-argument approximation is

$$L_{a,t}(\alpha) = \frac{1}{at^\alpha \Gamma(1-\alpha)}, \qquad at^\alpha \gg 1. \tag{7.5}$$

Direct differentiation of this *truncated expression* gives

$$\partial_\alpha L_{a,t}(\alpha) = L_{a,t}(\alpha)[\psi(1-\alpha) - \log t]. \tag{7.6}$$

Because $\psi(1-\alpha) < -\gamma$ on $(0,1)$, the leading truncation has no stationary order for $t \geq e^{-\gamma}$. Formally, it has one stationary order $1 - \psi^{-1}(\log t)$ when $0 < t < e^{-\gamma}$, where its second derivative is $-L_{a,t}\psi_1(1-\alpha) < 0$. This formal stationary order lies outside the large-time limit $t \to \infty$ with fixed a. The statement concerns the leading truncation, and does not assert absence of critical orders for the complete Mittag–Leffler function.

The large-argument expansion is not uniform as α approaches the endpoints of $(0,1)$, nor does a function-value approximation alone justify differentiating its error term. Consequently, the comparison in Figure 4 is explicitly numerical, restricted to the displayed interior orders, and is not used as a theorem on the critical points of the full function. In the short-argument regime, the truncation shifts the full critical order by the amount above and in the large-time regime, the leading truncation can conceal fine sensitivity features unless the approximation error in the order derivative is also controlled.

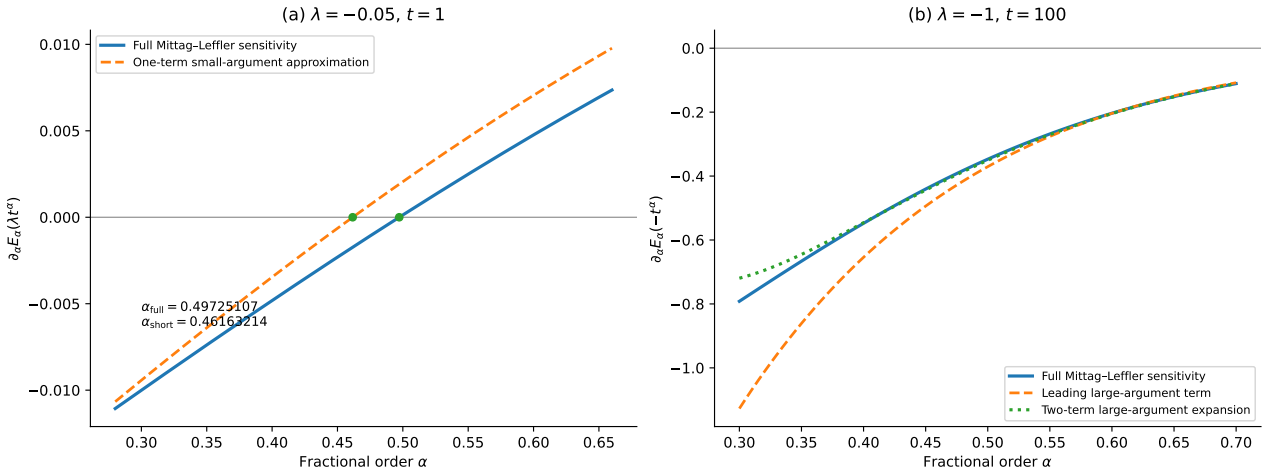


Figure 4. Order sensitivity of the full Mittag–Leffler response and its approximations. (a) Small-argument regime $\lambda = -0.05$, $t = 1$: the zero of the complete sensitivity is compared with the stationary order of the first nonconstant truncation. (b) Large-argument relaxation regime $\lambda = -1$, $t = 100$: the full order sensitivity is compared with derivatives of the one- and two-term asymptotic expressions for interior orders. The curves illustrate approximation differences and do not imply a global absence theorem for the full response.

7.3. Numerical comparison of the full and truncated sensitivities

The plotted curves were generated directly from the defining Mittag–Leffler series in the small-argument panel and from an integral representation of the relaxation response in the large-argument panel; the numerical procedure and independent checks are supplied with the figure-generation script. The plotted small-argument roots are approximately $\alpha_{\text{full}} = 0.49725107$ and $\alpha_{\text{short}} = 0.46163214$.

8. Conclusions

This paper investigated the dependence of Mittag–Leffler dynamics on the fractional order, with particular emphasis on critical orders at which first-order sensitivity with respect to the order vanishes. Rather than treating differentiability with respect to the fractional parameter as the main issue, the analysis focused on the local geometry of the order-to-observation map and on the emergence, uniqueness, and nondegeneracy of its critical points.

For the reference family

$$F_\lambda(\alpha) = E_\alpha(\lambda),$$

we showed that, for sufficiently small nonzero $|\lambda|$, there exists a unique critical order $\alpha_*(\lambda) \in (0, 1)$ emerging from the distinguished value α_0 determined by

$$\psi(\alpha_0 + 1) = 0.$$

The critical branch was obtained through a uniform C^1 perturbation argument, which also excludes additional critical points in the full interval $(0, 1)$. The critical point is nondegenerate: it is a strict minimum for $\lambda < 0$ and a strict maximum for $\lambda > 0$. Consequently, the usual first-order dependence

on perturbations of the fractional order is lost at $\alpha_*(\lambda)$ and is replaced locally by the quadratic law

$$|F_\lambda(\alpha_*(\lambda) + h) - F_\lambda(\alpha_*(\lambda))| \asymp |h|^2.$$

The first-order asymptotic displacement of the critical branch with respect to λ was also determined explicitly.

The analysis was subsequently extended to the dynamical observation map

$$F_{\lambda,t}(\alpha) = E_\alpha(\lambda t^\alpha).$$

For the unperturbed problem, the critical orders form the explicit curve

$$\alpha_0(t) = \psi^{-1}(\log t) - 1, \qquad e^{-\gamma} < t < e^{1-\gamma},$$

which is strictly increasing with the observation time. On every compact subinterval of this time range, we proved that the curve persists, for sufficiently small $|\lambda|$, as a unique smooth critical surface $\alpha_*(\lambda, t)$. Nondegeneracy and quadratic order sensitivity persist along this surface. Thus the vanishing of first-order order sensitivity is not confined to a single Mittag–Leffler profile or to the particular observation time $t = 1$, but survives under a genuine dynamical perturbation.

The numerical experiments were designed as independent checks of these analytical predictions. They reproduced the monotone critical curve, confirmed the first-order displacement

$$\alpha_*(\lambda, t) = \alpha_0(t) + \kappa(t)\lambda + O(\lambda^2),$$

verified the minimum–maximum classification, and recovered an observed local convergence exponent approaching 2 at the critical order.

Several questions remain open. The present perturbative theory is deliberately restricted to sufficiently small $|\lambda|$. A natural next problem is to determine how far the critical surface can be continued in parameter space and whether additional critical orders, degenerate turning points, or changes of multiplicity can occur away from the perturbative regime. It would also be useful to investigate whether analogous critical-order structures arise for generalized Mittag–Leffler functions and for broader classes of fractional evolution equations. In such settings, higher-order degeneracies could lead to sensitivity laws of order $|h|^m$ with $m > 2$. These questions would extend the present local theory without altering its central conclusion: at a nondegenerate critical fractional order, the order-to-observation map loses its linear sensitivity and exhibits a genuinely quadratic local response.

Use of generative-AI tools declaration

The author declares that no generative artificial intelligence (AI) tools were used in the preparation of this manuscript.

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Conflict of interest

The author declares that he has no competing interests.

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