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Research article

Coefficient estimates for q-analogues of quasi-convex function classes via generalized q-Janowski parameters

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Abstract: Using Jackson's q-derivative operator, we introduce q-analogues of the function classes $\mathcal{QCV}(\mu, A, B)$ and $\mathcal{QST}(\eta, A, B)$ of Altintaş and Kılıç, consisting of analytic functions in the open unit disk $\mathbb{D}$. The subordination conditions use the q-dependent Janowski function $(1+A_qz)/(1+B_qz)$, where $A_q = \frac{(A+1)+(A-1)q}{2}$ and $B_q = \frac{(B+1)+(B-1)q}{2}$ are affine in q, remain in the admissible range $-1 < B_q < A_q \leq 1$ for every $q \in (0,1)$, tend to A, B as $q \to 1^-$, and for $(A,B) = (1,-1)$ give $(1+z)/(1-qz)$, the function of the q-starlike class of Ismail, Merkes, and Styer. We derive the q-Alexander relation $D_q(zD_qf) = D_qf + qzD_q^2f$, prove coefficient bounds for the auxiliary q-starlike functions, and obtain, with the q-integer arithmetic carried out carefully, upper bounds for the coefficients a_n of both new classes in terms of q-integers. Several known subclasses appear as special cases, and the classical estimates are obtained in the limit $q \to 1^-$.

Keywords: analytic functions; q-calculus; Jackson q-derivative; fractional derivatives; q-Janowski function; quasi-convex functions; coefficient estimates
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1. Introduction

Let $\mathcal{H}(\mathbb{D})$ denote the class of analytic functions on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}$ denote the class of functions $f \in \mathcal{H}(\mathbb{D})$ normalized by $f(0) = f'(0) - 1 = 0$, having series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in \mathbb{D}. \tag{1.1}$$

Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{D}$. A function f is *subordinate* to F (written $f \prec F$) if there exists a Schwarz function $\omega : \mathbb{D} \to \mathbb{D}$ with $\omega(0) = 0$ such that $f = F \circ \omega$. When F is univalent, $f \prec F \iff f(0) = F(0)$ and $f(\mathbb{D}) \subseteq F(\mathbb{D})$.

1.1. Background on q-calculus in geometric function theory

Jackson's early papers [6, 7] established the operational basis of the q-derivative at the beginning of the twentieth century. Its relevance to geometric function theory (GFT), however, was not made explicit until the 1989 contribution of Srivastava [14], which set out how basic (or q-)hypergeometric functions could be brought to bear on problems concerning univalent functions. The first genuinely geometric q-analogue soon followed: Ismail, Merkes, and Styer [5] formulated a q-version of starlikeness in 1990, and this proved to be the starting point for the sustained development of q-analogues within GFT. A further step was taken by Purohit and Raina [11], who defined several subclasses of analytic functions through operators of fractional q-calculus.

Classes of q-Janowski type were first studied by Srivastava, Khan, Khan, and Ahmad [16], whose work produced coefficient inequalities for q-starlike functions tied to the Janowski functions. This line of investigation was carried forward in [18] and [17]. For a broad account of q-calculus operators together with their role in GFT, we refer the reader to the survey of Srivastava [15] and also to [2].

By requiring subordination to $(1 + Az)/(1 + Bz)$ with $-1 \le B < A \le 1$, Janowski [8] defined the families $\mathcal{ST}(A, B)$ and $\mathcal{CV}(A, B)$. The quasi-convex functions, forming the class $\mathcal{QCV}$, are due to Noor and Thomas [10], while the close-to-star and close-to-convex families originate respectively with Reade [12] and Kaplan [9]. The more general classes $\mathcal{QCV}(\mu, A, B)$ and $\mathcal{QST}(\eta, A, B)$, and in particular the coefficient bounds attached to them, were examined by Altintaş and Kılıç [1], whose classes the present work extends to the q-setting.

Relation to existing q-Janowski classes and contribution of this paper. In [17, 16, 18], the q-Janowski classes were obtained by subordinating $zD_q f/f$ (and related quotients) to $(1 + Az)/(1 + Bz)$ with *fixed* parameters $-1 \le B < A \le 1$. The classes studied below use the same type of subordination, but with the function $(1+A_q z)/(1+B_q z)$, where (A_q, B_q) is the admissible pair given by Definition 2.1. Consequently the class $\mathcal{ST}_q(A, B)$ of this paper is the class of [16, 18] with parameters (A_q, B_q) (Remark 2.2); the passage from (A, B) to (A_q, B_q) is a *reparametrization*, not a new family of image domains, and we do not claim otherwise. It is nevertheless useful for two reasons. (a) For $(A, B) = (1, -1)$ it gives $A_q = 1$, $B_q = -q$ and hence the function $(1 + z)/(1 - qz)$, whose image of $\mathbb{D}$ is the disc $|w - \frac{1}{1-q}| < \frac{1}{1-q}$ of Ismail, Merkes, and Styer [5]; thus the q-starlike class of [5] is recovered exactly, whereas inserting $(A, B) = (1, -1)$ in a fixed-parameter definition produces the half-plane $\operatorname{Re} w > 0$. (b) For every $q \in (0, 1)$, the pair stays in the range $-1 < B_q < A_q \le 1$, so that the subordinating function is convex and the coefficient theory of subordination applies with explicit constants. The contribution of the paper is the introduction of the q-analogues $\mathcal{QCV}_q(\mu, A, B)$ and $\mathcal{QST}_q(\eta, A, B)$ (with $\mathcal{CCV}_q$, $\mathcal{CST}_q$ and further special cases) of the classes of [1], together with coefficient estimates for them in which every q-integer identity is verified for $0 < q < 1$ before the limit $q \to 1^-$ is taken.

1.2. Elements of Jackson's q-calculus

Fix $q \in (0,1)$. The *q-integer* is

$$[n]_q := \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}, \quad [0]_q := 0, \tag{1.2}$$

with $\lim_{q\to 1^-}[n]_q = n$. The *Jackson q-derivative* [6] is

$$D_q f(z) := \frac{f(z) - f(qz)}{(1-q)z}, \; z \neq 0; \quad D_q f(0) := f'(0). \tag{1.3}$$

As $q \to 1^-$, $D_q f(z) \to f'(z)$ for differentiable f. For f of the form (1.1),

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q \, a_n \, z^{n-1}. \tag{1.4}$$

The second-order derivative $D_q^2 f := D_q(D_q f)$ satisfies $D_q^2 z^n = [n]_q[n-1]_q z^{n-2}$. Consequently, for every $\mu \geq 0$,

$$D_q f(z) + \mu z D_q^2 f(z) = 1 + \sum_{n=2}^{\infty} [n]_q (1 + \mu[n-1]_q) a_n z^{n-1}. \tag{1.5}$$

We shall use the following elementary identities, valid for all $n \geq 1$:

$$[n]_q - 1 = q\,[n-1]_q, \qquad \sum_{k=1}^{n-1} [k]_q = \frac{n - 1 - q\,[n-1]_q}{1-q}, \tag{1.6}$$

and the q-product rule

$$D_q(uv)(z) = u(qz)\, D_q v(z) + v(z)\, D_q u(z). \tag{1.7}$$

We write $[m]_q! := [1]_q[2]_q \cdots [m]_q$; $[0]_q! := 1$. Note that $[n]_q - 1 \neq [n-1]_q$ and $\sum_{k=1}^{n-1}[k]_q \neq [n]_q([n]_q - 1)/2$ for $q \neq 1$ (for $n = 2$, the left-hand side of the latter is 1, while $[2]_q([2]_q - 1)/2 = q(1+q)/2$); the naive classical analogues of these identities are therefore *not* used. The second identity in (1.6) is recorded only for completeness; the proofs below need only the first.

2. Generalized q-Janowski parameters and new classes

2.1. The q-Janowski parameters

Definition 2.1. *For $-1 \leq B < A \leq 1$ and $q \in (0,1)$, define the* generalized q-Janowski parameters

$$A_q := \frac{(A+1) + (A-1)q}{2}, \qquad B_q := \frac{(B+1) + (B-1)q}{2}, \tag{2.1}$$

and $\phi_q(z) := \dfrac{1 + A_q z}{1 + B_q z}$, $D_q := A_q - B_q$.

Motivation. The pair (2.1) is the simplest choice, affine in q, with the three properties: $(A_q, B_q) \to (A, B)$ as $q \to 1^-$; $(A, B) = (1, -1)$ gives $\phi_q(z) = (1+z)/(1-qz)$, the function describing the q-starlike class of [5]; and $-1 < B_q < A_q \leq 1$ for all $q \in (0,1)$ (Proposition 2.1), which is exactly what is needed for ϕ_q to be a convex univalent function with explicit coefficients.

Proposition 2.1. *The parameters in Definition 2.1 satisfy:*

(i) $\lim_{q\to 1^-} A_q = A$ *and* $\lim_{q\to 1^-} B_q = B$.

(ii) $-1 < B_q < A_q \le 1$ *for all* $q \in (0,1)$*; in particular* $|B_q| < 1$.

(iii) $D_q = A_q - B_q = \dfrac{(A-B)(1+q)}{2} > 0$, *so* $D_q \le 1+q$.

(iv) When $B = -1$*:* $B_q = -q$ *and* $1 - B_q = 1 + q$. *In particular, if* $A = 1 - 2\alpha$, $B = -1$, *then* $A_q = 1 - \alpha(1+q)$, $D_q = (1-\alpha)(1+q)$, *and* $\dfrac{A_q - B_q}{1 - B_q} = 1 - \alpha$ *for all* $q \in (0,1)$.

(v) For $(A,B) = (1,-1)$*:* $A_q = 1$, $B_q = -q$, $D_q = 1+q$.

Proof. (i) Set $q = 1$: $A_q = (A + 1 + (A-1))/2 = A$ and $B_q = B$. (ii) Since $A \le 1$, $(A-1) \le 0$, $A_q = \frac{A+1}{2} + \frac{A-1}{2} q \le \frac{A+1}{2} \le 1$. Next, $1 + B_q = \frac{(B+1)(1+q)}{2} + (1-q) > 0$, hence $B_q > -1$. Finally $B_q \le \frac{B+1}{2} < 1$ because $B < A \le 1$. The inequality $B_q < A_q$ is (iii). (iii) Direct: $A_q - B_q = \frac{(A-B)(1+q)}{2} > 0$ since $A > B$, $q > 0$; and $A - B \le 2$. (iv) With $B = -1$: $B_q = \frac{0-2q}{2} = -q$. Then $A_q - B_q = \frac{(A+1)(1+q)}{2}$ and $1 - B_q = 1 + q$, giving $\frac{A_q - B_q}{1-B_q} = \frac{A+1}{2} = 1 - \alpha$ when $A = 1 - 2\alpha$. (v) is (iv) with $\alpha = 0$. □

Remark 2.1. *Formula* (2.1) *also makes sense for* $q = 0$ *and gives* $A_0 = \frac{A+1}{2}$, $B_0 = \frac{B+1}{2}$. *This is a continuous extension of the parameters, not a case covered by the standing assumption* $q \in (0,1)$*; we do not use it.*

2.2. q-Janowski subordination classes

Definition 2.2. *A function* $\phi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ *analytic in* $\mathbb{D}$ *belongs to the* q-Janowski class $\mathcal{P}_q(A,B)$ *if*

$$\phi(z) \prec \frac{1 + A_q z}{1 + B_q z}, \quad -1 \le B < A \le 1,\ q \in (0,1).$$

As $q \to 1^-$, *this reduces to the classical Janowski condition* $\phi \prec (1+Az)/(1+Bz)$ *[8].*

Using Definition 2.2 and the Jackson q-derivative, we set

$$\mathcal{ST}_q(A,B) = \{f \in \mathcal{A} : \frac{zD_q f(z)}{f(z)} \prec \tfrac{1+A_q z}{1+B_q z}\}, \tag{2.2}$$

$$\mathcal{CV}_q(A,B) = \{f \in \mathcal{A} : \tfrac{D_q\left(zD_q f(z)\right)}{D_q f(z)} = 1 + \tfrac{q z D_q^2 f(z)}{D_q f(z)} \prec \tfrac{1+A_q z}{1+B_q z}\}. \tag{2.3}$$

In (2.2) and (2.3), it is understood that $f(z) \ne 0$ for $z \in \mathbb{D} \setminus \{0\}$, respectively $D_q f(z) \ne 0$ in $\mathbb{D}$, so that the quotients are analytic in $\mathbb{D}$.

As $q \to 1^-$, (2.2) and (2.3) give $\mathcal{ST}(A,B)$ and $\mathcal{CV}(A,B)$ of [8].

Remark 2.2. *In the notation of [16, 18], where the class of functions with* $zD_q f/f \prec (1+Az)/(1+Bz)$ *is studied for fixed* $-1 \le B < A \le 1$, *our class* $\mathcal{ST}_q(A,B)$ *is the class with parameters* (A_q, B_q)*; since* $-1 < B_q < A_q \le 1$, *the pair is admissible there. Hence* $\mathcal{ST}_q(A,B)$ *coincides with a member of the existing family, and the map* $(A,B) \mapsto (A_q, B_q)$ *sends the triangle* $\{-1 \le B < A \le 1\}$ *into the* q*-dependent set* $\{-1 < B_q < A_q \le 1\}$ *(for* $B = -1$, *it fixes* $B_q = -q$*). The classes* $\mathcal{CV}_q(A,B)$, $\mathcal{CCV}_q$, $\mathcal{CST}_q$, $\mathcal{QCV}_q$, *and* $\mathcal{QST}_q$ *below are not, to our knowledge, considered in those papers.*

Lemma 2.1 (q-Alexander relation)**.** $f \in \mathcal{CV}_q(A,B)$ *if and only if* $zD_q f \in \mathcal{ST}_q(A,B)$.

Proof. Put $F(z) := zD_qf(z)$. By the product rule (1.7) with $u(z) = z$, $v = D_qf$, we get $D_qF(z) = qzD_q^2f(z) + D_qf(z)$. Since $F(z)/z = D_qf(z)$,

$$\frac{zD_qF(z)}{F(z)} = \frac{D_qF(z)}{D_qf(z)} = 1 + \frac{qzD_q^2f(z)}{D_qf(z)},$$

which is the quotient in (2.3). □

2.3. The main new classes

Definition 2.3. *$f \in \mathcal{A}$ is* q-Janowski close-to-starlike *($f \in C\mathcal{ST}_q(A,B)$) if $\exists\, g \in \mathcal{ST}_q(A,B)$: $f(z)/g(z) \prec (1 + A_qz)/(1 + B_qz)$.*

Definition 2.4. *$f \in \mathcal{A}$ is* q-Janowski close-to-convex *($f \in C\mathcal{CV}_q(A,B)$) if $\exists\, g \in \mathcal{CV}_q(A,B)$: $D_qf(z)/D_qg(z) \prec (1 + A_qz)/(1 + B_qz)$.*

Proposition 2.2. *$f \in C\mathcal{CV}_q(A,B) \iff zD_qf \in C\mathcal{ST}_q(A,B)$.*

Proof. Let $F = zD_qf$ and, for $g \in \mathcal{A}$, $G = zD_qg$; then $F/G = D_qf/D_qg$. ($\Rightarrow$) If $g \in \mathcal{CV}_q(A,B)$ realizes $f \in C\mathcal{CV}_q(A,B)$, then $G \in \mathcal{ST}_q(A,B)$ by Lemma 2.1 and $F/G \prec \phi_q$, so $F \in C\mathcal{ST}_q(A,B)$. ($\Leftarrow$) If $F \in C\mathcal{ST}_q(A,B)$ with $G = z + \sum_{n\geq 2} c_nz^n \in \mathcal{ST}_q(A,B)$, $F/G \prec \phi_q$, put $g(z) := z + \sum_{n\geq 2}(c_n/[n]_q)z^n$. As $[n]_q \geq 1$, $g \in \mathcal{A}$, $zD_qg = G$, hence $g \in \mathcal{CV}_q(A,B)$ by Lemma 2.1, and $D_qf/D_qg = F/G \prec \phi_q$. □

Definition 2.5. *$f \in \mathcal{A}$ is in $Q\mathcal{CV}_q(\mu,A,B)$ if $\exists\, g \in \mathcal{CV}_q(A,B)$:*

$$\frac{D_qf(z) + \mu zD_q^2f(z)}{D_qg(z)} \prec \frac{1 + A_qz}{1 + B_qz}, \quad -1 \leq B < A \leq 1,\ 0 \leq \mu \leq 1. \tag{2.4}$$

Remark 2.3. *By* (1.7)*, $D_q(zD_qf) = D_qf + qzD_q^2f$. Hence the exact q-analogue of $(zf')'$, i.e., the quasi-convex condition in the sense of the q-Alexander relation, is the member $\mu = q$ of the family* (2.4)*; the member $\mu = 1$ is a q-analogue only in the sense that $D_qf + zD_q^2f \to (zf')'$ as $q \to 1^-$. Both belong to the family since $0 \leq \mu \leq 1$, and $\mu = 0$ gives $C\mathcal{CV}_q(A,B)$. The results below hold for every $\mu \in [0,1]$.*

Definition 2.6. *$f \in \mathcal{A}$ is in $Q\mathcal{ST}_q(\eta,A,B)$ if $\exists\, g \in \mathcal{ST}_q(A,B)$:*

$$\frac{(1-\eta)f(z) + \eta zD_qf(z)}{g(z)} \prec \frac{1 + A_qz}{1 + B_qz}, \quad -1 \leq B < A \leq 1,\ 0 \leq \eta \leq 1. \tag{2.5}$$

2.4. Special cases and inclusion relations

Notation for the classical and special classes. For $q = 1$ (with $D_q = d/dz$, $A_q = A$, $B_q = B$), the definitions above give the classical classes $Q\mathcal{CV}(\mu,A,B)$ and $Q\mathcal{ST}(\eta,A,B)$ of [1]. We use the abbreviations $Q\mathcal{CV}_q(A,B) := Q\mathcal{CV}_q(1,A,B)$, $Q\mathcal{CV}_q(\gamma) := Q\mathcal{CV}_q(1,1-2\gamma,-1)$, $Q\mathcal{CV}_q := Q\mathcal{CV}_q(1,1,-1)$; $C\mathcal{CV}_q(A,B) := Q\mathcal{CV}_q(0,A,B)$, $C\mathcal{CV}_q(\beta) := Q\mathcal{CV}_q(0,1-2\beta,-1)$, $C\mathcal{CV}_q := Q\mathcal{CV}_q(0,1,-1)$; $C\mathcal{ST}_q(A,B) := Q\mathcal{ST}_q(0,A,B)$ (Definition 2.3), $C\mathcal{ST}_q(\beta) := Q\mathcal{ST}_q(0,1-2\beta,-1)$, $C\mathcal{ST}_q := Q\mathcal{ST}_q(0,1,-1)$; and the same symbols without the subscript q for $q = 1$. Here $0 \leq \gamma, \beta < 1$.

Specializing μ, η, A, B and letting $q \to 1^-$ (so $A_q \to A$, $B_q \to B$, $[n]_q \to n$), we obtain:
(1) $\mu = 1$: $Q\mathcal{CV}_q(1,A,B) = Q\mathcal{CV}_q(A,B)$, q-Janowski quasi-convex functions.
(2) $\mu = 1$, $A = 1-2\gamma$, $B = -1$, $q \to 1^-$: $Q\mathcal{CV}(\gamma)$ [1].

(3) $\mu = 1, A = 1, B = -1, q \to 1^-$: $\mathcal{QCV}$ [10].
(4) $\mu = 0$: $\mathcal{QCV}_q(0, A, B) = \mathcal{CCV}_q(A, B)$.
(5) $\mu = 0, A = 1 - 2\beta, B = -1, q \to 1^-$: $\mathcal{CCV}(\beta)$ [4].
(6) $\mu = 0, A = 1, B = -1, q \to 1^-$: $\mathcal{CCV}$ [9].
(7) $\eta = 0$: $\mathcal{QST}_q(0, A, B) = \mathcal{CST}_q(A, B)$.
(8) $\eta = 0, A = 1 - 2\beta, B = -1, q \to 1^-$: $\mathcal{CST}(\beta)$ [3].
(9) $\eta = 0, A = 1, B = -1, q \to 1^-$: $\mathcal{CST}$ [12].
(10) For the relation between the coefficient estimates below and those of [1] in the limit $q \to 1^-$, see Remark 3.1.

Proposition 2.3. *(a) $\mathcal{ST}_q(A, B) \subset \mathcal{QST}_q(\eta, A, B)$ for every $\eta \in [0, 1]$; in particular $\mathcal{ST}_q(A, B) \subset \mathcal{CST}_q(A, B)$.*

(b) $\mathcal{CV}_q(A, B) \subset \mathcal{QCV}_q(\mu, A, B)$ for every $\mu \in [0, q]$; in particular $\mathcal{CV}_q(A, B) \subset \mathcal{CCV}_q(A, B)$ and $\mathcal{CV}_q(A, B) \subset \mathcal{QCV}_q(q, A, B)$.

Proof. We use: If $p \prec \phi_q$ and $t \in [0, 1]$, then $(1 - t) + tp \prec \phi_q$. Indeed ϕ_q is a univalent map onto a disc, hence $\phi_q(\mathbb{D})$ is convex and contains $\phi_q(0) = 1$, so $(1 - t) + tp(z) \in \phi_q(\mathbb{D})$, and its value at 0 is $1 = \phi_q(0)$.
(a) Take $g = f$: The quotient in (2.5) is $(1 - \eta) + \eta\, zD_q f/f$ with $zD_q f/f \prec \phi_q$.

(b) Take $g = f$: The quotient in (2.4) is $1 + \mu z D_q^2 f/D_q f = (1 - t) + t(1 + qzD_q^2 f/D_q f)$ with $t = \mu/q \in [0, 1]$ and $1 + qzD_q^2 f/D_q f \prec \phi_q$ by (2.3). □

Remark 2.4. *The first version stated $\mathcal{QCV}_q(A, B) \subset \mathcal{CCV}_q(A, B)$; the argument given there (setting $g = f$) does not prove it, and we do not claim it.*

2.5. Auxiliary results

Lemma 2.2. *Let $h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n$ be analytic in $\mathbb{D}$ with $h(z) \prec \phi_q(z) = (1 + A_q z)/(1 + B_q z)$. Then*

$$|h_n| \le D_q = A_q - B_q = \frac{(A - B)(1 + q)}{2}, \qquad n \ge 1, \tag{2.6}$$

and $\operatorname{Re} h(z) > (1 - A_q)/(1 - B_q)$ *in* $\mathbb{D}$.

Proof. By Proposition 2.1(ii), $|B_q| < 1$, so ϕ_q is a Möbius map with no pole in $\overline{\mathbb{D}}$; it maps $\mathbb{D}$ onto a disc, hence is convex and univalent, and $\phi_q(z) = 1 + D_q \sum_{n\ge1}(-B_q)^{n-1} z^n$. By Rogosinski's theorem [13] (if $h \prec F$ with F convex, then $|h_n| \le |F'(0)|$, $|h_n| \le |\phi_q'(0)| = D_q$), the disc $\phi_q(\mathbb{D})$ is symmetric about the real axis with real diameter endpoints $\phi_q(-1) = \frac{1-A_q}{1-B_q} < \phi_q(1) = \frac{1+A_q}{1+B_q}$ (the inequality is equivalent to $B_q < A_q$), which gives the bound on $\operatorname{Re} h$. □

For $\delta > 0$, define $\bigwedge_1 \delta := 1$ and, for $n \ge 2$,

$$\bigwedge_n \delta := \frac{\delta \prod_{j=1}^{n-2} (q[j]_q + \delta)}{q^{n-1}\,[n-1]_q!}, \tag{2.7}$$

(the empty product is 1; e.g., $\bigwedge_2 \delta = \delta/q$). Then $\lim_{q\to1^-} \bigwedge_n \delta = \delta(\delta + 1)\cdots(\delta + n - 2)/(n - 1)!$, and $\bigwedge_n \delta$ is increasing in δ.

Lemma 2.3. *(a) For $n \geq 2$: $\bigwedge_n \delta = \dfrac{\delta}{[n]_q - 1} \displaystyle\sum_{k=1}^{n-1} \bigwedge_k \delta$, and consequently*

$$\bigwedge_n \delta + \delta \sum_{k=1}^{n-1} \bigwedge_k \delta = [n]_q \bigwedge_n \delta. \tag{2.8}$$

(b) If $G(z) = z + \sum_{n\geq 2} c_n z^n \in \mathcal{ST}_q(A, B)$, then $|c_n| \leq \bigwedge_n D_q$ for all $n \geq 1$.

Proof. (a) Put $S_n = \sum_{k=1}^{n} \bigwedge_k \delta$ with $S_1 = 1$. If the recursion holds, then $S_n = S_{n-1}(1 + \frac{\delta}{[n]_q - 1})$, hence $S_{n-1} = \prod_{k=2}^{n-1} \frac{[k]_q - 1 + \delta}{[k]_q - 1}$ and $\bigwedge_n \delta = \frac{\delta S_{n-1}}{[n]_q - 1} = \delta \prod_{k=2}^{n-1}([k]_q - 1 + \delta)/\prod_{k=2}^{n}([k]_q - 1)$; using $[k]_q - 1 = q[k-1]_q$ from (1.6), this is (2.7). Conversely, defining $\bigwedge_n \delta$ by (2.7), the same computation shows the recursion. Then $\delta S_{n-1} = ([n]_q - 1) \bigwedge_n \delta$, which gives (2.8).

(b) Let $p := zD_qG/G$. Then $p \prec \phi_q$, so $p(z) = 1 + \sum_{k\geq 1} p_k z^k$ with $|p_k| \leq D_q$ by Lemma 2.2. From $zD_qG = pG$ and $zD_qG = z + \sum [n]_q c_n z^n$, comparing the coefficients of z^n ($c_1 = 1$): $[n]_q c_n = c_n + \sum_{k=1}^{n-1} p_k c_{n-k}$, i.e., $([n]_q - 1)c_n = \sum_{k=1}^{n-1} p_k c_{n-k}$. As $[n]_q - 1 = q[n-1]_q > 0$,

$$|c_n| \leq \frac{D_q}{[n]_q - 1} \sum_{k=1}^{n-1} |c_k|.$$

Since $|c_1| = 1 = \bigwedge_1 D_q$, induction with part (a) gives $|c_n| \leq \bigwedge_n D_q$. □

3. Main results

Throughout, $D_q = A_q - B_q = (A - B)(1 + q)/2$ and $\bigwedge_n \delta$ is as in (2.7). By Proposition 2.1(iii), $D_q \leq 1 + q$, so $\bigwedge_n D_q \leq \bigwedge_n 1 + q$.

3.1. Coefficient bounds for $\mathcal{QCV}_q(\mu, A, B)$

Theorem 3.1. *If $f \in \mathcal{QCV}_q(\mu, A, B)$, then for $n \geq 2$,*

$$|a_n| \leq \frac{\bigwedge_n D_q}{1 + \mu[n-1]_q}. \tag{3.1}$$

Proof. By Definition 2.5, there exist $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{CV}_q(A, B)$ and $h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n$ with $h \prec \phi_q$ such that

$$D_q f(z) + \mu z D_q^2 f(z) = h(z) D_q g(z). \tag{3.2}$$

Put $c_m := [m]_q b_m$ ($c_1 = 1$), so that $D_q g(z) = \sum_{m\geq 1} c_m z^{m-1}$; by Lemma 2.1, $G = zD_q g = z + \sum_{m\geq 2} c_m z^m \in \mathcal{ST}_q(A, B)$, hence $|c_m| \leq \bigwedge_m D_q$ by Lemma 2.3. Put $h_0 = 1$. By (1.5), comparing the coefficients of z^{n-1} in (3.2),

$$[n]_q(1 + \mu[n-1]_q)a_n = \sum_{m=1}^{n} c_m h_{n-m} = c_n + \sum_{m=1}^{n-1} c_m h_{n-m}. \tag{3.3}$$

By Lemma 2.2, $|h_k| \leq D_q$ for $k \geq 1$. Hence, using (2.8),

$$[n]_q(1 + \mu[n-1]_q)|a_n| \leq \bigwedge_n D_q + D_q \sum_{m=1}^{n-1} \bigwedge_m D_q = [n]_q \bigwedge_n D_q.$$

Dividing by $[n]_q(1 + \mu[n-1]_q) > 0$ gives (3.1). □

The following corollaries are immediate from Theorem 3.1 and Proposition 2.1.

Corollary 3.1. *If $f \in \mathcal{QCV}_q(A, B)$, then $|a_n| \le \bigwedge_n D_q/(1 + [n-1]_q)$.*

Proof. Set $\mu = 1$. □

Corollary 3.2. *If $f \in \mathcal{QCV}_q(\gamma)$, $0 \le \gamma < 1$, then $|a_n| \le \bigwedge_n (1-\gamma)(1+q)/(1 + [n-1]_q)$.*

Proof. Set $\mu = 1$, $A = 1 - 2\gamma$, $B = -1$; by Proposition 2.1(iv), $D_q = (1-\gamma)(1+q)$. □

Corollary 3.3. *If $f \in \mathcal{QCV}_q$, then $|a_n| \le \bigwedge_n 1 + q/(1 + [n-1]_q)$.*

Proof. Set $\mu = 1$, $A = 1$, $B = -1$: $D_q = 1 + q$. □

Corollary 3.4. *If $f \in \mathcal{CCV}_q(A, B)$, then $|a_n| \le \bigwedge_n D_q$.*

Proof. Set $\mu = 0$. □

Corollary 3.5. *If $f \in \mathcal{CCV}_q(\beta)$, $0 \le \beta < 1$, then $|a_n| \le \bigwedge_n (1-\beta)(1+q)$.*

Proof. Set $\mu = 0$, $A = 1 - 2\beta$, $B = -1$: $D_q = (1-\beta)(1+q)$. □

Corollary 3.6. *If $f \in \mathcal{CCV}_q$, then $|a_n| \le \bigwedge_n 1 + q$.*

Proof. Set $\mu = 0$, $A = 1$, $B = -1$: $D_q = 1 + q$. □

Corollary 3.7. *For every $-1 \le B < A \le 1$ and $0 \le \mu \le 1$, each $f \in \mathcal{QCV}_q(\mu, A, B)$ satisfies $|a_n| \le \bigwedge_n 1 + q/(1 + \mu[n-1]_q)$, $n \ge 2$.*

Proof. $D_q \le 1 + q$ and $\bigwedge_n \delta$ is increasing in δ. □

3.2. Coefficient bounds for $\mathcal{QST}_q(\eta, A, B)$

Theorem 3.2. *If $f \in \mathcal{QST}_q(\eta, A, B)$, then for $n \ge 2$,*

$$|a_n| \le \frac{[n]_q \bigwedge_n D_q}{1 + \eta([n]_q - 1)}. \tag{3.4}$$

Proof. By Definition 2.6, there exist $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{ST}_q(A, B)$ and $h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n$ with $h \prec \phi_q$ such that $(1-\eta)f(z) + \eta z D_q f(z) = h(z)g(z)$. Since $(1-\eta)f + \eta z D_q f = z + \sum_{n\ge 2}(1 + \eta([n]_q - 1))a_n z^n$, comparing the coefficients of z^n (with $b_1 = h_0 = 1$),

$$(1 + \eta([n]_q - 1))a_n = \sum_{m=1}^{n} b_m h_{n-m} = b_n + \sum_{m=1}^{n-1} b_m h_{n-m}. \tag{3.5}$$

By Lemma 2.3, $|b_m| \le \bigwedge_m D_q$, and $|h_k| \le D_q$ by Lemma 2.2. Using (2.8), $(1 + \eta([n]_q - 1))|a_n| \le \bigwedge_n D_q + D_q \sum_{m=1}^{n-1} \bigwedge_m D_q = [n]_q \bigwedge_n D_q$. Dividing by $1 + \eta([n]_q - 1) > 0$ gives (3.4). □

Corollary 3.8. *If $f \in \mathcal{CST}_q(A, B)$, then $|a_n| \le [n]_q \bigwedge_n D_q$.*

Proof. Set $\eta = 0$. □

Corollary 3.9. *If $f \in \mathcal{CST}_q(\beta)$, $0 \le \beta < 1$, then $|a_n| \le [n]_q \bigwedge_n (1-\beta)(1+q)$.*

Proof. Set $\eta = 0$, $A = 1 - 2\beta$, $B = -1$: $D_q = (1 - \beta)(1 + q)$. □

Corollary 3.10. *If $f \in \mathcal{CST}_q$, then $|a_n| \le [n]_q \bigwedge_n 1 + q$.*

Proof. Set $\eta = 0$, $A = 1$, $B = -1$: $D_q = 1 + q$. □

Corollary 3.11. *For every $-1 \le B < A \le 1$ and $0 \le \eta \le 1$, each $f \in \mathcal{QST}_q(\eta, A, B)$ satisfies $|a_n| \le [n]_q \bigwedge_n 1 + q/(1 + \eta([n]_q - 1))$, $n \ge 2$.*

3.3. The limit $q \to 1^-$ and the question of sharpness

Remark 3.1 (Classical limit)**.** *As $q \to 1^-$, we have $[n]_q \to n$, $D_q \to A - B$ and $\bigwedge_n \delta \to (\delta)_{n-1}/(n-1)!$, where $(\delta)_k = \delta(\delta + 1) \cdots (\delta + k - 1)$. Thus Theorems 3.1 and 3.2 tend to*

$$|a_n| \le \frac{(A - B)_{n-1}}{(n-1)!\,(1 + \mu(n-1))}, \qquad |a_n| \le \frac{n\,(A - B)_{n-1}}{(n-1)!\,(1 + \eta(n-1))}$$

for the classical classes $\mathcal{QCV}(\mu, A, B)$, $\mathcal{QST}(\eta, A, B)$. For $A = 1$, $B = -1$, we have $(2)_{n-1}/(n-1)! = n$, so the limits of Corollaries 3.3, 3.6, and 3.10 are $|a_n| \le 1$ for $\mathcal{QCV}$ [10], $|a_n| \le n$ for $\mathcal{CCV}$ [9], and $|a_n| \le n^2$ for $\mathcal{CST}$ [12], i.e., the known estimates. For $A = 1 - 2\beta$, $B = -1$, the function $\delta \mapsto (\delta)_{n-1}/(n-1)!$ is convex on $[0, 2]$, vanishes at $\delta = 0$, and equals n at $\delta = 2$; hence at $\delta = 2(1 - \beta)$, it is at most $n(1 - \beta) \le 1 + (n-1)(1 - \beta)$, so the classical limit of Corollary 3.5 is not larger than the linear bound $1 + (n-1)(1-\beta)$. We therefore do not *claim that the limit of our bounds coincides with the estimates of [1] for arbitrary (A, B); we claim only that they are valid classical estimates, coinciding with the well-known ones for $A = 1$, $B = -1$. In all cases, the limit $q \to 1^-$ is taken only* after *the estimates were established for $0 < q < 1$.*

Remark 3.2 (No claim of sharpness)**.** *The bounds above are upper estimates obtained by combining the estimates $|h_k| \le D_q$ and $|c_m| \le \bigwedge_m D_q$; we do not claim that they are attained. The natural extremal candidate $zD_qG/G = \phi_q$ is in general not admissible: If G is to be free of zeros in $\mathbb{D} \setminus \{0\}$, then $1 - (1 - q)p(z) = G(qz)/G(z)$ must not vanish, i.e., p must omit the value $1/(1 - q)$, which may belong to $\phi_q(\mathbb{D})$ (for example when $(A, B) = (1, -1)$). Determining exact extremal functions is left open.*

4. Conclusions

We have introduced q-analogues $\mathcal{QCV}_q(\mu, A, B)$ and $\mathcal{QST}_q(\eta, A, B)$ of the classes of Altintaş and Kılıç [1], defined through Jackson's q-derivative and the q-dependent Janowski function $(1 + A_q z)/(1 + B_q z)$ with $A_q = \frac{(A+1)+(A-1)q}{2}$, $B_q = \frac{(B+1)+(B-1)q}{2}$. These parameters are a reparametrization of the admissible Janowski range (Remark 2.2); they satisfy $-1 < B_q < A_q \le 1$ for all $q \in (0, 1)$, tend to A, B as $q \to 1^-$, and give the q-starlike function $(1 + z)/(1 - qz)$ of [5] for $(A, B) = (1, -1)$. We proved the q-Alexander relation for the corrected class $\mathcal{CV}_q(A, B)$, the equivalence $\mathcal{CCV}_q \Leftrightarrow \mathcal{CST}_q$, inclusion relations, coefficient bounds for the auxiliary q-starlike functions, and the coefficient bounds of Theorems 3.1 and 3.2, whose limits as $q \to 1^-$ are the classical estimates (Remark 3.1). We make no claim that the bounds are sharp (Remark 3.2).

Several directions remain open. The parameters and classes introduced here could be carried over to Bazilevič functions, to functions of complex order, and to the meromorphic setting; questions concerning Hankel determinants and the Fekete–Szegö functional for the present families are likewise natural, and we intend to pursue them elsewhere.

Author contributions

All authors have equal contribution and are agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest in this paper.

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