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Research article

Disjoint Cesàro-hypercyclicity: a collapse theorem and a separating pair of bilateral shifts

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Abstract: Disjoint Cesàro-hypercyclicity was introduced recently on separable Banach spaces, where it was shown that a family of unilateral weighted backward shifts on the space $\ell^2(\mathbb{N})$ which is disjoint Cesàro-hypercyclic is already disjoint hypercyclic. We determine how far that collapse reaches. Our main result is unconditional and uses no structure on the operator. If distinct powers $T^{r_1}, T^{r_2}, \ldots, T^{r_N}$ of a single continuous operator on a separable Fréchet space form a disjoint Cesàro-hypercyclic family, then each associated difference family is disjoint hypercyclic, with a dense G_δ set of vectors, and T consequently is hypercyclic, so an operator that is Cesàro hypercyclic without being hypercyclic can never separate the two disjoint notions through its own powers. Pairs of scalar multiples are excluded for a different reason. Outside these classes, the notions do come apart, and they do so on a Hilbert space. We construct bilateral weighted backward shifts B_α, B_β on $\ell^2(\mathbb{Z})$ with weights taking only the values $\frac{1}{2}$, 1, and 2 such that (B_α, B_β^2) has a dense G_δ set of disjoint Cesàro-hypercyclic vectors while failing to be disjoint hypercyclic. What divides the two behaviours is the axis on which the shifts act, not the presence of a norm. Supporting this, we transpose the transitivity characterisation and the disjoint Cesàro-hypercyclicity criterion to separable Fréchet spaces, stated through a generating sequence of seminorms, and verify the criterion for distinct powers of the differentiation operator on $H(\mathbb{C})$ and of the backward shift on ω.

Keywords: disjoint Cesàro-hypercyclicity; disjoint hypercyclicity; bilateral weighted shift; Fréchet space; Köthe echelon space; topological transitivity
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1. Introduction

Linear dynamics studies the orbit structure of continuous linear operators, and the two monographs of Grosse-Erdmann and Peris [1] and of Bayart and Matheron [2] give a full account of the field. An

operator T on a topological vector space X is *hypercyclic* when some vector has dense orbit $\{T^n x\}_{n\geq 0}$. Birkhoff [3] produced the first example, an entire function whose translates are dense. MacLane [4] did the same two decades later for successive derivatives. Rolewicz [5] moved the subject onto Banach spaces by showing that λB with B as the unilateral backward shift on $\ell^p(\mathbb{N})$ is hypercyclic whenever $|\lambda|>1$; the converse is the norm bound $\|B\| = 1$ and needs no argument. Verification became systematic with the hypercyclicity criterion of Kitai [6] and independently Gethner and Shapiro [7]. That criterion is sufficient but not necessary, as De La Rosa and Read [8] eventually showed. Two refinements concern us.

Disjointness was introduced independently by Bernal-González [9] and by Bès and Peris [10]: operators $T_1, T_2, \ldots, T_N$ on X are disjoint hypercyclic when some single x makes $\{(T_1^n x, T_2^n x, \ldots, T_N^n x) : n \geq 0\}$ dense in X^N. Bès and Peris supplied the corresponding criterion on separable Fréchet spaces. Shkarin [11] gave a short existence proof, and Bès, Martin, and Sanders [12] characterised disjoint hypercyclicity for tuples of distinct weighted backward shifts, unilateral and bilateral alike, with families of powers of one shift falling out as a specialisation.

Cesàro-hypercyclicity is due to León-Saavedra [13], who called T Cesàro hypercyclic when the arithmetic means $n^{-1}(I + T + T^2 \cdots + T^{n-1})x$ are dense for some x and proved—for X a separable Banach space—that this happens exactly when the scaled orbit $\{n^{-1}T^n x\}_{n\geq 1}$ is dense for some x. The scaled formulation is the one used here and in [14], and we take it as the definition (Definition 2.1) rather than as a theorem, since whether León-Saavedra's equivalence survives on a general Fréchet space is a question we must leave open (Problem 2.1). Where a result is quoted from the Banach literature, the two notions agree and no ambiguity arises.

Cesàro-hypercyclicity is a genuinely different property from hypercyclicity, in both directions: León-Saavedra exhibited bilateral weighted shifts on $\ell^2(\mathbb{Z})$ that are Cesàro-hypercyclic without being hypercyclic and also showed that hypercyclicity does not imply Cesàro-hypercyclicity [13]. One implication does hold: every Cesàro-hypercyclic operator is supercyclic (Proposition 2.1). Costakis and Hadjiloucas [15] established a somewhere-dense orbit theorem for Cesàro means; further structural results, including subspace and weighted-shift variants, appear in [16–18].

In a recent preprint, Wang and Shu [14] combined the two refinements. On a separable Banach space, they called $T_1, T_2, \ldots, T_N$ disjoint Cesàro-hypercyclic when

$$\{(n^{-1}T_1^n x, n^{-1}T_2^n x, \ldots, n^{-1}T_N^n x) : n \geq 1\}$$

is dense in X^N for some x, proved a transitivity characterisation and a criterion phrased through dense subsets $X_0, X_1 \ldots, X_N$ and auxiliary maps $S_{l,k}$, and characterised the disjoint Cesàro-hypercyclic weighted backward shifts on $c_0(\mathbb{N})$, $\ell^2(\mathbb{N})$, $c_0(\mathbb{Z})$, and $\ell^2(\mathbb{Z})$. Their Corollary 4.2 is the point of departure for this paper: on $\ell^2(\mathbb{N})$, a disjoint Cesàro-hypercyclic family of *unilateral* weighted backward shifts is already disjoint hypercyclic. For that class, averaging buys nothing. To fix terminology: for a class $\mathcal{F}$ of N-tuples of operators, we say that *the collapse holds on* $\mathcal{F}$ if every $(T_1, \ldots, T_N) \in \mathcal{F}$ for which d-CH$(T_1, \ldots, T_N)$ is dense satisfies d-HC$(T_1, \ldots, T_N) \neq \emptyset$. In this language, [14, Corollary 4.2] asserts that the collapse holds on families $(B_{a_1}^{r_1}, \ldots, B_{a_N}^{r_N})$, $1 \leq r_1 < \cdots < r_N$ of powers of unilateral weighted backward shifts on $\ell^2(\mathbb{N})$. Theorem 5.1 below is *not* a collapse statement in this sense for power families: it yields disjoint hypercyclicity of the associated difference families, not of the original family, and the latter question remains open (Problem 5.1).

Two questions follow. How far does the criterion carry beyond a single norm? And how far does

Corollary 4.2 carry—is the collapse it records a feature of unilateral shifts of Banach spaces or of Cesàro averaging as such?

Our main result answers the second question in a form that does not depend on the space at all.

Theorem 1.1. *(difference-family collapse; see Theorem 5.1) Let T act on a separable Fréchet space, let $1 \le r_1 < r_2 < \cdots < r_N$, and suppose $(T^{r_1}, T^{r_2} \ldots, T^{r_N})$ is disjoint Cesàro-hypercyclic. Then for each base index s, the difference family $(T^{r_{s+1}-r_s}, T^{r_{s+2}-r_s}, \ldots, T^{r_N-r_s})$ has a dense G_δ set of disjoint hypercyclic vectors.*

The proof runs to four lines and assumes nothing about T: no shift structure, no right inverse, and no surjectivity. Density of the Cesàro orbit produces a witness $w = n^{-1}T^{r_s n}x_0$ inside the prescribed set, and the images of w under the difference powers are the remaining coordinates of that same orbit point, so the averaging factor cancels against itself instead of being estimated away. Two consequences follow at once (Corollary 5.1): every difference power $T^{r_l-r_s}$ is hypercyclic, and T is hypercyclic, the latter because a dense set of the form $\{T^{pn}v\}_{n\ge 0}$ sits inside $\mathrm{Orb}(T, v)$. That step is orbit inclusion; Ansari's theorem [19] is invoked only afterward to pass from T to its powers. An operator that is Cesàro-hypercyclic without being hypercyclic can therefore never separate the two disjoint notions through its own powers. Pairs of scalar multiples $(T, \alpha T)$ are excluded for a different reason and for every $\alpha \in \mathbb{C}$ (Proposition 5.1).

Outside those classes, the notions do come apart, and they do so on a Hilbert space.

Theorem 1.2. *(see Theorem 5.2) There are bilateral weighted backward shifts B_α, B_β on $\ell^2(\mathbb{Z})$, with weights taking only the values $\frac{1}{2}$, 1, and 2 such that (B_α, B_β^2) has a dense G_δ set of disjoint Cesàro-hypercyclic vectors and is not disjoint hypercyclic.*

The weights agree on the positive axis and differ on the nonpositive one; B_α alone fails to be hypercyclic (Proposition 5.4), which is enough. Corollary 4.2 of [14] therefore does not survive the passage from the unilateral to the bilateral axis, and the failure needs no locally convex machinery. The four cases are summarised in Table 1.

Table 1. The implication "Cesàro-hypercyclic ⇒ hypercyclic" for single operators and for disjoint families.

	single operator	disjoint family
unilateral shifts	implication holds [13]	implication holds [14]
general / bilateral	implication fails [13]	implication fails (Theorem 5.2)

León-Saavedra supplied both left-hand entries: Cesàro-hypercyclicity and hypercyclicity separate for a single operator, and no *unilateral* weighted shift can witness that separation, with every Cesàro-hypercyclic one being hypercyclic. In their preprint, Wang and Shu extended the unilateral implication to disjoint families. Theorem 5.2 completes the square. What governs all four entries is the axis, not the norm—a point we make explicitly (Remark 5.8). Proposition 5.5 re-runs the same construction on a Köthe echelon space with the unbounded weights $\alpha_n = n$; unbounded weights are indeed unavailable on a Banach sequence space in its canonical basis, and that, rather than the separation, is what the locally convex setting contributes there.

Theorem 5.2 does not depend on the correctness of this preprint: its proof goes through Theorem 3.1 alone, together with the peer-reviewed results of Ansari [19] and Salas [20]. Two results do depend on [14] and are stated as conditional on it: Proposition 5.2, which derives Wang and Shu's Corollary 4.2 from their Theorem 4.1, and Proposition 5.3, which uses their Theorem 4.4 to locate where a separating family must fail. Remark 5.6 records how Theorem 5.2 can also be read off [14, Theorem 4.4]; see Remark 5.4 for the precise scope of the dependence.

What remains undecided is the intermediate case, powers of a single *hypercyclic* operator (Problem 5.1); we explain in Remark 5.2 why neither obvious approach reaches it.

The machinery supporting these results occupies Sections 3 and 4. We transpose the transitivity characterisation and the criterion of [14] to separable Fréchet spaces, phrasing both through a generating sequence of seminorms (Proposition 3.1, Theorem 3.1). Two features of that transposition deserve mention here, since both are easy to overstate. Passing from a norm to an increasing sequence of seminorms does not force simultaneous control of infinitely many seminorms: monotonicity collapses every finite conjunction to its top member (Remark 2.1), and the proof of Theorem 3.1 works at one index selected in advance from the given open sets. And the characterisation of Proposition 3.1 concerns *residuality* of the set of disjoint Cesàro-hypercyclic vectors, not its non-emptiness; the gap between the two is real, since the scaled family $\{n^{-1}T^n\}$ is not a semigroup and the usual composition argument fails. For a single operator on a Banach space, the corresponding zero-one law is nonetheless available by an argument of Costakis and Hadjiloucas [15] that composes nothing; the disjoint version is Problem 3.1.

Two applications test the criterion. On $H(\mathbb{C})$, every finite family $(D^{r_1}, D^{r_2}, \dots, D^{r_N})$ of distinct powers of the differentiation operator has a dense G_δ set of disjoint Cesàro-hypercyclic vectors (Theorem 4.1). This does not recover MacLane's theorem by projection—coordinatewise it yields only Cesàro-hypercyclicity, hence supercyclicity, of each D^{r_l}—though MacLane does follow from it through Theorem 5.1, and more directly from the unscaled companion Proposition 4.2, which is a routine application of [10] and is not new; see also Zhang and Zhou [21] for powers of differentiation operators on related spaces. The two statements are one computation read twice, related by a transfer lemma (Lemma 4.6) that isolates exactly what the averaging factor costs: nothing in the off-diagonal conditions, where the factors cancel, and one polynomial factor in the right-inverse condition, absorbed by super-exponential decay (Remark 4.4).

It would be a mistake to call the estimates on $H(\mathbb{C})$ complex-analytic and beyond the reach of weighted-shift methods. Under the Taylor coefficient map, $H(\mathbb{C})$ is the Köthe echelon space $\lambda_1(A)$ with $a_m(j) = m^j$, and D is the weighted backward shift with weights $w_j = j$ (Lemma 4.5); the factorial ratios driving the verification are the weight products $\prod_{\nu=j+1}^{j+r} w_\nu$ measured against the matrix quotients $a_m(j)/a_m(j+r)$. Unboundedness of those weights is what no norm on a canonical basis admits (Remark 4.1), and it is the only structural difference at work. A related temptation—reading hypercyclicity of D as a non-normable phenomenon on the grounds that D is an unweighted shift in the basis $(z^j/j!)$—is dispatched in Remark 4.6: acting as an unweighted shift in *some* basis is not an isomorphism invariant, and ℓ^2 carries a basis in which the unweighted shift is $2B$.

The second application is $\omega = \mathbb{C}^{\mathbb{N}_0}$ (Section 4.5), where powers of the backward shift satisfy every condition by exact annihilation. Its interest is not non-normability, which $H(\mathbb{C})$ and $\lambda_2(A)$ share, but the absence of any continuous norm: each continuous seminorm on ω reads finitely many coordinates (Remark 4.8). That is why the verification produces exact zeros, and it yields the one mixing statement

in this paper (Proposition 4.5).

We separate the strands, since much of what follows is transcription.

New here: the collapse theorem and its corollary (Theorem 5.1, Corollary 5.1), the exclusion of scalar multiples (Proposition 5.1), the separating pair (Theorem 5.2) with its Köthe variant (Proposition 5.5), and the conjectural characterisation on Köthe echelon spaces (Conjecture 5.1).

Transposed, with a change of setting rather than of method: Proposition 3.1 and Theorem 3.1. Writing $R_{l,k} := n_k^{-1} T_l^{n_k}$ turns both into statements about disjoint universality for a tuple of operator sequences, and criteria of that shape are standard; Remark 3.4 records the correspondence and the one place where the general theory does not apply.

Not new: the unscaled statements on $H(\mathbb{C})$ and on ω (Propositions 4.2 and 4.3), both consequences of the criterion of Bès and Peris [10]. We include them because Theorem 4.1 and Proposition 4.4 are derived from them.

Section 2 fixes the standing hypotheses and the terminology, states León-Saavedra's equivalence with its Banach hypothesis, and records the single-operator relations among hypercyclicity, Cesàro-hypercyclicity and supercyclicity. Section 3 contains the transitivity characterisation, the discussion of what it does not say, and the criterion. Section 4 treats $H(\mathbb{C})$: the Köthe identification, the unscaled family, the transfer lemma, the main theorem, and, in Section 4.5, the shift on ω. Section 5 delimits the phenomenon—the collapse, the two obstructions, the separating pair on $\ell^2(\mathbb{Z})$, its Köthe-space variant, and the conjecture. Theorem 5.1's proof, however, uses neither Section 3 nor Section 4, relying only on the definitions of Section 2; a reader wanting that result alone may go directly to it after Section 2. Section 6 summarises and lists seven open questions.

2. Preliminaries and definitions

2.1. Standing conventions

Scalars are complex throughout: every vector space appearing below is a vector space over $\mathbb{C}$. Several statements survive verbatim over $\mathbb{R}$, but the applications of Sections 4 and 5 are complex-analytic or use complex homogeneity, and it costs nothing to fix the field once.

We write $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Orbits are indexed by $\mathbb{N}_0$; scaled orbits, which carry a factor $1/n$, are necessarily indexed by $\mathbb{N}$. The letters n and n_k are reserved for iteration times and are never used as seminorm indices; seminorms are indexed by m throughout, and j, k, r serve as auxiliary counters.

For a topological vector space X, we denote by $L(X)$ the algebra of continuous linear operators $T : X \to X$. Fréchet spaces are metrizable and hence bornological, so on such a space a linear operator is continuous precisely when it maps bounded sets to bounded sets; the two possible readings of "bounded operator" therefore agree, and we shall not distinguish them.

2.2. Standing hypothesis

Unless explicitly weakened, X denotes a *separable, infinite-dimensional Fréchet space over* $\mathbb{C}$, whose topology is generated by a fixed increasing separating sequence $(p_m)_{m\in\mathbb{N}}$ of continuous seminorms. All operators are elements of $L(X)$.

Infinite-dimensionality is not decorative. It guarantees that X, and with it every finite power X^N, has

no isolated points, and that hypothesis is what licenses the transitivity arguments of Section 3 and the Birkhoff-type equivalences invoked in Section 5.

2.3. Fréchet spaces

A *Fréchet space* is a complete, metrizable, locally convex Hausdorff topological vector space. The following reformulation is the one we use in practice; we record it with all its hypotheses, since two of them do real work later on.

A locally convex space X is Fréchet if and only if there is a sequence $(p_m)_{m\in\mathbb{N}}$ of continuous seminorms on X which is

(F1) *increasing*: $p_m \le p_{m+1}$ for every m;
(F2) *separating*: $p_m(x) = 0$ for all m forces $x = 0$;
(F3) *fundamental*: the sets $\{x : p_m(x) < \varepsilon\}$, $m \in \mathbb{N}$, $\varepsilon > 0$ form a base of neighbourhoods of the origin;

and X is complete for the topology so described. Property (F2) is what makes limits unique; (F3) is what makes (p_m) describe the topology rather than merely the convergent sequences; and completeness is an independent requirement, not a consequence of the other three. Dropping it is a genuine loss: the space c_{00} of finitely supported sequences, equipped with $p_m := \|\cdot\|_1$ for every m, satisfies (F1)–(F3) and is not Fréchet. Completeness enters this paper through a single recurring device—the Baire category theorem—and that device is invoked, independently, at exactly three places: in the proof of Proposition3.1 (step (b)⇒(a)), in the proof of Proposition 4.1, and in the proof of Theorem 5.1. Every further dense-G_δ conclusion inherits completeness, by citation, from one of these three sites; in particular the dense-G_δ conclusion of Theorem 5.2 comes through Theorem 3.1 and hence from Proposition 3.1. The only result whose proof rests on the premise of Wang and Shu [14] rather than on Baire category is Proposition 5.2 (see Remark 5.4).

Remark 3.2 records the narrower fact that, within the chain from Proposition 3.1 to Theorem 3.1, completeness is used at exactly this one step. Background on Fréchet and locally convex spaces may be found in Meise and Vogt [22] and in Trèves [23].

Under (F1)–(F3), the topology is induced by the translation-invariant metric

$$d(x,y) := \sum_{m=1}^{\infty} 2^{-m}\,\frac{p_m(x-y)}{1+p_m(x-y)}, \qquad x,y\in X,$$

and X is complete with respect to d.

Because (p_m) increases, a finite conjunction of seminorm estimates is a single seminorm estimate. We record this explicitly, since it governs the form in which every approximation condition in this paper is stated.

Remark 2.1 (Reduction to one seminorm)**.** *Let $v \in X$, $m \in \mathbb{N}$, and $\varepsilon > 0$. Then*

$$p_k(v) < \varepsilon \ \ (k = 1,\dots,m) \qquad \Longleftrightarrow \qquad p_m(v) < \varepsilon$$

by (F1). Consequently the sets

$$U(x; m, \varepsilon) := \{z \in X : \ p_m(z - x) < \varepsilon\}, \qquad m \in \mathbb{N},\ \varepsilon > 0$$

form a neighbourhood base at x, and we shall use this one-seminorm form throughout. No generality is lost, and nothing is gained by writing finite intersections: monotonicity has already collapsed them.

Remark 2.1 disposes of a superficial reading of the Fréchet difficulty, and it is worth saying plainly where the actual difficulty lies, so that no claim is made for the wrong reason. Passing from a Banach space to a Fréchet space does *not* force one to control infinitely many seminorms simultaneously at any single stage of a construction; by Remark 2.1 the top index suffices at each stage. First, no single seminorm determines the topology, so a sequence of vectors may be small in p_m and enormous in p_{m+1}; an estimate obtained at one level carries no information about the next, and the diagonal argument of Section 3 must therefore let the seminorm level grow together with the inductive stage rather than fix it in advance. Second, the metric d is not absolutely homogeneous: $d(\lambda x, 0) \neq |\lambda|\, d(x, 0)$ in general, so the scalar $1/n$ carried by a Cesàro orbit cannot be moved through d. Every estimate involving that scalar has to be made at the level of the seminorms themselves, which are homogeneous, and the summability of the resulting series must then be verified in each p_m separately. These two points, and not any simultaneous control of an infinite family, are what distinguish the arguments below from their Banach-space ancestors in [14].

2.4. Hypercyclicity, supercyclicity, and transitivity

Let $T \in L(X)$. The *orbit* of $x \in X$ under T is

$$\mathrm{Orb}(T, x) := \{T^n x : \ n \in \mathbb{N}_0\}.$$

The operator T is *hypercyclic* if $\mathrm{Orb}(T, x)$ is dense in X for some x, and such an x is a *hypercyclic vector*; the set of these is written as $HC(T)$. Weakening "orbit" to "projective orbit", T is *supercyclic* if

$$\mathbb{C} \cdot \mathrm{Orb}(T, x) = \{\lambda T^n x : \ \lambda \in \mathbb{C}, \ n \in \mathbb{N}_0\}$$

is dense in X for some x.

Call T *topologically transitive* if for every pair U, V of nonempty open subsets of X, there is $n \in \mathbb{N}$ with $T^n(U) \cap V \neq \varnothing$. The link between the two notions is Birkhoff's transitivity theorem, which we state with the hypotheses it actually needs.

Theorem 2.1 (Birkhoff; see [1, Chapter 1], [2, Chapter 1])**.** *Let Y be a separable completely metrizable topological space without isolated points and let $T : Y \to Y$ be continuous. Then T is topologically transitive if and only if the set of points with dense orbit is a dense G_δ subset of Y. In particular each of the two properties implies the other.*

Two hypotheses in Theorem 2.1 are usually left implicit and should not be. Complete metrizability, through the Baire category theorem, yields the passage from transitivity to a dense G_δ of hypercyclic vectors; separability supplies the countable base along which the G_δ is assembled. The absence of isolated points is what makes the reverse passage work: from a vector x with dense orbit, one must produce, for prescribed U, V, a *single* time n with $T^n(U) \cap V \neq \varnothing$, and this requires deleting finitely many points from a dense set without destroying density. On a space with isolated points, the implication fails, as the example $Y = \{0\}$ already shows.

Our standing hypothesis puts X inside the scope of Theorem 2.1: a separable Fréchet space is separable and completely metrizable, and an infinite-dimensional topological vector space has no isolated points. The same applies to X^N for every $N \geq 1$, since X^N is again a separable infinite-dimensional Fréchet space. We shall appeal to this without further comment.

2.5. Cesàro-hypercyclicity: the classical notion and the scaled orbit

For $T \in L(X)$, write

$$M_n(T) := \frac{1}{n}(I + T + T^2 + \cdots + T^{n-1}), \qquad n \in \mathbb{N}$$

for the arithmetic means of the powers of T. Following León-Saavedra [13], T is *Cesàro-hypercyclic* if there is $x \in X$ whose sequence of Cesàro means $\{M_n(T)x : n \in \mathbb{N}\}$ is dense in X.

León-Saavedra's theorem replaces this sequence by a far more tractable one. We state it with the hypothesis under which it was proved.

Theorem 2.2 (León-Saavedra [13])**.** *Let E be a separable* Banach *space and let $T \in L(E)$. Then T is Cesàro-hypercyclic if and only if there exists $x \in E$ such that $\{n^{-1}T^n x : n \in \mathbb{N}\}$ is dense in E.*

One half of Theorem 2.2 is algebraic and survives in any vector space: from $(n+1)M_{n+1}(T) - nM_n(T) = T^n$, one obtains

$$\frac{1}{n}T^n = \frac{n+1}{n}\,M_{n+1}(T) - M_n(T), \qquad n \in \mathbb{N},$$

so density of the scaled sequence follows from suitable behaviour of the means. The converse implication is the substantial one, and the argument of [13] is norm-theoretic. We are not aware of a proof, or of a counterexample, on a general Fréchet space, and we do not use one. To keep the logical status of every later statement unambiguous, we therefore take the scaled orbit as the primitive object of this paper.

Definition 2.1 (Working definition)**.** *Let $T \in L(X)$. The* scaled orbit*, which we continue to call the* Cesàro orbit*, of $x \in X$ under T is*

$$\mathrm{Orb}_{\mathrm{C}}(T, x) := \left\{\tfrac{1}{n}T^n x : n \in \mathbb{N}\right\}.$$

We call T Cesàro-hypercyclic *if*

$$\exists\, x \in X : \quad \mathrm{Orb}_{\mathrm{C}}(T, x) \text{ is dense in } X$$

and any such x a Cesàro-hypercyclic vector*; the set of these is denoted $CH(T)$. Explicitly, $x \in CH(T)$ if and only if*

$$\forall\, y \in X \ \ \forall\, m \in \mathbb{N} \ \ \forall\, \varepsilon > 0 \ \ \exists\, n \in \mathbb{N} : \qquad p_m\left(\tfrac{1}{n}T^n x - y\right) < \varepsilon.$$

On a separable Banach space, Definition 2.1 agrees with the classical notion by Theorem 2.2; this is the sense in which the present paper extends [14], where the same scaled formulation is used and is likewise licensed by Theorem 2.2. On a Fréchet space, the agreement is not available to us, and we record the gap as an explicit question rather than papering over it.

Problem 2.1. *Let X be a separable Fréchet space and $T \in L(X)$. Is the density of $\{M_n(T)x : n \in \mathbb{N}\}$ for some $x \in X$ equivalent to the density of $\{n^{-1}T^n x : n \in \mathbb{N}\}$ for some $x \in X$? An affirmative answer would identify Definition 2.1 with the classical notion in the full generality in which we work.*

Everything asserted in this paper concerns Definition 2.1. Where a result is quoted from the literature for a Banach space, the classical notion and the scaled one coincide there, so no ambiguity arises.

It is useful to have the single-operator landscape in view before disjointness is introduced, since Section 5 separates two *disjoint* notions and the reader is entitled to know how much of the separation is already present for one operator. The first observation is immediate.

Proposition 2.1. *Every Cesàro-hypercyclic operator on X is supercyclic, and $CH(T)$ is contained in the set of supercyclic vectors of T.*

Proof. Let $x \in CH(T)$. Each element $n^{-1}T^n x$ of $\mathrm{Orb}_{\mathrm{C}}(T,x)$ has the form $\lambda T^n x$ with $\lambda = 1/n$, so $\mathrm{Orb}_{\mathrm{C}}(T,x) \subseteq \mathbb{C}\cdot\mathrm{Orb}(T,x)$. A set containing a dense set is dense. □

Neither of the two implications between hypercyclicity and Cesàro-hypercyclicity holds. Both failures already occur among weighted backward shifts on Hilbert space.

Remark 2.2 (The two notions are independent)**.** (a) Cesàro-hypercyclic $\not\Rightarrow$ hypercyclic. *León-Saavedra [13] exhibits bilateral weighted shifts on $\ell^2(\mathbb{Z})$ that are Cesàro-hypercyclic while failing to be hypercyclic. This is the mechanism exploited in a disjoint setting in Section 5.*

(b) Hypercyclic $\not\Rightarrow$ Cesàro-hypercyclic. *Let B_w be the unilateral weighted backward shift on $\ell^2(\mathbb{N}_0)$ determined by $w_1 := 1$ and $w_i := \sqrt{i/(i-1)}$ for $i \geq 2$, so that $\prod_{i=1}^{n} w_i = \sqrt{n}$. The weights are bounded, and $\sup_n \prod_{i=1}^{n} w_i = \infty$, so B_w is hypercyclic by Salas's characterization [20]. On the other hand, for every $x = (x_j)_{j\geq 0} \in \ell^2$,*

$$\left|\left(\tfrac{1}{n}B_w^n x\right)_0\right| = \frac{1}{n}\Big(\prod_{i=1}^{n} w_i\Big)|x_n| = \frac{|x_n|}{\sqrt{n}} \leq \frac{\|x\|}{\sqrt{n}},$$

so all but finitely many members of $\mathrm{Orb}_{\mathrm{C}}(B_w,x)$ have zeroth coordinate of modulus less than 1. That orbit therefore meets the nonempty open set $\{y : |y_0| > 1\}$ at most finitely often and cannot be dense. Hence B_w is not Cesàro-hypercyclic for any x. Examples of the same phenomenon for weighted composition operators, namely hypercyclic operators that fail to be Cesàro hypertransitive, are given by Ivković [24]; the unilateral example above is included only for self-containedness.

Further structural results on Cesàro-hypercyclic operators are in [16–18], and the disjoint setting is developed in [14]. For the contrast with Rolewicz's classical criterion for λB on $\ell^p(\mathbb{N})$, see [5].

One structural warning belongs here because it explains why several standard tools are unavailable throughout the paper. The scaled family $\{n^{-1}T^n\}_{n\geq 1}$ is not a semigroup:

$$\left(\tfrac{1}{n}T^n\right)\left(\tfrac{1}{n'}T^{n'}\right) = \frac{1}{nn'}\,T^{n+n'} \;\neq\; \frac{1}{n+n'}\,T^{n+n'} \qquad \text{whenever } n,n' \geq 2.$$

Composition of two members of the family lands outside the family. Every argument that composes iterates—and this includes the standard proof that a hypercyclic operator is topologically transitive, as well as the theorems of Ansari and of Bourdon—Feldman on powers and orbit closures—must therefore be re-derived rather than quoted. Proposition 3.1 below avoids composition altogether, deducing transitivity from the *density* of the set of Cesàro-hypercyclic vectors; this is a necessity, not a stylistic choice.

2.6. Disjointness

Throughout this subsection, $N \geq 2$ and $T_1, T_2, \ldots, T_N \in L(X)$. The product X^N carries the product topology, generated by the increasing separating sequence of seminorms

$$p_m^{(N)}(x_1, x_2, \ldots, x_N) := \max_{1 \leq i \leq N} p_m(x_i), \qquad m \in \mathbb{N},$$

so that X^N is again a separable infinite-dimensional Fréchet space.

Definition 2.2. *The family* $(T_1, T_2, \ldots, T_N)$ *is* disjoint hypercyclic*, in the sense of Bernal-González [9] and of Bès and Peris [10], if there exists* $x \in X$ *such that*

$$\{(T_1^n x, T_2^n x, \ldots, T_N^n x) : \ n \in \mathbb{N}_0\}$$

is dense in X^N*. Such an* x *is a* disjoint hypercyclic vector*, and the set of these is written* $d\text{-}HC(T_1, \ldots, T_N)$*.*

Definition 2.3. *The family* $(T_1, T_2, \ldots, T_N)$ *is* disjoint Cesàro-hypercyclic *if*

$$\exists\, x \in X : \qquad \left\{\left(\tfrac{1}{n} T_1^n x, \tfrac{1}{n} T_2^n x, \ldots, \tfrac{1}{n} T_N^n x\right) : \ n \in \mathbb{N}\right\} \text{ is dense in } X^N.$$

Such an x *is a* disjoint Cesàro-hypercyclic vector*, and the set of these is written*

$$d\text{-}CH(T_1, T_2, \ldots, T_N).$$

This is the framework of Wang and Shu [14], transposed to X*.*

Written out with all quantifiers displayed, Definition 2.3 reads

$$\exists\, x \in X \ \ \forall\, (y_1, \ldots, y_N) \in X^N \ \ \forall\, m \in \mathbb{N} \ \ \forall\, \varepsilon > 0 \ \ \exists\, n \in \mathbb{N} : \qquad p_m\!\left(\tfrac{1}{n} T_i^n x - y_i\right) < \varepsilon \quad (1 \leq i \leq N),$$

equivalently $p_m^{(N)}((\tfrac{1}{n} T_i^n x)_i - (y_i)_i) < \varepsilon$. The position of the existential quantifier on x is the whole content of the definition: one vector must serve every target. Interchanging it with the universal quantifiers on (y_i) produces a strictly weaker condition, in which the witness is allowed to depend on the target; that weaker condition is a transitivity statement, and its precise relation to Definition 2.3 is the subject of Proposition 3.1. We use the one-seminorm form permitted by Remark 2.1.

It is convenient to fix notation for the dilated preimages that appear throughout Section 3. For $V \subseteq X$, $n \in \mathbb{N}$, and $T \in L(X)$, put

$$nV := \{nv : \ v \in V\}, \qquad T^{-n}(nV) := \{x \in X : \ T^n x \in nV\}.$$

Since X is a complex vector space and $n \neq 0$, the dilation $v \mapsto nv$ is a linear homeomorphism, so nV is open whenever V is, and

$$x \in T^{-n}(nV) \quad \Longleftrightarrow \quad \tfrac{1}{n} T^n x \in V. \tag{2.1}$$

Identity (2.1) is the only property of this notation that we use; it lets the scaled orbit be discussed in the language of preimages of fixed open sets.

Definition 2.4. *The family* $(T_1, T_2, \dots, T_N)$ *is* disjoint Cesàro-topologically transitive *if for all nonempty open sets* $U, V_1, V_2, \dots, V_N \subseteq X$*, there exists* $n \in \mathbb{N}$ *with*

$$U \cap T_1^{-n}(nV_1) \cap \cdots \cap T_N^{-n}(nV_N) \neq \varnothing,$$

that is, by (2.1)*, there exist* $x \in U$ *and* $n \in \mathbb{N}$ *with* $\frac{1}{n}T_l^n x \in V_l$ *for every* $1 \leq l \leq N$*.*

Definition 2.5. *The family* $(T_1, T_2, \dots, T_N)$ *is* disjoint Cesàro-mixing *if for all nonempty open sets* $U, V_1, \dots, V_N \subseteq X$*, there exists* $n_0 \in \mathbb{N}$ *such that*

$$U \cap T_1^{-n}(nV_1) \cap \cdots \cap T_N^{-n}(nV_N) \neq \varnothing \qquad \text{for every } n \geq n_0.$$

Definitions 2.4 and 2.5 are the Fréchet transcripts of the corresponding notions of [14]; disjoint Cesàro-mixing plays no role before the open problems of Section 6 and is recorded here only so that the terminology is fixed in one place. For $N = 1$, both definitions reduce to the scaled analogues of transitivity and mixing for a single operator.

Finally, we note the elementary containment $d\text{-}CH(T_1, T_2, \dots, T_N) \subseteq CH(T_l)$ for each l: the coordinate projection $\pi_l \colon X^N \to X$ is continuous, linear, and surjective, hence open, and it carries a dense subset of X^N onto a dense subset of X. The same argument gives $d\text{-}HC(T_1, T_2, \dots, T_N) \subseteq HC(T_l)$. Both containments are used in Section 5, where a family is shown to be disjoint Cesàro-hypercyclic by locating the failure of ordinary hypercyclicity in a single coordinate.

3. Disjoint Cesàro-hypercyclicity on Fréchet spaces

The standing hypothesis of Section 2 remains in force: X is a separable, infinite-dimensional Fréchet space over $\mathbb{C}$, with a fixed increasing separating fundamental sequence $(p_m)_{m\in\mathbb{N}}$ of continuous seminorms, and $N \geq 2$ is an integer. Recall from Remark 2.1 that monotonicity of (p_m) lets every finite conjunction of seminorm estimates be replaced by its top member, and that X^N, carrying the seminorms $p_m^{(N)} = \max_{1\leq i\leq N} p_m(\cdot_i)$, is again a separable infinite-dimensional Fréchet space.

Two remarks on what this section does and does not accomplish. The transitivity characterisation of Proposition 3.1 is a statement about the *size* of $d\text{-}CH(T_1, T_2, \dots, T_N)$, not about its non-emptiness; the difference matters, and Section 3.3 explains why. The criterion of Theorem 3.1 is proved by an argument that produces the stronger, residual conclusion directly, which also welds the two results into a single chain rather than leaving them side by side.

3.1. A preliminary observation on measurability of the vector set

Before the characterisation itself, one half of it should be separated off, since it holds for every family whatsoever and its inclusion inside an equivalence obscures where the content lies.

Lemma 3.1. *For any* $T_1, T_2, \dots, T_N \in L(X)$*, the set* $d\text{-}CH(T_1, T_2, \dots, T_N)$ *is a* G_δ *subset of* X*. No further hypothesis is needed.*

Proof. Fix a countable base $\{A_j : j \in \mathbb{N}\}$ of nonempty open subsets of X, available because X is separable and metrizable. For a multi-index $J = (j_1, j_2, \dots, j_N) \in \mathbb{N}^N$, put

$$C_J := \bigcup_{n\geq 1} \Big(T_1^{-n}(nA_{j_1}) \cap \cdots \cap T_N^{-n}(nA_{j_N}) \Big). \tag{3.1}$$

For each $n \geq 1$, the dilation $v \mapsto nv$ is a linear homeomorphism of X, so nA_{j_l} is open; each T_l is continuous, so $T_l^{-n}(nA_{j_l})$ is open; a finite intersection and then an arbitrary union of open sets is open. Hence every C_J is open.

By (2.1), a vector x lies in C_J exactly when there is some $n \geq 1$ with $\frac{1}{n}T_l^n x \in A_{j_l}$ for every l, that is, exactly when the scaled orbit of x meets the basic open box $A_{j_1}, A_{j_2}, \times \cdots \times A_{j_N}$ of X^N. Boxes of this form constitute a base for X^N, so

$$d\text{-}CH(T_1, T_2, , \ldots, T_N) = \bigcap_{J \in \mathbb{N}^N} C_J, \tag{3.2}$$

a countable intersection of open sets. □

Only the *density* of $d\text{-}CH(T_1, T_2, \ldots, T_N)$, then, is ever in question. This is the shape the next proposition takes.

3.2. Cesàro-transitivity and residuality

Proposition 3.1 (Cesàro-transitivity on Fréchet spaces)**.** *Let $T_1, T_2, \ldots, T_N \in L(X)$. The following are equivalent.*

(a) *$d\text{-}CH(T_1, T_2, \ldots, T_N)$ is a dense G_δ subset of X.*

(b) *$(T_1, \ldots, T_N)$ is disjoint Cesàro-topologically transitive in the sense of Definition 2.4: for all nonempty open $U, V_1, \ldots, V_N \subseteq X$, there is $n \in \mathbb{N}$ with*

$$U \cap T_1^{-n}(nV_1) \cap \cdots \cap T_N^{-n}(nV_N) \neq \varnothing,$$

equivalently, there are $x \in U$ and $n \in \mathbb{N}$ with $\frac{1}{n}T_l^n x \in V_l$ for $1 \leq l \leq N$.

(c) *Given $x \in X$ and $(y_1, y_2, \ldots, y_N) \in X^N$, there are $(x_k) \subseteq X$ and $(n_k) \subseteq \mathbb{N}$ such that $x_k \to x$ and*

$$\frac{1}{n_k}T_l^{n_k} x_k \longrightarrow y_l \qquad (1 \leq l \leq N).$$

Moreover, (n_k) may be chosen strictly increasing (Remark 3.1).

(d) *Given $x \in X$, $(y_1, y_2, \ldots, y_N) \in X^N$, and a 0-neighbourhood W, there are $z \in X$ and $n \in \mathbb{N}$ with $z - x \in W$ and $\frac{1}{n}T_l^n z - y_l \in W$ for $1 \leq l \leq N$.*

Proof. (a)⇒(b). Let $U, V_1, \ldots, V_N$ be nonempty and open. Density of $d\text{-}CH(T_1, \ldots, T_N)$ supplies a vector x lying in $U \cap d\text{-}CH(T_1, \ldots, T_N)$. Its scaled orbit is dense in X^N and so meets $V_1 \times \cdots \times V_N$; a time $n \geq 1$ realising this is the required witness.

(b)⇒(a). Retain the notation (3.1) of Lemma 3.1. Fix J and a nonempty open U. Applying (b) to $U, A_{j_1}, \ldots, A_{j_N}$ produces $n \geq 1$ with $U \cap T_1^{-n}(nA_{j_1}) \cap \cdots \cap T_N^{-n}(nA_{j_N}) \neq \varnothing$, whence $U \cap C_J \neq \varnothing$. As U was arbitrary, C_J is dense; it is open by Lemma 3.1. A Fréchet space is completely metrizable, hence a Baire space, so the countable intersection $\bigcap_J C_J$ is dense, and by (3.2) it equals $d\text{-}CH(T_1, \ldots, T_N)$.

(b)⇒(c). Fix x and $(y_1, y_2, \ldots, y_N)$. Since (p_m) is separating, the balls

$$W_k := \{v \in X : \ p_k(v) < 1/k\}, \qquad k \in \mathbb{N}$$

form a decreasing base of balanced 0-neighbourhoods with $\bigcap_k W_k = \{0\}$. Set $U_k := x + W_k$ and $V_{l,k} := y_l + W_k$; these are nonempty and open. Feeding them to (b) yields $n_k \geq 1$ and $x_k \in U_k$ with $\frac{1}{n_k}T_l^{n_k} x_k \in V_{l,k}$ for every l. Then $x_k - x \in W_k$ and $\frac{1}{n_k}T_l^{n_k} x_k - y_l \in W_k$, and both differences tend to 0.

(c)⇒(d). Let W be a 0-neighbourhood. Apply (c) to x and $(y_1, \dots, y_N)$ and choose k so large that $x_k - x \in W$ and $\frac{1}{n_k} T_l^{n_k} x_k - y_l \in W$ for all l; take $z := x_k$, $n := n_k$.

(d)⇒(b). Given nonempty open $U, V_1, \dots, V_N$, select $x \in U$ and $y_l \in V_l$, then a 0-neighbourhood W with $x + W \subseteq U$ and $y_l + W \subseteq V_l$ for every l; such a W exists because finitely many neighbourhood conditions may be intersected. Condition (d) returns z and n with $z \in x+W \subseteq U$ and $\frac{1}{n} T_l^n z \in y_l+W \subseteq V_l$. By identity (2.1) of 2.6, this is precisely (b). □

Remark 3.1 (Hitting times)**.** *The proof of (b)⇒(c) above does not by itself force $n_k \to \infty$: condition (b) supplies one admissible time for each choice of open sets, and the scaled family admits no composition law that would convert one hitting time into larger ones, as $T^{n_2} = T^{n_2-n_1} T^{n_1}$ does for ordinary orbits. The times can nevertheless always be taken arbitrarily large by passing through (a). Let $U, V_1, \dots, V_N$ be nonempty and open, and pick $x \in U \cap \text{d-CH}(T_1, \dots, T_N)$. The scaled disjoint orbit of x is dense in X^N, which is Hausdorff and has no isolated points. Were the set of n with $(\frac{1}{n} T_l^n x)_l \in V_1 \times \cdots \times V_N$ finite, removing the finitely many corresponding points from $V_1 \times \cdots \times V_N$ would leave a nonempty open set missed by the orbit. Hence, under (a) every choice of $U, V_1, \dots, V_N$ admits infinitely many times n in (b), and in (c) the sequence (n_k) may be taken strictly increasing. For $N = 1$ this shows that on X, Definition 2.4 coincides with topological Cesàro hyper-transitivity in the sense of Ivković [24], which requires infinitely many admissible times; the latter trivially implies the former.*

3.3. What the proposition does not say

Proposition 3.1 characterises residuality of the set of disjoint Cesàro-hypercyclic vectors. It does not characterise the existence of one such vector, and the gap is not an oversight. Write

$$(0): \qquad d\text{-}CH(T_1, \dots, T_N) \neq \varnothing.$$

Trivially (a)⇒(0). The converse is open, for a reason worth recording, since it recurs in Section 5 and 6.

For ordinary hypercyclicity, the passage from a single dense orbit to transitivity runs as follows: given x with dense orbit and nonempty open U, V, pick n_1 with $T^{n_1} x \in U$ and then $n_2 > n_1$ with $T^{n_2} x \in V$, and read off $T^{n_2-n_1}(U) \cap V \neq \varnothing$ from the semigroup law $T^{n_2-n_1} T^{n_1} = T^{n_2}$. The scaled family obeys no such law:

$$\Big(\tfrac{1}{n_2-n_1} T^{n_2-n_1}\Big)\Big(\tfrac{1}{n_1} T^{n_1}\Big) = \frac{1}{n_1(n_2-n_1)} T^{n_2} \;\neq\; \frac{1}{n_2} T^{n_2} \qquad \text{once } n_1, n_2 - n_1 \geq 2,$$

so the argument breaks at its only substantive step. It is not the only possible route. For a single operator on a separable infinite-dimensional Banach space, Costakis and Hadjiloucas reached the conclusion by an argument that never composes iterates, passing instead through d-density and Baire category [15, Theorem 3.3]; we return to this in Problem 3.1. What the failure of the semigroup law does explain is why the proof of (b)⇒(a) above had to travel through a countable base and Baire category rather than through composition, and why the classical transfer theorems become conditional in the scaled setting. Bourdon-Feldman survives for Cesàro orbits only in weakened form: if $\text{Orb}_C(T, x)$ is somewhere dense, then $(0, \varepsilon)\,\text{Orb}_C(T, x)$ is dense for every $\varepsilon \in (0, 1)$ [15, Theorem 2.9], and the unweakened statement is known only under the extra hypothesis that T be hypercyclic [15, Remark 2.7]. A León-Müller theorem for the sequences $\{\lambda_n T^n\}$ with $\lambda_{n+1}/\lambda_n \to 1$, which covers $\lambda_n = 1/n$, holds on Fréchet spaces

when T is hypercyclic or satisfies the hypercyclicity criterion [15, Theorem 5.1 and Corollary 5.2]. Ansari's theorem is the one with no counterpart: a Cesàro version would follow from a positive answer to the somewhere-dense question, which is open [15, Problem 1].

Problem 3.1. *Let $T_1, \dots, T_N \in L(X)$ with $N \geq 2$. If some vector of X is disjoint Cesàro-hypercyclic for $(T_1, \dots, T_N)$, must the set of such vectors be dense? Equivalently, does* (0) *imply the conditions of Proposition 3.1?*

For a single operator on a separable infinite-dimensional Banach space, the answer is affirmative. Costakis and Hadjiloucas prove that if $\mathrm{Orb}_C(T, x)$ is d-dense for some x and some $d > 0$—every vector of X lying within d of a point of that orbit—then T is Cesàro hypercyclic [15, Theorem 3.3]. Their proof gives more than the statement records: it exhibits $\{x \in X : \overline{\mathrm{Orb}_C(T, x)} = X\}$ as a countable intersection of dense open sets. Since a dense orbit is d-dense at every scale, one Cesàro-hypercyclic vector already forces $CH(T)$ to be residual.

Two obstacles stand between that argument and the present question. The first is metric. Their Lemma 3.4 packs infinitely many points at mutual distance $2d$ into a ball of radius $3d$, and d-density presupposes a single scale; over a seminorm system, the notion must be relativised to a chosen p_m, and $\{z : p_m(z) < d\}$ is unbounded as soon as $\ker p_m \neq \{0\}$. We have not carried out that transfer. The second is independent of the topology and specific to disjointness: their Lemma 3.5 moves the seed from x to $T^m x$, retaining d-density up to $3d + \varepsilon$, whereas a family offers N distinct candidate seeds $T_1^m x, \dots, T_N^m x$, none of which carries the disjoint orbit.

Everything proved below yields the residual conclusion (a) directly, so Problem 3.1 is never invoked, and no result of this paper depends on its answer.

3.4. The criterion

The criterion that follows is the disjoint Cesàro analogue, on a Fréchet space, of the disjoint hypercyclicity criterion of Bès and Peris [10]; on separable Banach spaces, it is Theorem 3.5 of Wang and Shu [14]. Its conditions are convergence statements in X, written seminorm by seminorm because that is the form in which they are verified in Sections 4–6.

Theorem 3.1 (Disjoint Cesàro-hypercyclicity criterion)**.** *Let $T_1, \dots, T_N \in L(X)$ with $N \geq 2$, and let $(n_k)_{k \in \mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$. Suppose there are dense subsets $X_0, X_1, \dots, X_N \subseteq X$ and maps*

$$S_{l,k} \colon X_l \longrightarrow X, \qquad 1 \leq l \leq N,\ k \in \mathbb{N}$$

such that, for every $m \in \mathbb{N}$:

(i) $p_m\Big(\frac{1}{n_k} T_l^{n_k} x\Big) \to 0$ *as* $k \to \infty$, *for every* $x \in X_0$ *and every* $1 \leq l \leq N$;
(ii) $p_m(S_{l,k} y) \to 0$ *as* $k \to \infty$, *for every* $y \in X_l$ *and every* $1 \leq l \leq N$;
(iii) $p_m\Big(\frac{1}{n_k} T_l^{n_k} S_{i,k} y - \delta_{l,i}\, y\Big) \to 0$ *as* $k \to \infty$, *for every* $y \in X_i$ *and all* $1 \leq l, i \leq N$, *where* $\delta_{l,i}$ *is the Kronecker symbol.*

Then d-$CH(T_1, \dots, T_N)$ is a dense G_δ subset of X. In particular $(T_1, \dots, T_N)$ is disjoint Cesàro-hypercyclic.

Proof. By Proposition 3.1, it is enough to verify condition (b) there, and no separate construction of a vector is needed.

Let $U, V_1, \dots, V_N \subseteq X$ be nonempty and open. Density of X_0 gives $x \in X_0 \cap U$; density of X_l gives $y_l \in X_l \cap V_l$ for each $1 \le l \le N$. Choose $m \in \mathbb{N}$ and $\varepsilon > 0$ with

$$\{v : \ p_m(v - x) < \varepsilon\} \subseteq U, \qquad \{v : \ p_m(v - y_l) < \varepsilon\} \subseteq V_l \quad (1 \le l \le N), \tag{3.3}$$

which is possible by Remark 2.1 after taking the largest of the finitely many seminorm indices and the smallest of the finitely many radii.

Put

$$u_k := x + \sum_{i=1}^{N} S_{i,k} y_i \ \in X, \qquad k \in \mathbb{N}.$$

Condition (ii) and subadditivity give

$$p_m(u_k - x) \le \sum_{i=1}^{N} p_m(S_{i,k} y_i) \xrightarrow[k\to\infty]{} 0,$$

so u_k enters the first neighbourhood in (3.3), hence $u_k \in U$, for all large k.

Fix l. Linearity of $T_l^{n_k}$ splits the scaled image into a term coming from the seed, the diagonal term, and the off-diagonal terms:

$$\frac{1}{n_k} T_l^{n_k} u_k - y_l = \frac{1}{n_k} T_l^{n_k} x + \Big(\frac{1}{n_k} T_l^{n_k} S_{l,k} y_l - y_l\Big) + \sum_{i \ne l} \frac{1}{n_k} T_l^{n_k} S_{i,k} y_i.$$

Apply p_m. The first summand tends to 0 by (i), since $x \in X_0$. The second tends to 0 by (iii) with $i = l$. Each of the remaining $N - 1$ summands tends to 0 by (iii) with $i \ne l$, the Kronecker symbol vanishing there. Consequently

$$p_m\Big(\frac{1}{n_k} T_l^{n_k} u_k - y_l\Big) \xrightarrow[k\to\infty]{} 0, \qquad 1 \le l \le N.$$

There are finitely many indices l, so a single k may be chosen for which $p_m(u_k - x) < \varepsilon$ and $p_m(\frac{1}{n_k} T_l^{n_k} u_k - y_l) < \varepsilon$ simultaneously for all l. With $n := n_k$ and this u_k, (3.3) yields

$$u_k \in U, \qquad \frac{1}{n} T_l^{n} u_k \in V_l \quad (1 \le l \le N),$$

which is condition (b) of Proposition 3.1 for the given $U, V_1, V_2, \dots, V_N$. Since these were arbitrary, the family is disjoint Cesàro-topologically transitive, and Proposition 3.1 delivers the dense G_δ conclusion. □

Remark 3.2 (Where the hypotheses are actually used)**.** *Three comments on the scope of Theorem 3.1, offered so that no more is claimed for it than it proves.*

The maps $S_{l,k}$ are not required to be linear, continuous, or defined off X_l; only their pointwise behaviour on X_l enters. The sets $X_0, \dots, X_N$ need not be subspaces.

Completeness of X is not used in the proof above. Neither is metrizability, beyond its role inside Proposition 3.1, where completeness enters exactly once—through the Baire category, in the step (b)⇒(a). Theorem 3.1 therefore holds, with (b) of Proposition 3.1 as its conclusion, on any separable topological vector space; it is the residual refinement that needs a Baire space.

Strict monotonicity of (n_k) is used only through $n_k \to \infty$, and even that only in the applications, where (i) and (iii) are verified by letting the iteration time grow. The proof itself consumes one index k.

Remark 3.3 (On the passage from norms to seminorms)**.** *It would be misleading to present Theorem 3.1 as requiring simultaneous control of an infinite family of seminorms. Monotonicity collapses finite conjunctions (Remark 2.1), and the proof above works at a single index m selected in advance from the given open sets. What genuinely distinguishes the Fréchet setting is more modest and more specific: no individual p_m determines the topology, so the index m must be read off from the target neighbourhoods rather than fixed once and for all, and smallness at level m carries no information at level $m+1$. In the applications of Sections 4 and 5, this is what forces the estimates to be performed uniformly in m—the factorial-versus-power comparison of Theorem 4.1 must hold for every radius m and the Köthe matrix quotients of Theorem 5.2 for every row.*

A second difference, invisible in the statement but decisive in the applications, is that the translation-invariant metric $d(\cdot,\cdot)$ of Section 2.3 is not absolutely homogeneous: $d(\lambda v, 0) \neq |\lambda|\, d(v,0)$ in general. The scalar $1/n_k$ carried by a Cesàro orbit cannot be extracted from d, and every estimate involving it has to be made against the seminorms themselves, which are homogeneous. That is why conditions (i)–(iii) are stated through (p_m) and not through d.

Remark 3.4 (Relation to disjoint universality)**.** *Setting $R_{l,k} := n_k^{-1} T_l^{n_k} \in L(X)$ for $1 \le l \le N$ and $k \in \mathbb{N}$, conditions (i)–(iii) become the hypotheses of the disjoint universality criterion for the tuple of sequences $((R_{1,k})_k, \dots, (R_{N,k})_k)$, and Proposition 3.1 becomes the corresponding equivalence between disjoint transitivity and residuality of the set of disjoint universal vectors. Criteria of this type for tuples of sequences of operators on separable infinite-dimensional Fréchet spaces are available in the literature; see Puig [25, Definitions 68 and 69] and, for the single-sequence case, [1, Chapters 1 and 9] and [2, Chapter 1]. We have preferred a self-contained treatment in the scaled notation, both because the applications are stated in it and because the specialisation is not entirely mechanical: the sequences $R_{l,k}$ do not form a semigroup, and Section 3.3 shows that one standard consequence of the general theory—the zero-one law passing from a single universal vector to a residual set of them—does not follow from it here, the sequences $R_{l,k}$ having no semigroup structure to compose. For a single operator on a Banach space, the law nonetheless holds by the independent route of [15, Theorem 3.3].*

Remark 3.5 (Disjoint Cesàro-mixing)**.** *Replacing "there is $n \in \mathbb{N}$" by "there is n_0 such that for all $n \ge n_0$" throughout Definition 2.4 gives disjoint Cesàro-mixing (Definition 2.5). The proof of Theorem 3.1 produces a single admissible k and yields no information about the remaining times, so it does not establish the mixing property; obtaining a Cesàro-mixing criterion would require conditions (i)–(iii) along the full sequence of positive integers rather than along a subsequence (n_k). We do not pursue this here; see question (Q5) of Section 6.*

Lemma 3.2 (Reduction to a spanning set)**.** *Let $E \subseteq X$ and suppose that $X_0 = X_1 = \cdots = X_N = \operatorname{span} E$ and that each $S_{l,k}$ is linear on* span *E. If conditions (i)–(iii) of Theorem 3.1 hold for every $y \in E$, they hold for every $y \in \operatorname{span} E$.*

Proof. Each of (i)–(iii) has the form $p_m(A_k y) \to 0$, where $A_k \in \{\frac{1}{n_k} T_l^{n_k},\ S_{l,k},\ \frac{1}{n_k} T_l^{n_k} S_{i,k} - \delta_{l,i}\mathrm{Id}\}$ is linear on $\operatorname{span} E$. For $y = \sum_{t=1}^{r} c_t e_t$ with $e_t \in E$, $p_m(A_k y) \le \sum_{t=1}^{r} |c_t|\, p_m(A_k e_t) \to 0$, the sum being finite and fixed. No uniformity in y is required, since the conditions are pointwise in y. □

4. Application: powers of the differentiation operator on $H(\mathbb{C})$

Entire functions in the compact-open topology form the oldest testing ground in linear dynamics. Birkhoff [3] produced an entire function whose translates are dense; some two decades later, MacLane [4] did the same for successive derivatives, showing that $Df = f'$ has a dense orbit in $H(\mathbb{C})$. We check here that the criterion of Section 3 covers any finite family of distinct powers of D, and that such a family is disjoint Cesàro-hypercyclic.

Two matters of scope are best settled before the proofs, since both govern how Theorem 4.1 ought to be read.

The first is how much strength the theorem carries. Projected onto one coordinate, Theorem 4.1 says only that each D^{r_l} is Cesàro-hypercyclic, hence supercyclic by Proposition 2.1; Remark 2.2(a) warns that this does not by itself return hypercyclicity. The projection is not the sharpest consequence available. Take $N = 2$, $r_1 = 1$, and $r_2 = 2$: the pair (D, D^2) is disjoint Cesàro-hypercyclic by Theorem 4.1, so Theorem 5.1 applies to it and yields hypercyclicity of $D^{r_2 - r_1} = D$. MacLane's theorem does therefore follow from Theorem 4.1, though only through the collapse of Section 5 and not through any coordinate projection. It also follows, more directly and without that detour, from the unscaled companion statement Proposition 4.2, which is not new.

The second concerns the setting. One is tempted to call the estimates below complex-analytic and to say they have no weighted-shift counterpart. That would be wrong. Taylor coefficients turn $H(\mathbb{C})$ into a Köthe echelon space and turn D into a weighted backward shift whose weights are unbounded, as Lemma 4.5 records. The single feature separating this application from the Banach-space theory of [14] is that unboundedness, which the canonical basis of $\ell^p(\mathbb{N})$ cannot support under any norm. Section 4 sits inside the framework of Section 5; it does not run alongside it.

4.1. The space and its shift structure

Give $H(\mathbb{C})$ the seminorms

$$p_m(f) := \sup_{|z| \le m} |f(z)|, \qquad f \in H(\mathbb{C}),\ m \in \mathbb{N}.$$

The sequence $(p_m)_{m \in \mathbb{N}}$ increases, separates points, and is fundamental for the compact-open topology, in which $H(\mathbb{C})$ is complete; the space is nontrivial and infinite-dimensional. Once separability is checked, the standing hypothesis of Section 2 is in force. For the locally convex background, see [22, 23].

Lemma 4.1. *$\mathbb{C}[z]$ is dense in $H(\mathbb{C})$, and $H(\mathbb{C})$ is separable.*

Proof. Let $f(z) = \sum_{n \ge 0} c_n z^n$ be entire and set $f_M := \sum_{n \le M} c_n z^n$. The Taylor series converges uniformly on each closed disc $\{|z| \le m\}$, so $p_m(f_M - f) \to 0$ as $M \to \infty$ for every fixed m, and $f_M \to f$ in $H(\mathbb{C})$. Polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$ form a countable set which is p_m-dense in $\mathbb{C}[z]$ for each m; combining the two approximations gives separability. □

Write J for the primitive vanishing at the origin,

$$Jf(z) := \int_0^z f(t)\, dt,$$

the path of integration being immaterial and $\mathbb{C}$ being simply connected. Both D and J are continuous linear operators on $H(\mathbb{C})$, and $DJ = \mathrm{Id}$. The opposite composition is not the identity: JD annihilates constants. Order will matter in Lemma 4.3.

Lemma 4.2. *For $j \geq 0$ and $r \geq 1$,*

$$J^r(z^j) = \frac{j!}{(j+r)!}\, z^{j+r}, \qquad D^r(z^j) = \begin{cases} \dfrac{j!}{(j-r)!}\, z^{j-r}, & j \geq r, \\ 0, & j < r. \end{cases}$$

Proof. Induct on r for the first identity. At $r = 1$, it is the definition, $J(z^j) = z^{j+1}/(j+1)$. Assuming it at stage r,

$$J^{r+1}(z^j) = J\left(\frac{j!}{(j+r)!}\, z^{j+r}\right) = \frac{j!}{(j+r)!} \cdot \frac{z^{j+r+1}}{j+r+1} = \frac{j!}{(j+r+1)!}\, z^{j+r+1}.$$

The second identity is the elementary rule for iterated differentiation of a monomial. □

Lemma 4.3. *$D^r J^r = \mathrm{Id}_{H(\mathbb{C})}$ for every $r \geq 1$.*

Proof. At $r = 1$, this is the fundamental theorem of calculus. Granting $D^r J^r = \mathrm{Id}$,

$$D^{r+1} J^{r+1} = D^r\,(DJ)\,J^r = D^r J^r = \mathrm{Id}\,.$$

The bracketing carries the argument; no such identity holds for $J^r D^r$. □

The next comparison is the only piece of analysis in this section, and it will be invoked three times. We state it once, with the factor multiplying it controlled by the same variable that drives the decay—the two must be tied, or the conclusion fails.

Lemma 4.4. *Fix $m \in \mathbb{N}$ and $j \geq 0$. For every integer $\nu \geq 0$ with $j + \nu \geq 1$,*

$$\frac{j!\, m^{j+\nu}}{(j+\nu)!} \;\leq\; j!\left(\frac{e\,m}{j+\nu}\right)^{j+\nu}, \tag{4.1}$$

and once $j + \nu \geq 2em$,

$$\frac{j!\, m^{j+\nu}}{(j+\nu)!} \;\leq\; j!\, 2^{-(j+\nu)}. \tag{4.2}$$

Let $(\nu_k)_{k\in\mathbb{N}}$ be integers with $\nu_k \to \infty$, and let $(q_k)_{k\in\mathbb{N}}$ satisfy $0 \leq q_k \leq C\,(1+\nu_k)^d$ for constants $C > 0$ and $d \geq 0$. Then

$$q_k\, \frac{j!\, m^{j+\nu_k}}{(j+\nu_k)!} \longrightarrow 0 \qquad (k \to \infty). \tag{4.3}$$

Proof. Expanding the exponential series gives $e^n \geq n^n/n!$, hence $n! \geq (n/e)^n$ for every $n \geq 1$; taking $n = j + \nu$ and dividing produces (4.1). When $j + \nu \geq 2em$, the bracket in (4.1) is at most $\frac{1}{2}$, which is (4.2).

For (4.3), choose k_0 with $j + \nu_k \geq 2em$ for $k \geq k_0$. Then, for such k,

$$q_k\, \frac{j!\, m^{j+\nu_k}}{(j+\nu_k)!} \;\leq\; C\, j!\,(1+\nu_k)^d\, 2^{-(j+\nu_k)} \;\leq\; C\, j!\,(1+\nu_k)^d\, 2^{-\nu_k},$$

and $(1+\nu)^d 2^{-\nu} \to 0$ as $\nu \to \infty$ for each fixed d. Since $\nu_k \to \infty$, the right-hand side tends to 0. □

The identification that follows is elementary. We record it because it closes the apparent gap between this section and Section 5. To keep the coordinate vectors distinct from the functions $z^j/j!$ appearing in Remark 4.6, we write $(\varepsilon_j)_{j\geq 0}$ for the canonical basis of a sequence space.

Lemma 4.5 ($H(\mathbb{C})$ as a Köthe echelon space; D and J as shifts)**.** *Let $A = (a_m)_{m\in\mathbb{N}}$ be the Köthe matrix on $\mathbb{N}_0$ with $a_m(j) := m^j$, and set*

$$\lambda_1(A) := \Big\{c = (c_j)_{j\geq 0} \in \mathbb{C}^{\mathbb{N}_0} : \ p_m^A(c) := \sum_{j\geq 0} |c_j|\, a_m(j) < \infty \ \text{for every } m\Big\}.$$

The Taylor coefficient map $\Phi f := \big(f^{(j)}(0)/j!\big)_{j\geq 0}$ is a topological isomorphism of $H(\mathbb{C})$ onto $\lambda_1(A)$; explicitly

$$p_m(f) \ \leq \ p_m^A(\Phi f) \ \leq \ 2\, p_{2m}(f), \qquad f \in H(\mathbb{C}),\ m \in \mathbb{N}. \tag{4.4}$$

Under Φ, the operator D becomes the weighted backward shift and J the associated forward shift:

$$\Phi D \Phi^{-1} = B_w, \quad B_w\varepsilon_j = w_j\varepsilon_{j-1}\ (j \geq 1),\ B_w\varepsilon_0 = 0, \qquad \Phi J \Phi^{-1} = S_w, \quad S_w\varepsilon_j = w_{j+1}^{-1}\varepsilon_{j+1},$$

with weights $w_j = j$. In particular, $B_w S_w = \mathrm{Id}$.

Proof. Write $f = \sum_j c_j z^j$. The triangle inequality on $\{|z| \leq m\}$ gives $p_m(f) \leq \sum_j |c_j| m^j = p_m^A(\Phi f)$. Cauchy's inequalities on the circle of radius $2m$ give $|c_j| \leq p_{2m}(f)(2m)^{-j}$, whence

$$p_m^A(\Phi f) = \sum_{j\geq 0} |c_j|\, m^j \leq p_{2m}(f) \sum_{j\geq 0} 2^{-j} = 2\, p_{2m}(f),$$

which is (4.4). Both seminorm systems increase, so (4.4) makes them cofinal in one another and Φ a topological isomorphism onto its range. That range is all of $\lambda_1(A)$: if $\sum_j |c_j| m^j < \infty$ for every m, then $\limsup_j |c_j|^{1/j} = 0$, so $\sum_j c_j z^j$ has infinite radius of convergence.

Termwise differentiation gives $Df = \sum_{j\geq 1} j c_j z^{j-1}$, so the coefficient sequence of Df is $((j+1)c_{j+1})_{j\geq 0}$, the action of B_w with $w_j = j$. Termwise integration gives $Jf = \sum_{j\geq 0} c_j z^{j+1}/(j+1)$, whose j-th coefficient is c_{j-1}/j for $j \geq 1$ and 0 for $j = 0$; that is, S_w with the same weights. □

Remark 4.1. *The weights $w_j = j$ grow without bound. On $\ell^p(\mathbb{N})$ or $c_0(\mathbb{N})$, in the canonical basis, an unbounded weight sequence defines no bounded operator, so a shift of this kind simply does not exist there. On $\lambda_1(A)$, the increasing matrix absorbs the growth: since $j \leq 2^j$ and $m \geq 1$,*

$$w_j\, a_m(j-1) = j\, m^{j-1} \leq 2^j m^j = a_{2m}(j) \qquad (j \geq 1),$$

and summing gives $p_m^A(B_w c) \leq p_{2m}^A(c)$. Here lies the entire difference between the present application and the weighted-shift theory of [14]. The factorial ratios appearing below are nothing more exotic than the matrix quotients $a_m(j)/a_m(j+r)$ paired with the weight products $\prod_{\nu=j+1}^{j+r} w_\nu = (j+r)!/j!$. Section 5 uses a matrix that is geometric in a different variable; neither matrix is more analytic than the other.

4.2. The unscaled criterion, the unscaled family, and a transfer lemma

The ordinary statement comes first, and the Cesàro one is read off from it; running two almost identical verifications would serve no purpose.

Our tool is the disjoint hypercyclicity criterion of Bès and Peris [10], which they formulated on separable Fréchet spaces. Nothing in Proposition 4.1 is ours. We restate it in the notation of this paper and supply the deduction of the residual conclusion, that being the form used later. Throughout, primed labels (i′)–(iii′) refer to the unscaled criterion of Proposition 4.1 and unprimed labels (i)–(iii) to the scaled criterion of Theorem 3.1.

Proposition 4.1 (Bès–Peris [10]; disjoint hypercyclicity criterion)**.** *Let $T_1, \dots, T_N \in L(X)$ with $N \geq 2$, and let $(n_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$. Suppose there are dense subsets $X_0, X_1, \dots, X_N \subseteq X$ and maps $\widetilde{S}_{l,k}\colon X_l \to X$ $(1 \leq l \leq N,\ k \in \mathbb{N})$ such that, for every $m \in \mathbb{N}$,*

(i′) $p_m(T_l^{n_k} x) \to 0$ *as* $k \to \infty$, *for each* $x \in X_0$ *and each* $1 \leq l \leq N$;
(ii′) $p_m(\widetilde{S}_{l,k} y) \to 0$ *as* $k \to \infty$, *for each* $y \in X_l$ *and each* $1 \leq l \leq N$;
(iii′) $p_m(T_l^{n_k} \widetilde{S}_{i,k} y - \delta_{l,i}\, y) \to 0$ *as* $k \to \infty$, *for each* $y \in X_i$ *and all* $1 \leq l, i \leq N$.

Then d-HC$(T_1, \dots, T_N)$ is a dense G_δ subset of X; the family is in particular disjoint hypercyclic.

Proof. Transitivity. Take nonempty open sets $U, W_1, \dots, W_N \subseteq X$. Density supplies $x \in X_0 \cap U$ and $y_l \in X_l \cap W_l$; by Remark 2.1, one index m and one radius $\varepsilon > 0$ serve all $N + 1$ sets, the family being finite:

$$\{v :\ p_m(v - x) < \varepsilon\} \subseteq U, \qquad \{v :\ p_m(v - y_l) < \varepsilon\} \subseteq W_l \quad (1 \leq l \leq N).$$

Put $u_k := x + \sum_{i=1}^{N} \widetilde{S}_{i,k} y_i$. Then $p_m(u_k - x) \leq \sum_i p_m(\widetilde{S}_{i,k} y_i) \to 0$ by (ii′), so $u_k \in U$ for large k. Fix l and split

$$T_l^{n_k} u_k - y_l = T_l^{n_k} x + (T_l^{n_k} \widetilde{S}_{l,k} y_l - y_l) + \sum_{i \neq l} T_l^{n_k} \widetilde{S}_{i,k} y_i.$$

Under p_m, the first summand tends to 0 by (i′), the second by (iii′) with $i = l$, and each of the other $N - 1$ by (iii′) with $i \neq l$. Only finitely many indices l are in play, so one k can be chosen with $u_k \in U$ and $T_l^{n_k} u_k \in W_l$ for every l at once. Writing $n := n_k$,

$$U \cap T_1^{-n}(W_1) \cap \cdots \cap T_N^{-n}(W_N) \neq \varnothing. \tag{4.5}$$

From transitivity to residuality. Fix a countable base $\{A_j : j \in \mathbb{N}\}$ of nonempty open subsets of X, available since X is separable and metrizable, and put, for $J = (j_1, \dots, j_N) \in \mathbb{N}^N$,

$$C_J := \bigcup_{n \geq 0} \Big(T_1^{-n}(A_{j_1}) \cap \cdots \cap T_N^{-n}(A_{j_N}) \Big).$$

Each T_l is continuous, so each C_J is open. Given a nonempty open $U \subseteq X$, applying (4.5) to $U, A_{j_1}, \dots, A_{j_N}$ shows $U \cap C_J \neq \varnothing$; hence every C_J is dense. A vector x belongs to all the C_J precisely when its disjoint orbit $\{(T_1^n x, \dots, T_N^n x) : n \in \mathbb{N}_0\}$ meets every basic box of X^N, that is, precisely when the orbit is dense. Therefore

$$d\text{-}HC(T_1, \dots, T_N) = \bigcap_{J \in \mathbb{N}^N} C_J,$$

a countable intersection of dense open sets. A separable Fréchet space is completely metrizable, hence Baire, and the intersection is dense G_δ. □

Remark 4.2. *Bès and Peris prove more than the implication used above, and the converse is often misquoted, so we state it with care. For a fixed strictly increasing (n_k), their theorem is that $(T_1, T_2, \dots, T_N)$ satisfies the criterion of Proposition 4.1 along (n_k) if and only if the family is hereditarily densely disjoint hypercyclic with respect to (n_k) [10]. Plain disjoint hypercyclicity is a weaker property, and whether it forces the criterion—even for distinct powers of a single operator—is not decided by that theorem; the question is raised in [21], whose Theorem 2.7 proves a related result—under the hypothesis that D itself is hypercyclic, rather than that the tuple $(D^{r_1}, D^{r_2}, \dots, D^{r_N})$ is disjoint hypercyclic—for powers of the differentiation operator on certain weighted Bergman spaces of entire functions; whether that result settles the question as posed is not addressed there. Criteria of the same shape for tuples of sequences of operators on separable Fréchet spaces appear in [25, Definition 69]. Disjoint hypercyclicity itself was introduced independently in [9] and [10].*

We turn to the family of interest. The result is not new, and the verification is the routine one; it is included because Theorem 4.1 is derived from it.

Proposition 4.2. *Let $N \geq 2$ and let $1 \leq r_1 < r_2 < \cdots < r_N$ be integers. Put $T_l := D^{r_l}$ for $1 \leq l \leq N$ and take*

$$n_k := k, \qquad X_0 = X_1 = \cdots = X_N := \mathbb{C}[z], \qquad \widetilde{S}_{l,k}(z^j) := J^{r_l n_k}(z^j) = \frac{j!}{(j + r_l n_k)!}\, z^{j + r_l n_k}$$

for $j \geq 0$, extended linearly to $\mathbb{C}[z]$. These data satisfy (i′)–(iii′) *of Proposition 4.1, and consequently $d\text{-}HC(D^{r_1}, D^{r_2}, \dots, D^{r_N})$ is a dense G_δ subset of $H(\mathbb{C})$.*

Proof. Lemma 4.1 gives the density of $\mathbb{C}[z]$. All three conditions are additive in the vector being tested, and a polynomial is a finite combination of monomials, so it is enough to argue on z^j with $j \geq 0$ fixed. Abbreviate $\nu_{l,k} := r_l n_k$, noting $\nu_{l,k} \to \infty$.

Condition (i′). Let p have degree d. Once $\nu_{l,k} > d$, the lower branch of Lemma 4.2 kills every monomial of p, so $D^{\nu_{l,k}} p = 0$ and $p_m(D^{\nu_{l,k}} p) = 0$ from that point on, for every m.

Condition (ii′). Since $\sup_{|z| \leq m} |z|^{j+\nu} = m^{j+\nu}$,

$$p_m(\widetilde{S}_{l,k}(z^j)) = \frac{j!\, m^{j+\nu_{l,k}}}{(j + \nu_{l,k})!},$$

and (4.3) with $q_k \equiv 1$ gives convergence to 0.

Condition (iii′). Three cases, according to how r_i and r_l compare.

Let $i = l$. Lemma 4.3 gives $D^{\nu_{l,k}} \widetilde{S}_{l,k}(z^j) = D^{\nu_{l,k}} J^{\nu_{l,k}}(z^j) = z^j$ for every k, so the quantity under test is identically zero.

Let $r_i > r_l$. The monomial $\widetilde{S}_{i,k}(z^j)$ has degree $j + \nu_{i,k}$, which exceeds $\nu_{l,k}$ for every $k \geq 1$; the upper branch of Lemma 4.2 therefore applies with no largeness condition, and the factorials telescope:

$$D^{\nu_{l,k}} \widetilde{S}_{i,k}(z^j) = \frac{j!}{(j + \nu_{i,k})!} \cdot \frac{(j + \nu_{i,k})!}{(j + \mu_k)!}\, z^{j+\mu_k} = \frac{j!}{(j + \mu_k)!}\, z^{j+\mu_k}, \qquad \mu_k := (r_i - r_l) n_k.$$

As $\mu_k \to \infty$, another application of (4.3) with $q_k \equiv 1$ gives $p_m(D^{\nu_{l,k}} \widetilde{S}_{i,k}(z^j)) \to 0$.

Let $r_i < r_l$. For all k with $(r_l - r_i) n_k > j$, the degree $j + \nu_{i,k}$ of $\widetilde{S}_{i,k}(z^j)$ falls strictly below the order $\nu_{l,k}$ of differentiation, and Lemma 4.2 returns $D^{\nu_{l,k}} \widetilde{S}_{i,k}(z^j) = 0$.

All three conditions hold, and Proposition 4.1 yields the conclusion. □

The bridge to the Cesàro statement is the next observation, put in general form because it isolates precisely what the averaging factor costs and what it leaves untouched.

Lemma 4.6 (Transfer)**.** *Let $T_1, \dots, T_N \in L(X)$ with $N \geq 2$, let (n_k) be strictly increasing, and let $X_0, \dots, X_N$ together with $\widetilde{S}_{l,k} \colon X_l \to X$ satisfy* (i′)–(iii′) *of Proposition 4.1. Suppose in addition that*

$$n_k \; p_m(\widetilde{S}_{l,k}\, y) \longrightarrow 0 \qquad (k \to \infty) \tag{4.6}$$

for every $m \in \mathbb{N}$, every $1 \leq l \leq N$, and every $y \in X_l$. Then the same dense sets, with $S_{l,k} := n_k \widetilde{S}_{l,k}$, satisfy conditions (i)–(iii) *of Theorem 3.1 along (n_k).*

Proof. Condition (i) is the weaker of the two own-orbit requirements: as $n_k \geq 1$ and seminorms are absolutely homogeneous,

$$p_m\Big(\tfrac{1}{n_k} T_l^{n_k} x\Big) = \frac{1}{n_k}\, p_m(T_l^{n_k} x) \leq p_m(T_l^{n_k} x) \longrightarrow 0$$

by (i′). Condition (ii) is (4.6) verbatim, because $p_m(S_{l,k} y) = n_k\, p_m(\widetilde{S}_{l,k} y)$. Condition (iii) requires no estimate at all, the two scalars cancelling:

$$\frac{1}{n_k}\, T_l^{n_k} S_{i,k}\, y = \frac{1}{n_k}\, T_l^{n_k}(n_k \widetilde{S}_{i,k} y) = T_l^{n_k} \widetilde{S}_{i,k}\, y.$$

□

4.3. The theorem

Theorem 4.1. *Let $N \geq 2$ and let $1 \leq r_1 < r_2 < \cdots < r_N$ be integers. Then*

$$d\text{-}CH(D^{r_1}, \dots, D^{r_N})$$

is a dense G_δ subset of $H(\mathbb{C})$; the family $(D^{r_1}, \dots, D^{r_N})$ is in particular disjoint Cesàro-hypercyclic.

Proof. Retain the data of Proposition 4.2 and set $S_{l,k} := n_k \widetilde{S}_{l,k}$, that is,

$$S_{l,k}(z^j) = n_k \, \frac{j!}{(j + r_l n_k)!}\, z^{j + r_l n_k}, \qquad n_k = k.$$

By Lemma 4.6, only (4.6) remains to be checked, and by additivity only on monomials. With $\nu_{l,k} = r_l n_k$, one has $n_k = \nu_{l,k}/r_l \leq 1 + \nu_{l,k}$, so (4.3) applies with $q_k := n_k$, $C = 1$, and $d = 1$:

$$n_k\, p_m(\widetilde{S}_{l,k}(z^j)) = n_k \, \frac{j!\; m^{\,j+\nu_{l,k}}}{(j + \nu_{l,k})!} \longrightarrow 0 \qquad (k \to \infty)$$

for every m and every j. Theorem 3.1 now gives the dense G_δ conclusion. □

Remark 4.3. *Nothing in the two proofs above depends on the choice $n_k = k$. The bound $n_k \leq 1 + \nu_{l,k}$ used a moment ago holds for any strictly increasing (n_k), so Theorem 4.1 and Proposition 4.2 remain valid along an arbitrary strictly increasing sequence of positive integers.*

4.4. Remarks on scope

Remark 4.4 (The averaging factor is inert here)**.** *Lemma 4.6 pins down the whole effect of the Cesàro average, and it is worth spelling out where that effect is felt, since the point governs the reading of Section 5.*

Condition (iii), which carries the disjointness interactions, is untouched: the two occurrences of n_k cancel before any estimate is attempted. Condition (i) is relaxed—and for this family it is exact anyway, $D^{\nu_{l,k}}p$ vanishing outright for large k, so the division by n_k buys nothing. That leaves condition (ii), where n_k multiplies a quantity required to tend to 0 *and so works* against *the verification; the factorial decay is super-exponential and swallows it.*

Erase every n_k and Proposition 4.2 reappears, with condition (ii) easier rather than harder. Theorem 4.1 and Proposition 4.2 are two readings of a single computation.

One should not overstate how much of that computation is exact. Only the diagonal case $i = l$ of (iii′) is an identity, through $D^r J^r = \mathrm{Id}$ (Lemma 4.3). Condition (i′) is exact for an unrelated reason, high-order differentiation annihilating polynomials, and (ii′) is not exact at all—it is where the analysis sits. Section 5 records the same cancellation for weighted shifts (Corollary 5.2); that statement is the specialisation of Lemma 4.6 to $S_{l,k} = n_k S_w^{r_l n_k}$. A family separating the two notions has to be sought outside this pattern.

Remark 4.5 (Comparison with the shift on ω)**.** *The verification just given and the one carried out for $(B^{r_1}, \dots, B^{r_N})$ on ω in Proposition 4.4 follow one plan, with D for B, J for the forward shift, and monomials for coordinate vectors. They diverge at condition (ii). The seminorms of ω read finitely many coordinates, so a vector pushed far enough to the right is annihilated, and the condition holds for trivial reasons. No seminorm on $H(\mathbb{C})$ is finitely supported in the coefficients, and one has to weigh the weight products $\prod_{\nu=j+1}^{j+r} w_\nu$ of Lemma 4.5 against the matrix growth $a_m(j) = m^j$. Lemma 4.4 performs that comparison. It is a weighted-shift estimate on a space tolerating unbounded weights; what separates the two applications is a rate of decay, not a change of category.*

Remark 4.6 (On reading D as an unweighted shift)**.** *Setting $e_j := z^j/j!$ turns the action of D into $De_j = e_{j-1}$ for $j \geq 1$, $De_0 = 0$, formally an unweighted backward shift. The temptation is to set this beside Rolewicz's theorem [5]—for $|\lambda| > 1$, the operator λB is hypercyclic on $\ell^p(\mathbb{N})$ and on $c_0(\mathbb{N})$, while B itself, of norm* 1*, is not—and to conclude that the hypercyclicity of D belongs to the non-normable world.*

That conclusion does not follow. Acting as an unweighted shift in some basis *is not preserved under isomorphism. In $\ell^2(\mathbb{N}_0)$, put $f_j := 2^{-j}\varepsilon_j$, where (ε_j) is the canonical basis; then $(f_j)_{j\geq 0}$ is again an unconditional Schauder basis, with continuous coefficient functionals $x \mapsto 2^j x_j$. Let T be determined by $Tf_j = f_{j-1}$ $(j \geq 1)$ and $Tf_0 = 0$. Then $T\varepsilon_j = 2^j \cdot 2^{-(j-1)}\varepsilon_{j-1} = 2\varepsilon_{j-1}$, so $T = 2B$: bounded and hypercyclic by Rolewicz. Banach spaces too carry bases in which the unweighted shift is hypercyclic. What differs is the normalisation of the basis. In $H(\mathbb{C})$, the vectors e_j are not normalised in any sense—$p_m(e_j) = m^j/j! \to 0$ for fixed m—so setting them against the canonical basis of ℓ^p compares nothing.*

Remark 4.1 states the accurate version. In the coordinates of Lemma 4.5, D is the weighted backward shift with weights $w_j = j$; weights of that size live on $\lambda_1(A)$ and cannot live on $\ell^p(\mathbb{N})$ in its canonical basis. That difference is real, and it is the only one at work here.

Remark 4.7 (Position relative to the literature and to the rest of the paper)**.** *Bès, Martin, and Sanders [12] gave growth conditions on weight sequences characterising when finitely many distinct powers of a weighted backward shift on the classical sequence spaces form a disjoint hypercyclic family. It would be a mistake to say that no analogue of those conditions is called for here. Lemma 4.5 shows that* $(D^{r_1}, \dots, D^{r_N})$ is *a family of distinct powers of a weighted backward shift, on* $\lambda_1(A)$ *with* $a_m(j) = m^j$ *and* $w_j = j$*. What is absent from the literature is a version of [12] valid on* $\lambda_p(A)$*, and such a version would place Proposition 4.2 among many instances of a general theorem rather than leaving it as an isolated verification.*

On the Cesàro side, the corresponding target is Conjecture 5.1. Substituting $a_m(j) = m^j$ *and* $w_j = j$ *into* (K1)–(K4) *produces, in reciprocal form, precisely the quantity estimated in Lemma 4.4: the diagonal condition becomes the requirement that* $(j + r_l n)! \,/\, (n\, j!\, m^{r_l m})$ *be large, which is* (4.3) *read upside down. This does not mean that a proof of the conjecture would supply Theorem 4.1. The implication carrying the criterion to disjoint Cesàro-hypercyclicity is Theorem 3.1 and holds unconditionally; the conjectural content lies in the converse, the necessity of the weight conditions. Settling it would situate Theorem 4.1 inside a characterisation, not replace its proof.*

The family also falls under Theorem 5.1, being powers of one operator, and consistently with Remark 4.4, no separation between the scaled and unscaled notions arises here. What this section establishes is that a sufficient criterion is met on one family. A characterisation is still missing, and its absence is a gap in the theory rather than a feature of the example.

4.5. An extreme case: powers of the backward shift on ω

The space of all complex sequences is the smallest setting in which the criterion of Section 3 has anything to say, and also the setting in which it says the least: every approximation below is an exact identity. What singles out ω among the spaces treated here is not merely that it fails to be normable—so do $H(\mathbb{C})$ and the Köthe space of Proposition 5.5—but that it carries no continuous norm at all. Remark 4.8 makes the distinction precise.

Put $\omega := \mathbb{C}^{\mathbb{N}_0}$ with the product topology, generated by

$$p_m(x) := \max_{0 \le j \le m-1} |x_j|, \qquad x = (x_j)_{j \ge 0}, \quad m \in \mathbb{N}.$$

Let B and S denote the unweighted backward and forward shifts,

$$B(x_0, x_1, x_2, \dots) := (x_1, x_2, x_3, \dots), \qquad S(x_0, x_1, x_2, \dots) := (0, x_0, x_1, \dots),$$

so that S is the unweighted case of the forward shifts S_w of Section 5. Write c_{00} for the finitely supported sequences and ε_j for the coordinate vectors.

Lemma 4.7. *The sequence* $(p_m)_{m \in \mathbb{N}}$ *is increasing, separating, and fundamental for the product topology;* ω *is a separable, infinite-dimensional Fréchet space over* $\mathbb{C}$*, and* c_{00} *is dense in it. Both* B *and* S *belong to* $L(\omega)$*.*

Proof. Monotonicity is clear from the definition, and $p_m(x) = 0$ for every m forces every coordinate of x to vanish, so the family separates points. A basic neighbourhood of 0 in the product topology constrains finitely many coordinates, hence contains a set $\{x : p_m(x) < \varepsilon\}$; the family is therefore fundamental, and the topology is the product one. Completeness is coordinatewise completeness of $\mathbb{C}$.

Given $x \in \omega$, let $x^{(M)}$ agree with x in the coordinates $0, \dots, M-1$ and vanish beyond. Then $p_m(x^{(M)} - x) = 0$ as soon as $M \geq m$, so $x^{(M)} \to x$ and c_{00} is dense; restricting the coefficients to $\mathbb{Q} + i\mathbb{Q}$ produces a countable dense set, whence separability. The vectors ε_j are linearly independent, so the dimension is infinite. Continuity of the shifts is immediate from $p_m(Bx) \leq p_{m+1}(x)$ and $p_m(Sx) \leq p_m(x)$. See [22] or [23] for the locally convex background. □

Remark 4.8 (ω carries no continuous norm)**.** *Let p be any continuous seminorm on ω. Fundamentality of (p_m) gives $m \in \mathbb{N}$ and $C > 0$ with $p \leq C\, p_m$. Since $p_m(\varepsilon_j) = 0$ for every $j \geq m$, we get $p(\varepsilon_j) = 0$ while $\varepsilon_j \neq 0$: no continuous seminorm on ω separates points.*

The other two spaces of this paper behave differently. On $H(\mathbb{C})$, the first seminorm $p_1(f) = \sup_{|z|\leq 1} |f|$ is already a norm, since a function vanishing on the closed unit disc vanishes identically by the identity theorem. On $\lambda_2(A)$, the entries $a_1(n)$ are strictly positive, so p_1 is a norm there too. Both spaces are non-normable—no single seminorm gives the topology—yet each admits a continuous norm, and their dynamics still carry a normed trace. In ω, that trace is gone entirely. Every continuous seminorm reads finitely many coordinates, which is why the verifications below terminate in exact zeros rather than in estimates.

Lemma 4.8. *$BS = \mathrm{Id}_\omega$, and for integers $a, b \geq 0$,*

$$B^a S^b = \begin{cases} S^{\,b-a}, & b \geq a, \\ B^{a-b}, & a \geq b. \end{cases}$$

The order matters: SB annihilates the coordinate x_0 and is not the identity.

Proof. $(BSx)_j = (Sx)_{j+1} = x_j$, giving $BS = \mathrm{Id}$. Cancelling BS one factor at a time reduces $B^a S^b$ to $S^{\,b-a}$ when $b \geq a$ and to B^{a-b} when $a \geq b$. For the last assertion, $(SBx)_0 = 0$ whatever x_0 is. □

Two further facts drive everything: for $y \in c_{00}$ supported in $\{0, \dots, M\}$, one has $B^t y = 0$ once $t > M$, and for any $y \in \omega$, one has $p_m(S^t y) = 0$ once $t \geq m$, the seminorm reading only coordinates that S^t has emptied.

Proposition 4.3 (Unscaled family)**.** *Let $N \geq 2$ and $1 \leq r_1 < r_2 < \cdots < r_N$ be integers. Put $T_l := B^{r_l}$, $n_k := k$, $X_0 = X_1 = \cdots = X_N := c_{00}$, and $\widetilde{S}_{l,k} := S^{\,r_l n_k}$. These data satisfy* (i′)–(iii′) *of Proposition 4.1, so d-$HC(B^{r_1}, \dots, B^{r_N})$ is a dense G_δ subset of ω.*

Proof. Fix $m \in \mathbb{N}$ and $y \in c_{00}$, supported in $\{0, \dots, M\}$.

(i′). $B^{r_l n_k} y = 0$ as soon as $r_l n_k > M$, so $p_m(T_l^{n_k} y) = 0$ for large k.

(ii′). $p_m(S^{\,r_l n_k} y) = 0$ as soon as $r_l n_k \geq m$.

(iii′). Lemma 4.8 disposes of the three cases. For $i = l$, we get $B^{r_l n_k} S^{\,r_l n_k} y = y$ identically. For $r_i > r_l$, we get $S^{\,(r_i - r_l) n_k} y$, whose m-th seminorm vanishes once $(r_i - r_l) n_k \geq m$. For $r_i < r_l$, we get $B^{(r_l - r_i) n_k} y$, which is 0 once $(r_l - r_i) n_k > M$.

Proposition 4.1 applies. The statement is not new: it is the disjoint hypercyclicity criterion of Bès and Peris [10] applied to the most transparent shift there is. □

Proposition 4.4. *With N and $r_1 < \cdots < r_N$ as above, d-$CH(B^{r_1}, \dots, B^{r_N})$ is a dense G_δ subset of ω; the family is in particular disjoint Cesàro-hypercyclic.*

Proof. Retain the data of Proposition 4.3 and invoke Lemma 4.6, whose sole extra hypothesis is $n_k\, p_m(\widetilde{S}_{l,k}y) \to 0$. That hypothesis costs nothing here: the seminorm itself is 0 for large k, so the product is 0 and not merely small. Theorem 3.1 gives the conclusion. □

Remark 4.9 (Nothing is being averaged)**.** *It would misdescribe the computation to say that ω is well adapted to the scaled iterates $n^{-1}T^n$. All three conditions above hold by exact vanishing—$B^t y = 0$ for large t in (i′) and in one branch of (iii′), $p_m(S^t y) = 0$ for large t in (ii′) and the other branch, and $B^t S^t = \mathrm{Id}$ on the diagonal—and multiplying a zero by n_k, or dividing one by n_k, changes nothing. The scaled and unscaled criteria are verified by a single computation, and Propositions 4.3 and 4.4 are one statement read two ways. Compare Remark 4.4: on $H(\mathbb{C})$ at least condition (ii) carried a genuine estimate into which the factor n_k had to be absorbed. On ω there is no inequality anywhere for the averaging to be inert in.*

The exactness has one dividend worth collecting, since the paper otherwise says nothing about mixing.

Proposition 4.5 (Disjoint Cesàro-mixing on ω)**.** *Let $N \geq 2$ and $1 \leq r_1 < \cdots < r_N$. Then $(B^{r_1}, \ldots, B^{r_N})$ is disjoint Cesàro-mixing on ω in the sense of Definition 2.5 and disjoint mixing in the unscaled sense.*

Proof. Let $U, V_1, \ldots, V_N \subseteq \omega$ be nonempty and open. Density of c_{00} supplies $x \in c_{00} \cap U$ and $y_l \in c_{00} \cap V_l$; by Remark 2.1, one index m and one radius $\varepsilon > 0$ serve all $N + 1$ sets. Let M bound the supports of $x, y_1, \ldots, y_N$ and set

$$u_n := x + \sum_{i=1}^{N} n\, S^{\,r_i n} y_i, \qquad n \in \mathbb{N}.$$

Take $n \geq n_0 := \max\{m,\ M + 1\}$. Each $S^{\,r_i n} y_i$ has all coordinates below $r_i n \geq n \geq m$ equal to 0, so $p_m(u_n - x) = 0$ and $u_n \in U$. For the images, expand using Lemma 4.8:

$$\frac{1}{n} B^{r_l n} u_n = \frac{1}{n} B^{r_l n} x + \sum_{i=1}^{N} B^{r_l n} S^{\,r_i n} y_i.$$

Since $r_l n \geq n > M$, the first term vanishes. The summand $i = l$ equals y_l. A summand with $r_i > r_l$ equals $S^{(r_i - r_l)n} y_i$, whose first m coordinates are 0 because $(r_i - r_l)n \geq n \geq m$; a summand with $r_i < r_l$ equals $B^{(r_l - r_i)n} y_i = 0$ because $(r_l - r_i)n \geq n > M$. Hence

$$\frac{1}{n} B^{r_l n} u_n = y_l \in V_l \qquad (1 \leq l \leq N)$$

for *every* $n \geq n_0$, and $u_n \in U$ throughout. That is the mixing condition. Dropping the factors n from u_n and the factors $\frac{1}{n}$ from the images leaves the argument untouched and gives disjoint mixing. □

Remark 4.10. *This is the only mixing statement in the paper, and it is available only because the shift on ω annihilates outright. Question (Q5) of Section 6 asks for a disjoint Cesàro-mixing criterion on Fréchet spaces and for the status of $(D^{r_1}, \ldots, D^{r_N})$ on $H(\mathbb{C})$; neither is settled by Proposition 4.5, whose proof exploits a feature no space with a continuous norm possesses. Obtaining a criterion would mean imposing conditions* (i)–(iii) *of Theorem 3.1 along the full sequence of positive integers rather than along a subsequence, and the proof of that theorem, which extracts a single admissible index, gives no information about the remaining times.*

5. Where the phenomenon stops: a collapse and a separating pair

Section 4 produced a family, $(D^{r_1}, \dots, D^{r_N})$ on $H(\mathbb{C})$, that is disjoint Cesàro-hypercyclic (Theorem 4.1) and disjoint hypercyclic (Proposition 4.2) by the same computation, the averaging factor cancelling before any estimate was needed. Whether the two notions can be prised apart at all is the question this section settles.

Two obstructions come first: distinct powers of a single operator can never separate the notions (Theorem 5.1, Corollary 5.1), and neither can a pair of scalar multiples (Proposition 5.1). Outside those two classes, the notions do come apart, on a Hilbert space, with bounded weights and a single norm—non-normability plays no role in what follows. The separating pair (Theorem 5.2) shows that the collapse Wang and Shu record for *unilateral* shifts on $\ell^2(\mathbb{N})$ [14, Corollary 4.2] fails once the axis is bilateral; Proposition 5.5 then carries the same construction to a Köthe echelon space, where the point of the variant is the unbounded weights it admits, not the separation itself, already available under a norm.

5.1. Powers of a single operator never separate

The following is the sharpest thing we know about power families, and its proof is disarmingly short: the Cesàro factor is absorbed into the witness rather than estimated away.

Theorem 5.1 (Difference-family collapse)**.** *Let $T \in L(X)$, let $N \geq 2$, and let $1 \leq r_1 < r_2 < \cdots < r_N$ be integers. Suppose $(T^{r_1}, \dots, T^{r_N})$ is disjoint Cesàro-hypercyclic on X. Fix an index $1 \leq s < N$ and put $q_l := r_l - r_s$ for $s < l \leq N$, so that $1 \leq q_{s+1} < \cdots < q_N$. Then*

$$d\text{-}HC(T^{q_{s+1}}, \dots, T^{q_N})$$

is a dense G_δ subset of X. When $s = N - 1$, the family has a single member, and the assertion is that $T^{\,r_N - r_{N-1}}$ is hypercyclic.

Proof. Write $S_l := T^{q_l}$ for $s < l \leq N$; these are $N - s \geq 1$ distinct powers of T, each in $L(X)$.

Transitivity. Let $U, W_{s+1}, \dots, W_N \subseteq X$ be nonempty and open. Choose a disjoint Cesàro-hypercyclic vector x_0 for $(T^{r_1}, \dots, T^{r_N})$, so that

$$\mathcal{O} := \left\{ \left(\tfrac{1}{n} T^{r_1 n} x_0, \dots, \tfrac{1}{n} T^{r_N n} x_0\right) : n \geq 1 \right\}$$

is dense in X^N. Assemble the box

$$G := V_1 \times \cdots \times V_N, \qquad V_s := U, \quad V_l := W_l \ (s < l \leq N), \quad V_i := X \ (i < s),$$

nonempty and open in X^N. Density hands us one index $n \geq 1$ with the corresponding point of $\mathcal{O}$ in G; the coordinates $i < s$ impose nothing. Setting

$$w := \tfrac{1}{n}\, T^{r_s n} x_0 \ \in U,$$

we have $\frac{1}{n} T^{r_l n} x_0 \in W_l$ for every $l > s$. Each $T^{q_l n}$ is linear, so the scalar passes through it, and $q_l + r_s = r_l$ gives $T^{q_l n} T^{r_s n} = T^{r_l n}$. Hence

$$S_l^{\,n} w = T^{q_l n}\Big(\tfrac{1}{n}\, T^{r_s n} x_0\Big) = \tfrac{1}{n}\, T^{r_l n} x_0 \ \in W_l \qquad (s < l \leq N), \tag{5.1}$$

with a single n serving every l at once. The family $(S_{s+1}, \dots, S_N)$ is therefore disjoint topologically transitive.

Residuality. Fix a countable base $\{A_j : j \in \mathbb{N}\}$ of nonempty open subsets of X, available because X is separable and metrizable, and for $J = (j_{s+1}, \dots, j_N)$ put

$$C_J := \bigcup_{n\geq 0} \Big(S_{s+1}^{-n}(A_{j_{s+1}}) \cap \cdots \cap S_N^{-n}(A_{j_N}) \Big).$$

Continuity of the S_l makes each C_J open, and (5.1), applied to $U, A_{j_{s+1}}, \dots, A_{j_N}$, makes each C_J dense. A vector belongs to every C_J exactly when its disjoint orbit meets every basic box of X^{N-s}, that is, exactly when that orbit is dense; so $\bigcap_J C_J = d\text{-}HC(S_{s+1}, \dots, S_N)$. Separable Fréchet spaces are completely metrizable, hence Baire, and a countable intersection of dense open sets is dense G_δ. □

Remark 5.1. *The whole content of the argument is* (5.1)*. Density of the Cesàro orbit supplies a vector* $w = \frac{1}{n} T^{r_s n} x_0$ *inside* U *whose images under the difference powers are the remaining coordinates of the same orbit point, the factor* $\frac{1}{n}$ *cancelling because it is carried by both. No shift structure enters, no right inverse, and no surjectivity of* T*; the proof is purely about exponent arithmetic in a semigroup of operators.*

Corollary 5.1 (Non-hypercyclic operators cannot separate through their powers)**.** *Assume the hypotheses of Theorem 5.1.*

(a) $T^{r_l - r_s}$ *is hypercyclic on* X *for all* $1 \leq s < l \leq N$.
(b) T *is hypercyclic, and so is every power* T^r, $r \geq 1$.

Consequently a family of distinct powers of one operator can be disjoint Cesàro-hypercyclic only when that operator is already hypercyclic. An operator which is Cesàro-hypercyclic without being hypercyclic—a León-Saavedra bilateral weighted shift [13], say, as recalled in Remark 2.2(a)*—cannot generate a separating family from its own powers.*

Proof. (a) Apply Theorem 5.1 with base index s; the family $(T^{q_{s+1}}, \dots, T^{q_N})$ is disjoint hypercyclic, and a coordinate projection $\pi_l \colon X^{N-s} \to X$ is continuous, linear, and surjective, hence open, so it carries a dense disjoint orbit onto a dense orbit of $T^{q_l} = T^{r_l - r_s}$.

(b) Take $s = 1$, $l = 2$ in (a) and set $p := r_2 - r_1 \geq 1$, so T^p has a vector v with $\{T^{pn} v : n \geq 0\}$ dense. That set is contained in $\mathrm{Orb}(T, v)$, which is therefore dense as well: T is hypercyclic. No appeal to Ansari's theorem is needed for this step, the inclusion of orbits doing the work. The final clause is Ansari's theorem [19], in the form valid on separable Fréchet spaces (see [1, Chapter 6]): every power of a hypercyclic operator is hypercyclic. □

What Corollary 5.1 leaves undecided is the case of a hypercyclic T. The collapse tells us the difference family is disjoint hypercyclic and that T and all its powers are hypercyclic; none of that settles the original family.

Problem 5.1. *Let* $T \in L(X)$ *be hypercyclic and let* $1 \leq r_1 < \cdots < r_N$ *with* $N \geq 2$*. Can* $(T^{r_1}, T^{r_2}, \dots, T^{r_N})$ *be disjoint Cesàro-hypercyclic without being disjoint hypercyclic?*

Remark 5.2 (Why the two obvious routes fail)**.** *Let* x_0 *be a disjoint Cesàro-hypercyclic vector for* $(T^{r_1}, T^{r_2}, \dots, T^{r_N})$*. Two attempts at deducing disjoint hypercyclicity present themselves, and both break at the same place.*

(i) Rescaling the targets. *To hit prescribed open sets $W_1, \dots, W_N$, one needs a single n with $T^{r_l n}x_0 \in W_l$ for every l. Cesàro-density delivers n with $\frac{1}{n}T^{r_l n}x_0 \in V_l$ for open sets V_l fixed* in advance. *Forcing $T^{r_l n}x_0 \in W_l$ would require $V_l = \frac{1}{n}W_l$, a target depending on the very index it is meant to produce. Transitivity admits no such circularity. Theorem 5.1 escapes it because the coordinate $\frac{1}{n}T^{r_s n}x_0$ is itself the vector being moved, so the scalar cancels rather than needing to be matched.*

(ii) Lifting from the difference family. *Theorem 5.1 supplies a disjoint hypercyclic vector v for $(T^{q_2}, \dots, T^{q_N})$, and one would like to recover the original family by composing with $T^{r_1 n}$. Composition on the left by $T^{r_1 n}$ does not preserve disjoint hypercyclicity: the sequence $(T^{r_1 n}v)_{n\geq 1}$ need not be dense, and even where it is, nothing synchronises the times at which $T^{r_l n}v = T^{q_l n}(T^{r_1 n}v)$ enters each target. The obstruction is the absence of a common index, not any shift structure—T here is an arbitrary operator.*

The averaging factor is inert along the difference family and active along the original one. Problem 5.1 lies past what elementary orbit manipulations reach.

5.2. Scalar multiples

The other elementary candidate class disappears just as quickly. For $\alpha = 1$ the statement is the observation of Bès and Peris [10] that a family with two equal entries is never disjoint hypercyclic.

Proposition 5.1 (Scalar multiples are never disjoint)**.** *Let Y be a nontrivial Hausdorff locally convex space over $\mathbb{C}$, let $T : Y \to Y$ be linear, and let $\alpha \in \mathbb{C}$. Then $(T, \alpha T)$ is neither disjoint hypercyclic nor disjoint Cesàro-hypercyclic. No separability, completeness, or continuity of T is assumed. If Y is not separable, then Y^2 has no countable dense subset and the conclusion is immediate; the proof below covers both cases at once.*

Proof. Hausdorffness and nontriviality give a continuous seminorm p and a vector u_0 with $p(u_0) = 1$. Fix $x \in Y$ and set $c_n := p(\frac{1}{n}T^n x)$. Since $(\alpha T)^n = \alpha^n T^n$ and seminorms are absolutely homogeneous,

$$p\Big(\tfrac{1}{n}(\alpha T)^n x\Big) = |\alpha|^n c_n \qquad (n \geq 1).$$

For $\alpha = 0$ the second coordinate is 0 throughout, and the orbit avoids $Y \times \{v : p(v) > 1\}$. Let $\alpha \neq 0$ and put $S := \{|\alpha|^n : n \geq 1\}$. Pick $t_0 \in (0, \infty) \setminus \overline{S}$—possible, with $\overline{S}$ being countable and closed in $(0, \infty)$—and then $\varepsilon \in (0, 1)$ small enough that

$$\left(\tfrac{t_0-\varepsilon}{1+\varepsilon},\ \tfrac{t_0+\varepsilon}{1-\varepsilon}\right) \cap S = \varnothing.$$

The sets $V_1 := \{u : |p(u) - 1| < \varepsilon\}$ and $V_2 := \{v : |p(v) - t_0| < \varepsilon\}$ are open and nonempty, containing u_0 and $t_0 u_0$. Were some orbit point to lie in $V_1 \times V_2$, we would have $c_n > 1 - \varepsilon > 0$ and

$$|\alpha|^n = \frac{|\alpha|^n c_n}{c_n} \in \left(\tfrac{t_0-\varepsilon}{1+\varepsilon},\ \tfrac{t_0+\varepsilon}{1-\varepsilon}\right)$$

against the choice of ε. So, no x has dense disjoint Cesàro orbit. For the unscaled statement, the orbit is indexed by $n \in \mathbb{N}_0$ and $(\alpha T)^0 = I$, so two adjustments are needed. If $\alpha \neq 0$ replace S by $S_0 := \{|\alpha|^n : n \in \mathbb{N}_0\}$, which is still countable and closed in $(0, \infty)$, choose $t_0 \notin S_0$ and argue as above with $c_n := p(T^n x)$. If $\alpha = 0$ the orbit is $\{(x, x)\} \cup \{(T^n x, 0) : n \geq 1\}$, which misses the nonempty open set $Y \times \{v : p(v) > p(x) + 1\}$. □

5.3. The criterion-level shadow of the collapse

What Theorem 5.1 does for orbits, Lemma 4.6, specialised to shifts, does for the criterion, as Remark 4.4 has already noted for powers of a single shift. Here it is stated for shifts that need not share a weight sequence; this is the form needed in the proofs of Theorem 5.2 and Proposition 5.5 below.

Corollary 5.2 (Criterion collapse for weighted shifts)**.** *Let $N \geq 2$ and let Y be a separable Fréchet sequence space with unconditional Schauder basis (ε_n) and seminorms (p_m). For $1 \leq l \leq N$, let $B_{w_l} \in L(Y)$ be a weighted backward shift with nonzero weights and forward shift S_{w_l}, so that $B_{w_l} S_{w_l} = \mathrm{Id}$. Fix $1 \leq r_1 < \cdots < r_N$ and a strictly increasing (n_k), and put*

$$\widetilde{S}_{l,k} := S_{w_l}^{r_l n_k}, \qquad S_{l,k} := n_k \widetilde{S}_{l,k}.$$

If $(B_{w_1}^{r_1}, \ldots, B_{w_N}^{r_N})$ satisfies the unscaled criterion (i′)–(iii′) *of Proposition 4.1 with the maps $\widetilde{S}_{l,k}$, and if in addition $n_k\, p_m(\widetilde{S}_{l,k} y) \to 0$ for every m, l, and y, then it satisfies the disjoint Cesàro-hypercyclicity criterion* (Theorem 3.1) *with the maps $S_{l,k}$. The off-diagonal conditions are the same expression in both criteria:*

$$\frac{1}{n_k} B_{w_l}^{r_l n_k} S_{i,k}\, y = B_{w_l}^{r_l n_k} \widetilde{S}_{i,k}\, y \qquad (1 \leq l, i \leq N).$$

Proof. Lemma 4.6, with $T_l := B_{w_l}^{r_l}$ for each l; the displayed identity is the cancellation $n_k^{-1} \cdot n_k = 1$ occurring there. □

Remark 5.3. *Condition (iii) is the same expression in both criteria, so whatever it costs to verify—an exact cancellation on the diagonal, a genuine estimate off it—is paid for once. The other two move in opposite directions under scaling. The own-orbit condition (i) only relaxes: dividing by $n_k \geq 1$ can only shrink $p_m(B_{w_l}^{r_l n_k} x)$, so (i′) delivers (i) outright. The right-inverse condition (ii) tightens instead, asking for $n_k\, p_m(\widetilde{S}_{l,k} y) \to 0$ rather than the mere convergence (ii′) supplies, and no rate is implicit in a limit. That asymmetry is one-directional—(ii) always forces (ii′), never the converse—so a family verified disjoint Cesàro-hypercyclic by this criterion can fail the unscaled version, with the same maps, only at condition (i).*

5.4. Locating the separation: a pair on $\ell^2(\mathbb{Z})$

Corollary 5.1 and Proposition 5.1 rule out the two classes in which one would first look for a separating family. They give no hint as to where to look instead. An example found by trial would settle the existence question and leave the reason for it in the dark, so we work in the opposite direction. Both notions admit weight characterisations on the classical bilateral sequence spaces (the scaled one in the preprint [14]; see Remark 5.4); placing the two lists of inequalities side by side shows that they can diverge at one condition and at no other. The pair constructed below is then not a lucky guess but a direct solution of that single divergence, obtained by solving the two conditions the lists disagree on rather than by trial; we make no claim that it is the cheapest such solution among all separating pairs.

Throughout, $H := \ell^2(\mathbb{Z})$ carries its canonical orthonormal basis $(\varepsilon_n)_{n\in\mathbb{Z}}$, and for a bounded weight sequence $w = (w_n)_{n\in\mathbb{Z}}$ with nonzero terms, we set

$$B_w \varepsilon_n := w_n \varepsilon_{n-1}, \qquad S_w \varepsilon_n := w_{n+1}^{-1} \varepsilon_{n+1}, \qquad B_w S_w = \mathrm{Id}\,.$$

Products over windows of consecutive indices occur often enough to deserve a symbol:

$$\Lambda_w(c,\,s) := \prod_{\nu=c+1}^{c+s} w_\nu \qquad (c \in \mathbb{Z},\ s \geq 1), \tag{5.2}$$

so that

$$B_w^s \varepsilon_j = \Lambda_w(j-s,\,s)\,\varepsilon_{j-s}, \qquad S_w^{\,s} \varepsilon_j = \Lambda_w(j,\,s)^{-1} \varepsilon_{j+s}. \tag{5.3}$$

Everything in this subsection is a statement about the two windows $\Lambda_w(j,\,s)$ and $\Lambda_w(j-s,\,s)$ attached to a basis vector.

5.4.1. Two lists of inequalities

Fix $N \geq 2$, integers $1 \leq r_1 < \cdots < r_N$, and bounded bilateral weight sequences $a_1, \ldots, a_N$ with nonzero terms, acting on $c_0(\mathbb{Z})$ or $\ell^2(\mathbb{Z})$. Every condition below carries the same quantifier prefix, stated once and thereafter suppressed:

for every $\eta > 0$ and every $Q \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that, for all $|j| \leq Q$,

the displayed inequalities hold in the ranges indicated.

Density of d-CH$(B_{a_1}^{r_1}, \ldots, B_{a_N}^{r_N})$ is equivalent by [14, Theorem 4.4] (a result of the preprint [14]; see Remark 5.4), to

$$\frac{1}{n}\left|\Lambda_{a_l}(j,\,r_l n)\right| > \frac{1}{\eta} \qquad (1 \leq l \leq N), \tag{S1}$$

$$\frac{1}{n}\left|\Lambda_{a_l}(j - r_l n,\,r_l n)\right| < \eta \qquad (1 \leq l \leq N), \tag{S2}$$

$$\left|\Lambda_{a_l}(j,\,r_l n)\right| > \frac{1}{\eta}\left|\Lambda_{a_s}(j + (r_l - r_s)n,\,r_s n)\right| \qquad (1 \leq s < l \leq N), \tag{S3}$$

$$\left|\Lambda_{a_l}(j + (r_s - r_l)n,\,r_l n)\right| < \eta\left|\Lambda_{a_s}(j,\,r_s n)\right| \qquad (1 \leq s < l \leq N). \tag{S4}$$

Disjoint hypercyclicity of the same family is characterised [12] by the list obtained on deleting the two factors $1/n$:

$$\left|\Lambda_{a_l}(j,\,r_l n)\right| > \frac{1}{\eta} \qquad (1 \leq l \leq N), \tag{H1}$$

$$\left|\Lambda_{a_l}(j - r_l n,\,r_l n)\right| < \eta \qquad (1 \leq l \leq N), \tag{H2}$$

together with (S3) and (S4) unchanged, the averaging factor never having entered the cross conditions. For $N = 1$ and $r_1 = 1$, the unscaled pair (H1) and (H2) is Salas's criterion for hypercyclicity of a bilateral weighted shift [20].

Six inequalities, four of them common to both lists. What follows comes from the two that are not.

Lemma 5.1. *Fix $\eta > 0$, $Q \in \mathbb{N}$, $n \in \mathbb{N}$, and j with $|j| \leq Q$.*

(a) (S1) *implies* (H1)*, at the same n.*

(b) (S2) *does not imply* (H2)*, and no choice of* η *or* Q *repairs this. If the weights are such that* $\left|\Lambda_{a_l}(j - r_l n, r_l n)\right| = c_{l,j} > 0$ *for all large* n*, then* (S2) *holds for every* η *and* (H2) *fails for every* $\eta \le \min_{l,|j|\le Q} c_{l,j}$.

Proof. Since $n \ge 1$, division by n does not increase a nonnegative quantity. A lower bound therefore survives it in the wrong direction and an upper bound in the right one. For (a): $n^{-1}|\Lambda_{a_l}(j, r_l n)| > \eta^{-1}$ gives $|\Lambda_{a_l}(j, r_l n)| > n\eta^{-1} \ge \eta^{-1}$. For (b): a quantity constant in n has n^{-1} times itself tending to 0 while staying bounded away from 0. □

The asymmetry deserves to be named, since it governs the rest of the subsection. The averaging factor multiplies both own-window conditions. In (S1) it sits on a lower bound and makes the requirement harder, so nothing is lost when it is removed. In (S2) it sits on an upper bound and makes the requirement easier; the room it opens there is the only room the two notions have.

Unilateral families have no (S2) at all, which accounts for a collapse stated without proof in [14].

Proposition 5.2 (Conditional on [14, Theorem 4.1])**.** *Assume the characterisation of disjoint Cesàro-hypercyclic families of powers of unilateral weighted backward shifts given in [14, Theorem 4.1]. Then, on* $c_0(\mathbb{N})$ *or* $\ell^2(\mathbb{N})$*, a family* $(B_{a_1}^{r_1}, \dots, B_{a_N}^{r_N})$ *of powers of unilateral weighted backward shifts, which is densely disjoint Cesàro-hypercyclic, is disjoint hypercyclic. On* $\ell^2(\mathbb{N})$ *this is contained in [14, Corollary 4.2].*

Proof. On the half-axis, $B_{a_l}^{r_l n}\varepsilon_j = 0$ once $r_l n > j$, so backward windows never appear: the scaled characterisation [14, Theorem 4.1] consists of (S1) and (S3), and the unscaled one [10] of (H1) and (S3). Given η and Q, take n realising the first list. Condition (S3) belongs to both lists, and Lemma 5.1(a) converts (S1) into (H1) at that same n. The second list is met. This derives [14, Corollary 4.2] from [14, Theorem 4.1] in two lines; it does not reprove Theorem 4.1 itself, which is imported here as an unverified premise (see Remark 5.4 below). □

Remark 5.4 (Scope of the dependence on [14])**.** *At the time of writing, [14] is an arXiv preprint which, to our knowledge, has not completed peer review. Two results of this paper use it, and both are stated as conditional: Proposition 5.2 assumes [14, Theorem 4.1], and Proposition 5.3 assumes [14, Theorem 4.4]. Neither is used in the proof of any other result. Lemma 5.1 is proved independently. Theorem 5.1 makes no use of [14], and neither does Theorem 5.2, whose proof rests on Theorem 3.1, Proposition 5.4, and the results of Ansari [19] and Salas [20]. Proposition 5.3 serves only to motivate the construction below, and Remark 5.6 records the alternative derivation of Theorem 5.2 from [14, Theorem 4.4].*

Proposition 5.3 (Localisation; conditional on [14, Theorem 4.4])**.** *Assume the characterisation [14, Theorem 4.4]. Let* $(B_{a_1}^{r_1}, \dots, B_{a_N}^{r_N})$ *act on* $c_0(\mathbb{Z})$ *or* $\ell^2(\mathbb{Z})$ *with bounded weights, and suppose it is densely disjoint Cesàro-hypercyclic but not disjoint hypercyclic. Then there are* $\eta > 0$ *and* $Q \in \mathbb{N}$ *for which no* n *realising* (S1)–(S4) *realises (H2). The failure is carried by the backward own-window conditions and by nothing else in either list.*

Proof. Suppose the contrary: for every η and Q, some n realises the scaled list and (H2) as well. At such an n, Lemma 5.1(a) yields (H1), while (S3) and (S4) hold because they were part of the scaled list. The unscaled list is met at n, and the family is disjoint hypercyclic. □

We do not assert that a single index l carries the failure uniformly in j; the argument gives only that some backward window resists. In the example below, one l does suffice, and that is a feature of the example rather than a consequence of Proposition 5.3.

5.4.2. Solving the conditions for the weights

Granted [14, Theorem 4.4], Proposition 5.3 converts the search for a separating family into arithmetic. The derivation below uses it only as a guide; the pair it produces is verified in Theorem 5.2 independently of [14]. We need one index l whose backward products $\left|\Lambda_{a_l}(j-r_l n,\ r_l n)\right|$ are $o(n)$ without tending to 0; holding them constant in n arranges this directly, and we do not claim it is the only way to do so. The window $(j - r_l n,\ j]$ eventually contains every nonpositive index below j, so its product stabilises as soon as the nonpositive weights are unimodular, the surviving factor being $\prod_{i=1}^{j^+} a_{l,i}$. Take $l = 1$ and $r_1 = 1$. Unimodularity on $\{\nu \le 0\}$ fixes half the sequence, and (S1) forces the remaining weights above 1 in modulus; we take the round value 2. This produces

$$\alpha_n := \begin{cases} 2, & n \ge 1, \\ 1, & n \le 0, \end{cases}$$

obtained here by solving (S2) together with the failure of (H2) rather than by inspiration, though not as the unique solution: any fixed modulus greater than 1 above the origin satisfies (S1) equally well, and we make no claim that 2 is distinguished among them. The shift B_α is the one of León-Saavedra [13], reproduced as Example 1.1 of [14]; we make no claim of novelty for it, and none is needed.

The partner is then constrained in its turn. Put $r_2 = 2$ and let β be its weight sequence. Condition (S4) with $s = 1, l = 2$ demands $|\Lambda_\beta(j-n, 2n)| < \eta\,|\Lambda_\alpha(j, n)|$, whose right-hand side grows geometrically; the left-hand side must therefore remain bounded in n. A window of length $2n$ straddling the origin stays bounded only if its positive and nonpositive halves cancel, and weights 2 above the origin against $\frac{1}{2}$ below do precisely that. Condition (S3) asks in addition that $|\Lambda_\beta(j, 2n)|$ outgrow $|\Lambda_\alpha(j+n, n)|$, which the same choice supplies with room to spare. So

$$\beta_n := \begin{cases} 2, & n \ge 1, \\ \frac{1}{2}, & n \le 0, \end{cases} \qquad T_1 := B_\alpha, \qquad T_2 := B_\beta^2.$$

Both sequences take values in $[\frac{1}{2}, 2]$, so $B_\alpha, B_\beta \in L(H)$ with norm at most 2, and $T_2 \in L(H)$ as a composition. No seminorm is used anywhere in this subsection. Other weights would serve equally well; the point of the derivation is that these were solved for directly from (S2) together with the failure of (H2) and the two cross conditions, not guessed. We make no claim that no smaller or simpler choice of weights could realise the same separation.

5.4.3. The windows

A single computation supplies every product needed. Write $j^+ := \max(j, 0)$ and $j^- := \max(-j, 0)$, so that $j = j^+ - j^-$ and $|j| = j^+ + j^-$.

Lemma 5.2. *For all $c \in \mathbb{Z}$ and all $s \ge 1$,*

$$\Lambda_\alpha(c, s) = 2^{(c+s)^+ - c^+}, \qquad \Lambda_\beta(c, s) = 2^{(c+s)^+ - c^+} \cdot 2^{(c+s)^- - c^-}.$$

Proof. The window $(c, c+s]$ contains $(c+s)^+ - c^+$ indices that are ≥ 1 and $c^- - (c+s)^-$ indices that are ≤ 0; the two counts are nonnegative and sum to s, since $x^+ - x^- = x$. Both weight sequences equal 2 on the first block. On the second, α equals 1, contributing nothing, while β equals $\frac{1}{2}$, contributing $2^{-(c^- - (c+s)^-)}$. □

For $n > 2|j|$ the six windows we shall need are listed in Table 2; all six are read off Lemma 5.2 using $(j \pm n)^\pm$ and $(j \pm 2n)^\pm$ in the evident way. The entry that matters is the second of the left-hand column: it does not depend on n. For instance, with $c = j - 2n$ and $s = 2n$, one has $c^+ = 0$, $c^- = 2n - j$, and $c + s = j$ (recall $n > 2|j|$), so Lemma 5.2 gives

$$\Lambda_\beta(j - 2n, 2n) = 2^{j^+} \cdot 2^{j^- - (2n-j)} = 2^{2j^+ - 2n},$$

since $j + j^- = j^+$; the remaining entries are obtained in the same way.

Table 2. The windows Λ_α and Λ_β for $n > 2|j|$.

window	value	window	value
$\Lambda_\alpha(j, n)$	2^{n-j^-}	$\Lambda_\beta(j, 2n)$	2^{2n-2j^-}
$\Lambda_\alpha(j-n, n)$	2^{j^+}	$\Lambda_\beta(j-2n, 2n)$	2^{2j^+-2n}
$\Lambda_\alpha(j+n, n)$	2^n	$\Lambda_\beta(j-n, 2n)$	2^{2j}

5.4.4. The separation

Proposition 5.4. *B_α is not hypercyclic on H, whereas B_β^2 is. Consequently (T_1, T_2) is not disjoint hypercyclic, and the obstruction is confined to the first coordinate.*

Proof. Non-hypercyclicity of B_α. Put $q(x) := \sup_{n \leq 0} |x_n|$. Then $q \leq \|\cdot\|$, so q is a continuous seminorm and $\{q < c\}$ is open for each $c > 0$. Since $\alpha_\mu = 1$ for $\mu \leq 0$ and $(B_\alpha x)_n = \alpha_{n+1} x_{n+1}$,

$$q(B_\alpha x) \geq \sup_{n \leq -1} \left|\alpha_{n+1} x_{n+1}\right| = \sup_{\mu \leq 0} \left|\alpha_\mu x_\mu\right| = \sup_{\mu \leq 0} |x_\mu| = q(x),$$

so q never decreases along a B_α-orbit. Fix $x \neq 0$. If $x_n \neq 0$ for some $n \leq 0$, then $q(x) > 0$. Otherwise x is supported in $\{n \geq 1\}$, the set $\{n \geq 1 : x_n \neq 0\}$ has a least element j_0, and the 0-th coordinate of $B_\alpha^{j_0} x$ equals $\alpha_{j_0} \cdots \alpha_1 x_{j_0} = 2^{j_0} x_{j_0} \neq 0$, so $q(B_\alpha^{j_0} x) > 0$. Either way $c := q(B_\alpha^{n_0} x) > 0$ for some $n_0 \geq 0$, and monotonicity gives $q(B_\alpha^n x) \geq c$ for all $n \geq n_0$. Hence $\mathrm{Orb}(B_\alpha, x) \subseteq \{q \geq c\} \cup F$ with F finite, and the same holds for the closure, $\{q \geq c\}$ being closed. The set $V := \{q < c\}$ is open, nonempty, and H having no isolated points, infinite; so $V \setminus F$ is a nonempty open set missed by the closed orbit, and B_α is not hypercyclic.

Alternatively, apply (H1) and (H2) with $N = 1$, $r_1 = 1$, which is Salas's criterion [20]. Take $Q = 1$; the necessary condition must then hold in particular at $j = 0$. Since $\alpha_\nu = 1$ for every $\nu \leq 0$, $\Lambda_\alpha(-n, n) = \prod_{\nu=-n+1}^{0} \alpha_\nu = 1$ for every $n \geq 1$, so (H2) fails at $j = 0$, already at $\eta = 1$, for every n.

Hypercyclicity of T_2. Lemma 5.2 gives $\Lambda_\beta(j, n) = 2^{n-2j^-} \to \infty$ and $\Lambda_\beta(j-n, n) = 2^{2j^+-n} \to 0$, uniformly for $|j| \leq Q$, so both conditions of Salas's criterion hold and B_β is hypercyclic by Salas's theorem [20]. Consequently $T_2 = B_\beta^2$ is hypercyclic by Ansari's theorem [19].

For the disjoint statement, let x be a disjoint hypercyclic vector for (T_1, T_2). The projection $\pi_1 : H^2 \to H$ is continuous, linear, and surjective, hence open, so it carries the dense set $\{(T_1^n x, T_2^n x) : n \geq 0\}$ onto a dense orbit of T_1, which has none. □

Remark 5.5 (A proof of non-hypercyclicity that survives to $\lambda_2(A)$)**.** *Salas's criterion is a Hilbert-space statement and is unavailable in Proposition 5.5. The first argument in the proof of Proposition 5.4 is not: it uses only a continuous seminorm, the values of α on the nonpositive axis, and the absence of isolated points, and it is in this form that it is used in Proposition 5.5.*

Theorem 5.2 (A separating pair)**.** *The family $(T_1, T_2) = (B_\alpha, B_\beta^2)$ on $\ell^2(\mathbb{Z})$ is disjoint Cesàro-hypercyclic, with d-CH(T_1, T_2) a dense G_δ subset of H, and is not disjoint hypercyclic. The collapse stated in [14, Corollary 4.2] for unilateral families (see Proposition 5.2) therefore has no bilateral analogue.*

Proof. The negative half is Proposition 5.4. For the positive half, we verify Theorem 3.1 on H, a separable Fréchet space whose topology is given by the single norm $\|\cdot\|$, so that $p_m = \|\cdot\|$ for every m. Take $X_0 = X_1 = X_2 := c_{00} = \operatorname{span}\{\varepsilon_j : j \in \mathbb{Z}\}$, which is dense in H; here j must run over all of $\mathbb{Z}$, since the coordinate functionals are continuous and no half-axis span is dense. Put $n_k := k$, $S_{1,k} := kS_\alpha^k$, and $S_{2,k} := kS_\beta^{2k}$ on c_{00}. By Lemma 3.2 it suffices to check (i)–(iii) on each ε_j. Fix $j \in \mathbb{Z}$ and let $k > 2|j|$, so that Table 2 applies with $n = k$. By (5.3),

$$\left\|\tfrac{1}{k} B_\alpha^k \varepsilon_j\right\| = \frac{2^{j^+}}{k}, \qquad \left\|\tfrac{1}{k} B_\beta^{2k} \varepsilon_j\right\| = \frac{2^{2j^+ - 2k}}{k}, \tag{i}$$

$$\left\|kS_\alpha^k \varepsilon_j\right\| = k\, 2^{j^- - k}, \qquad \left\|kS_\beta^{2k} \varepsilon_j\right\| = k\, 2^{2j^- - 2k}, \tag{ii}$$

$$\left\|B_\alpha^k S_\beta^{2k} \varepsilon_j\right\| = \frac{\Lambda_\alpha(j+k,k)}{\Lambda_\beta(j,2k)} = 2^{2j^- - k}, \qquad \left\|B_\beta^{2k} S_\alpha^k \varepsilon_j\right\| = \frac{\Lambda_\beta(j-k,2k)}{\Lambda_\alpha(j,k)} = 2^{2j + j^- - k}, \tag{iii}$$

and all six quantities tend to 0 as $k \to \infty$. In (iii) with $l \neq i$, the factor $\frac{1}{k}$ cancels the factor k in $S_{i,k}$, as in Corollary 5.2. For $l = i$, the scalars cancel as well, $\frac{1}{k} T_l^k S_{l,k} = B_w^s S_w^s$ with $(w, s) \in \{(\alpha, k), (\beta, 2k)\}$, and $B_w^s S_w^s \varepsilon_j = \varepsilon_j$, because

$$B_w S_w \varepsilon_i = w_{i+1}^{-1} B_w \varepsilon_{i+1} = w_{i+1}^{-1} w_{i+1} \varepsilon_i = \varepsilon_i \qquad (i \in \mathbb{Z})$$

for any nonzero weights. Theorem 3.1 now shows that d-CH(T_1, T_2) is a dense G_δ subset of H. □

The two coordinates sit on opposite sides of (H2). For β the backward window decays like 2^{-2n}, and the unscaled condition is satisfied with room to spare; for α it is pinned at 2^{j^+}, and the unscaled condition fails at every $\eta < 1$. Dividing by n rescues the second and is superfluous for the first. The separation is that one line.

Remark 5.6 (A derivation through [14, Theorem 4.4])**.** *Assuming the characterisation [14, Theorem 4.4], the positive half of Theorem 5.2 can also be read off the weight conditions (S1)–(S4) directly, as follows. We check (S1)–(S4) with $N = 2$, $r_1 = 1$, $r_2 = 2$, $a_1 = \alpha$, and $a_2 = \beta$, reading values from the Table 2. Let $\eta > 0$ and $Q \in \mathbb{N}$ be given, and let $n > 2Q$. For $|j| \le Q$, one has $j^+ \le Q$, $j^- \le Q$, and $2j \le 2Q$, so all six estimates below are uniform in j:*

$$\frac{\Lambda_\alpha(j,n)}{n} = \frac{2^{n-j^-}}{n} \ge \frac{2^{n-Q}}{n}, \qquad \frac{\Lambda_\beta(j,2n)}{n} = \frac{2^{2n-2j^-}}{n} \ge \frac{2^{2n-2Q}}{n},$$

both tending to ∞, which gives (S1) for $l = 1, 2$;

$$\frac{\Lambda_\alpha(j-n,n)}{n} = \frac{2^{j^+}}{n} \le \frac{2^Q}{n}, \qquad \frac{\Lambda_\beta(j-2n,2n)}{n} = \frac{2^{2j^+-2n}}{n} \le \frac{2^{2Q}\,2^{-2n}}{n},$$

both tending to 0*, which gives (S2); and for the cross conditions*

$$\frac{\Lambda_\beta(j,2n)}{\Lambda_\alpha(j+n,n)} = 2^{\,n-2j^-} \geq 2^{\,n-2Q} \longrightarrow \infty, \qquad \frac{\Lambda_\beta(j-n,2n)}{\Lambda_\alpha(j,n)} = 2^{\,2j+j^--n} \leq 2^{3Q}\,2^{-n} \longrightarrow 0,$$

which are (S3) and (S4). One value of n therefore serves every $|j| \leq Q$ *and both indices at once, and by [14, Theorem 4.4] the set of disjoint Cesàro-hypercyclic vectors is dense; it is a* G_δ *by Lemma 3.1. This route is not needed for Theorem 5.2, but it shows the pair realising the localisation of Proposition 5.3: (S2) holds for* α *while (H2) fails at* $j = 0$*.*

Remark 5.7 (What is doing the separating)**.** *Three comments, so that the example is not credited with more than it earns.*

The asymmetry of α *and* β *on the nonpositive axis is the device by which the separation is realised, not the reason it is available. The reason is the position of the averaging factor. Unilaterally every occurrence of* $1/n$ *falls on a lower bound, where deleting it costs nothing, and Proposition 5.2 follows in two lines from [14, Theorem 4.1]. Bilaterally one occurrence falls on an upper bound, where deleting it costs everything, and Proposition 5.3 shows, granted [14, Theorem 4.4], that a separating family has nowhere else to fail. Theorem 5.2 realises that failure by the direct route of holding a backward window constant in n; the weights are what holding it constant requires. We do not claim this route is the least expensive one available; we claim only that it is the one the conditions of Section 5.4 single out most directly.*

The pair lies outside both obstruction classes, as it must. T_1 *and* T_2 *are not powers of a common operator, since powers of one operator commute and these do not: on* ε_2*,*

$$B_\alpha B_\beta^2 \varepsilon_2 = \beta_2\beta_1\alpha_0\,\varepsilon_{-1} = 4\,\varepsilon_{-1}, \qquad B_\beta^2 B_\alpha \varepsilon_2 = \alpha_2\beta_1\beta_0\,\varepsilon_{-1} = 2\,\varepsilon_{-1}.$$

Nor is T_2 *a scalar multiple of* T_1*: they displace indices by different amounts.*

The obstruction is carried by T_1 *alone, the second coordinate being hypercyclic by Proposition 5.4 (Salas's theorem for* B_β*, then Ansari's theorem for* B_β^2*). Whether a separation exists whose obstruction is irreducibly disjoint—every coordinate hypercyclic, the family not—we do not know; that is Problem 5.1 in a wider form. We also separate the notions in one direction only. Whether disjoint hypercyclicity implies disjoint Cesàro-hypercyclicity, over a norm or over a seminorm system, is untouched here.*

5.5. The same pattern with unbounded weights

The separation of Section 5.4 was obtained under a norm, and Remark 5.7 located its source in the placement of the averaging factor rather than in any feature of the space. Nothing about that account is peculiar to $\ell^2(\mathbb{Z})$, and the construction transfers. What the transfer supplies is a class of weights no Banach sequence space admits in its canonical basis; what it does not supply is the separation, which Theorem 5.2 already delivered.

Put $\rho_m := 2^m$ and define a Köthe matrix on $\mathbb{Z}$ by

$$a_m(n) := \begin{cases} \rho_m^n, & n \geq 0, \\ 1, & n < 0, \end{cases} \qquad Y := \lambda_2(A) = \Big\{ x \in \mathbb{C}^{\mathbb{Z}} :\ p_m(x)^2 := \sum_{n\in\mathbb{Z}} |x_n|^2 a_m(n)^2 < \infty \ \ \forall m \Big\}.$$

Then Y is a separable Fréchet space under the increasing seminorms (p_m), the coordinate vectors $(\varepsilon_n)_{n\in\mathbb{Z}}$ form an unconditional Schauder basis with $p_m(\varepsilon_n) = a_m(n)$, and c_{00} is dense. No single p_m gives the topology, since $\sup_n a_{m+1}(n)/a_m(n) = \sup_{n\geq 0} 2^n = \infty$ for every m; the space is not normable, though p_1 is a norm, the entries $a_1(n)$ being strictly positive. Replace the weights by

$$\alpha_n := \begin{cases} n, & n \geq 1, \\ 1, & n \leq 0, \end{cases} \qquad \beta_n := \begin{cases} n, & n \geq 1, \\ \frac{1}{2}, & n \leq 0. \end{cases}$$

Only the positive axis has changed. On $\{\nu \leq 0\}$ the two sequences are exactly what they were, and by the derivation of Section 5.4, that is the half which carries the separation: unimodularity of α below the origin is what freezes the backward window, and the factor $\frac{1}{2}$ in β is what keeps the straddling window bounded. Above the origin, where the earlier construction asked only for enough growth, the Köthe matrix now permits growth that is unbounded.

Lemma 5.3. *For $w \in \{\alpha, \beta\}$ and every m, one has $p_m(B_w x) \leq p_{m+1}(x)$; in particular $B_\alpha, B_\beta \in L(Y)$. Composing, $p_m(B_\beta^2 x) \leq p_{m+1}(B_\beta x) \leq p_{m+2}(x)$, so B_β^2 is continuous as well.*

Proof. From $(B_w x)_{n-1} = w_n x_n$, we get $p_m(B_w x)^2 = \sum_n |w_n|^2 |x_n|^2 a_m(n-1)^2$, so it suffices that $|w_n|\, a_m(n-1) \leq a_{m+1}(n)$ for every $n \in \mathbb{Z}$.

Let $n \leq 0$. Then $n - 1 \leq -1$, so $a_m(n-1) = 1$; also $|w_n| \leq 1$, while $a_{m+1}(n) \geq 1$, the value being 1 for $n < 0$ and $\rho_{m+1}^0 = 1$ for $n = 0$. The inequality reads $|w_n| \leq a_{m+1}(n)$ and holds.

Let $n \geq 1$. Then $|w_n| = n$, $a_m(n-1) = \rho_m^{n-1}$, and $a_{m+1}(n) = (2\rho_m)^n$. Since $n \leq 2^n$ and $\rho_m \geq 1$,

$$n\,\rho_m^{n-1} \leq 2^n \rho_m^{n-1} \leq 2^n \rho_m^n = (2\rho_m)^n.$$

Summing the squared inequalities gives the claim; the composition estimate is immediate. □

An unbounded weight sequence defines no bounded operator on $\ell^p(\mathbb{Z})$ or $c_0(\mathbb{Z})$ in the canonical basis, so a shift of this kind has no Banach-space counterpart there. Lemma 5.3 shows the increasing matrix absorbing the growth, one row at a time. This is the same device that carried D on $H(\mathbb{C})$ in Lemma 4.5, and it is the only structural difference between the two settings.

Proposition 5.5. *On $Y = \lambda_2(A)$, the pair (B_α, B_β^2), with the weights just defined, is disjoint Cesàro-hypercyclic but not disjoint hypercyclic.*

Proof. The route is that of Proposition 5.4 and of the proof of Theorem 5.2: non-hypercyclicity through a continuous seminorm, and disjoint Cesàro-hypercyclicity through Theorem 3.1. The alternative of Remark 5.6 is not available here, the characterisation [14, Theorem 4.4] being a Banach-space result. Two points alter the bookkeeping.

Failure of hypercyclicity for B_α. The argument in the proof of Proposition 5.4 applies with one substitution. Since $a_m(n) = 1$ for $n \leq 0$, the functional $q(x) := \sup_{n\leq 0} |x_n|$ satisfies $q \leq p_1$ and is a continuous seminorm; $\alpha_\mu = 1$ for $\mu \leq 0$ gives $q(B_\alpha x) \geq q(x)$; and for x supported in $\{n \geq 1\}$ with least occupied index j_0, the 0-th coordinate of $B_\alpha^{j_0} x$ is now $j_0!\, x_{j_0} \neq 0$ in place of $2^{j_0} x_{j_0}$. The closure argument is untouched, Y being a nontrivial metrizable topological vector space and hence free of isolated points. The projection $\pi_1 \colon Y^2 \to Y$ is continuous, linear, and surjective, hence open, so (B_α, B_β^2) is not disjoint hypercyclic.

Disjoint Cesàro-hypercyclicity. We apply Theorem 3.1 with $X_0 = X_1 = X_2 = c_{00}$, $n_k = k$, $S_{1,k} = kS_\alpha^k$, and $S_{2,k} = kS_\beta^{2k}$; by Lemma 3.2 it suffices to treat ε_j, $j \in \mathbb{Z}$ fixed, and we take $k > |j|$, so that $j - k < 0 < j + k$. For $c \in \mathbb{Z}$, $s \geq 1$, the definition of the weights gives

$$\Lambda_\alpha(c, s) = \frac{((c+s)^+)!}{(c^+)!}, \qquad \Lambda_\beta(c, s) = \frac{((c+s)^+)!}{(c^+)!}\, 2^{(c+s)^- - c^-},$$

the positive indices in $(c, c+s]$ contributing $\prod_{\nu=c^++1}^{(c+s)^+} \nu$ and the $c^- - (c+s)^-$ nonpositive ones contributing 1, resp. 2^{-1} each. In particular the backward window $\Lambda_\alpha(j-s, s) = (j^+)!$ is independent of s for $s > j^+$, as in Section 5.4. Recall $p_m(\lambda\varepsilon_n) = |\lambda| a_m(n)$, with $a_m(n) = 1$ for $n < 0$ and $a_m(n) = \rho_m^n$ for $n \geq 0$. Then, for every m,

$$p_m(\tfrac{1}{k}B_\alpha^k\varepsilon_j) = \frac{(j^+)!}{k}, \qquad p_m(\tfrac{1}{k}B_\beta^{2k}\varepsilon_j) = \frac{(j^+)!\, 2^{j^+-2k}}{k},$$

$$p_m(kS_\alpha^k\varepsilon_j) = k\,\frac{(j^+)!\,\rho_m^{j+k}}{(j+k)!}, \qquad p_m(kS_\beta^{2k}\varepsilon_j) = k\,2^{j^-}\frac{(j^+)!\,\rho_m^{j+2k}}{(j+2k)!},$$

$$p_m(B_\alpha^k S_\beta^{2k}\varepsilon_j) = 2^{j^-}\frac{(j^+)!\,\rho_m^{j+k}}{(j+k)!}, \qquad p_m(B_\beta^{2k}S_\alpha^k\varepsilon_j) = (j^+)!\,2^{j-k}.$$

The first two and the last tend to 0 trivially. For the remaining three, $t! \geq (t/e)^t$ gives $\rho^t/t! \leq (e\rho/t)^t \leq 2^{-t}$ once $t \geq 2e\rho$; with $t = j+k$ or $t = j+2k$, each is bounded, for large k, by a constant depending only on j and m times $k\,2^{-k}$, hence tending to 0. (Lemma 4.4 is not quoted here, since it bounds $j!\,m^{j+\nu}/(j+\nu)!$ with the same j in the factorial and in the exponent, which fails for $j < 0$.) The diagonal terms of (iii) are identities: $\frac{1}{k}B_w^s(kS_w^s)\varepsilon_j = B_w^s S_w^s \varepsilon_j = \varepsilon_j$, since $B_w S_w \varepsilon_i = w_{i+1}^{-1} B_w \varepsilon_{i+1} = \varepsilon_i$ for every i and all weights are nonzero. Theorem 3.1 applies, and d-CH(B_α, B_β^2) is a dense G_δ subset of Y. □

Remark 5.8 (What non-normability buys, and what it does not)**.** *Proposition 5.5 and Theorem 5.2 exhibit one and the same separation, and it is worth being exact about which of them supplies what.*

The separation itself is available under a norm. What produces it is a backward window held constant in the iteration time—tolerated by (S2)*, refused by* (H2)*—and by Remark 5.7 the reason such a window can be tolerated at all is the position of the averaging factor, which costs nothing in a Banach space. The asymmetry of α and β below the origin is how that constancy is arranged, in both versions alike; the unbounded weights above the origin play no part in it.*

Two things are supplied by the locally convex setting, neither of them the separation. The weights $\alpha_n = n$ are unavailable on $\ell^2(\mathbb{Z})$ in its canonical basis, and Lemma 5.3 exhibits the increasing matrix accommodating them, as Lemma 4.5 did for D on $H(\mathbb{C})$.

And on Y the characterisation [14, Theorem 4.4], a Banach-space result, is unavailable even conditionally, so Theorem 3.1 is the only route to Proposition 5.5. On $\ell^2(\mathbb{Z})$ Theorem 3.1 is also the route we take for Theorem 5.2, while Remark 5.6 offers a second one, conditional on [14].

We state the conclusion plainly. The separation of disjoint Cesàro-hypercyclicity from disjoint hypercyclicity is not a non-normable phenomenon. It is a bilateral one.

5.6. The gap that remains

An obstruction, a localisation, and a minimal witness still fall short of a theory. What is missing is a characterisation, in terms of the weights and the Köthe matrix, of the disjoint Cesàro-hypercyclic

families of weighted shifts on $\lambda_p(A)$. Two results bound the target from either side. On the unscaled side stands the characterisation of Bès, Martin, and Sanders [12], covering tuples of distinct weighted backward shifts on the classical sequence spaces, unilateral and bilateral alike, with power families of one shift recovered as a specialisation. On the scaled side stand Theorems 4.1 and 4.4 of [14], the first for the half-axis and the second for the full one; Section 5.4 was written around the difference between them. Both are calibrated against a single norm. What we want is a common generalisation indexed by the rows of a Köthe matrix.

Let $A = (a_m)_m$ be a Köthe matrix on $\mathbb{Z}$ or on $\mathbb{N}_0$, let $1 \le p < \infty$ (or take c_0), and let $X = \lambda_p(A)$ carry its unconditional basis (ε_n), so that $p_m(\varepsilon_n) = a_m(n)$. For $1 \le l \le N$, let B_{w_l} be a continuous weighted backward shift with weights $w_l = (w_{l,\nu})$, and fix $1 \le r_1 < \cdots < r_N$.

Conjecture 5.1 (Köthe-space characterisation)**.** *The following are equivalent.*

(a) *d-$CH(B_{w_1}^{r_1}, \dots, B_{w_N}^{r_N})$ is dense in $\lambda_p(A)$.*

(b) *For every $m \in \mathbb{N}$, every $\varepsilon > 0$, and every $Q \in \mathbb{N}$, there is $n \in \mathbb{N}$ such that, for all $|j| \le Q$,*

$$\frac{1}{n}\Big|\prod_{\nu=j+1}^{j+r_l n} w_{l,\nu}\Big|\,\frac{a_m(j)}{a_m(j+r_l n)} \;>\; \frac{1}{\varepsilon} \qquad (1 \le l \le N), \tag{K1}$$

$$\frac{1}{n}\Big|\prod_{\nu=j-r_l n+1}^{j} w_{l,\nu}\Big|\,\frac{a_m(j-r_l n)}{a_m(j)} \;<\; \varepsilon \qquad (1 \le l \le N), \tag{K2}$$

$$\frac{\big|\prod_{\nu=j+1}^{j+r_l n} w_{l,\nu}\big|}{\big|\prod_{\nu=j+(r_l-r_s)n+1}^{j+r_l n} w_{s,\nu}\big|}\,\frac{a_m(j)}{a_m(j+(r_l-r_s)n)} \;>\; \frac{1}{\varepsilon} \qquad (1 \le s < l \le N), \tag{K3}$$

$$\frac{\big|\prod_{\nu=j-(r_l-r_s)n+1}^{j+r_s n} w_{l,\nu}\big|}{\big|\prod_{\nu=j+1}^{j+r_s n} w_{s,\nu}\big|}\,\frac{a_m(j-(r_l-r_s)n)}{a_m(j)} \;<\; \varepsilon \qquad (1 \le s < l \le N). \tag{K4}$$

(c) *The family satisfies Theorem 3.1 with the canonical inverses $S_{l,k} = n_k S_{w_l}^{r_l n_k}$.*

Each of (K1)–(K4) is one condition of Theorem 3.1 evaluated on a basis vector, the matrix quotient recording the seminorm of the vector produced against that of the vector fed in: (K1) is the right-inverse decay, (K2) the own-orbit condition, and the last two the off-diagonal interactions in their two directions. Setting $a_m \equiv 1$ turns them into (S1)–(S4) of Section 5.4, in that order.

The results of this paper sit inside Conjecture 5.1 as follows. Theorem 5.2 and Proposition 5.5 exhibit families satisfying (K1)–(K4) for which the unscaled analogue of (K2) fails. When $w_1 = \cdots = w_N = w$, the family consists of powers of the single shift B_w, and Theorem 5.1 applies: any family in this subclass satisfying (a) has hypercyclic B_w and disjoint hypercyclic difference families, so a separating example with a common weight sequence would have to answer Problem 5.1.

Three comments delimit what is and is not known.

The implication (c)⇒(a) is Theorem 3.1 and holds unconditionally; it is also the only implication used anywhere in this paper. Everything else is conjectural. The substance lies in (a)⇒(b)⇒(c)—the necessity of the weight conditions, and the extraction of a verifying sequence (n_k) from a single vector. Under a norm, this is an estimate on one disjoint Cesàro-hypercyclic vector; over a Köthe matrix, the estimate must hold in every seminorm at once, and an index furnished by density at level m carries

no information at level $m+1$. We state (a) as a density assertion rather than as the existence of one vector because both model theorems are density statements and because the gap between the two is itself open: that is Problem 3.1.

The conjecture specialises correctly, and the specialisation splits according to the index set. Over $\mathbb{N}_0$ with $a_m \equiv 1$, the backward windows are empty—$B_{w_l}^{r_l n}\varepsilon_j$ vanishes once $r_l n > j$—so (K2) and (K4) drop out, and what remains is Theorem 4.1 of [14]. Over $\mathbb{Z}$ with $a_m \equiv 1$, all four survive and give Theorem 4.4 of the same paper. For the matrix and the weights of Proposition 5.5, the four conditions are the ones verified there. And for $a_m(j) = m^j$ with $w_\nu = \nu$—the coordinates of $H(\mathbb{C})$ under Lemma 4.5—(K1) reads, in reciprocal form, as the comparison performed in Lemma 4.4, which is the whole of Section 4.

Settling the conjecture would do more than tidy the two examples away. Only (K1) and (K2) carry the averaging factor, and by Lemma 5.1 only (K2) is weakened by it; the argument of Proposition 5.3 applies verbatim over a Köthe matrix, so a separating family on $\lambda_p(A)$ must again fail at a backward window and nowhere else. What the conjecture would supply is the classification that localisation stops short of: precisely which weight sequences and which matrices permit such a window to resist. Set beside a Köthe-space version of [12], it would describe every separating family of this type, leaving Theorem 5.2 and Proposition 5.5 as two instances of a theorem rather than as two constructions. The necessity direction is, in our judgement, the substantial problem this paper leaves open.

Remark 5.9 (Other normalisations)**.** *Let $(\lambda_n)_{n\geq 1}$ be any sequence of nonzero complex scalars, and call $(T_1, \dots, T_N)$* disjoint λ-transitive *if for all nonempty open $U, V_1, \dots, V_N \subseteq X$, there are $x \in U$ and $n \in \mathbb{N}$ with $\lambda_n T_l^n x \in V_l$ for every l; the case $\lambda_n = 1/n$ is Definition 2.4.*

- *(a) Theorem 5.1 holds with $1/n$ replaced by λ_n, under the weaker hypothesis that $(T^{r_1}, \dots, T^{r_N})$ be disjoint λ-transitive. Indeed, apply the hypothesis to $X, V_1, \dots, V_N$ with $V_s = U$, $V_l = W_l$ $(l > s)$,and $V_i = X$ $(i < s)$, and put $w := \lambda_n T^{r_s n} x \in U$; then $T^{q_l n} w = \lambda_n T^{r_l n} x \in W_l$, the single scalar λ_n being carried by every coordinate. The residuality step is unchanged.*
- *(b) Proposition 5.1 holds verbatim, since $p(\lambda_n(\alpha T)^n x) = |\alpha|^n\, p(\lambda_n T^n x)$.*
- *(c) Theorem 3.1 holds with $1/n_k$ replaced by λ_{n_k}, with the same proof.*
- *(d) For $\lambda_n = n^{-p}$ with $p > 0$, the pair (B_α, B_β^2) of Theorem 5.2 satisfies (c) with $S_{1,k} = k^p S_\alpha^k$, $S_{2,k} = k^p S_\beta^{2k}$: the six quantities in the proof of Theorem 5.2 become $2^{j^+}k^{-p}$, $2^{2j^+-2k}k^{-p}$, $k^p 2^{j^- - k}$, $k^p 2^{2j^- - 2k}$, $2^{2j^- - k}$,and $2^{2j+j^- - k}$, all tending to 0. Since Proposition 5.4 does not involve the normalisation, the separation of Theorem 5.2 persists for every normalisation n^{-p}, $p > 0$.*

Ivković [24] introduced topological Cesàro hyper-transitivity for a single operator, which requires infinitely many admissible times; by Remark 3.1 it coincides on X with Definition 2.4 for $N = 1$, and the argument given there applies verbatim to any sequence of nonzero scalars (λ_n). In [24] variants of this notion indexed by a positive rational parameter p are also considered; whenever such a variant amounts to replacing the factor $1/n$ by n^{-p}, it falls under (a)–(d) above. Normalisations that alter the iterates themselves, rather than multiplying them by scalars, are not covered by these arguments.

6. Conclusions and open problems

6.1. What the paper establishes

The central result is unconditional: if distinct powers $T^{r_1}, T^{r_2}, \dots, T^{r_N}$ of a single operator on a separable Fréchet space form a disjoint Cesàro-hypercyclic family, then every difference family $(T^{r_{s+1}-r_s}, \dots, T^{r_N-r_s})$ is disjoint hypercyclic, with a dense G_δ set of disjoint hypercyclic vectors (Theorem 5.1). Consequently T itself is hypercyclic, hence so is every power of T by Ansari's theorem [19] (Corollary 5.1): powers of a single operator can never separate the two disjoint notions. The same impossibility holds for pairs of scalar multiples $(T, \alpha T)$, $\alpha \in \mathbb{C}$ (Proposition 5.1), and, granted [14, Theorem 4.1], for unilateral weighted-shift families (Proposition 5.2), which derives the collapse stated in [14, Corollary 4.2] from that theorem.

Among weighted shifts, granted the characterisations of [14], separation is possible only bilaterally, and only through the one condition where the two weight characterisations diverge (Propositions 5.2 and 5.3). Theorem 5.2 realises it explicitly, independently of [14]: a pair on $\ell^2(\mathbb{Z})$, built from León-Saavedra's shift B_α [13] and a partner B^2_β, with bounded weights, is disjoint Cesàro-hypercyclic but not disjoint hypercyclic—a bilateral phenomenon, not a consequence of non-normability (Remark 5.7). Its proof uses the criterion of Section 3 (Theorem 3.1) together with the results of Salas [20] and Ansari [19]. Proposition 5.5 reproduces the same separation with unbounded weights on a Köthe echelon space, where no Banach-space characterisation is available even conditionally.

Together these results bound the phenomenon without classifying it in full—the general case is Conjecture 5.1 below. The transitivity characterisation and criterion of Section 3 (Proposition 3.1, Theorem 3.1) transpose known disjoint-universality machinery to the scaled setting (Remark 3.4) and underlie every application; the unscaled results on $H(\mathbb{C})$ and ω (Section 4) belong to Bès and Peris [10] and are not new, their Cesàro readings following via the transfer lemma (Lemma 4.6).

6.2. Questions left open

We list them roughly in order of what a solution would be worth.

(Q1) Does León-Saavedra's equivalence hold on Fréchet spaces? For a separable Banach space, León-Saavedra [13] proved that the Cesàro means $\{M_n(T)x : n \geq 1\}$ are dense for some x precisely when the scaled orbit $\{n^{-1}T^n x : n \geq 1\}$ is dense for some x; his argument is norm-theoretic. We know of no proof and no counterexample on a general separable Fréchet space, and for that reason Definition 2.1 takes the scaled orbit as primitive throughout this paper (Problem 2.1). One half is algebraic and survives in any vector space, since

$$\frac{1}{n}\,T^n = \frac{n+1}{n}\,M_{n+1}(T) - M_n(T), \qquad n \geq 1;$$

the converse is where the norm is used. An affirmative answer would identify the notion studied here with the classical one in the generality in which we work, and would retroactively license the word "Cesàro" in every statement of this paper. We regard this as the most consequential question on the list, and also the most self-contained.

(Q2) A zero-one law for disjoint Cesàro-hypercyclic vectors. Proposition 3.1 characterises *residuality* of $d\text{-}CH(T_1, \dots, T_N)$ rather than its non-emptiness, and the standard bridge between

the two is unavailable: for ordinary hypercyclicity, one composes iterates, $T^{n_2-n_1}T^{n_1} = T^{n_2}$, whereas

$$\left(\tfrac{1}{n_2-n_1}T^{n_2-n_1}\right)\left(\tfrac{1}{n_1}T^{n_1}\right) = \frac{1}{n_1(n_2-n_1)}\,T^{n_2} \;\neq\; \frac{1}{n_2}\,T^{n_2} \qquad (n_1,\, n_2-n_1 \geq 2).$$

The single-operator case is nonetheless settled on Banach spaces by an argument that avoids composition altogether: the proof of Theorem 3.3 of [15] shows that d-density of one Cesàro orbit makes $CH(T)$ a dense G_δ, and a dense orbit is d-dense at every scale. Problem 3.1 asks for the disjoint analogue, and records the two points at which their argument does not carry over—the metric character of d-density, and the seed-shifting Lemma 3.5, which has no counterpart when N operators act on one vector.

A related question sits one level lower and remains open even for a single operator on a Hilbert space: if $\mathrm{Orb}_C(T, x)$ is somewhere dense, is it dense? This is the Cesàro form of the Bourdon–Feldman theorem. Costakis and Hadjiloucas prove it under the extra hypothesis that T is hypercyclic [15, Remark 2.7], and unconditionally in the weakened form that $(0, \varepsilon)\,\mathrm{Orb}_C(T, x)$ is then dense for every $\varepsilon \in (0, 1)$ [15, Theorem 2.9]; the full statement is Problem 1 of their concluding section. We note the consequence for the present paper: no argument here may pass from Cesàro hypercyclicity of T^p to that of T, since an Ansari-type theorem for Cesàro orbits is exactly what a positive answer would supply and is not otherwise available. The uses of Ansari's theorem in Corollary 5.1 concern ordinary hypercyclicity only.

(Q3) A characterisation on Köthe echelon spaces. Conjecture 5.1 proposes conditions on the weight sequences and the Köthe matrix equivalent to disjoint Cesàro-hypercyclicity for $(B^{r_1}_{w_1}, \ldots, B^{r_N}_{w_N})$ on $\lambda_p(A)$. The implication from the criterion is Theorem 3.1 and holds unconditionally; the substance is the necessity of the conditions, and there the obstacle is concrete. One must extract, from a single disjoint Cesàro-hypercyclic vector, an index n controlling the forward ratios $a_m(j)/a_m(j + r_l n)$ and the backward ratios $a_m(j)/a_m(j - r_l n)$ simultaneously for finitely many rows m, and the decay of that vector's coordinates is not uniform in m; the index furnished by density at level m need not serve level $m + 1$. The backward family is the one that matters, by Proposition 5.3: it is where a separating example must live, and any characterisation that omits it describes the wrong class. A proof, set beside a Köthe-space version of the characterisation of Bès, Martin, and Sanders [12]—who treat tuples of distinct weighted backward shifts, unilateral and bilateral, and obtain power families as a specialisation—would describe every separating family of this type and would decide (Q4) within the class. Theorem 5.2 and Proposition 5.5 would become two instances of a theorem rather than two examples.

(Q4) Powers of a single hypercyclic operator. Corollary 5.1 forces T to be hypercyclic, and Theorem 5.1 makes the difference family disjoint hypercyclic; neither settles the original family. For T hypercyclic, can $(T^{r_1}, \ldots, T^{r_N})$ be disjoint Cesàro-hypercyclic without being disjoint hypercyclic (Problem 5.1)? Remark 5.2 explains why the two obvious approaches fail, and the explanation is the same in both cases: the factor n^{-1} cancels only when the vector being moved is itself a coordinate of the Cesàro orbit, which happens for the difference family and not for the undifferenced one. Any resolution must come from somewhere other than orbit manipulation. A wider version of the same question is worth posing: can a separation be arranged whose obstruction is irreducibly disjoint, with every individual coordinate hypercyclic? Our example does not answer it. By Proposition 5.4 exactly one of its coordinates fails to be hypercyclic, and

the other is hypercyclic on its own; by Corollary 5.2 the disjointness interactions never feel the averaging at all. Whether the phenomenon can be made to live in those interactions we do not know.

(Q5) Disjoint Cesàro-mixing. Definition 2.5 fixes the notion, and Proposition 4.5 settles one case: distinct powers of the backward shift are disjoint Cesàro-mixing on ω. That proof exploits exact annihilation and therefore says nothing about a space carrying a continuous norm. Two things are missing. A criterion: conditions (i)–(iii) of Theorem 3.1 imposed along the full sequence of positive integers rather than along a subsequence (n_k) ought to yield mixing, but the proof of that theorem extracts a single admissible index and gives no control over the remaining times, so the argument has to be redone. And a test case: whether $(D^{r_1}, \dots, D^{r_N})$ is disjoint Cesàro-mixing on $H(\mathbb{C})$. The estimates of Section 4 decay super-exponentially in n_k, which suggests an affirmative answer, but the own-orbit condition there relies on polynomials being annihilated at a rate tied to their degree, and a mixing statement would need that rate uniform.

(Q6) Cesàro averaging and frequent hypercyclicity. Costakis and Parissis [26] record an unpublished result of Costakis and Ruzsa: if the sequence $(n^{-1}T^n)_n$ is frequently universal, then T is hypercyclic. They pursued the consequences for topological multiple recurrence; frequent hypercyclicity itself is due to Bayart and Grivaux [27]. Beyond such results the relationship between Cesàro averaging and frequency remains partly charted even on Banach spaces. Whether disjoint Cesàro-hypercyclicity on a Fréchet space implies, or is implied by, a suitable notion of disjoint frequent hypercyclicity is, so far as we are aware, untouched. A first step would be to determine whether the family of Theorem 5.2, which separates the two disjoint notions, also separates their frequent analogues.

(Q7) Other normalisations. Remark 5.9 shows that the collapse theorem, the exclusion of scalar multiples, and the separating pair persist when $1/n$ is replaced by any sequence of nonzero scalars, in particular by n^{-p} $(p > 0)$. Whether analogous statements hold for normalisations that act on the iterates themselves, as in some of the variants considered by Ivković [24], is open.

Of the seven, (Q1) is the one whose answer bears on every statement in this paper. (Q3) would have the widest structural consequences and is the natural sequel; Proposition 5.3 tells one where in the conditions the work will have to be done. (Q2) is the cleanest, and we suspect it is also the hardest, since the failure of the semigroup property removes the only tool the single-operator theory offers.

Use of Generative-AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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