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Research article

Conditional Park–Pham bounds under positive correlation

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Abstract: We record a conditional form of the ϵ-dependent Park–Pham theorem. If a monotone property $\mathcal{F} \subseteq 2^X$ is positively correlated with a conditioning event $B \subseteq 2^X$ under the product measure μ_p, then the usual Park–Pham lower bound for $\mu_p(\mathcal{F})$ transfers to the conditional probability $\mathbb{P}(X_p \in \mathcal{F} \mid X_p \in B)$. This gives, in particular, conditional Park–Pham bounds for increasing conditioning events by Harris's inequality, and for nonmonotone conditioning events that are independent of the target property. We also formulate the transfer principle for finite posets embedded in Boolean lattices and illustrate it with pattern-containment upper sets in permutation classes. Finally, we prove that, for every finite poset P and every nonempty proper upper set $U \subseteq \mathcal{I}(P)$, the lattice of order ideals $\mathcal{I}(P)$ is positively correlated under μ_p with the Boolean upper set $\langle U \rangle \subseteq 2^P$ generated by U. Via Birkhoff's representation theorem, this yields a Park–Pham bound on every finite distributive lattice for the natural one-parameter probability measure induced by conditioning μ_p on its canonical order-ideal representation.

Keywords: Park–Pham; Kahn-Kalai; thresholds; positive correlation; finite posets; order ideals; distributive lattices
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1. Introduction and preliminaries

Threshold phenomena for monotone properties form a central theme in probabilistic combinatorics [1]. Let X be a finite set, and write 2^X for the Boolean lattice ordered by inclusion. A family $\mathcal{F} \subseteq 2^X$ is called an upper set, or increasing event, if $S \in \mathcal{F}$ and $S \subseteq T \subseteq X$ implies $T \in \mathcal{F}$. We say that $\mathcal{F}$ is nontrivial if $\mathcal{F} \neq \emptyset, 2^X$.

For $p \in [0,1]$, let μ_p be the product measure on 2^X defined by $\mu_p(S) = p^{|S|}(1-p)^{|X|-|S|}$ for $S \subseteq X$. For a family $\mathcal{A} \subseteq 2^X$, write $\mu_p(\mathcal{A}) = \sum_{S \in \mathcal{A}} \mu_p(S)$. Equivalently, if X_p is a random subset of X with distribution μ_p, then $\mu_p(\mathcal{A}) = \mathbb{P}(X_p \in \mathcal{A})$. For clarity, here X denotes the fixed ground set, whereas X_p denotes the random subset of X sampled according to μ_p. For a nontrivial upper set $\mathcal{F}$, the critical

probability $p_c(\mathcal{F})$ is the unique value of p for which $\mu_p(\mathcal{F}) = \frac{1}{2}$.

The Park–Pham theorem [2, 3], formerly the Kahn-Kalai conjecture [4], bounds $p_c(\mathcal{F})$ in terms of the expectation threshold $q(\mathcal{F})$. If $\mathcal{G} \subseteq 2^X$, write $\langle\mathcal{G}\rangle = \{T \subseteq X : S \subseteq T \text{ for some } S \in \mathcal{G}\}$. A family $\mathcal{G}$ is a cover of $\mathcal{F}$ if $\mathcal{F} \subseteq \langle\mathcal{G}\rangle$. The upper set $\mathcal{F}$ is called p-small if there exists a cover $\mathcal{G}$ of $\mathcal{F}$ such that $\sum_{S\in\mathcal{G}} p^{|S|} \leq \frac{1}{2}$. The expectation threshold $q(\mathcal{F})$ is the supremum of the values p for which $\mathcal{F}$ is p-small. Let $\mathcal{F}_0$ denote the set of minimal elements of $\mathcal{F}$, and write $\ell(\mathcal{F}) = \max\{|S| : S \in \mathcal{F}_0\}$.

We will use the following ϵ-dependent form of the Park–Pham theorem, due to Bell [5]. We write logarithms in base 2.

Theorem 1.1 (Bell's ϵ-dependent Park–Pham theorem [5])**.** *Let X be a finite set, let $\mathcal{F} \subseteq 2^X$ be a nontrivial upper set, and let $\epsilon \in (0, 1)$. If $p > 48q(\mathcal{F}) \log_2\left(\frac{\ell(\mathcal{F})}{\epsilon}\right)$, then $\mathbb{P}(X_p \in \mathcal{F}) > 1 - \epsilon$.*

This formulation follows from Bell's original statement [5] by applying it with any $q' > q(\mathcal{F})$ sufficiently close to $q(\mathcal{F})$.

The expectation-threshold program began with Kahn and Kalai [4]; a fractional version was proved by Frankston, Kahn, Narayanan, and Park [6] before Park and Pham proved the integral conjecture [2]. Subsequent work includes Bell's optimal dependence on the error probability [5], the short proof of Tran and Vu [3], the nonuniform product-measure theorem of Park and Vondrák [7], and the cloning refinement of Przybyłowski and Riordan [8]. There is also a sharp-threshold theory for dependent monotonic measures based on influences [9], but it addresses a different question from the comparison considered here. The requirement that the target be an upper set is structural rather than cosmetic: Logarithmic expectation-threshold bounds can fail polynomially for down-sets [10].

The purpose of this note is to record a conditional consequence of Theorem 1.1. Suppose that one does not sample from the full Boolean lattice 2^X, but instead samples from μ_p conditioned on some event $B \subseteq 2^X$. We ask when the Park–Pham lower bound for an upper set $\mathcal{F}$ transfers to the conditional probability $\mathbb{P}(X_p \in \mathcal{F} \mid X_p \in B)$. The transfer mechanism is elementary, but useful: The precise hypothesis needed for conditioning to preserve the Park–Pham lower bound is positive correlation with the target upper set. This observation gives a convenient way to move Park–Pham estimates from the full Boolean lattice to conditional models, including monotone conditioning events, independent-coordinate restrictions, and finite-poset models embedded in Boolean lattices.

In general, this is not the same as applying the Park–Pham theorem directly on the conditioned space. Conditioning usually produces a dependent law supported on a family that need not be isomorphic to the full powerset of a finite set, whereas Theorem 1.1 is a product-measure theorem on 2^X. Merely defining a conditional analogue of the expectation threshold would not supply a theorem relating that parameter to the conditional probability. Our approach therefore retains the ambient parameters $q(\mathcal{F})$ and $\ell(\mathcal{F})$, for which Bell's theorem is available, and compares the conditional probability with the ambient one. When the conditioned space has a tractable product representation, a direct analysis may instead be possible and may give a sharper estimate.

This framing is deliberately conditional. Not every conditioning event B is favorable: If B is negatively correlated with $\mathcal{F}$, or if B is disjoint from $\mathcal{F}$, then no such transfer can be expected. However, the positive-correlation condition is checkable in several natural cases. If B is itself an upper set, then Harris's inequality gives the required positive correlation. If B depends on coordinates disjoint from those determining $\mathcal{F}$, then independence gives the same conclusion, even when B is not monotone.

We also record a poset-valued formulation. If P is a nonempty finite poset, $U \subseteq P$ is an upper set, and $f : P \to 2^X$ is an order embedding, then conditioning X_p on the image $f(P)$ induces a P-valued random variable. Under the corresponding positive-correlation hypothesis, the Park–Pham bound for the Boolean upper set generated by $f(U)$ yields a conditional threshold estimate for U.

Finally, we give a structural result that greatly extends the situations in which a Park–Pham-type bound is available. Specifically, we show that, for every finite poset P, the generally nonmonotone event of its associated lattice of order ideals, $\mathcal{I}(P)$, is positively correlated in 2^P with the canonical Boolean lift $\langle U \rangle$ of any upper set $U \subseteq \mathcal{I}(P)$. Using Birkhoff's representation theorem, this produces a form of the Park–Pham bound on every finite distributive lattice for the natural one-parameter probability measure induced by conditioning μ_p on the lattice's order-ideal representation.

Thus, the note is both organizational and technical. We first isolate the exact positive-correlation condition under which the ϵ-dependent Park–Pham estimate survives conditioning. This gives a reusable criterion for conditional threshold models, especially in settings where the natural probability space is not the full Boolean lattice but a conditioned subspace or an embedded finite poset. Using this, we then carry the Park–Pham bounds to the far larger setting of finite distributive lattices. Finally, we conclude with a number of examples.

2. Conditional transfer under positive correlation

We begin by isolating the elementary transfer mechanism.

Definition 2.1. *Let $\mathcal{A}, \mathcal{B} \subseteq 2^X$. We say that $\mathcal{A}$ and $\mathcal{B}$ are positively correlated at p if*

$$\mu_p(\mathcal{A} \cap \mathcal{B}) \geq \mu_p(\mathcal{A})\mu_p(\mathcal{B}).$$

Theorem 2.2 (Conditional Park–Pham transfer)**.** *Let X be a finite set, let $p \in [0,1]$, let $\mathcal{F} \subseteq 2^X$ be a nontrivial upper set, and let $B \subseteq 2^X$ satisfy $\mu_p(B) > 0$. Suppose that $\mathcal{F}$ and B are positively correlated at p. If $\epsilon \in (0,1)$ and $p > 48q(\mathcal{F}) \log_2 \left(\frac{\ell(\mathcal{F})}{\epsilon}\right)$, then $\mathbb{P}(X_p \in \mathcal{F} \mid X_p \in B) > 1 - \epsilon$.*

Proof. By positive correlation,

$$\mathbb{P}(X_p \in \mathcal{F} \mid X_p \in B) = \frac{\mu_p(\mathcal{F} \cap B)}{\mu_p(B)} \geq \mu_p(\mathcal{F}).$$

Theorem 1.1 gives $\mu_p(\mathcal{F}) > 1 - \epsilon$ under the displayed hypothesis on p. □

Remark 2.3. *The positive-correlation hypothesis in Theorem 2.2 is pointwise in p. Thus, in applications, the conclusion holds only for values of p for which both the Park–Pham lower bound and the correlation inequality*

$$\mu_p(\mathcal{F} \cap B) \geq \mu_p(\mathcal{F})\mu_p(B)$$

hold simultaneously. The sufficient conditions given below—namely, Harris's inequality for increasing conditioning events and independence for disjoint coordinate restrictions—have the advantage of holding uniformly for all $p \in (0,1)$.

Remark 2.4. *When $\mu_p(B) > 0$, positive correlation is equivalent to the comparison*

$$\mathbb{P}(X_p \in \mathcal{F} \mid X_p \in B) \geq \mu_p(\mathcal{F}).$$

It is therefore the exact hypothesis needed for the transfer argument in Theorem 2.2. It is not necessary for the bare conclusion $\mathbb{P}(X_p \in \mathcal{F} \mid X_p \in B) > 1-\epsilon$, which may hold for other reasons, and the theorem does not assert that every conditioned model has its own Park–Pham theorem.

Remark 2.5. *The comparison step is not specific to product measure: For any probability measure ν, positive correlation of $\mathcal{F}$ and B gives $\nu(\mathcal{F} \mid B) \geq \nu(\mathcal{F})$. What is specific here is the ambient estimate supplied by Theorem 1.1. Correlation alone does not create an analogous expectation-threshold theorem for an arbitrary dependent measure. Consequently, any Park–Pham-type ambient probability estimate under a measure ν transfers unchanged to $\nu(\mathcal{F} \mid B)$ whenever $\mathcal{F}$ and B are positively correlated under ν. In particular, a nonuniform product-measure result can be transferred in this way once the corresponding positive-correlation inequality has been verified.*

Example 2.6 (Failure without positive correlation)**.** *Fix p such that $0 < \mu_p(\mathcal{F}) < 1$ and take $B = 2^X \setminus \mathcal{F}$. Then $\mu_p(B) > 0$, but*

$$\mathbb{P}(X_p \in \mathcal{F} \mid X_p \in B) = 0.$$

Thus no unconditional lower bound for $\mu_p(\mathcal{F})$ can transfer to arbitrary conditioning events without a favorable comparison hypothesis.

3. Sources of positive correlation

The simplest source of positive correlation is monotonicity. When both the target event and the conditioning event are upper sets, Harris's inequality shows that the conditioning can only increase the probability of the target event. The more flexible point of Theorem 2.2 is that the same transfer also applies to nonmonotone conditioning events whenever positive correlation can be verified by some other means.

Theorem 3.1 (Harris's inequality [11])**.** *If $\mathcal{A}, \mathcal{B} \subseteq 2^X$ are upper sets, then $\mu_p(\mathcal{A} \cap \mathcal{B}) \geq \mu_p(\mathcal{A})\mu_p(\mathcal{B})$ for every $p \in [0,1]$.*

We will also use the functional form of Harris's inequality: If $g, h : 2^X \to \mathbb{R}$ are nondecreasing, then

$$\mathbb{E}[g(X_p)h(X_p)] \geq \mathbb{E}[g(X_p)]\mathbb{E}[h(X_p)].$$

Equivalently, $\mathrm{Cov}(g(X_p), h(X_p)) \geq 0$. On the finite Boolean lattice, this follows from the event form given in Theorem 3.1. Indeed, after subtracting their respective minimum values, g and h can each be written as a nonnegative linear combination of indicators of the superlevel sets of g and h (i.e., the sets $g^{-1}([t,\infty))$ and $h^{-1}([t,\infty))$, which are upper sets since g and h are nondecreasing), so the result follows by applying Harris's inequality term-by-term to this decomposition.

Remark 3.2. *When both $\mathcal{F}$ and B are upper sets, Harris's inequality gives*

$$\mathbb{P}(X_p \in \mathcal{F} \mid X_p \in B) \geq \mu_p(\mathcal{F}),$$

so Bell's theorem (i.e., Theorem 1.1) immediately yields the same conditional bound. The point of Theorem 2.2 is that the same argument also applies when positive correlation is obtained by other means, including nonmonotone conditioning events.

Harris's inequality says that product measures are positively associated [12]. More generally, the Fortuin–Kasteleyn–Ginibre (FKG) inequality gives association for log-supermodular measures on finite distributive lattices [13]. Such inequalities can verify the correlation input in Theorem 2.2, but they do not by themselves supply an unconditional Park–Pham estimate for a new dependent measure.

The positive-correlation hypothesis is not limited to monotone conditioning events. The following simple case shows that the same conclusion can hold for nonmonotone restrictions.

Proposition 3.3 (Independent-coordinate conditioning)**.** *Let $X = Y \sqcup Z$ be a disjoint union. Let $\mathcal{G} \subseteq 2^Y$ be a nontrivial upper set, and define $\mathcal{F} = \{S \subseteq X : S \cap Y \in \mathcal{G}\}$. Let $\mathcal{H} \subseteq 2^Z$ be nonempty, and define $B = \{S \subseteq X : S \cap Z \in \mathcal{H}\}$. Then, $\mathcal{F}$ and B are independent under μ_p for every $p \in (0,1)$. In particular, if $\epsilon \in (0,1)$ and $p > 48q(\mathcal{F}) \log_2\left(\frac{\ell(\mathcal{F})}{\epsilon}\right)$, then $\mathbb{P}(X_p \in \mathcal{F} \mid X_p \in B) > 1 - \epsilon$.*

Proof. The event $\mathcal{F}$ depends only on the coordinates in Y, while B depends only on the coordinates in Z. Since μ_p is a product measure and $Y \cap Z = \emptyset$, the two events are independent and $\mu_p(\mathcal{F} \cap B) = \mu_p(\mathcal{F})\mu_p(B)$. The desired conditional bound follows from Theorem 2.2. □

Remark 3.4. *Because $\mathcal{H}$ is arbitrary, the preceding proposition permits the conditioning event B to be highly nonmonotone. For instance, one may take*

$$\mathcal{H} = \{T \subseteq Z : |T| \text{ is even}\},$$

so that B is the event that an even number of Z-coordinates are selected. This event is not an upper set (when Z is nonempty), but it is still independent of every event determined only by the Y-coordinates.

4. Poset-valued formulations and Boolean lifts

We now translate the preceding conditional bound into a finite-poset setting. First, we record a general, flexible formulation where we embed a poset into a Boolean lattice such that the image of the poset is positively correlated with the upper set $\mathcal{F}$.

Theorem 4.1 (Boolean lifts of poset upper sets)**.** *Let $p \in (0,1)$, let P be a nonempty finite poset, let $f : P \to 2^X$ be an injection, and let $U \subseteq P$. Suppose that $\mathcal{F} \subseteq 2^X$ is a nontrivial upper set satisfying $\mathcal{F} \cap f(P) = f(U)$. Let Y_p be the P-valued random variable obtained by conditioning X_p on $f(P)$ and pulling back along f. If $\epsilon \in (0,1)$, $\mathcal{F}$ and $f(P)$ are positively correlated at p, and $p > 48q(\mathcal{F}) \log_2\left(\frac{\ell(\mathcal{F})}{\epsilon}\right)$, then $\mathbb{P}(Y_p \in U) > 1 - \epsilon$.*

Proof. Since $\mathcal{F} \cap f(P) = f(U)$, we have

$$\mathbb{P}(Y_p \in U) = \frac{\mu_p(f(U))}{\mu_p(f(P))} = \frac{\mu_p(\mathcal{F} \cap f(P))}{\mu_p(f(P))}.$$

By positive correlation, this is at least $\mu_p(\mathcal{F})$. Bell's ϵ-dependent Park–Pham theorem gives the result. □

A particularly useful way to produce Boolean lifts is through order embeddings. Let P be a nonempty finite poset and recall that an injective map $f : P \to 2^X$ is an order embedding if, for all $x, y \in P$, $x \leq_P y \iff f(x) \subseteq f(y)$. If $f : P \to 2^X$ is an order embedding and $U \subseteq P$ is an upper set, then the Boolean upper set $\langle f(U) \rangle$ cuts out $f(U)$ inside $f(P)$.

Lemma 4.2. *Let P be a nonempty finite poset, let $U \subseteq P$ be an upper set, and let $f : P \to 2^X$ be an order embedding. Put $\mathcal{F} = \langle f(U) \rangle$. Then, $\mathcal{F} \cap f(P) = f(U)$.*

Proof. The inclusion $f(U) \subseteq \mathcal{F} \cap f(P)$ is immediate. Conversely, suppose $T \in \mathcal{F} \cap f(P)$. Then $T = f(y)$ for some $y \in P$, and since $T \in \langle f(U) \rangle$, there exists $x \in U$ such that $f(x) \subseteq f(y)$. Since f is an order embedding, $x \leq_P y$. Because U is an upper set, $y \in U$. Hence, $T = f(y) \in f(U)$. □

Lemma 4.3 (Nontriviality of Boolean lifts)**.** *Under the hypotheses of Lemma 4.2, if U is nonempty and proper, then $\langle f(U) \rangle$ is nontrivial.*

Proof. Since U is nonempty, its Boolean lift is nonempty. If $\langle f(U) \rangle = 2^X$, then $\emptyset \in \langle f(U) \rangle$, so $f(u) = \emptyset$ for some $u \in U$. For every $x \in P$ we then have $f(u) \subseteq f(x)$, and the order-embedding property makes u a least element of P. Since U is upper and contains u, this would force $U = P$, contrary to properness. □

If $f : P \to 2^X$ is an injection and $p \in (0, 1)$, then conditioning μ_p on the event $X_p \in f(P)$ induces a probability measure ν_p on P by

$$\nu_p(A) := \frac{\mu_p(f(A))}{\mu_p(f(P))} \qquad (A \subseteq P).$$

Equivalently, if Y_p has distribution ν_p, then $\mathbb{P}(Y_p \in A) = \mathbb{P}(X_p \in f(A) \mid X_p \in f(P))$.

Corollary 4.4 (Embedded finite posets under positive correlation)**.** *Let $p \in (0, 1)$, let P be a nonempty finite poset, let $U \subseteq P$ be a nonempty upper set, and let $f : P \to 2^X$ be an order embedding. Put $\mathcal{F} = \langle f(U) \rangle$, set $B = f(P)$, and assume that $\mathcal{F}$ is nontrivial. Suppose that $\mathcal{F}$ and B are positively correlated at p. If $\epsilon \in (0, 1)$ and $p > 48q(\mathcal{F}) \log_2 \left(\frac{\ell(\mathcal{F})}{\epsilon}\right)$, then $\mathbb{P}(Y_p \in U) > 1 - \epsilon$.*

Proof. By Lemma 4.2, $f(U) = \mathcal{F} \cap f(P) = \mathcal{F} \cap B$. Therefore,

$$\mathbb{P}(Y_p \in U) = \frac{\mu_p(f(U))}{\mu_p(f(P))} = \frac{\mu_p(\mathcal{F} \cap B)}{\mu_p(B)}.$$

Since $\mathcal{F}$ and B are positively correlated at p, this is at least $\mu_p(\mathcal{F})$. Theorem 1.1 now gives $\mu_p(\mathcal{F}) > 1-\epsilon$ under the displayed hypothesis. □

The following corollary gives a structural sufficient condition for the positive-correlation hypothesis in Corollary 4.4.

Corollary 4.5 (Upper-image embedded posets)**.** *Let P be a nonempty finite poset, let $U \subseteq P$ be a nonempty upper set, and let $f : P \to 2^X$ be an order embedding such that $f(P)$ is an upper set in 2^X. Put $\mathcal{F} = \langle f(U) \rangle$, and assume that $\mathcal{F}$ is nontrivial. If $p \in (0, 1)$, $\epsilon \in (0, 1)$, and $p > 48q(\mathcal{F}) \log_2 \left(\frac{\ell(\mathcal{F})}{\epsilon}\right)$, then $\mathbb{P}(Y_p \in U) > 1 - \epsilon$.*

Proof. The family $\mathcal{F} = \langle f(U) \rangle$ is an upper set by construction, and $f(P)$ is an upper set by assumption. Harris's inequality implies that $\mathcal{F}$ and $f(P)$ are positively correlated for every $p \in [0, 1]$. The result follows from Corollary 4.4. □

Remark 4.6. *The assumption that $f(P)$ is an upper set is restrictive, but it is a clean, checkable condition. More generally, Corollary 4.4 applies whenever the image $f(P)$ is positively correlated with the Boolean upper set generated by $f(U)$. In applications where $f(P)$ is not monotone, this correlation condition may still hold by independence, symmetry, or direct computation.*

5. Order-ideal models and distributive lattices

We can, in fact, take all of this further and obtain a version of the Park–Pham theorem for a natural one-parameter family of probability measures on every finite distributive lattice. To get there, we need a structural positive correlation result which showcases the full utility of Theorem 2.2.

Recall that an order ideal in a finite poset P is a lower set in P (possibly, an empty one) and that the lattice of order ideals of a partially ordered set P, denoted $\mathcal{I}(P)$, is the sublattice of 2^P consisting of order ideals of P. For $A \subseteq P$, write A_0 for the set of minimal elements of the induced subposet on A and for $x \in P$, write $\downarrow x$ to denote the principal lower set generated by x; i.e., $\downarrow x := \{y \in P : y \leq x\}$. If L is a finite lattice with least element $\widehat{0}$, an element $j \in L$ is join-irreducible if $j \neq \widehat{0}$ and $j = a \vee b$ implies that $j = a$ or $j = b$. We write $J(L)$ for the subposet of join-irreducible elements of L. Recall also that Birkhoff's representation theorem [14] states that every finite distributive lattice L is isomorphic to

$$\mathcal{I}(J(L)) \subseteq 2^{J(L)}$$

under the Birkhoff map $f(x) = \{j \in J(L) : j \leq x\}$.

Theorem 5.1. *Let P be a finite poset, let $\mathcal{I}(P) \subseteq 2^P$ be its lattice of order ideals, and let $U \subseteq \mathcal{I}(P)$ be a nonempty proper upper set. Then, for every $p \in [0,1]$, the events $\mathcal{I}(P)$ and $\langle U \rangle$ are positively correlated under μ_p; i.e.,*

$$\mu_p(\mathcal{I}(P) \cap \langle U \rangle) \geq \mu_p(\mathcal{I}(P))\mu_p(\langle U \rangle).$$

Equivalently, if X_p is the product-measure random subset of P, then

$$\mathbb{P}(X_p \in \langle U \rangle \mid X_p \in \mathcal{I}(P)) \geq \mathbb{P}(X_p \in \langle U \rangle).$$

Proof. For $S \in 2^P$, define $\mathrm{core}(S)$ to be

$$\mathrm{core}(S) := \{x \in P : \downarrow x \subseteq S\}.$$

That is, $\mathrm{core}(S)$ is the largest order ideal contained in S (i.e., it is an order ideal contained in S, and every order ideal $I \subseteq S$ is contained in $\mathrm{core}(S)$). We first claim $S \in \langle U \rangle$ if and only if $\mathrm{core}(S) \in U$. To see this, one direction is easy: If $\mathrm{core}(S) \in U$, then $\mathrm{core}(S) \in \langle U \rangle$. Since $\langle U \rangle$ is an upper set, the containment $\mathrm{core}(S) \subseteq S$ implies that $S \in \langle U \rangle$ as desired. In the other direction, if $S \in \langle U \rangle$, then there exists some $I \in U$ such that $I \subseteq S$. Since I is an order ideal, this implies $I \subseteq \mathrm{core}(S)$, and hence $\mathrm{core}(S) \in U$ because U is upper.

Likewise, for $I \in \mathcal{I}(P)$, we claim that $\mathrm{core}(S) = I$ if and only if both $I \subseteq S$ and $S \cap (P \setminus I)_0 = \emptyset$. Indeed, suppose first that $\mathrm{core}(S) = I$. Then, $I \subseteq S$. If some $m \in (P \setminus I)_0$ also belonged to S, then every $y < m$ would belong to $I \subseteq S$, by the minimality of m in $P \setminus I$. Consequently, $\downarrow m \subseteq S$, which would imply that $m \in \mathrm{core}(S) = I$, a contradiction. Therefore, $S \cap (P \setminus I)_0 = \emptyset$. Conversely, suppose that $I \subseteq S$ and $S \cap (P \setminus I)_0 = \emptyset$. Since I is an order ideal, $I \subseteq S$ implies that $I \subseteq \mathrm{core}(S)$. On the other hand, if $x \in P \setminus I$, then, because P is finite, there exists some $m \in (P \setminus I)_0$ such that $m \leq x$. Since $m \notin S$, we have $\downarrow x \nsubseteq S$, and hence $x \notin \mathrm{core}(S)$. Therefore, $\mathrm{core}(S) \subseteq I$ as desired.

Consequently, for $0 < p < 1$, we have

$$\mathbb{P}(X_p = I \mid X_p \in \mathcal{I}(P)) = \frac{p^{|I|}(1-p)^{|P|-|I|}}{\mu_p(\mathcal{I}(P))}, \qquad \mathbb{P}(\mathrm{core}(X_p) = I) = p^{|I|}(1-p)^{|(P\setminus I)_0|}. \tag{5.1}$$

The second distribution in (5.1) is the homogeneous special case of the conjunctive Bayesian-network distribution studied in [15]. Here we compare this distribution with product measure conditioned on the event $X_p \in \mathcal{I}(P)$.

Defining $r_p(I)$ by

$$r_p(I) := \frac{\mathbb{P}(X_p = I \mid X_p \in \mathcal{I}(P))}{\mathbb{P}(\mathrm{core}(X_p) = I)},$$

we get from (5.1) that

$$r_p(I) = \frac{(1-p)^{|(P\setminus I)\setminus(P\setminus I)_0|}}{\mathbb{P}(X_p \in \mathcal{I}(P))}. \tag{5.2}$$

We show that r_p is nondecreasing on $\mathcal{I}(P)$. To do so, we claim that if $I \subseteq J$, then

$$(P \setminus J) \setminus (P \setminus J)_0 \subseteq (P \setminus I) \setminus (P \setminus I)_0.$$

To see this, note that $P \setminus I \supseteq P \setminus J$ whenever $I \subseteq J$, so if $x \in (P \setminus J) \setminus (P \setminus J)_0$, then there exists some $y \in P \setminus J$ such that $y < x$. However, $P \setminus J \subseteq P \setminus I$, so $y, x \in P \setminus I$ and $x \notin (P \setminus I)_0$, yielding $x \in (P \setminus I) \setminus (P \setminus I)_0$ whenever $x \in (P \setminus J) \setminus (P \setminus J)_0$.

Hence, the exponent in (5.2) for J is no larger than the exponent for I. Since $0 < 1 - p < 1$, it follows that $r_p(I) \leq r_p(J)$. That is, r_p is nondecreasing on $\mathcal{I}(P)$. Since the map core : $2^P \to \mathcal{I}(P)$ is order-preserving, and since the indicator $\mathbf{1}_U$ is nondecreasing on $\mathcal{I}(P)$ as well, this yields that $\mathbf{1}_U \circ \mathrm{core}$ and $r_p \circ \mathrm{core}$ are both nondecreasing real-valued functions on 2^P. We will apply Harris's inequality to these functions. First, by the definition of r_p, we have

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_U(\mathrm{core}(X_p))r_p(\mathrm{core}(X_p))\right] &= \sum_{I\in U} \mathbb{P}(\mathrm{core}(X_p) = I) r_p(I) \\
&= \sum_{I\in U} \mathbb{P}(X_p = I \mid X_p \in \mathcal{I}(P)) \\
&= \mathbb{P}(X_p \in U \mid X_p \in \mathcal{I}(P)),
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\left[r_p(\mathrm{core}(X_p))\right] &= \sum_{I\in\mathcal{I}(P)} \mathbb{P}(\mathrm{core}(X_p) = I) r_p(I) \\
&= \sum_{I\in\mathcal{I}(P)} \mathbb{P}(X_p = I \mid X_p \in \mathcal{I}(P)) = 1.
\end{aligned}$$

Together, the above and Harris's inequality yield:

$$\begin{aligned}
\mathbb{P}(X_p \in U \mid X_p \in \mathcal{I}(P)) &= \mathbb{E}[\mathbf{1}_U(\mathrm{core}(X_p))r_p(\mathrm{core}(X_p))] \\
&\geq \mathbb{E}[\mathbf{1}_U(\mathrm{core}(X_p))]\mathbb{E}[r_p(\mathrm{core}(X_p))] = \mathbb{P}(\mathrm{core}(X_p) \in U).
\end{aligned}$$

Finally, since $\mathrm{core}(S) = S$ whenever $S \in \mathcal{I}(P)$, it follows that $\mathcal{I}(P) \cap \langle U\rangle = U$. As a consequence, we get the desired result for $0 < p < 1$:

$$\mathbb{P}(X_p \in \langle U\rangle \mid X_p \in \mathcal{I}(P)) = \mathbb{P}(X_p \in U \mid X_p \in \mathcal{I}(P)) \geq \mathbb{P}(\mathrm{core}(X_p) \in U) = \mathbb{P}(X_p \in \langle U\rangle).$$

It remains to consider the endpoints. At $p = 0$, we have $X_p = \emptyset$. Since U is proper and upper, $\emptyset \notin U$, and hence $\emptyset \notin \langle U\rangle$. At $p = 1$, we have $X_p = P$. Since U is nonempty and upper, $P \in U \subseteq \langle U\rangle$ and we are done. □

As a consequence, we get the following corollary via Birkhoff:

Corollary 5.2. *Let L be a finite distributive lattice, let $U \subseteq L$ be a nonempty proper upper set, and let $f : L \longrightarrow \mathcal{I}(J(L))$ be the Birkhoff isomorphism. Define $\widehat{U} := \langle f(U) \rangle \subseteq 2^{J(L)}$, fix $p \in (0,1)$, let X_p be the product-measure random subset of $J(L)$, and define the L-valued random variable Y_p by*

$$\mathbb{P}(Y_p = x) = \frac{p^{|f(x)|}(1-p)^{|J(L)|-|f(x)|}}{\mu_p(\mathcal{I}(J(L)))}, \qquad x \in L.$$

Then, for every $\epsilon \in (0,1)$, $\mathbb{P}(Y_p \in U) > 1 - \epsilon$ whenever $p > 48q(\widehat{U}) \log_2\left(\frac{\ell(\widehat{U})}{\epsilon}\right)$.

Proof. Since f is an order isomorphism, $f(U)$ is a nonempty proper upper set in $\mathcal{I}(J(L))$. Moreover, $f(U) = \mathcal{I}(J(L)) \cap \widehat{U}$. To see this, suppose $f(x) \in \mathcal{I}(J(L)) \cap \widehat{U}$. Then, $f(u) \subseteq f(x)$ for some $u \in U$. Since f is an order isomorphism, $u \leq x$, and hence $x \in U$. The reverse inclusion is immediate. Accordingly, by the definition of Y_p,

$$\mathbb{P}(Y_p \in U) = \mathbb{P}(X_p \in \widehat{U} \mid X_p \in \mathcal{I}(J(L))).$$

Applying Theorem 5.1 to the poset $J(L)$ and the upper set $f(U)$ gives $\mathbb{P}(Y_p \in U) \geq \mu_p(\widehat{U})$. Since $\widehat{U}$ is a nontrivial upper set in $2^{J(L)}$ (it is nonempty because U is nonempty, while $\emptyset \notin \widehat{U}$ because the proper upper set U does not contain the least element of L), Bell's theorem (i.e., Theorem 1.1) applies, thereby delivering the result. □

The conditioning event in Theorem 5.1 is generally nonmonotone and the resulting conditional law is generally dependent; it is a product law on the individual coordinates only when P is an antichain.

Remark 5.3 (Nonvacuity)**.** *The upper sets covered by Theorem 5.1 are not merely formal. If $I_0 \in \mathcal{I}(P)$ is nonempty, then the upper set U generated by I_0 in $\mathcal{I}(P)$ (i.e., $U := \{I \in \mathcal{I}(P) : I_0 \subseteq I\}$) is nonempty and proper in $\mathcal{I}(P)$ and the upper set it generates in 2^P (i.e., $\langle U \rangle = \{S \subseteq P : I_0 \subseteq S\}$) is nontrivial. Consequently, the theorem gives*

$$\mathbb{P}(I_0 \subseteq X_p \mid X_p \in \mathcal{I}(P)) \geq p^{|I_0|}.$$

Moreover, if P contains a comparable pair, then $\mathcal{I}(P)$ is neither an upper nor a lower set in 2^P, so the event form of Harris's inequality does not directly give this conclusion.

Remark 5.4 (Rank-biased interpretation)**.** *The measure in the preceding corollary has a simple interpretation. The function $\rho(x) := |f(x)|$ is a rank function of the finite distributive lattice L, and, for $0 < p < 1$,*

$$\mathbb{P}(Y_p = x) = \frac{\left(\frac{p}{1-p}\right)^{\rho(x)}}{\sum_{z \in L} \left(\frac{p}{1-p}\right)^{\rho(z)}}.$$

Accordingly, $Y_{1/2}$ is uniform on L, while taking $p > 1/2$ biases the measure toward elements of larger rank and taking $p < 1/2$ biases it toward elements of smaller rank. In this sense, Y_p mimics X_p.

6. Examples

We give examples illustrating the preceding mechanisms as well as a more concrete permutation-profile example that can be computed directly without the results presented here. This serves as a calibration test, thereby allowing the resulting conclusions obtained through the framework here to be verified by alternative means.

First, we give an example of how Proposition 3.3 can be used with a nonmonotone independent conditioning event.

Example 6.1 (A nonmonotone independent conditioning event)**.** *Let $X = Y \sqcup Z$, and let $\mathcal{G} \subseteq 2^Y$ be a nontrivial upper set. Define $\mathcal{F} = \{S \subseteq X : S \cap Y \in \mathcal{G}\}$ and condition on the event $B = \{S \subseteq X : |S \cap Z| \text{ is even}\}$. The event B is generally not an upper set. However, $\mathcal{F}$ depends only on the Y-coordinates and B depends only on the Z-coordinates, so the two events are independent under μ_p. Hence, Proposition 3.3 gives $\mathbb{P}(X_p \in \mathcal{F} \mid X_p \in B) > 1 - \epsilon$ whenever $p > 48q(\mathcal{F}) \log_2\left(\frac{\ell(\mathcal{F})}{\epsilon}\right)$. Thus the Park–Pham bound transfers unchanged even though the conditioning event is not monotone.*

Next, we give a worked example of how Theorem 5.1 can be applied to an explicit, genuinely dependent model.

Example 6.2 (Coupled chains with a common maximum)**.** *Fix integers $k \geq 1$ and $m \geq 2$. Let $P = P_{m,k}$ consist of m pairwise disjoint k-element chains $C_i = \{c_{i,1} < c_{i,2} < \cdots < c_{i,k}\}$ for $1 \leq i \leq m$, together with an element b lying above every element of every C_i. Define the nonempty proper upper set $U_{m,k}$ in $\mathcal{I}(P)$ to be the one generated by $C_1, \ldots, C_m$; i.e., $U_{m,k} := \{I \in \mathcal{I}(P) : C_i \subseteq I \text{ for some } i\}$. Its Boolean lift is then*

$$\mathcal{F}_{m,k} := \langle U_{m,k} \rangle = \{S \subseteq P : C_i \subseteq S \text{ for some } i\}.$$

The minimal elements of $\mathcal{F}_{m,k}$ are precisely $C_1, \ldots, C_m$, so $\ell(\mathcal{F}_{m,k}) = k$. The cover $\{C_1, \ldots, C_m\}$ has covering sum mp^k, so $\mathcal{F}_{m,k}$ is p-small whenever $mp^k \leq \frac{1}{2}$. Conversely, suppose that $\mathcal{F}_{m,k}$ is p-small, and let $\mathcal{G}$ be a cover satisfying $\sum_{G \in \mathcal{G}} p^{|G|} \leq \frac{1}{2}$. Then $\emptyset \notin \mathcal{G}$. Since $C_i \in \mathcal{F}_{m,k} \subseteq \langle \mathcal{G} \rangle$, for each i there exists a nonempty set $G_i \in \mathcal{G}$ such that $G_i \subseteq C_i$. These sets G_i are distinct because the chains C_i are pairwise disjoint. Since $|G_i| \leq k$ and $0 \leq p \leq 1$, we have

$$\sum_{G \in \mathcal{G}} p^{|G|} \geq \sum_{i=1}^{m} p^{|G_i|} \geq mp^k.$$

Thus, $\mathcal{F}_{m,k}$ can be p-small only if $mp^k \leq \frac{1}{2}$. Combining, this yields that $q(\mathcal{F}_{m,k}) = (2m)^{-1/k}$.

Let Y_p have the law of X_p conditioned on the event $X_p \in \mathcal{I}(P)$. By Theorem 5.1,

$$\mathbb{P}(Y_p \in U_{m,k}) \geq \mu_p(\mathcal{F}_{m,k}) = 1 - (1 - p^k)^m.$$

In particular, Bell's theorem gives $\mathbb{P}(Y_p \in U_{m,k}) > 1 - \epsilon$ whenever

$$p > 48(2m)^{-1/k} \log_2\left(\frac{k}{\epsilon}\right).$$

Thus, the Park–Pham criterion admits values $p \in (0, 1)$ whenever

$$m > \frac{1}{2}\left(48 \log_2\left(\frac{k}{\epsilon}\right)\right)^k.$$

The common maximum b is included to create dependence between the chains: Without b, the conditioned order-ideal law factorizes across the chains and can be analyzed directly. With b, this factorization fails: If $b \in Y_p$, then the lower-set property forces $x \in Y_p$ for every $x \le b$. Since b is the maximum of P, every $x \in P$ satisfies $x \le b$, and hence $Y_p = P$.

Counterexample 6.3 (Why the canonical Boolean lift matters)**.** *The conclusion of Theorem 5.1 does not extend to an arbitrary Boolean upper set. Let $P = \{a < b\}$. Then*

$$\mathcal{I}(P) = \{\emptyset, \{a\}, \{a, b\}\}.$$

Consider the Boolean upper set

$$\mathcal{A} := \{S \subseteq P : b \in S\} = \{\{b\}, \{a, b\}\}.$$

At $p = \frac{1}{2}$,

$$\mathbb{P}(X_{1/2} \in \mathcal{A} \mid X_{1/2} \in \mathcal{I}(P)) = \frac{1}{3} < \frac{1}{2} = \mathbb{P}(X_{1/2} \in \mathcal{A}).$$

This does not contradict Theorem 5.1. The trace

$$U := \mathcal{A} \cap \mathcal{I}(P) = \{\{a, b\}\}$$

is an upper set in $\mathcal{I}(P)$, but its canonical Boolean lift is $\langle U \rangle = \{\{a, b\}\}$, not $\mathcal{A}$. Indeed,

$$\mathbb{P}(X_{1/2} \in \langle U \rangle \mid X_{1/2} \in \mathcal{I}(P)) = \frac{1}{3} \ge \frac{1}{4} = \mathbb{P}(X_{1/2} \in \langle U \rangle).$$

The remaining examples concern principal pattern profiles. Unlike the full ideal lattice covered by Theorem 5.1, the family of principal ideals requires an additional correlation hypothesis. We therefore regard these examples as calibrations of the general embedded-poset formulation rather than as the principal application of the paper.

Example 6.4 (Permutation pattern classes)**.** *Let C be a permutation class, and let $C_{\le n}$ be the nonempty finite poset of permutations in C of length at most n, ordered by pattern containment. Define*

$$\Phi(\sigma) = \{\tau \in C_{\le n} : \tau \le \sigma\}$$

for $\sigma \in C_{\le n}$. Thus, $\Phi(\sigma)$ is the principal pattern profile of σ: It records exactly the patterns from $C_{\le n}$ that occur in σ, and the embedding Φ converts pattern containment into set inclusion. Then $\Phi : C_{\le n} \to 2^{C_{\le n}}$ is an order embedding. Let $\mathcal{U} \subseteq C_{\le n}$ be a nontrivial upper set, and let M be the set of minimal elements of $\mathcal{U}$. Define

$$\mathcal{E}_M = \{T \subseteq C_{\le n} : T \cap M \neq \emptyset\}.$$

Thus, $\mathcal{E}_M$ is the Boolean event that a subset of patterns contains at least one member of the minimal target antichain M. $\mathcal{E}_M$ is an upper set in the Boolean lattice $2^{C_{\le n}}$, and $\mathcal{E}_M \cap \Phi(C_{\le n}) = \Phi(\mathcal{U})$. Indeed, for $\sigma \in C_{\le n}$,

$$\Phi(\sigma) \in \mathcal{E}_M \quad \Longleftrightarrow \quad \sigma \text{ contains some } \pi \in M \quad \Longleftrightarrow \quad \sigma \in \mathcal{U}.$$

Write $m = |M|$. The minimal elements of $\mathcal{E}_M$ are precisely the singletons $\{\pi\}$ with $\pi \in M$, so $\ell(\mathcal{E}_M) = 1$. Moreover, $\mathcal{E}_M$ is covered by these m singletons, giving $q(\mathcal{E}_M) \ge 1/(2m)$. Conversely, if $\mathcal{G}$

covers $\mathcal{E}_M$, then for each $\pi \in M$, the singleton $\{\pi\} \in \mathcal{E}_M$ must contain some $G \in \mathcal{G}$. Since the only subsets of $\{\pi\}$ are $\emptyset$ and $\{\pi\}$, and including $\emptyset$ would contribute 1 to the covering sum, any p-small cover must contain $\{\pi\}$ for each $\pi \in M$. Hence, the covering sum is at least mp, so $q(\mathcal{E}_M) = 1/(2m)$.

Therefore, if $\mathcal{E}_M$ and $\Phi(C_{\leq n})$ are positively correlated at p, then $\mathbb{P}(Y_p \in \mathcal{U}) > 1 - \epsilon$ whenever

$$p > 48q(\mathcal{E}_M) \log_2\left(\frac{\ell(\mathcal{E}_M)}{\epsilon}\right) = \frac{24}{m} \log_2\left(\frac{1}{\epsilon}\right).$$

In this formulation,

$$\mathbb{P}(Y_p = \sigma) = \frac{p^{|\Phi(\sigma)|}(1-p)^{|C_{\leq n}|-|\Phi(\sigma)|}}{\mu_p(\Phi(C_{\leq n}))}.$$

Thus the conclusion says that a profile-weighted random permutation contains at least one of the minimal target patterns in M. At $p = 1/2$, the induced law is uniform on $C_{\leq n}$.

Remark 6.5. *In the permutation-pattern setting, the positive-correlation hypothesis is a genuine additional condition. Determining when this holds for natural permutation classes appears to be an interesting problem in its own right. The image $\Phi(C_{\leq n})$ is typically far from an upper set in $2^{C_{\leq n}}$, so Harris's inequality does not apply directly. Thus, the conditional Park–Pham transfer reduces the problem to understanding when the set of valid pattern profiles is positively correlated with the Boolean event of hitting the minimal pattern antichain M, rather than the stronger condition of containing an entire prescribed pattern profile.*

The preceding example leaves the positive-correlation condition as an additional hypothesis. The next example shows that this hypothesis is not vacuous by verifying it in a natural permutation class.

Example 6.6 (Stack-sortable permutations)**.** *Let $C = \mathrm{Av}(231)$ be the class of one-stack-sortable permutations, where $\mathrm{Av}(231)$ denotes the permutations avoiding the pattern 231. Write $C_r := \{\sigma \in C : |\sigma| = r\}$, so that $|C_r| = \mathrm{Cat}_r$, the r-th Catalan number [16]. We take the convention that C_0 consists of the empty permutation. Fix $1 \leq k \leq n$, and let $C_{\leq n} = C_0 \cup \ldots \cup C_n$ be ordered by pattern containment. Define*

$$\mathcal{U}_{k,n} = \{\sigma \in C_{\leq n} : |\sigma| \geq k\}.$$

Then $\mathcal{U}_{k,n}$ is an upper set, and its minimal elements are exactly $M = C_k$. In particular, $m = |M| = \mathrm{Cat}_k$. Let $\Phi(\sigma)$ and $\mathcal{E}_M$ be as before, so that $\mathcal{E}_M \cap \Phi(C_{\leq n}) = \Phi(\mathcal{U}_{k,n})$.

At $p = 1/2$, the measure on $2^{C_{\leq n}}$ is uniform. Hence, we get

$$\mathbb{P}(Y_{1/2} \in \mathcal{U}_{k,n}) = \frac{\mu_{1/2}(\mathcal{E}_M \cap \Phi(C_{\leq n}))}{\mu_{1/2}(\Phi(C_{\leq n}))} = \frac{|\mathcal{U}_{k,n}|}{|C_{\leq n}|} = \frac{\sum_{r=k}^{n} \mathrm{Cat}_r}{\sum_{r=0}^{n} \mathrm{Cat}_r}$$

and $\mu_{1/2}(\mathcal{E}_M) = 1 - \left(1 - \frac{1}{2}\right)^{\mathrm{Cat}_k} = 1 - 2^{-\mathrm{Cat}_k}$. Therefore, $\mathcal{E}_M$ and $\Phi(C_{\leq n})$ are positively correlated at $p = 1/2$ whenever

$$\frac{\mu_{1/2}\left(\mathcal{E}_M \cap \Phi(C_{\le n})\right)}{\mu_{1/2}(\Phi(C_{\le n}))} \ge \mu_{1/2}(\mathcal{E}_M)$$
$$\frac{\sum_{r=k}^{n} \mathrm{Cat}_r}{\sum_{r=0}^{n} \mathrm{Cat}_r} \ge 1 - 2^{-\mathrm{Cat}_k}$$
$$2^{-\mathrm{Cat}_k} \sum_{r=0}^{n} \mathrm{Cat}_r \ge \left(\sum_{r=0}^{n} \mathrm{Cat}_r\right) - \left(\sum_{r=k}^{n} \mathrm{Cat}_r\right) = \sum_{r=0}^{k-1} \mathrm{Cat}_r\,.$$

The first line is precisely the positive-correlation condition. Substituting the two probabilities computed above gives the second line, while multiplying through by the positive denominator and rearranging gives the final line. For fixed k, the right-hand side of the final inequality is independent of n, whereas its left-hand side tends to infinity because $2^{-\mathrm{Cat}_k} > 0$ *and* $\sum_{r=0}^{n} \mathrm{Cat}_r \to \infty$. *Hence, the inequality holds for all sufficiently large n. Since* $q(\mathcal{E}_M) = (2\,\mathrm{Cat}_k)^{-1}$ *and* $\ell(\mathcal{E}_M) = 1$, *the conditional Park–Pham transfer result (Theorem 4.1) gives* $\mathbb{P}(Y_{1/2} \in \mathcal{U}_{k,n}) > 1 - \epsilon$ *whenever* $\epsilon > 2^{-\mathrm{Cat}_k/48}$ *and n is sufficiently large.*

Remark 6.7. *The preceding example is meant as a calibration example rather than a deep application: The conclusion can also be obtained directly from the Catalan enumeration. Its purpose is to show that the positive-correlation hypothesis in the permutation-profile formulation is not vacuous, even for a natural non-Boolean permutation poset. More interesting applications would require verifying the same correlation inequality for upper sets defined by a proper subcollection of patterns, rather than by all patterns of a fixed length.*

7. Conclusions

The preceding results show that the ϵ-dependent Park–Pham theorem admits a simple conditional form under positive correlation. The point is not that every conditioning event preserves a Park–Pham-type bound; rather, the positive-correlation condition is exactly the hypothesis ensuring that conditioning does not reduce the probability of the target upper set. Harris's inequality supplies one general source of such conditioning events, while independent-coordinate restrictions supply another, including nonmonotone examples.

The consequences of this are substantial: Theorem 5.1 gives an automatic favorable comparison for a global, generally nonmonotone conditioning event on every finite poset. Through Birkhoff's representation theorem, this yields a canonical rank-biased family of conditional Park–Pham estimates on every finite distributive lattice in Corollary 5.2.

The scope of the comparison should nevertheless be kept in view. For arbitrary embedded posets, including principal permutation-profile families, positive correlation remains an additional hypothesis. Even for order-ideal conditioning, the automatic comparison applies to canonical Boolean lifts rather than arbitrary Boolean upper sets, as Counterexample 6.3 shows. Finally, the method retains the ambient quantities $q(\mathcal{F})$ and $\ell(\mathcal{F})$; if the resulting numerical lower bound on p is at least 1, the estimate is vacuous (though some work has been done to understand when nontrivial information occurs [17]).

In light of Example 6.4, it would also be interesting to determine for which permutation classes C, integers n, and upper sets $\mathcal{U} \subseteq C_{\le n}$ with minimal antichain M, the inequality

$$\mu_p(\mathcal{E}_M \cap \Phi(C_{\le n})) \ge \mu_p(\mathcal{E}_M)\mu_p(\Phi(C_{\le n}))$$

holds. Even special cases would give conditional Park–Pham bounds for pattern-containment properties, or equivalently for complements of pattern-avoidance properties.

Author contributions

Bryce Alan Christopherson: Conceptualization, methodology, writing–original draft, writing–review and editing. Darian Colgrove: Conceptualization, methodology, writing–review and editing. Both authors read and approved the final manuscript.

Use of Generative-AI tools declaration

During the preparation of a revision of the manuscript, the first author used ChatGPT (OpenAI) as an interactive aid for language editing, organization, identifying potentially relevant literature, and exploring mathematical arguments, including the development and refinement of the proof strategy for Theorem 5.1 and related examples. All AI-generated suggestions were critically evaluated by the authors. The authors verified the final mathematical statements, proofs, and citations, made all final decisions concerning the manuscript, and take full responsibility for its content.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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