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Research article

Distance eccentric connectivity and d-connectivity indices of circulant networks: exact formulas and generalizations

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Abstract: Graphical invariants play a fundamental role in characterizing the structural and distance-related properties of complex networks, with significant applications in graph theory, chemical graph theory, and communication systems. Among these, the distance eccentric connectivity index (DECI) and the d-connectivity index (DCI) have emerged as effective descriptors that integrate vertex degree, eccentricity, and distance information. In this paper, we present an exact analytical study of these two distance-based graphical invariants for several families of circulant networks. Closed-form expressions are established for the DECI and DCI of 3-, 4-, and 5-regular circulant networks. Furthermore, unified general formulas for the DCI are derived for generalized circulant networks with arbitrary even and odd regularities. The obtained results reveal the influence of graph order, regularity, and parity on the behavior of these invariants, providing a comprehensive characterization of their structural properties. The proposed formulas not only extend the theoretical understanding of distance-based graphical invariants for highly symmetric networks but also offer efficient computational tools for the analysis of communication networks, chemical graph structures, and other complex, interconnected systems.

Keywords: circulant networks; regular circulant graphs; distance eccentric connectivity index; d-connectivity index; graphical invariants; distance-based topological indices; exact formulas; graph invariants; chemical graph theory; complex networks
Mathematics Subject Classification: 05C12, 05C90, 05C76, 05C09, 05C82

1. Introduction

Wiener, in 1947, initiated the study of connectivity indices of a connected graph by defining a distance-based index (called the Wiener index), which is the best source to predict the boiling point of alkanes with almost 89% accuracy [31]. After that, Dobrynin and Kochetova [8] and Gutman [15], independently studied a weighted version of the Wiener index. An unceasing employment on the study of various connectivity indices of a graph have been made by a number of researchers to date. For instance, Estrada et al. defined the atom bond connectivity index to predict the heat of the formation of alkanes [9]; Ali et al. provided the sum version of atom bond connectivity index [1]; Zhou and Trinajstíc introduced the sum connectivity index [33]; Yu and Feng defined the connective eccentricity index [32]; Sharma et al. supplied the eccentric connectivity index for structure property and the structure activity studies of molecular objects [30]; Gupta et al. studied the eccentric distance sum index to predict biological and physical properties of molecular graphs [12]; Sardana and Madan introduced an adjacent eccentric distance sum index to predict anti-HIV activity of TIBO derivatives [29]; Ramane and Yalnaik proposed two status connectivity indices having the best correlation to predict the boiling point of benzenoid hydrocarbons [27]; and Gutman and Lepović introduced and studied the m-connectivity index [14].

Graph indices based on notions of degree and distance are extensively applied, as graph invariants, in theoretical chemistry to develop correlations between molecular structure properties [10, 21]. They provide a variety of correlations with physical, chemical, and thermodynamic properties of chemical compounds [17, 19]. In [30], the authors proposed the eccentric connectivity index, which has found successful applications toward the construction of numerous mathematical models for predicting biological activities of different types. The eccentric connectivity index yields good correlations toward both physical and biological properties [20]. The ease combined with the good correlating potential of this index can be readily utilized in quantitative structure studies of property activity relationship [10, 11]. These types of studies give useful leads for the synthesis of possible therapeutic drugs. For the mathematical aspect of eccentric connectivity index, the reader is referred to [16, 23]. The authors in [13], developed a topological descriptor known as the connective eccentric index while exploring the antihypertensive activity of N-benzylimidazole derivatives [22]. The values found with this descriptor were superior to corresponding values found using Balaban's mean square distance index [3, 4]. Within this framework, Alkqesmah) et al. [2] introduced the distance eccentric connectivity index (DECI), which incorporates eccentricity into degree-distance interactions, providing valuable insights into extremal structural properties. Building on the existing framework of degree and distance-based connectivity descriptors, Noreen and Salman introduced the d-connectivity index (DCI) as a novel degree-distance descriptor based on d-neighbors [25], where its basic properties were established and its values were computed for hexa-organic molecular structures. It was subsequently studied for chain polygonal cactus networks, with applications to structural connectivity, network accessibility, and resilience [26]. These indices complement each other: whereas DECI emphasizes global efficiency and extremal communication delays, DCI highlights localized degree-distance interactions and resilience under partial failures. The remarkable rotational symmetry and vertex-transitive nature of circulant networks substantially simplify the computation of global graph invariants. Because every vertex belongs to the same automorphism orbit, local structural properties extend uniformly throughout the network, allowing exact analytical expressions to be derived without exhaustive vertex-by-vertex

computations. This inherent symmetry not only reduces computational complexity but also provides deeper insight into the relationship between graph structure, eccentricity, and connectivity, making circulant networks an ideal class of graphs for investigating distance-based topological descriptors.

Main contributions

The principal contributions of this paper are summarized as follows:

- Exact closed-form expressions for the distance eccentric connectivity index (DECI) of 3-, 4-, and 5-regular circulant networks are established.
- Exact formulas for the d-connectivity index (DCI) are derived for these families of circulant networks.
- General expressions for the d-connectivity index are obtained for generalized circulant networks with arbitrary even and odd regularity.
- The influence of graph order, regularity, and congruence classes on the proposed indices is analyzed.
- The obtained results enrich the theory of distance-based graph invariants and provide useful analytical tools for applications in graph theory, chemical graph theory, and complex network analysis.
- All closed-form expressions are validated against direct computation on explicit circulant graphs.

2. Preliminaries and basic results

Let $G = (V(G), E(G))$ be a simple and connected graph. Two vertices of G are called neighbors if an edge exists between them. The number of vertices adjacent to a vertex u is called its degree and is denoted by $d(u)$. A path is defined as a walk in which no vertex is repeated, and the shortest path between two vertices is termed a geodesic. The distance between two vertices u and v, denoted by $d(u, v)$, is the length (in edges) of a geodesic joining them. The eccentricity of a vertex u, written as $ecc(u)$, is defined as the maximum distance from u to any other vertex in G. The minimum and maximum eccentricities among all vertices of G are known as the radius $rad(G)$ and the diameter $diam(G)$ of the graph, respectively [7]. Vertices lying at distance $d(u)$ from u are referred to as the d-neighbors of u, and vertices at distance $ecc(u)$ from u are called the eccentric neighbors of u. The distance eccentricity neighborhood of a vertex u is denoted by $N_{De}(u) = \{v \in V(G) \mid d(u, v) = ecc(u)\}$, and its cardinality, $|N_{De}(u)| = deg^{De}(u)$, is called the distance eccentricity degree of u in G [2]. The distance eccentric connectivity index of G in [2] is defined by

$$\xi^{De}(G) = \sum_{v \in V(G)} deg^{De}(v) ecc(v). \tag{2.1}$$

Overall, connectivity indices have been constructed by exploiting the interplay of distance, degree, and eccentricity in graphs. Several indices emphasize degree, that is, the number of 1-neighbors (vertices at distance one); others focus on 2-neighbors (vertices at distance two), and still others incorporate eccentric neighbors (vertices at eccentric distance). In this paper, we extend this framework by concentrating on d-neighbors, vertices at distance d from a given vertex. Throughout the paper, $d = d(u)$ denotes the degree of the vertex u, and not a free distance parameter. For an integer $k \geq 0$, the

k-distance neighborhood of a vertex u is $N_k(u) = \{v \in V(G) : d(u, v) = k\}$ [24, 28]; thus, $N_1(u) = N(u)$ is the open neighborhood of u. The *d-neighborhood* of u is the particular k-distance neighborhood obtained for $k = d(u)$, $N_{d(u)}(u) = \{v \in V(G) : d(u, v) = d(u)\}$, and its cardinality $n_d(u) = |N_{d(u)}(u)|$ is called the *d-number* of u (see Figures 2–4). Note that $N_{d(u)}(u)$ consists of the vertices at distance *exactly* $d(u)$ from u, not of those within distance $d(u)$, and that the distance used depends on the degree of u itself. In an r-regular, graph every vertex has $d(u) = r$, so $N_{d(u)}(u) = N_r(u)$, and $n_d(u) = |N_r(u)|$. Following [25], the *d-connectivity index* of G is the sum, over all vertices, of the degree of a vertex multiplied by its d-number:

$$C_d(G) = \sum_{v \in V(G)} n_d(v)\, d(v). \tag{2.2}$$

Figure 1 summarizes all of these concepts with pictorial view. From [25], we have the following straightforward proposition.

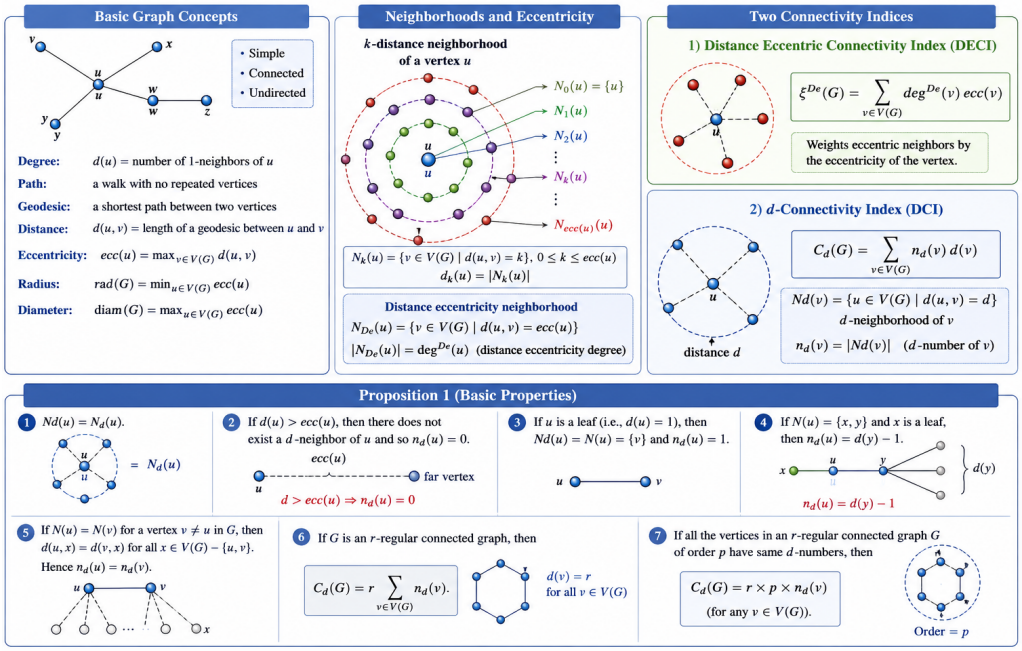


Figure 1. Preliminaries and basic results.

Proposition 1. *[25] Let u be a vertex of G.*

(1) Then, $Nd(u) = N_{d(u)}(u)$.
(2) There does not exist a d-neighbor of u if $d(u) > ecc(u)$, and so $n_d(u) = 0$.
(3) For a leaf u (a vertex of degree one), $Nd(u) = N(u) = \{x\}$, and $n_d(u) = 1$.
(4) If $N(u) = \{x, y\}$, and x is a leaf, then $n_d(u) = d(y) - 1$.
(5) If $N(u) = N(v)$ for a vertex $v \neq u$ in G, then $d(u, x) = d(v, x)$ for all $x \in V(G) - \{u, v\}$, and so $n_d(u) = n_d(v)$.
(6) If G is an r-regular connected graph, then $d(v) = r$ for every $v \in V(G)$. Consequently, the d-neighborhood of v coincides with its r-neighborhood, denoted by $N_r(v)$, and the corresponding d-number is $n_r(v)$. Accordingly, $C_d(G) = r \sum_{v \in V(G)} n_r(v)$.

(7) If all the vertices in an r-regular connected graph G of order p have same d-numbers, then $C_d(G) = r \times p \times n_r(v)$*, where v is any vertex of G.*

3. Circulant networks

Circulant networks, owing to their symmetry, fault-tolerance, and efficient routing properties, provide a natural setting for testing and comparing these indices [5, 6]. Such networks are widely applied in parallel computing, memory systems, telecommunication networks, and molecular modeling. In this section, we undertake a comparative study of DECI and DCI in circulant networks. Our objective is twofold: to highlight their complementary sensitivities in measuring global performance and local robustness, and to demonstrate their potential applications in both computational and chemical domains. This network can be described more formally as being circulant. A circulant network with n-vertices is a Cayley graph of Z_n (the additive group of integers under modulo n) with vertex set $\{v_{i+1} : i \in Z_n\}$ and edge set $\{v_i v_{((i+s) \bmod n)} : 1 \leq i \leq n, s \in S\}$, where $S \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$ is known as a generating set. If S satisfies the additive inverse property, we have an undirected circulant network; otherwise, it is directed. An undirected circulant network is denoted by $C_n(a_1, a_2, \ldots, a_m)$, where $1 \leq a_i \leq \lfloor n/2 \rfloor$ are called generators. Accordingly, the edge set of $C_n(a_1, a_2, \ldots, a_m)$ is $\{v_i v_{i+a_j} : 1 \leq i \leq n, 1 \leq j \leq m\}$ [18].

Due to their high symmetry and regular structure, circulant networks are particularly well-suited for analyzing distance-degree based indices such as DECI and DCI. For a 2-regular family of circulant network $C_p(1)$, the DECI calculated by Alkqesmah et al. in [2] is as follows: For all $p \geq 3$,

$$\xi^{De}(C_p) = \begin{cases} \frac{p^2}{2}, & \text{when p is even,} \\ 2p\lfloor \frac{p}{2} \rfloor, & \text{when p is odd.} \end{cases}$$

The DCI of $C_p(1)$ is then $4p$ for $p \geq 5$. In what follows, we compute exact values of the DECI and DCI for different families of circulant networks.

3.1. A family of 3*-regular circulant networks*

In this section, we derive closed-form expressions for the DECI and the DCI of the 3-regular family of circulant networks $C_n(1, n/2)$, for even $n \geq 6$. For $n = 4$, the generator $n/2 = 2$ gives $C_4(1, 2) = K_4$, a complete graph whose indices are already known [2, 25]; this degenerate case is therefore excluded, and the family $C_n(1, n/2)$ is considered for even $n \geq 6$. Remember that $ecc(v) = \lfloor (n-1)/4 \rfloor + 1$ for each vertex v of $C_n(1, n/2)$.

Theorem 2. *For even* $n \geq 6$*, let* $G = C_n(1, n/2)$*. Then,*

$$\xi^{De}(G) = \begin{cases} n^2, & n \equiv 0 \pmod 4, \\ \dfrac{n(n+2)}{2}, & n \equiv 2 \pmod 4. \end{cases}$$

Proof. Because G is rotationally symmetric, it suffices to analyze a fixed vertex v (Figure 2). When $n \equiv 0 \pmod 4$, $\mathrm{ecc}(v) = n/4$, and there are four eccentric neighbors $v_{i+ecc(v_i)}, v_{i-ecc(v_i)}, v_{i+ecc(v_i)+1}$, and $v_{i-ecc(v_i)-1}$, so $deg^{De}(v) = 4$, yielding

$$\xi^{De}(G) = n \cdot 4 \cdot n/4 = n^2.$$

When $n \equiv 2 \pmod 4$, $\text{ecc}(v) = (n+2)/4$, and there are two eccentric neighbors $v_{i+ecc(v_i)}$ and $v_{i-ecc(v_i)}$, so $deg^{De}(v) = 2$. Hence,

$$\xi^{De}(G) = n \cdot 2 \cdot (n+2)/4 = n(n+2)/2.$$

□

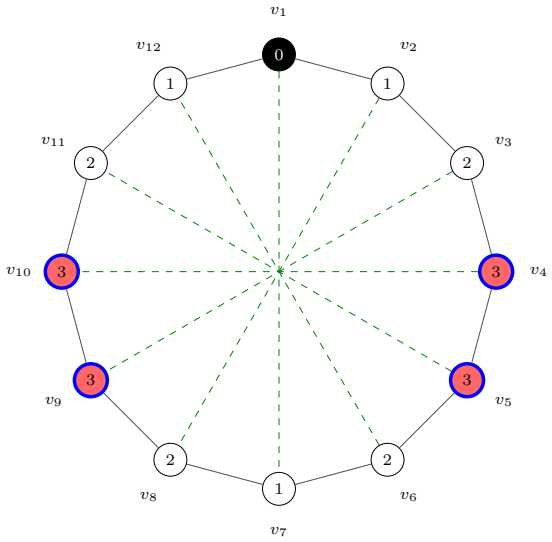


Figure 2. The circulant network $C_{12}(1, \frac{12}{2})$ with the distance $d(v_1, v)$ written inside each vertex. The fixed vertex v_1 is black, the eccentric vertices (at distance ecc $= 3$) are red, and the d-neighbors (at distance $d = 3$) have a thick blue border. Here, $\deg^{De}(v) = 4$, so $\xi^{De}(G) = 12 \cdot 4 \cdot 3 = 144$ (Theorem 2), and $n_d(v) = 4$, so $C_d(G) = 12 \cdot 4 \cdot 3 = 144$ (Theorem 3).

Theorem 3. *For even $n \geq 6$, let $G = C_n(1, n/2)$. Then,*

$$C_d(G) = \begin{cases} 0, & n \leq 8, \\ 60, & n = 10, \\ 12n, & n \geq 12. \end{cases}$$

Proof. Because G is rotationally symmetric and 3-regular, $d(v_i) = 3$, and it suffices to determine the d-number of a fixed arbitrary vertex v_i. Note that $ecc(v_i) < 3$ when $n \leq 8$, and so $n_3(v_i) = 0$, by Proposition 1(2). Moreover, when $n \geq 10$,

$$N_3(v_i) = \begin{cases} \{v_{i-3}, v_{i+3}\}, & \text{when } n = 10, \\ \{v_{i-3}, v_{i+3}, v_{i+n/2+2}, v_{i+n/2-2}\}, & \text{when } n \geq 12. \end{cases}$$

Hence, for each $1 \leq i \leq n$, we have

$$n_3(v_i) = \begin{cases} 0, & \text{when } n \leq 8, \\ 2, & \text{when } n = 10, \\ 4, & \text{when } n \geq 12. \end{cases}$$

By Proposition 1(7), the required index then follows. □

Remark 4. *By Proposition 1(2),* $n_d(v) = 0$ *until* ecc(v) *reaches* $d = r$*, which first happens at* $n = 10$, 14, *and* 32 *for* $r = 3, 4, 5$*. At these orders, the vertices at distance* d *coming from opposite sides of the network still coincide or overlap (for example,* $v_{i+7} = v_{i-7}$ *in* $C_{14}(1, 2)$*); they separate as* n *increases, giving the intermediate values of* n_d *and then the constant value* 4, 4, 8 *and the linear growth of* C_d*.*

3.2. A family of 4*-regular circulant networks*

In this section, we derive closed-form expressions for the DECI and DCI of the 4-regular family of circulant networks $C_n(1, 2)$ for $n \geq 5$. For each vertex v of this family,

$$ecc(v) = \begin{cases} \lceil \frac{n}{4} \rceil, & \text{when } n \equiv 0, 2, 3 \pmod 4 , \\ \lfloor \frac{n}{4} \rfloor, & \text{when } n \equiv 1 \pmod 4. \end{cases}$$

Theorem 5. *For* $n \geq 5$*, let* $G = C_n(1, 2)$*. Then,*

$$\xi^{De}(G) = \begin{cases} \dfrac{3n^2}{4}, & n \equiv 0 \pmod 4, \\ n(n-1), & n \equiv 1 \pmod 4, \\ \dfrac{n(n+2)}{4}, & n \equiv 2 \pmod 4, \\ \dfrac{n(n+1)}{2}, & n \equiv 3 \pmod 4. \end{cases}$$

Proof. Because G is rotationally symmetric, it suffices to analyze a fixed vertex v (Figure 3). When $n \equiv 0 \pmod 4$, ecc$(v) = n/4$, and there are three eccentric neighbors $v_{i+n/2}, v_{i+n/2+1}$, and $v_{i+n/2-1}$. Therefore, $deg^{De}(v) = 3$, implies that $\xi^{De}(G) = 3n^2/4$.

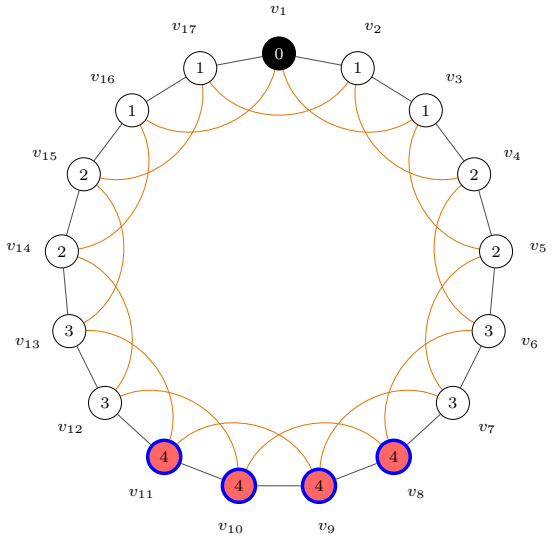


Figure 3. The circulant network $C_{17}(1, 2)$ with the distance $d(v_1, v)$ written inside each vertex. The fixed vertex v_1 is black, the eccentric vertices (at distance ecc $= 4$) are red, and the d-neighbors (at distance $d = 4$) have a thick blue border. Here, $\deg^{De}(v) = 4$, so $\xi^{De}(G) = 17 \cdot 4 \cdot 4 = 272$ (Theorem 5), and $n_d(v) = 4$, so $C_d(G) = 17 \cdot 4 \cdot 4 = 272$ (Theorem 6).

When $n \equiv 1 \pmod 4$, $\text{ecc}(v) = (n-1)/4$, and there are four eccentric neighbors $v_{i+\lfloor \frac{n}{2} \rfloor}$, $v_{i-\lfloor n/2 \rfloor}$, $v_{i+\lfloor n/2 \rfloor+1}$, and $v_{i+\lfloor n/2 \rfloor-1}$. Then, $deg^{De}(v) = 4$, implies that $\xi^{De}(G) = n(n-1)$.

When $n \equiv 2 \pmod 4$, $\text{ecc}(v) = (n+2)/4$, and there is only one eccentric neighbor $v_{i+n/2}$, so $deg^{De}(v) = 1$. Hence, $\xi^{De}(G) = n(n+2)/4$.

When $n \equiv 3 \pmod 4$, $\text{ecc}(v) = (n+1)/4$, and there are two eccentric neighbors $v_{i-\lfloor n/2 \rfloor}$ and $v_{i+\lfloor n/2 \rfloor}$. Therefore, $deg^{De}(v) = 2$, providing $\xi^{De}(G) = n(n+1)/2$. □

Theorem 6. *For $n \geq 5$, let $G = C_n(1,2)$. Then,*

$$C_d(G) = \begin{cases} 0, & \text{for } n \leq 13, \\ 56, & \text{for } n = 14, \\ 120, & \text{for } n = 15, \\ 192, & \text{for } n = 16, \\ 16n, & \text{for } n \geq 17. \end{cases}$$

Proof. Because G is rotationally symmetric and 4-regular, $d(v_i) = 4$, and it suffices to determine the d-number of a fixed arbitrary vertex v_i. Note that $ecc(v_i) < 4$ when $n \leq 13$, and so $n_4(v_i) = 0$, by Proposition 1(2). Moreover, when $n \geq 14$,

$$N_4(v_i) = \begin{cases} \{v_{i+7}\}, & \text{when } n = 14, \\ \{v_{i-7}, v_{i+7}\}, & \text{when } n = 15, \\ \{v_{i-7}, v_{i+7}, v_{i+8}\}, & \text{when } n = 16, \\ \{v_{i-7}, v_{i+7}, v_{i-8}, v_{i+8}\}, & \text{when } n \geq 17. \end{cases}$$

Hence, for each $1 \leq i \leq n$, we have

$$n_4(v_i) = \begin{cases} 0, & \text{when } n \leq 13, \\ 1, & \text{when } n = 14, \\ 2, & \text{when } n = 15, \\ 3, & \text{when } n = 16, \\ 4, & \text{when } n \geq 17. \end{cases}$$

By Proposition 1(7), the required index then follows. □

3.3. A family of 5-regular circulant networks

In this section, we derive closed-form expressions for the DECI and DCI of the 5-regular family of circulant networks $C_n(1,2,n/2)$, for even $n \geq 8$. Remember that, for each vertex v of this family, $ecc(v) = \lfloor n/8 \rfloor + 1$.

Theorem 7. *For even $n \geq 8$, let $G = C_n(1,2,n/2)$. Then,*

$$\xi^{De}(G) = \begin{cases} \dfrac{n(n+8)}{4}, & n \equiv 0 \pmod 8, \\ \dfrac{n(n+6)}{2}, & n \equiv 2 \pmod 8, \\ \dfrac{3n(n+4)}{4}, & n \equiv 4 \pmod 8, \\ n(n+2), & n \equiv 6 \pmod 8. \end{cases}$$

Proof. By vertex-transitivity, it is enough to check one vertex v. For each case, $\text{ecc}(v) = \lfloor n/8 \rfloor + 1$. The number of eccentric neighbors varies with $n \pmod 8$, as given in Table 1.

By using eccentricity and values of $deg^{De}(v)$ from Table 1 into the formula (2.1) of $\xi^{De}(G)$ completes the computation. □

Table 1. The distance eccentricity degree of v in $C_n(1, 2, n/2)$.

$n \pmod 8$	Eccentric neighbors	$deg^{De}(v)$
$n \equiv 0 \pmod 8$	$v_{i+(\frac{n}{4}+1)}, v_{i-(\frac{n}{4}+1)}$	2
$n \equiv 2 \pmod 8$	$v_{i+\lfloor \frac{n}{4} \rfloor+1}, v_{i-\lfloor \frac{n}{4} \rfloor-1}, v_{i+\lfloor \frac{n}{4} \rfloor+2}, v_{i-\lfloor \frac{n}{4} \rfloor-2}$	4
$n \equiv 4 \pmod 8$	$v_{i+\frac{n}{4}}, v_{i-\frac{n}{4}}, v_{i+(\frac{n}{4}+1)}, v_{i-(\frac{n}{4}+1)}, v_{i+(\frac{n}{4}+2)}, v_{i-(\frac{n}{4}+2)}$	6
$n \equiv 6 \pmod 8$	$v_{i+\lfloor \frac{n}{4} \rfloor}, v_{i-\lfloor \frac{n}{4} \rfloor}, v_{i+\lfloor \frac{n}{4} \rfloor+1}, v_{i-\lfloor \frac{n}{4} \rfloor-1}, v_{i+\lfloor \frac{n}{4} \rfloor+2}, v_{i-\lfloor \frac{n}{4} \rfloor-2}, v_{i+\lfloor \frac{n}{4} \rfloor+3}, v_{i-\lfloor \frac{n}{4} \rfloor-3}$	8

Theorem 8. *For even $n \geq 8$, let $G = C_n(1, 2, n/2)$. Then,*

$$C_d(G) = \begin{cases} 0, & \text{for } n \leq 30, \\ 320, & \text{for } n = 32, \\ 680, & \text{for } n = 34, \\ 1080, & \text{for } n = 36, \\ 40n, & \text{for } n \geq 38. \end{cases}$$

Proof. Because G is rotationally symmetric and 5-regular (see Figure 4), $d(v_i) = 5$, and it suffices to determine the d-number of a fixed arbitrary vertex v_i. Note that $ecc(v_i) < 5$ when $n \leq 30$, and so $n_5(v_i) = 0$, by Proposition 1(2). Moreover, when $n \geq 32$,

$$N_5(v_i) = \begin{cases} \{v_{i-9}, v_{i+9}\}, & \text{when } n = 32, \\ \{v_{i-9}, v_{i+9}, v_{i-10}, v_{i+10}\}, & \text{when } n = 34, \\ \{v_{i-9}, v_{i+9}, v_{i-10}, v_{i+10}, v_{i-11}, v_{i+11}\}, & \text{when } n = 36, \\ \{v_{i-9}, v_{i+9}, v_{i-10}, v_{i+10}, v_{i-11}, v_{i+11}, v_{i-12}, v_{i+12}\}, & \text{when } n \geq 38. \end{cases}$$

Hence, for each $1 \leq i \leq n$, we have

$$n_5(v_i) = \begin{cases} 0, & \text{when } n \leq 30, \\ 2, & \text{when } n = 32, \\ 4, & \text{when } n = 34, \\ 6, & \text{when } n = 36, \\ 8, & \text{when } n \geq 38. \end{cases}$$

By Proposition 1(7), the required index then follows. □

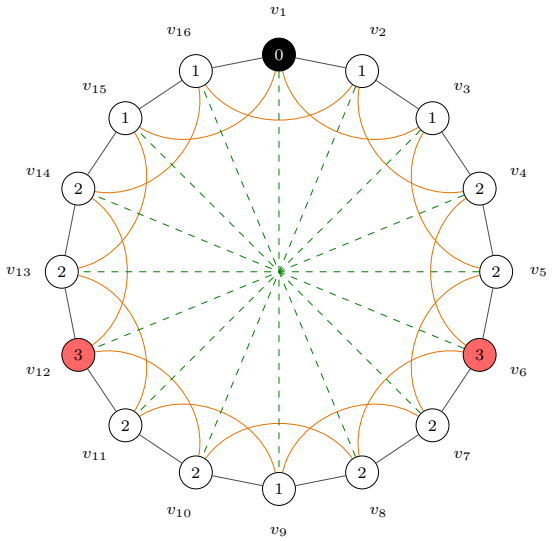


Figure 4. The circulant network $C_{16}(1, 2, \frac{16}{2})$ with the distance $d(v_1, v)$ written inside each vertex. The fixed vertex v_1 is black, and the eccentric vertices (at distance ecc = 3) are red. Here, $\deg^{De}(v) = 2$, so $\xi^{De}(G) = 16 \cdot 2 \cdot 3 = 96$ (Theorem 7). Because $d(v) = 5 > \text{ecc}(v)$, there is no d-neighbor, $n_d(v) = 0$, and $C_d(G) = 0$ (Theorem 8).

3.4. A generalization of the d-connectivity index

We now provide general formulas for $C_d(G)$ when $G \cong C_n(1, 2, 3, \ldots, r/2)$ is an r-regular circulant network for both even and odd values of r.

Theorem 9. *For even $r \geq 2$ and any $n > r$, let $G \cong C_n(1, 2, 3, \ldots, r/2)$ be an r-regular generalized circulant network. Then,*

$$C_d(G) = \begin{cases} 0, & n \leq r(r-1)+1, \\ nr(t+1), & n = r(r-1)+2+t \text{ with } 0 \leq t \leq r-2, \\ nr^2, & n \geq r^2+1. \end{cases}$$

Proof. Let v_i be any vertex of G. Then, $N(v_i) = \{v_{i+j}, v_{i-j;\ 1\leq j\leq r/2}\}$, and

$$ecc(v) = \begin{cases} \lceil \frac{n}{r} \rceil, & \text{whenever } n \equiv 0, 2, 3, \ldots, r-1 \pmod r, \\ \lfloor \frac{n}{r} \rfloor, & \text{whenever } n \equiv 1 \pmod r. \end{cases}$$

Now, we have the following three cases to discuss:

Case 1 By Proposition 1(2),

$$\begin{aligned} n_r(v_i) = 0 &\iff ecc(v_i) < d(v_i) = r, \\ &\iff \begin{cases} \lceil \frac{n}{r} \rceil < r, & \text{whenever } n \equiv 0, 2, 3, \ldots, r-1 \pmod r, \\ \lfloor \frac{n}{r} \rfloor < r, & \text{whenever } n \equiv 1 \pmod r. \end{cases} \\ &\iff \begin{cases} \frac{n}{r} \leq r-1, & \text{whenever } n \equiv 0, 2, 3, \ldots, r-1 \pmod r, \\ \frac{n}{r} < r, & \text{whenever } n \equiv 1 \pmod r. \end{cases} \end{aligned}$$

$$\iff \begin{cases} n \le r(r-1), & \text{whenever } n \equiv 0, 2, 3, \ldots, r-1 \pmod{r}, \\ n \le r^2 - 1, & \text{whenever } n \equiv 1 \pmod{r}. \end{cases}$$
$$\iff n \le r(r-1) + 1.$$

Now, by using Proposition 1(7), we get the required index.

Case 2 Let $n = r(r-1) + 2 + t$ for $0 \le t \le r-2$ so that $t = n - r(r-1) - 2$. Then, it can be seen that $n_r(v_i) = t + 1$, and Proposition 1(7) proposes the required index.

Case 3 For $n \ge r^2 + 1$, we have to find all the vertices in G lying at distance r from v_i. Now, any vertex belongs to $N_r(v_i)$ can be found by tracing the following geodesics each of length r that starts from v_i and ends at a vertex lying at distance r from v_i :

$$P_{1j} = v_i - v_{i+j} - v_{i+j+\frac{r}{2}} - v_{i+j+2(\frac{r}{2})} - \ldots - v_{i+j+(r-1)\frac{r}{2}},$$

$$P_{2j} = v_i - v_{i-j} - v_{i-j-\frac{r}{2}} - v_{i-j-2(\frac{r}{2})} - \ldots - v_{i-j-(r-1)\frac{r}{2}}.$$

Accordingly, $N_r(v_i) = \{v_{i+j+(r-1)\frac{r}{2}}, v_{i-j-(r-1)\frac{r}{2}}; 1 \le j \le r/2\}$ of cardinality r. Thus, $n_r(v_i) = r$, and Proposition 1(7) proposes the required index.

□

Theorem 10. *For odd $r \ge 3$ and even $n > r$, let $G \cong C_n(1, 2, 3, \ldots, (r-1)/2, n/2)$ be an r-regular generalized circulant network. Then,*

$$C_d(G) = \begin{cases} 0, & n \le 2r^2 - 5r + 5, \\ nr(t+2), & n = 2r^2 - 5r + 7 + t \text{ with } t \text{ even and } 0 \le t \le 2r-6, \\ 2nr(r-1), & n \ge 2r^2 - 3r + 3. \end{cases}$$

Proof. Let v_i be any vertex of G. Then, $N(v_i) = \{v_{i+j}, v_{i-j}, v_{i+n/2}; 1 \le j \le (r-1)/2\}$, and $ecc(v_i) = \lfloor (n + r - 5)/(2(r-1)) \rfloor + 1$. Now, we have the following three cases to discuss:

Case 1 By Proposition 1(2),

$$\begin{aligned} n_r(v_i) = 0 &\iff ecc(v) < d(v_i) = r, \\ &\iff \lfloor \frac{n + r - 5}{2(r-1)} \rfloor + 1 < r, \\ &\iff \lfloor \frac{n + r - 5}{2(r-1)} \rfloor < r - 1, \\ &\iff \frac{n + r - 5}{2(r-1)} < r - 1, \\ &\iff n < 2r^2 - 5r + 7, \\ &\iff n \le 2r^2 - 5r + 5, \text{ because } n \text{ is even.} \end{aligned}$$

Then, Proposition 1(7) yields the required index.

Case 2 Let $n = 2r^2 - 5r + 7 + t$ for even $0 \le t \le 2r - 6$ so that $t = n - (2r^2 - 5r + 7)$. Then, it can be seen that $n_r(v_i) = t + 2$, and according to Proposition 1(7), we get the required index.

Case 3 For $n \geq 2r^2 - 3r + 3$, we have to find all the vertices in G lying at distance r from v_i. Letting x be a vertex lying at distance r from v_i, x can be found by tracing following the geodesics which start from v_i and end at a vertex lying at distance r from v_i.

$$P_{1j} = v_i - v_{i+j} - v_{i+j+\frac{r-1}{2}} - v_{i+j+2(\frac{r-1}{2})} - \ldots - v_{i+j+(r-2)\frac{r-1}{2}},$$

$$P_{2j} = v_i - v_{i-j} - v_{i-j-\frac{r-1}{2}} - v_{i-j-2(\frac{r-1}{2})} - \ldots - v_{i-j-(r-2)\frac{r-1}{2}},$$

$$P_{3j} = v_i - v_{i+n/2} - v_{i+n/2+j} - v_{i+n/2+j+\frac{r-1}{2}} - v_{i+n/2+j+2(\frac{r-1}{2})} - \ldots - v_{i+n/2+j+(r-2)\frac{r-1}{2}},$$

$$P_{4j} = v_i - v_{i+n/2} - v_{i+n/2-j} - v_{i+n/2-j-\frac{r-1}{2}} - v_{i+n/2-j-2(\frac{r-1}{2})} - \ldots - v_{i+n/2-j-(r-2)\frac{r-1}{2}}.$$

The length of each geodesic in P_{1j}, P_{2j}, P_{3j}, and P_{4j} is r, which can easily be verified with help of the formula for finding the general term of an arithmetic sequence. Accordingly, $x \in X = \{v_{i+j+\frac{(r-1)^2}{2}}, v_{i-j-\frac{(r-1)^2}{2}}, v_{i+n/2+j+(r-2)\frac{r-1}{2}}, v_{i+n/2-j-(r-2)\frac{r-1}{2}}; 1 \leq j \leq (r-1)/2\}$, and $N_r(v_i) = X$ of cardinality $2(r-1)$. That is, $n_r(v_i) = 2(r-1)$ for each v_i in G, and Proposition 1(7) proposes the required index.

□

Although general expressions for $C_d(G)$ are obtained in Theorems 9 and 10, we do not derive a corresponding general formula for $\xi^{De}(G)$ for arbitrary r-regular circulant networks. This difference is mainly due to the nature of the two indices. The computation of $C_d(G)$ is governed by the d-neighborhood determined by the vertex degree, which admits a systematic characterization for the regular circulant families considered here. On the other hand, $\xi^{De}(G)$ involves the eccentricity $\mathrm{ecc}(v)$, which is determined by the global distance structure of the graph and is sensitive to the order, parity, and generating set of the circulant network. Consequently, obtaining a unified closed-form expression for the DECI would require additional case analysis of eccentricities and distance layers. We therefore restrict the present generalization to $C_d(G)$ and leave the derivation of a general DECI formula for broader classes of circulant networks as an interesting problem for future research.

3.5. Computational verification

For each family, we generated the circulant network, computed all distances from a vertex by breadth-first search (BFS), and evaluated $\xi^{De}(G) = \sum_{v \in V(G)} \deg^{De}(v)\,\mathrm{ecc}(v)$ and $C_d(G) = \sum_{v \in V(G)} n_d(v)\,d(v)$ directly from the definitions. Each of the Tables 2–6 shows the value from our closed-form theorem equals the computed value.

Table 2. Verification for the 3-regular family $C_n(1, n/2)$ (Theorems 2 and 3).

n	ξ^{De} (formula)	ξ^{De} (BFS)	C_d (formula)	C_d (BFS)	Match
6	24	24	0	0	✓
8	64	64	0	0	✓
10	60	60	60	60	✓
12	144	144	144	144	✓
14	112	112	168	168	✓
16	256	256	192	192	✓
18	180	180	216	216	✓
20	400	400	240	240	✓
22	264	264	264	264	✓
24	576	576	288	288	✓

Table 3. Verification for the 4-regular family $C_n(1, 2)$ (Theorems 5 and 6).

n	ξ^{De} (formula)	ξ^{De} (BFS)	C_d (formula)	C_d (BFS)	Match
5	20	20	0	0	✓
6	12	12	0	0	✓
7	28	28	0	0	✓
8	48	48	0	0	✓
9	72	72	0	0	✓
10	30	30	0	0	✓
11	66	66	0	0	✓
12	108	108	0	0	✓
13	156	156	0	0	✓
14	56	56	56	56	✓
15	120	120	120	120	✓
16	192	192	192	192	✓
17	272	272	272	272	✓
18	90	90	288	288	✓
19	190	190	304	304	✓
20	300	300	320	320	✓

Table 4. Verification for the 5-regular family $C_n(1, 2, n/2)$ (Theorems 7 and 8).

n	ξ^{De} (formula)	ξ^{De} (BFS)	C_d (formula)	C_d (BFS)	Match
8	32	32	0	0	✓
10	80	80	0	0	✓
12	144	144	0	0	✓
14	224	224	0	0	✓
16	96	96	0	0	✓
18	216	216	0	0	✓
20	360	360	0	0	✓
22	528	528	0	0	✓
24	192	192	0	0	✓
26	416	416	0	0	✓
28	672	672	0	0	✓
30	960	960	0	0	✓
32	320	320	320	320	✓
34	680	680	680	680	✓
36	1080	1080	1080	1080	✓

Table 5. Verification of Theorem 9 (even r, $G = C_n(1, \ldots, r/2)$).

r	n	C_d (formula)	C_d (BFS)	Match
4	5,6,7,8,9	0	0	✓
4	10,11,12,13	0	0	✓
4	14	56	56	✓
4	16	192	192	✓
4	17	272	272	✓
4	20	320	320	✓
6	30,31	0	0	✓
6	32	192	192	✓
6	33	396	396	✓
6	34	612	612	✓
6	35	840	840	✓
6	36	1080	1080	✓
6	37	1332	1332	✓
8	56,57	0	0	✓
8	58	464	464	✓
8	59	944	944	✓
8	60	1440	1440	✓
8	61	1952	1952	✓
8	62	2480	2480	✓

Table 6. Verification of Theorem 10 (odd r, $G = C_n(1, \ldots, (r-1)/2, n/2)$, n even).

r	n	C_d (formula)	C_d (BFS)	Match
3	6	0	0	✓
3	8	0	0	✓
3	10	60	60	✓
3	12	144	144	✓
3	14	168	168	✓
5	20,22	0	0	✓
5	24,26	0	0	✓
5	28,30	0	0	✓
5	32	320	320	✓
5	34	680	680	✓
5	36	1080	1080	✓
5	38	1520	1520	✓
5	40	1600	1600	✓
7	52,54	0	0	✓
7	56,58	0	0	✓
7	60,62	0	0	✓
7	64,66	0	0	✓
7	68	0	0	✓
7	70	980	980	✓
7	72	2016	2016	✓
7	74	3108	3108	✓
7	76	4256	4256	✓
7	78	5460	5460	✓
7	80	6720	6720	✓
7	82	6888	6888	✓

4. Applications of the DECI and DCI

In addition to their theoretical significance, connectivity indices have become indispensable tools in chemical graph theory and quantitative structure-property/activity relationship (QSPR/QSAR) studies. By encoding structural information into numerical descriptors, these indices facilitate the prediction of various physicochemical, thermodynamic, and biological properties of chemical compounds without the need for expensive laboratory experiments. The practical relevance of the d-connectivity index (DCI) has been explored in our previous studies on molecular and network structures [25]. In that work, the DCI was investigated for hexa-organic molecular structures, demonstrating its applicability as a structural descriptor in chemical graph theory. This chemical graph application motivates the consideration of the DCI for other structured network families. Later, it was studied for chain polygonal cactus networks, where the index was considered in connection with structural and network characteristics, as well as wireless communication [26]. In the present study, we derive closed-form

expressions that provide an explicit means of examining the behavior of the DECI and DCI as the order of a circulant network varies. Figure 5 summarizes these behaviors for the three principal circulant families considered in this work. In particular, the normalized quantities $\xi^{De}(G)/n^2$ and $C_d(G)/n$ highlight the dependence of the indices on the network order, parity, and underlying circulant structure.

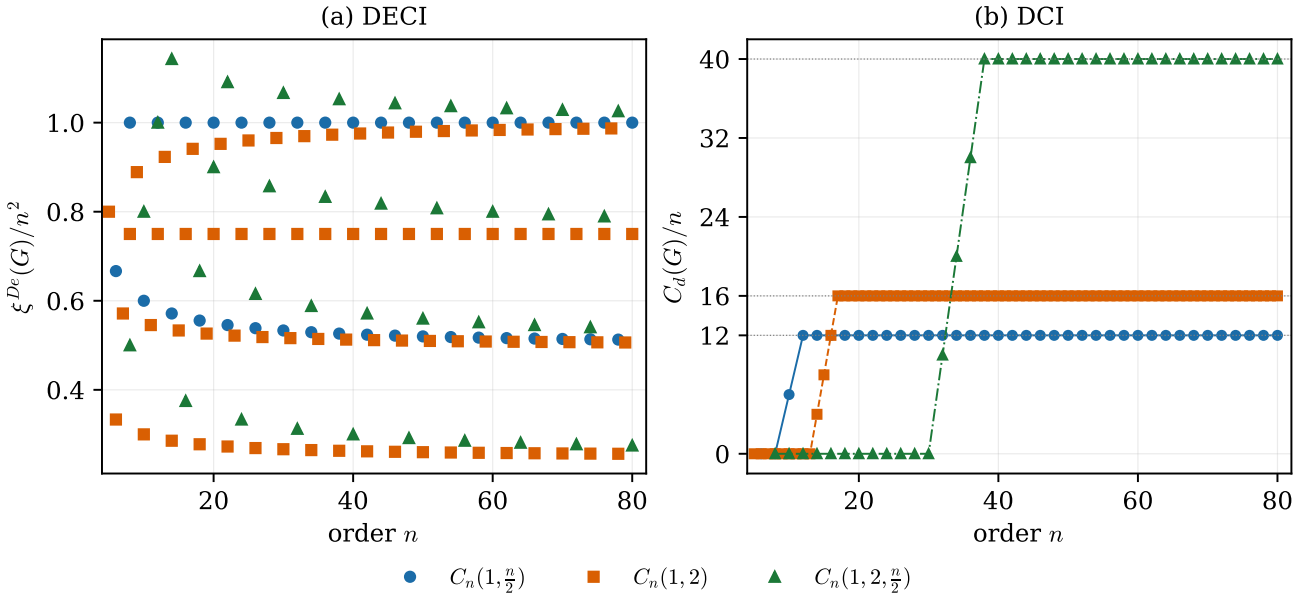


Figure 5. Behavior of the two indices on the three circulant networks, computed from Theorems 2–8. (a) $\xi^{De}(G)/n^2$: values group into levels determined by the residue class of n (mod 4 or mod 8). (b) $C_d(G)/n$: index vanishes below a family-dependent threshold and then stabilizes at 12, 16, and 40 for $C_n(1, n/2)$, $C_n(1, 2)$, and $C_n(1, 2, n/2)$, respectively.

Beyond chemical graph theory, distance and connectivity-based descriptors are also relevant to the structural analysis of communication and interconnection networks. Circulant graphs are particularly useful mathematical models in this context because of their regularity, vertex-transitivity, and highly structured distance properties. The results obtained here provide exact analytical expressions for theDECI and DCI on several important circulant families, thereby offering a direct means of evaluating and comparing their distance-based structural characteristics. In this sense, the present results extend the study of the DCI from molecular and cactus-like network structures to circulant networks and provide a foundation for further investigations of distance-based descriptors in structured communication networks.

5. Concluding remarks

In this work, we have undertaken a comparative study of two important degree-distance based topological indices, namely the distance eccentric connectivity index (DECI) and the d-connectivity index (DCI), within the framework of 3-, 4- and 5-regular circulant networks. By exploiting the inherent rotational symmetry of circulant graphs, we derived closed-form expressions of these indices and established their behavior for different network orders. For example, in $C_n(1, n/2)$, the DECI assumes values $n(n + 2)/2$ or n^2 depending on n (mod 4), whereas the DCI grows linearly with n.

Similar exact results are established for $C_n(1,2)$, and $C_n(1,2,n/2)$, and generalized results of the DCI were provided for r-regular circulant networks. All formulas were additionally confirmed by direct computation on explicit circulant graphs over wide ranges of n and r. These computations confirm that the DECI captures extremal eccentric distances, whereas the DCI reflects degree-distance uniformity, together providing a comprehensive description of circulant networks structure.

Collectively, these indices share a two-sided view: the DECI illuminates communication paths' worst-case scenarios, the DCI guarantees knowledge of the average-case resilience and structural stability. Together, their use goes beyond computer networks to chemical graph theory, where the DECI detects molecules with extremal interactions, and the DCI quantifies bonding robustness and stability. Hence, combining DECI and DCI analyses not only enhances structural knowledge about circulant networks but also enhances their applicability in designing optimized fault-tolerant and energy-efficient systems in computational and molecular fields.

Author contribution

Haoyu Kong: Validation, visualization, manuscript preparation and improvement, revision of the manuscript, preparation and refinement of figures and graphical illustrations; Muhammad Salman: Supervision and conceptualization; Tahira Noreen: Investigations, methodology; Writing-initial draft; Ziye Kong: final draft; Hanen Karamti: Formal analysis. All the authors have read and approved the final version of the article.

Use of Generative-AI tools declaration

During the preparation of this manuscript, the authors used ChatGPT solely for language editing and improving the clarity, grammar, and readability of the text. The authors carefully reviewed and revised all AI-assisted content and take full responsibility for the accuracy, originality, and integrity of the final manuscript.

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Conflict of interest

The authors stated that there were no competing interests in connection with the work reported in this article.

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