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Research article

Hyper-Wiener index of zero-divisor graphs of matrices: A computational study

Sahil Sharma[1], **Omaima Al Shanqiti**[2], **and Vijay Kumar Bhat**[1,*]

[1] School of Mathematics, Shri Mata Vaishno Devi University, Katra-182320, Jammu and Kashmir, India

[2] Department of Mathematics, Umm Al-Qura University, Makkah 24381, Saudi Arabia

* **Correspondence:** Email: vijaykumarbhat2000@yahoo.com.

Abstract: Topological indices, which are numerical values derived from the structure (topology) of a graph, were used to characterize its attributes. With applications in a variety of fields including network research, chemistry, and cryptography, the Hyper-Wiener index is a crucial topological descriptor for examining the structural characteristics of graphs. In this work, the Hyper-Wiener index for the zero-divisor graph of the ring of upper triangular matrices over a finite commutative ring and non-commutative ring was computed. In this study, we computed the Hyper-Wiener index of the directed and undirected zero-divisor graphs obtained from upper triangular matrix (M_3).

Keywords: upper triangular matrix; zero-divisor graphs; topological indices; Hyper-Wiener index
Mathematics Subject Classification: 05C10, 05C12, 05C40, 05C85, 05C92, 11C20

1. Introduction

The study of zero-divisor graphs has attracted a lot of research recently. The concept of linking the zero-divisors of a ring with a graph was first put forward by Beck [1]. Assume that two vertices x and y are adjacent if $x \cdot y = 0$ and that the vertices of G are the elements of R. The graph G is known as the zero-divisor graph of R. The first people to simplify Beck's zero-divisor graph were Anderson and Livingston [2]. Their purpose was to provide a clearer description of the ring's zero-divisor structure. When two distinct vertices a and b are adjacent to each other if, and only if, $ab = 0$, they defined the zero-divisor graph $\Gamma(R)$. This is known as the set of nonzero zero-divisors of R. In order to include non-commutative rings, Redmond expanded the concept of a zero-divisor graph [3]. For the non-commutative ring R, he demonstrated that $\Gamma(R)$ is connected if the set of left and right zero-divisors is equal. In 2002, DeMeyer et al. [4] expanded the concept of zero-divisor graphs of rings to semi-groups. Behboodi [5] then provides zero-divisor graphs for modules over commutative

rings.

In chemical graph theory, topological indices are crucial tools that are derived from the connectivity patterns of molecular graphs. Topological indices are based on the arrangement and connections of the elements in a molecular structure which can be described as a graph with atoms at the vertices and bonds at the edges. The degree of a vertex, which indicates the number of connections it has, is a crucial component in calculating these indices. While there are numerous varieties of topological indices, they can be generically classified into three groups: mixed, degree-based, and distance-based indices.

In distance-based indices, the distances between vertex pairs in a graph are considered. The Hyper-Wiener index is a crucial tool in quantitative structure-activity relationship (QSAR) research since it has been proven to relate well with a variety of molecular characteristics. Researchers can accurately describe and anticipate chemical responses due to its capacity to capture the complexities of molecular architecture. For example, it has been applied to chemical reactivity and logP (partition coefficient) prediction, demonstrating its applicability in cheminformatics. A crucial topological index in chemical graph theory, the Hyper-Wiener index was first proposed by Milan Randic in 1993 as an extension of the Wiener index (which was first described by Harry Wiener in 1947). This index is especially helpful in molecular modeling and biochemistry as a good descriptor of a molecule's structural characteristics. It is represented by the notation $WW(G)$.

The Hyper-Wiener index is calculated for a connected graph G by the formula:

$$WW(G) = \frac{1}{2} \sum_{u,v \in V(G)} \left(d(u,v) + (d(u,v))^2 \right),$$

where $d(u, v)$ represents the distance between vertices u and v within the graph. This distance function accounts for both the direct connections (or edges) and the squared distances providing a more complex view of molecular structure compared to the original Wiener index which only considers the sum of distances between vertex pairs.

According to Xing et al. [6], the n-vertex unicyclic graphs of cycle length r with the smallest and largest Hyper-Wiener indices for $3 \leq r \leq n$, and the n-vertex unicyclic graphs with the smallest, second smallest, largest, and second largest Hyper-Wiener indices for $n \geq 5$ were identified respectively. Yuan discovered the unique class of unicyclic graphs [7]. The maximal and least Hyper-Wiener index extremal bicyclic networks were reported by Feng et al., [8]. Several studies have examined the Wiener index and its applications in graph theory and chemical graph theory. Castro et al. [9] discussed modifications of the Wiener index, while Knor et al. [10] presented a survey of its mathematical properties. Gutman et al. [11] investigated the Hyper-Wiener index and its chemical applications, and Gutman and Furtula [12] compared the Wiener and Hyper-Wiener indices as molecular structure descriptors. Related studies have also considered topological indices of molecular graphs and extremal graph structures [13–15]. Graph theory also plays an important role in combinatorial design and information-security applications. El-Mesady, Hamed, and Abualnaja investigated mutually orthogonal graph squares and used them to construct graph-orthogonal arrays and authentication codes, demonstrating the application of graph-based combinatorial structures in secure communication and authentication systems [16]. In a related study, El-Mesady, Bazighifan, and Shabana introduced graph-transversal designs and graph-authentication codes based on mutually orthogonal graph squares and established constructions for several classes of graphs, including paths,

cycles, and disjoint unions of stars [17]. These studies further illustrate the role of graph structures in combinatorial design and cryptographic applications. Motivated by the broad applicability of graph-theoretic methods, the present work focuses on the structural and distance-based analysis of zero-divisor graphs associated with upper triangular matrices and investigates their Hyper–Wiener indices.

For $n = pq$ and $n = p^2$, Ahmadi and Jahani-Nezhad [18] investigated the Wiener index and energy of the graph of $\Gamma(\mathbb{Z}_n)$. Suthar [19] talked about the Wiener index and energy of the line graph of the $\Gamma(\mathbb{Z}_n)$ zero-divisor graph for $n = pq$ and $n = p^2$. Chattopadhyay et al. [20] investigated the Laplacian eigenvalues of $\mathbb{Z}_n$'s zero-divisor graphs. Singh and Bhat [21] discussed the wiener index of zero-divisor graph of $\mathbb{Z}_n$, i.e., $\Gamma(\mathbb{Z}_n)$. Readers can refer to [22–25] for more work.

The Hyper-Wiener index of the zero-divisors graph constructed from upper triangular matrices has several applications in areas like coding theory, optimization, network theory, and graph theory. It offers valuable insights into the relationships between zero-divisors within matrix rings helping to understand the connectivity of the graph. Therefore, it contributes to real-world uses including system modeling, matrix factorization, and cryptography.

Despite the growing interest in topological indices of zero-divisor graphs, the Hyper–Wiener index of zero-divisor graphs associated with upper triangular matrix rings has not been systematically studied, particularly in both directed and undirected settings. The present work addresses this gap by investigating the Hyper–Wiener index of the directed and undirected zero-divisor graphs of $M_3(\mathbb{Z}_2)$ and analyzing the influence of their underlying algebraic structures on the corresponding distance properties. The applications shown above motivated us to study the topological indices of zero-divisor graphs. In this study, we computed the Hyper-Wiener index of directed and undirected graphs derived from the zero divisor graph of upper triangular matrix (M_3) .

2. Materials and methods

In this section, we explore the fundamental concepts of the Hyper-Wiener index of zero-divisor graph obtained from the upper triangular matrix.

2.1. Definitions

Definition 1. *A zero-divisor graph is one in which the vertices represent the elements of the ring and pairs of zero-divisor elements are connected by edges. For example, $G = (V, E)$ is a zero-divisor graph, where V is the collection of nonzero zero-divisors such that vertices x_1 and x_2 are connected by an edge in V such that $x_1 \cdot x_2 = 0$. It is represented by $\Gamma(R)$. Throughout this manuscript, the term zero-divisor graph is abbreviated as ZDG.*

Definition 2. *An upper triangular matrix is a matrix in which all of the elements below the major diagonal of a square matrix are zero, i.e., $A = [a_{ij}]$, it is an upper triangular matrix if $a_{ij} = 0$ for all $i > j$, where i denotes row index and y denotes column index.*

Definition 3. *For a directed graph G with vertices $V = \{v_1, v_2, v_3, \ldots, v_n\}$, the Hyper-Wiener index is calculated by summing the shortest directed distances between all pairs of vertices, adding the square*

of those distances and then dividing by 2:

$$WW(G) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(d(v_i, v_j) + (d(v_i, v_j))^2\right),$$

where $d(v_i, v_j)$ is the directed distance from vertex v_i to vertex v_j.

Definition 4. *For an undirected graph G, the Hyper-Wiener index $WW(G)$ is defined as:*

$$WW(G) = \frac{1}{2}\sum_{1\le x<y\le n}\left(d(x, y) + (d(x, y))^2\right),$$

where $d(x, y)$ is the shortest path distance between the vertices x and y.

2.2. Construction of ZDG of an upper triangular matrix

The zero-divisor graphs of the upper triangular matrix $M_2(R)$ and $M_3(R)$ will be defined in this section. First, we will examine the upper triangular matrix $M_2(R)$'s zero-divisor graphs. For the general form of the upper triangular matrices in M_2, the space of 2×2 matrices is as follows:

$$A = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$$

where x, y, and z are elements from the field (for example, real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, or any field). For upper triangular matrices in M_2, the possibilities are infinite. In such a case, x, y, and z can each independently choose any value from the field.

However, we may calculate the finite number of possible upper triangular matrices in M_2 if we need to determine the number of matrices that can be constructed given a specific range of values, such as integer modulo n. For example: Every element can separately take two values if we limit x, y, and z to being in the set $\{0, 1\}$ (i.e., modulo 2). Therefore, possible upper triangular matrices in this situation would be:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and we gave the corresponding names to these upper triangular matrices $X_1, X_2, X_3, X_4, X_5, X_6$, and X_7. The upper triangular matrix of $M_2(Z_2)$ generates a ring that is not commutative. The zero-divisor graph of the top triangular matrix of $M_2(Z_2)$ is thus shown in the Figure 1.

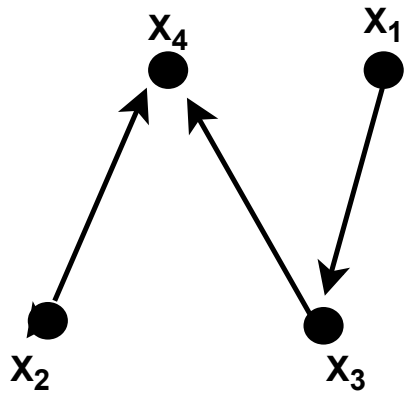


Figure 1. Zero-divisor graph $M_2(Z_2)$.

The zero-divisor graphs of the upper triangular matrix $M_3(R)$ will also be examined. For the general form of the upper triangular matrices in M_3, the space of 3×3 matrices is as follows:

$$A = \begin{pmatrix} p & q & r \\ 0 & s & t \\ 0 & 0 & u \end{pmatrix}$$

where p, q, r, s, t, and u are field elements such as complex numbers $\mathbb{C}$, real numbers $\mathbb{R}$, or any field. For upper triangular matrices in M_3, the possibilities are infinite. In such a case, p, q, r, s, t, and u can each independently choose any value from the field. However, we may calculate the finite number of possible upper triangular matrices in M_3 if we need to determine the number of matrices that can be constructed given a specific range of values, such as integer modulo n. For example, every element can separately take two values if we limit p, q, r, s, t, and u to being in the set $\{0, 1\}$ (i.e., modulo 2). Therefore, possible upper triangular matrices in this situation would be the set of all nonzero 3x3 upper triangular matrices over Z_2

$$\mathcal{U} = \left\{ \begin{pmatrix} p & q & r \\ 0 & s & t \\ 0 & 0 & u \end{pmatrix} \; : \; (p, q, r, s, t, u) \in \{0, 1\}^6 \setminus \{(0, 0, 0, 0, 0, 0)\} \right\}.$$

Vertex set and labeling: We label the vertices $X_1, \ldots, X_{63}$ via the fixed sequence

$$X_k = \begin{pmatrix} p_k & q_k & r_k \\ 0 & s_k & t_k \\ 0 & 0 & u_k \end{pmatrix} \qquad (k = 1, \ldots, 63),$$

where each $\begin{pmatrix} p_k & q_k & r_k \\ 0 & s_k & t_k \\ 0 & 0 & u_k \end{pmatrix}$ is upper triangular $\in \{0, 1\}^6$ and is the k-th tuple, e.g.,

$$X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, X_{63} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

2.3. Directed ZDG of an upper triangular matrix ZDG$[M_3(\mathbb{Z}_2)]$

The complete directed graph, denoted as A_D (Figure 2), contains too many edges to be clearly represented in a single figure. To avoid confusion and misinterpretation, the graph is separated into two distinct sections:

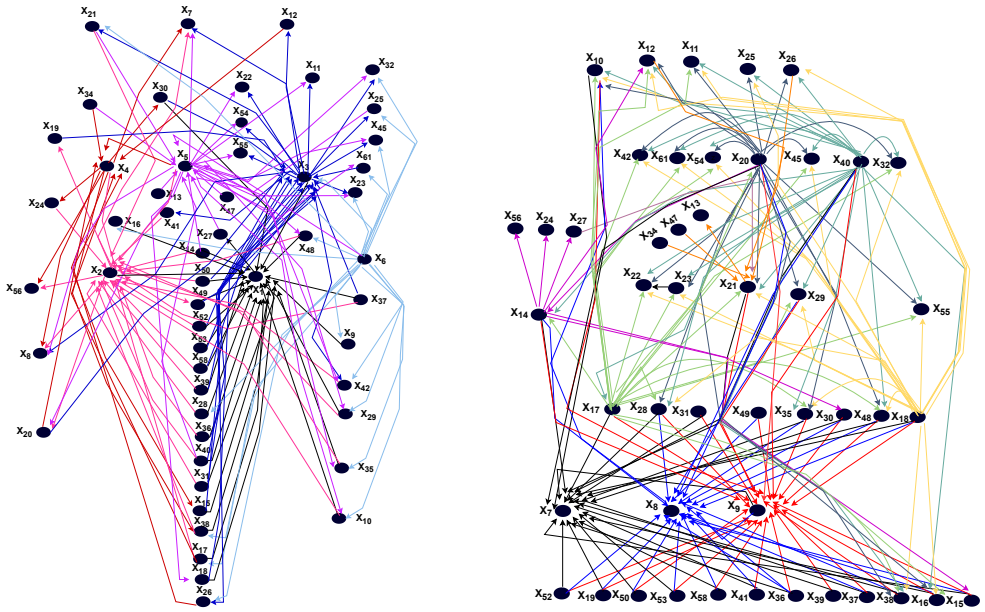
- D_a: A graph derived from a subset of vertices, specifically X_1 through X_6, and their associated edges.
- D_b: A graph derived from another subset of vertices, specifically X_7 through X_{63}, and their associated edges.

This separation into two figures, along with the use of different colors, is a clarity-driven design choice intended to make the complex relationships between the vertices easier for readers to understand. This division is made only to improve the readability of the graph and does not represent any particular algebraic decomposition of the ring.

Role of the Graphs

Both D_a and D_b serve the same core purpose: to visually represent the zero-divisor relationships within a specific set of upper triangular matrices. Specifically:

- The vertices in the graphs correspond to certain upper triangular matrices in the set $M_3(\mathbb{Z}_2)$.
- A directed edge $(X_i \to X_j)$ signifies that the product $X_i \cdot X_j = 0$, indicating that X_i and X_j are zero divisors.
- The absence of a reverse arrow $(X_j \to X_i)$ implies that the product is $X_j \cdot X_i \neq 0$. This highlights the non-commutative nature of matrix multiplication within the graph's relationships.



(a) D_a **(b)** D_b

Figure 2. Directed zero-divisor graph of the upper triangular matrix ring $(M_3(\mathbb{Z}_2))$, where an arc from (X_i) to (X_j) exists whenever $(X_i \cdot X_j = 0)$.

2.4. Undirected ZDG of an upper triangular matrix ZDG$[M_3(\mathbb{Z}_2)]$

We'll explore the undirected graph A_{un} (Figure 3), which is derived from the vertices $X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9, , X_{10}, X_{11}, X_{12}, X_{13}, X_{14}, X_{15}, X_{16}, X_{17}, X_{18}, X_{19}, X_{20}, X_{21}, X_{28}, X_{31}, X_{40}$, and X_{48}. The possible upper triangular matrix in $M_3(\mathbb{Z}_2)$ that is described above is $X_1, X_2, ...X_{63}$, along with the edges that are related to it. To ensure that there is no misunderstanding regarding the figures, we have represented it with a variety of colors.

This figure shows that their vertices have a commutative relationship. In other words, if there is an edge between X_1 and X_2, then $X_1 \cdot X_2 = 0$ and $X_2 \cdot X_1 = 0$. Therefore, we include only those elements that form such a relation with another element. Elements that do not form any such relation are excluded, as they do not contribute to the calculation of the Hyper–Wiener index. A zero divisor graph is created when these zero-divisor graphs are joined. As upper triangular matrix X_{22}, X_{23}, X_{24}, X_{25}, X_{26}, X_{27}, X_{29}, X_{30}, X_{32}, X_{33}, X_{34}, X_{36}, X_{37}, X_{39}, X_{41}, X_{42}, X_{43}, X_{44}, X_{45}, X_{46}, X_{47}, X_{49}, X_{50}, X_{51},

X_{52}, X_{53}, X_{54}, X_{55}, X_{56}, X_{57}, X_{58}, X_{59}, X_{60}, X_{61} X_{62}, X_{63} doesn't make any commutative zero-divisors, we remove these vertices from the graph. The given figure will be represented by A_{un}. Therefore, the zero-divisor graph of upper triangular matrix of $M_3(\mathbb{Z}_2)$ which is undirected is given below in Figure 3:

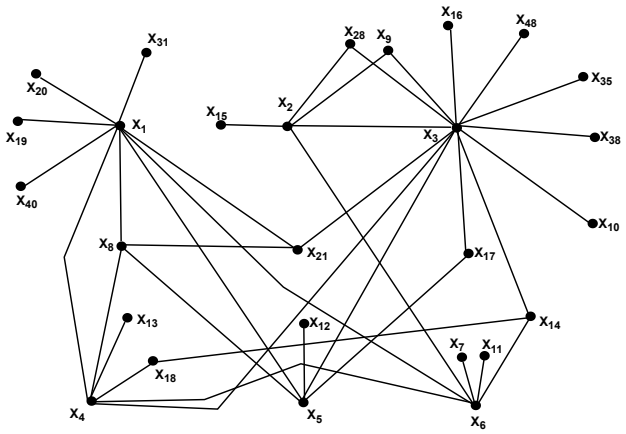


Figure 3. Undirected zero-divisor graph of the upper triangular matrix ring ($M_3(\mathbb{Z}_2)$), where two vertices (X_i) and (X_j) are adjacent whenever ($X_i \cdot X_j = X_j \cdot X_i = 0$).

3. Results

In this section, we study the Hyper-Wiener indices of the directed and undirected zero-divisor graphs associated with $M_3(\mathbb{Z}_2)$. By calculating the distances between vertices and applying the definition of the Hyper-Wiener index, we obtain the corresponding indices for both graphs.

Theorem 1. *Assume A_D (Figure 2) is the zero-divisor graph of the directed upper triangular matrix $M_3(\mathbb{Z}_2)$. Then, Hyper-Wiener index of the directed graph derived from the zero-divisor graph of the upper triangular matrix $M_3(\mathbb{Z}_2)$ is equal to 11258.*

Proof. Let $(X_1, X_2, \ldots, X_{63})$ be vertices in a graph. Each entry $d(x, y)$ denotes the distance between vertices v_x and v_y, where x and y range from 1 to 63. The Hyper-Wiener index is calculated by summing the shortest directed distances between all pairs of vertices, adding the square of those distances, and then dividing by 2:

$$WW(A_D) = \frac{1}{2} \sum_{x=1}^{n} \sum_{y=1}^{n} \Big(d(v_x, v_y) + d^2(v_x, v_y) \Big),$$

where $d(v_x, v_y)$ is the directed distance from vertex v_x to vertex v_y.

Here, $WW(A_D)$ represents the Hyper-Wiener index of the graph A_D.

To compute the Hyper-Wiener index:

(1) When we construct the table, the shortest path distance between vertices v_x and v_y is denoted by $d(x, y)$.
(2) The shortest directed distances between every pair of vertices are added up, their squares are added, and the result is divided by two to calculate the Hyper-Wiener index $WW(G)$.

(3) If no path exists between two vertices (v_x) and (v_y), we take $(d(v_x, v_y) = 0)$ for the purpose of the present computation. Consequently, such vertex pairs make zero contribution to the Hyper-Wiener index.

The distances between every pair of vertices are shown in Tables 1–5 given below.

Tables 1–5 represent the shortest path distances between all pairs of vertices in a directed graph. Each entry $d(v_x, v_y)$ indicates the length of the shortest directed path from vertex v_x to vertex v_y, where 0 denotes that no such path exists.

Therefore, the Hyper-Wiener index of the graph $WW(A_D)$ is

$$\begin{aligned} WW(A_D) &= \frac{1}{2}\left[\sum_{y=1}^{63}\left(d(1,y)+(d(1,y))^2\right)+\sum_{y=1}^{63}\left(d(2,y)+(d(2,y))^2\right)\right. \\ &\quad \left.+\cdots+\sum_{y=1}^{63}\left(d(63,y)+(d(63,y))^2\right)\right] \\ &= 11258. \end{aligned}$$

Hence, the Hyper-Wiener index of the directed zero-divisor graph derived from the upper triangular matrix ring $M_3(\mathbb{Z}_2)$ is given by 11258. □

Table 1. Shortest path distance between vertices v_x and v_y.

Possible pairwise Distances	Shortest path distance between vertices
$d(1, v_y),\ 1 \le y \le 63$	1 for $y = 27.$; 2 for $y = 21.$; 3 for $y = 2, 9, 13.$; 4 for $y = 5.$ 5 for $y = 4, 7, 8, 10, 11, 18, 20, 22, 23, 29, 35, 42, 45, 48, 54, 55, 61.$ 6 for $y = 3, 12, 14, 15, 16, 25, 26, 28, 32, 38.$ 7 for $y = 19, 24, 30, 49, 56$ and 0 for all other y.
$d(2, v_y),\ 1 \le y \le 63$	1 for $y = 1, 7, 8, 19, 20, 49.$ 2 for $y = 3, 4, 9, 10, 11, 12, 14, 16, 27, 28, 29, 32, 35, 38, 42, 45, 48, 54, 61.$ 3 for $y = 13, 15, 21, 22, 23, 24, 25, 26, 30, 55, 56.$ 4 for $y = 5.$; 5 for $y = 18.$ 0 for all other y.
$d(3, v_y),\ 1 \le y \le 63$	1 for $y = 1, 7, 8, 12, 21, 22, 23, 25, 26, 32, 42, 45, 54, 55$; 2 for $y = 4, 5, 13, 27.$ 3 for $y = 2, 9, 10, 11, 15, 18, 19, 20, 24, 35, 38, 48, 49, 56, 61.$ 4 for $y = 14, 16, 28, 29.$; 5 for $y = 30.$ 0 for all other y.

Table 2. Shortest path distance between vertices v_x and v_y

Possible pairwise Distances	Shortest path distance between vertices
$d(4, v_y),\ 1 \leq y \leq 63$	1 for $y = 7, 8, 15, 24, 30, 38, 56.$; 2 for $y = 1, 2, 3, 9.$ 3 for $y = 19, 20, 21, 22, 23, 25, 26, 27, 32, 42, 45, 49.$ 4 for $y = 5, 10, 13, 14, 16, 28, 29, 35, 48, 55, 61.$ 5 for $y = 11, 12, 18.$ and 0 for all other y.
$d(5, v_y),\ 1 \leq y \leq 63$	1 for $y = 2, 4, 7, 10, 11, 18, 20, 22, 23, 29, 35, 42, 45, 48, 54, 55, 61.$ 2 for $y = 1, 3, 9, 12, 14, 15, 16, 21, 25, 26, 28, 32, 38.$ 3 for $y = 19, 24, 27, 30, 49, 56.$; 6 for $y = 13.$ and 0 for all other y.
$d(6, v_y),\ 1 \leq y \leq 63$	1 for $y = 3, 5, 9, 10, 16, 21, 23, 25, 26, 28, 29, 32, 35, 38, 42, 45, 48, 61.$ 2 for $y = 1, 2, 4, 7, 8, 11, 12, 18, 20, 22, 54, 55.$; 3 for $y = 14.$ 4 for $y = 13, 15, 19, 24, 27, 30, 49, 56.$ and 0 for all other y.
$d(7, v_y),\ 1 \leq y \leq 63$	0 for $1 \leq y \leq 63$.
$d(8, v_y),\ 1 \leq y \leq 63$	1 for $y = 10.$; 2 for $y = 1, 2, 7, 9.$; 3 for $y = 19, 20, 27, 49.$ 4 for $y = 3, 4, 11, 12, 14, 16, 28, 29, 32, 35, 38, 42, 45, 48, 54, 61.$ 6 for $y = 5.$; 5 for $y = 13, 15, 21, 22, 23, 24, 25, 26, 30, 55, 56.$ 7 for $y = 18.$ and 0 for all other y.
$d(9, v_y),\ 1 \leq y \leq 63$	1 for $y = 7, 8.$; 2 for $y = 10.$; 3 for $y = 1, 2.$; 4 for $y = 19, 20, 27, 49.$ 5 for $y = 3, 4, 11, 12, 14, 16, 28, 29, 32, 35, 38, 42, 45, 48, 54, 61.$ 6 for $y = 13, 15, 21, 22, 23, 24, 25, 26, 30, 55, 56.$ 7 for $y = 5,.$; 8 for $y = 18.$ and 0 for all other y
$d(10, v_y),\ 1 \leq y \leq 63$	1 for $y = 1, 2, 7, 9.$; 2 for $y = 8, 19, 20, 27, 49.$ 3 for $y = 3, 4, 11, 12, 14, 16, 28, 29, 32, 35, 42, 45, 48, 54, 61.$ 4 for $y = 13, 15, 21, 22, 23, 24, 25, 26, 30, 55, 56.$ 5 for $y = 5.$; 6 for $y = 18.$ and 0 for all other y.
$d(11, v_y),\ 1 \leq y \leq 63$	0 for $1 \leq y \leq 63$.
$d(12, v_y),\ 1 \leq y \leq 63$	2 for $y = 9.$; 3 for $y = 7, 8.$; 4 for $y = 10.$; 5 for $y = 1, 2.$ 6 for $y = 19, 20, 27, 49.$ 7 for $3, 4, 11, 14, 16, 28, 29, 32, 35, 38, 42, 45, 48, 54, 61.$ 8 for $y = 13, 15, 21, 22, 23, 24, 25, 26, 30, 55, 56.$ 9 for $y = 5.$; 10 for $y = 18.$ and 0 for all other y.
$d(13, v_y),\ 1 \leq y \leq 63$	0 for $1 \leq y \leq 63$.
$d(14, v_y),\ 1 \leq y \leq 63$	2 for $y = 2.$ 3 for $y = 1, 3, 7, 8, 12, 19, 20, 21, 22, 23, 25, 26, 32, 42, 45, 49, 54, 55.$ 4 for $y = 4, 8, 10, 11, 13, 16, 27, 28, 29, 35, 38, 48, 61.$ 5 for $y = 15, 18, 24, 30, 56.$ and 0 for all other y.
$d(15, v_y),\ 1 \leq y \leq 63$	1 for $y = 8, 9.$; 2 for $y = 7, 10.$; 3 for $y = 1, 2.$; 4 for $y = 19, 20, 27, 49.$ 5 for $y = 3, 4, 11, 12, 14, 16, 28, 29, 32, 35, 38, 42, 45, 48, 54, 61.$ 6 for $y = 13, 21, 22, 23, 24, 25, 26, 30, 55, 56.$ 7 for $y = 5.$ and 0 for all other y.

Table 3. Shortest path distance between vertices v_x and v_y.

Possible pairwise Distances	Shortest path distance between vertices
$d(16, v_y),\ 1 \le y \le 63$	1 for $y = 8, 9.$; 2 for $y = 7, 10.$; 3 for $y = 1, 2.$ 4 for $y = 15, 19, 20, 27, 49.$ 5 for $y = 3, 4, 11, 12, 14, 28, 29, 32, 35, 38, 42, 45, 48, 54, 61.$ 6 for $y = 13, 21, 22, 23, 24, 25, 26, 30, 55, 56.$ 7 for $y = 5.$ and 0 for all other y.
$d(17, v_y),\ 1 \le y \le 63$	1 for $y = 2, 4, 7, 8, 9, 10, 11, 12, 14, 16, 20, 21, 22, 23, 28, 29,$ $32, 35, 38, 42, 45, 48, 54, 55, 61.$; 2 for $y = 1, 3, 13, 15, 24, 25, 26, 27, 30.$ 3 for $y = 18, 19, 49, 56.$; 4 for $y = 5.$ and 0 for all other y.
$d(18, v_y),\ 1 \le y \le 63$	1 for $y = 1, 3, 7, 8, 9, 10, 11, 12, 16, 21, 22, 23, 25, 26, 28, 29, 32$,$35, 38, 42, 45, 48, 54, 55, 61.$; 2 for $y = 2, 4, 5, 13.$; 3 for $y = 20.$ 5 for $y = 14.$; 6 for $y = 15, 24, 27, 30.$ and 0 for all other y.
$d(19, v_y),\ 1 \le y \le 63$	1 for $y = 7, 8, 9.$; 2 for $y = 10.$; 3 for $y = 1, 2.$; 4 for $y = 19, 20, 27, 49.$ 5 for $y = 3, 4, 11, 12, 14, 16, 28, 29, 32, 35, 38, 42, 45, 48, 54, 61.$ 6 for $y = 13, 21, 22, 23, 24, 25, 26, 30, 55, 56.$ 7 for $y = 5.$ and 0 for all other y.
$d(20, v_y),\ 1 \le y \le 63$	1 for $y = 3, 4, 7, 8, 9, 10, 11, 12, 14, 16, 21, 22, 23, 25, 26, 28, 29, 32, 35, 38$, $42, 45, 54, 55, 61.$; 2 for $y = 2, 13, 15, 24, 27, 30, 48.$ 3 for $y = 1, 5, 19.$; 4 for $y = 18.$ and 0 for all other y.
$d(21, v_y),\ 1 \le y \le 63$	1 for $y = 2, 5, 9, 13.$ 2 for $y = 4, 7, 8, 10, 11, 18, 19, 20, 22, 23, 29, 35, 42, 45, 48, 54$,$55, 61.$ 3 for $y = 1, 3, 12, 26, 28, 32, 38.$ 4 for $y = 14, 15, 16, 24, 27, 30, 49, 56.$ and 0 for all other y.
$d(v_x, v_y),\ 1 \le y \le 63$	0 for $1 \le y \le 63$, where $x \in \{22, 23, 24\}$.
$d(25, v_y),\ 1 \le y \le 63$	1 for $y = 5, 7.$ 2 for $y = 2, 4, 10, 11, 18, 20, 22, 23, 29, 35, 42, 45, 48, 54, 55, 61.$ 3 for $y = 1, 3, 9, 12, 14, 15, 16, 21, 25, 26, 28, 32, 38.$ 4 for $y = 19, 24, 27, 30, 49, 56.$; 7 for $y = 13.$ and 0 for all other y.
$d(26, v_y),\ 1 \le y \le 63$	1 for $y = 4, 15, 21.$; 2 for $y = 2, 5, 7, 8, 9, 13, 24, 30, 38, 56.$ 3 for $y = 1, 3, 11, 18, 19, 20, 22, 23, 29, 35, 42, 45, 48, 54, 55, 61.$ 4 for $y = 12, 25, 27, 28, 32, 49.$; 5 for $y = 14, 16.$ and 0 for all other y.
$d(27, v_y),\ 1 \le y \le 63$	1 for $y = 21$; 2 for $y = 2, 5, 9, 13.$; 3 for $y = 4, 7, 8, 10, 11, 18, 19, 20$, $22, 23, 29, 35, 42, 45, 48, 54, 55, 61.$; 4 for $y = 1, 3, 12, 26, 28, 32, 38.$ 5 for $y = 14, 15, 16, 24, 30, 49, 56.$ and 0 for all other y.
$d(28, v_y),\ 1 \le y \le 63$	1 for $y = 1, 2, 7, 8, 9.$; 2 for $y = 10, 19, 20, 27, 49.$ 3 for $y = 3, 4, 11, 12, 14, 16, 28, 29, 32, 35, 42, 45, 48, 54, 61.$ 4 for $y = 13, 15, 21, 22, 23, 24, 25, 26, 30, 55, 56.$ 5 for $y = 5.$; 6 for $y = 18.$ and 0 for all other y.

Table 4. Shortest path distance between vertices v_x and v_y.

Possible pairwise Distances	Shortest path distance between vertices
$d(29, v_y),\ 1 \le y \le 63$	1 for $y = 1, 2, 8, 9.$; 2 for $y = 7, 10, 19, 20, 27, 49.$ 3 for $y = 3, 4, 11, 12, 14, 16, 28, 29, 32, 35, 42, 45, 48, 54, 61.$ 4 for $y = 13, 15, 21, 22, 23, 24, 25, 26, 30, 55, 56.$ 5 for $y = 5.$; 6 for $y = 18.$ and 0 for all other y.
$d(30, v_y),\ 1 \le y \le 63$	1 for $y = 1, 3, 8, 9.$; 2 for $y = 7, 12, 21, 22, 23, 25, 26, 32, 42, 45, 54, 55.$ 3 for $y = 4, 5, 13, 27.$ 4 for $y = 2, 10, 11, 15, 18, 19, 20, 24, 35, 38, 48, 49, 56, 61.$ 5 for $y = 14, 16.$ and 0 for all other y.
$d(v_x, v_y),\ 1 \le y \le 63$	0 for $1 \le y \le 63$, where $x \in \{32, 33\}$.
$d(34, v_y),\ 1 \le y \le 63$	4 for $y = 27, 49.$; 1 for $y = 4, 5, 21.$ 3 for $y = 1, 3, 9, 12, 14, 16, 19, 25, 26, 28, 32, 35.$ 2 for $y = 2, 7, 8, 10, 11, 13, 15, 18, 20, 22, 23, 24, 29, 30, 38, 42$ $, 45, 48, 54, 55, 56, 61.$; 0 for all other y.
$d(35, v_y),\ 1 \le y \le 63$	1 for $y = 1, 2, 7, 8, 9.$; 2 for $y = 10, 19, 20, 27, 49.$ 3 for $y = 3, 4, 11, 12, 14, 16, 28, 29, 32, 42, 45, 48, 54, 61.$ 4 for $y = 13, 15, 21, 22, 23, 24, 25, 26, 30, 55, 56.$ 5 for $y = 5.$; 6 for $y = 18.$ and 0 for all other y.
$d(36, v_y),\ 1 \le y \le 63$	1 for $y = 1, 2, 3, 7, 8, 9.$; 2 for $y = 10, 12, 19, 2, 21, 22, 23, 25, 26, 27, 32, 42, 45$ $, 49, 54, 55.$; 3 for $y = 4, 5, 11, 13, 14, 16, 28, 29, 35, 38, 48, 61.$ 4 for $y = 15, 18, 24, 30, 56.$ and 0 for all other y.
$d(37, v_y),\ 1 \le y \le 63$	1 for $y = 1, 2, 3, 7, 8, 9.$; 2 for $y = 10, 12, 19, 2, 21, 22, 23, 25, 26, 27, 32, 42$ $, 45, 49, 54, 55.$ 3 for $y = 4, 5, 11, 13, 14, 16, 28, 29, 35, 38, 48, 61.$ 4 for $y = 15, 18, 24, 30, 56.$ and 0 for all other y.
$d(38, v_y),\ 1 \le y \le 63$	1 for $y = 1, 2, 8, 9.$; 2 for $y = 7, 10, 19, 20, 27, 49.$ 3 for $y = 3, 4, 11, 12, 14, 16, 28, 29, 32, 42, 45, 48, 54, 61.$ 4 for $y = 13, 15, 21, 22, 23, 24, 25, 26, 30, 55, 56.$ 5 for $y = 5.$; 6 for $y = 18.$ and 0 for all other y.
$d(39, v_y),\ 1 \le y \le 63$	1 for $y = 1, 2, 3, 7, 8, 9.$; 2 for $y = 10, 12, 19, 2, 21, 22, 23, 25, 26, 27, 32, 42$ $, 45, 49, 54, 55.$; 3 for $y = 4, 5, 11, 13, 14, 16, 28, 29, 35, 38, 48, 61.$ 4 for $y = 15, 18, 24, 30, 56.$ and 0 for all other y.
$d(40, v_y),\ 1 \le y \le 63$	1 for $y = 1, 2, 3, 5, 7.8, 10, 11, 12, 14, 16, 17, 21, 22, 23, 25, 26, 28, 29, 32, 35, 38$ $, 42, 45, 48, 54, 55, 61.$ 2 for $y = 4, 9, 13, 15, 18, 19, 20, 24, 27, 30, 49, 56.$ and 0 for all other y.
$d(41, v_y),\ 1 \le y \le 63$	1 for $y = 2, 3, 7, 8, 9.$ 2 for $y = 1, 10, 12, 19, 20, 21, 22, 23, 25, 26, 32, 42, 45, 49, 54, 55.$ 3 for $y = 4, 5, 11, 13, 14, 16, 27, 28, 29, 35, 38, 48, 61.$ 4 for $y = 15, 18, 24, 30, 56.$ and 0 for all other y.

Table 5. Shortest path distance between vertices v_x and v_y.

Possible pairwise Distances	Shortest path distance between vertices
$d(42, v_y),\ 1 \leq y \leq 63$	1 for $y = 2.$; 2 for $y = 1, 7, 8, 19, 20, 49.$ 3 for $y = 3, 4, 9, 10, 11, 12, 14, 16, 27, 28, 29, 32, 35, 38, 45, 48, 54, 61.$ 4 for $y = 13, 15, 21, 22, 23, 24, 25, 26, 30, 55, 56.$ 5 for $y = 5.$; 6 for $y = 18.$ and 0 for all other y.
$d(v_x, v_y),\ 1 \leq y \leq 63$	0 for $1 \leq y \leq 63$, where $x \in \{43, 44, 45, 46\}$.
$d(47, v_y),\ 1 \leq y \leq 63$	1 for $y = 2, 5, 21.$ 2 for $y = 1, 4, 7, 8, 9, 10, 11, 13, 18, 19, 20, 22, 23, 29, 35, 42, 45$ $, 48, 49, 54, 55, 61.$ 3 for $y = 3, 12, 14, 15, 16, 25, 26, 27, 28, 32, 38.$ 4 for $y = 24, 30, 56.$ and 0 for all other y.
$d(48, v_y),\ 1 \leq y \leq 63$	1 for $y = 1, 2, 7, 8, 9.$; 2 for $y = 10, 19, 20, 27, 49.$ 3 for $y = 3, 4, 11, 12, 14, 16, 28, 29, 32, 42, 45, 54, 61.$ 4 for $y = 13, 15, 21, 22, 23, 24, 25, 26, 30, 55, 56.$ 5 for $y = 5.$; 6 for $y = 18.$ and 0 for all other y.
$d(49, v_y),\ 1 \leq y \leq 63$	1 for $y = 3, 7, 8, 9.$; 2 for $y = 1, 10, 12, 21, 22, 23, 25, 26, 32, 42, 45, 54, 55.$ 3 for $y = 2, 4, 5, 13, 27.$; 4 for $y = 11, 15, 18, 19, 20, 24, 35, 38, 48, 61.$ 5 for $y = 14, 16, 28, 29.$; 6 for $y = 30$ and 0 for all other y.
$d(50, v_y),\ 1 \leq y \leq 63$	1 for $y = 3, 7, 8, 9.$; 2 for $y = 1, 10, 12, 21, 22, 23, 25, 26, 32, 42, 45, 54, 55.$ 3 for $y = 2, 4, 5, 13, 27.$; 4 for $y = 11, 15, 18, 19, 20, 24, 35, 38, 48, 61.$ 5 for $y = 14, 16, 28, 29.$; 6 for $y = 30.$ and 0 for all other y.
$d(51, v_y),\ 1 \leq y \leq 63$	0 for $1 \leq y \leq 63$.
$d(52, v_y),\ 1 \leq y \leq 63$	1 for $y = 2, 3, 7, 8, 9.$ 2 for $y = 1, 10, 12, 19, 20, 21, 22, 23, 25, 26, 32, 42, 45, 49, 54, 55.$ 3 for $y = 4, 5, 11, 13, 14, 16, 27, 28, 29, 35, 38, 48, 61.$ 4 for $y = 15, 18, 24, 30, 56.$ and 0 for all other y.
$d(53, v_y),\ 1 \leq y \leq 63$	1 for $y = 1, 2, 3, 7, 8, 9.$; 3 for $y = 4, 5, 11, 13, 14, 16, 28, 29, 35, 38, 48, 61.$ 2 for $y = 10, 12, 19, 2, 21, 22, 23, 25, 26, 27, 32, 42, 45, 49, 54, 55.$ 4 for $y = 15, 18, 24, 30, 56.$ and 0 for all other y.
$d(v_x, v_y),\ 1 \leq y \leq 63$	0 for $1 \leq y \leq 63$, where $x \in \{54, 55, 56, 57\}$.
$d(58, v_y),\ 1 \leq y \leq 63$	1 for $y = 1, 2, 3, 7, 8, 9.$ 2 for $y = 10, 12, 19, 2, 21, 22, 23, 25, 26, 27, 32, 42, 45, 49, 54, 55.$ 3 for $y = 4, 5, 11, 13, 14, 16, 28, 29, 35, 38, 48, 61.$ 4 for $y = 15, 18, 24, 30, 56.$ and 0 for all other y.
$d(v_x, v_y),\ 1 \leq y \leq 63$	0 for $1 \leq y \leq 63$, where $x \in \{59, 60\}$.
$d(61, v_y),\ 1 \leq y \leq 63$	1 for $y = 3.$ 2 for $y = 1, 7, 8, 12, 21, 22, 23, 25, 26, 32, 42, 45, 54, 55$ 3 for $y = 4, 5, 13, 27.$ 4 for $y = 2, 9, 10, 11, 15, 18, 19, 20, 24, 35, 38, 48, 49, 56.$ 5 for $y = 14, 16, 28, 29.$ 6 for $y = 30.$ and 0 for all other y.
$d(v_x, v_y),\ 1 \leq y \leq 63$	0 for $1 \leq y \leq 63$, where $x \in \{62, 63\}$.

Remark 1. *Let $A_D(M_n(\mathbb{Z}_m))$ denote the directed zero-divisor graph of the upper-triangular matrix ring $M_n(\mathbb{Z}_m)$, and let $WW(A_D(M_n(\mathbb{Z}_m)))$ be its Hyper–Wiener index. The computed value for the smallest nontrivial case, $WW(A_D(M_3(\mathbb{Z}_2))) = 11258$, serves as a baseline. More generally, the Hyper–Wiener index is monotone increasing in both the matrix size n and the modulus m. In particular, for integers $n_1 < n_2$ and $m_1 < m_2$, we have the chain of inequalities*

$$WW(A_D(M_{n_1}(\mathbb{Z}_{m_1}))) \;\le\; WW(A_D(M_{n_1}(\mathbb{Z}_{m_2}))) \;\le\; WW(A_D(M_{n_2}(\mathbb{Z}_{m_2}))).$$

Therefore,

$$WW(A_D(M_3(\mathbb{Z}_2))) \;=\; 11258 \;\le\; WW(A_D(M_n(\mathbb{Z}_m))) \qquad \text{for all } n \ge 3,\ m \ge 2.$$

The computations in this work show an increasing trend in the Hyper–Wiener index as the matrix size and modulus increase. However, this observation is based only on the cases considered in the present study and does not establish monotonicity for arbitrary values of n and m. A rigorous proof of monotonicity and the determination of tight bounds for $WW(A_{un}(M_n(\mathbb{Z}_m)))$ would require a detailed analysis of the vertex structure and distance distribution of the corresponding graphs. Hence, we restrict our conclusion to the computed cases and leave the investigation of general monotonicity and tight bounds for future work.

Sketch of justification

(1) *Vertex growth.* The number of upper-triangular entries is $\frac{n(n+1)}{2}$. As m or n increases, the ambient ring has strictly more elements and therefore (generically) more zero-divisor matrices; hence the vertex set of the zero-divisor graph does not decrease and in practice grows rapidly with n and m.

(2) *Edge / distance effect.* Increasing the number of vertices (and allowable nonzero zero-divisor patterns) increases the number of unordered vertex pairs and typically increases pairwise distances or yields additional longer paths; both effects increase the sums that define the Hyper–Wiener index (which aggregates distances and squared distances over all unordered pairs).

(3) *Monotonicity conclusion.* Combining the vertex increase with the nondecreasing contribution of pairwise distances yields the chain of inequalities above; hence the $M_3(\mathbb{Z}_2)$ value 11258 is a lower bound for the family $M_n(\mathbb{Z}_m)$ with larger n or m.

Remark 2. *The computational approach used to calculate the Hyper-Wiener index of the directed zero-divisor graph of $M_3(\mathbb{Z}_2)$ can potentially be extended to larger matrix sizes. However, the rapid growth in the number of vertices increases the computational complexity. For larger matrices, the computation may be optimized by exploiting the structural properties of the graph, reducing redundant distance calculations, and employing efficient graph algorithms. The development of such optimized computational techniques is a potential direction for future research.*

Theorem 2. *Consider the zero-divisor graph A_{un} (Figure 3) of the undirected upper triangular matrix $M_3(\mathbb{Z}_2)$. Therefore, the Hyper-Wiener index obtained via the algorithm that constructs the pairwise distance matrix of A_{un} is equal to* 1852.

Proof. Let $(X_1, X_2, \ldots, X_{63})$ be vertices in a graph. Each entry $d(x, y)$ denotes the distance between vertices v_x and v_y, where x and y range from 1 to 63. The Hyper-Wiener index for a graph G_{un} is given by formula:

$$WW(A_{un}) = \sum_{1 \le x < y \le n} \left(\frac{1}{2} d(x, y) + d(x, y)^2 \right),$$

where:

- $d(x, y)$ is the minimum distance between vertices x and y,
- The sum is taken over all pairs of vertices $x, y \in V$.
- If no path exists between two vertices (v_x) and (v_y), we take $(d(v_x, v_y) = 0)$ for the purpose of the present computation. Consequently, such vertex pairs make zero contribution to the Hyper–Wiener index.

The distances between every pair of vertices are shown in Tables 6–8.

Tables 6–8 represent the shortest path distances between all pairs of vertices in an undirected graph. Each entry $d(v_x, v_y)$ indicates the length of the shortest path from vertex v_x to vertex v_y, where 0 denotes that no such path exists.

Table 6. Minimum distance between vertices v_x and v_y.

Possible pairwise Distances	Minimum distance between vertices
$d(1, v_y),\ 2 \le y \le 63$	1 for $y = 4, 5, 6, 8, 19, 20, 21, 31, 40.$; 2 for $y = 2, 3, 7, 11, 12, 13, 14, 17, 18.$ 3 for $y = 9, 10, 16, 28, 35, 38, 48$ and 0 for all other y.
$d(2, v_y),\ 3 \le y \le 63$	1 for $y = 3, 6, 9, 15.$; 2 for $y = 4, 5, 7, 10, 11, 14, 16, 17, 28, 35, 38, 48.$ 3 for $y = 8, 12, 13, 18, 19, 20, 21, 31, 40$ and 0 for all other y.
$d(3, v_y),\ 4 \le y \le 63$	1 for $y = 4, 5, 9, 10, 14, 16, 17, 21, 28, 35, 38, 48.$; 2 for $y = 6, 8, 12, 13, 15, 18.$ 3 for $y = 7, 11, 19, 20, 31, 40$ and 0 for all other y.
$d(4, v_y),\ 5 \le y \le 63$	1 for $y = 8, 13, 18.$ 2 for $y = 5, 6, 9, 10, 14, 16, 17, 19, 20, 21, 28, 31, 35, 38, 40, 48$ 3 for $y = 7, 11, 12, 15$ and 0 for all other y.

Table 7. Minimum distance between vertices v_x and v_y.

Possible pairwise Distances	Minimum distance between vertices
$d(k, v_y),\ 21 \le k \le 27,\ k + 1 \le y \le 63$	0
$d(28, v_y),\ 29 \le y \le 63$	2 for $y = 35, 38, 48.$ 4 for $y = 31, 40$ and 0 for all other y.
$d(29, v_y),\ 30 \le y \le 63$	0 for $30 \le y \le 63$.
$d(30, v_y),\ 31 \le y \le 63$	0 for $31 \le y \le 63$.
$d(31, v_y),\ 32 \le y \le 63$	2 for $y = 35, 40.$ 4 for $y = 38, 48$ and 0 for all other y.
$d(k, v_y),\ 32 \le k \le 34,\ k + 1 \le y \le 63$	0
$d(35, v_y),\ 36 \le y \le 63$	2 for $y = 38, 48.$; 4 for $y = 40$ and 0 for all other y.
$d(36, v_y),\ 37 \le y \le 63$	0 for $37 \le y \le 63$.
$d(37, v_y),\ 38 \le y \le 63$	0 for $38 \le y \le 63$.
$d(38, v_y),\ 39 \le y \le 63$	2 for $y = 48.$ 4 for $y = 40$ and 0 for all other y.
$d(39, v_y),\ 40 \le y \le 63$	0 for $40 \le y \le 63$.
$d(40, v_y),\ 41 \le y \le 63$	4 for $y = 48$ and 0 for all other y.
$d(x, v_y),\ 41 \le x \le 63,\ 42 \le y \le 63$	0 for $y \ge x + 1$ and $y \le 63$

Table 8. Minimum distance between vertices v_x and v_y.

Possible pairwise Distances	Minimum distance between vertices
$d(5, v_y),\ 6 \le y \le 63$	1 for $y = 8, 12, 17$ 2 for $y = 6, 7, 9, 10, 16, 19, 20, 21, 28, 31, 35, 38, 40, 48.$ 3 for $y = 11, 13, 14, 15, 18$ and 0 for all other y.
$d(6, v_y),\ 7 \le y \le 63$	1 for $y = 11, 14.$; 2 for $y = 19, 20, 21, 31, 40.$ 3 for $y = 8, 12, 13, 17, 18.$; 4 for $y = 9, 10, 16, 28, 35, 38, 48.$ 5 for $y = 15$ and 0 for all other y.
$d(7, v_y),\ 8 \le y \le 63$	2 for $y = 11, 12, 14.$; 3 for $y = 8, 20, 21, 31, 40.$; 4 for $y = 13, 18.$ 5 for $y = 9, 16, 17, 28, 35, 38, 48.$; 6 for $y = 15$ and 0 for all other y.
$d(8, v_y),\ 9 \le y \le 63$	2 for $y = 21.$; 3 for $y = 9, 10, 14, 16, 17, 19, 20, 28, 31, 35, 38, 40, 48.$ 4 for $y = 11, 12, 13, 15, 18$ and 0 for all other y.
$d(9, v_y),\ 10 \le y \le 63$	2 for $y = 10, 14, 15, 16, 17, 19, 21, 28, 35, 38.$; 3 for $y = 11, 12, 13, 18.$ 4 for $y = 20, 31, 40$ and 0 for all other y.
$d(10, v_y),\ 11 \le y \le 63$	2 for $y = 14, 16, 17, 19, 21, 28, 35, 38, 48.$ 3 for $y = 12, 13, 15, 18.$ 4 for $y = 20, 31, 40$ and 0 for all other y.
$d(11, v_y),\ 12 \le y \le 63$	3 for $y = 13, 17, 18, 19, 20, 21, 31, 40.$ 4 for $y = 12, 14, 16, 28, 35, 38, 48.$ 5 for $y = 15$ and 0 for all other y.
$d(12, v_y),\ 13 \le y \le 63$	3 for $y = 14, 16, 17, 19, 20, 21, 28, 31, 35, 38, 40, 48.$ 4 for $y = 13, 15, 18$ and 0 for all other y.
$d(13, v_y),\ 14 \le y \le 63$	2 for $y = 18.$ 3 for $y = 14, 16, 17, 19, 20, 21, 28, 31, 35, 38, 40, 48.$ 4 for $y = 15$ and 0 for all other y.
$d(14, v_y),\ 15 \le y \le 63$	1 for $y = 18.$ 2 for $y = 16, 17, 20, 21, 28, 35, 38, 48.$ 3 for $y = 15, 19, 31, 40$ and 0 for all other y.
$d(15, v_y),\ 16 \le y \le 63$	3 for $y = 16, 17, 28, 35, 38, 48.$ 4 for $y = 18, 19, 20, 21, 40$ and 0 for all other y.
$d(16, v_y),\ 17 \le y \le 63$	2 for $y = 17, 21, 35, 38, 48.$ 3 for $y = 18.$ 4 for $y = 19, 20, 28, 31, 40$ and 0 for all other y.
$d(17, v_y),\ 18 \le y \le 63$	2 for $y = 21, 28, 35, 38, 48.$ 3 for $y = 18, 19, 20, 31, 40$ and 0 for all other y.
$d(18, v_y),\ 19 \le y \le 63$	3 for $y = 19, 20, 21, 28, 31, 35, 38, 40, 48$ and 0 for all other y.
$d(19, v_y),\ 20 \le y \le 63$	2 for $y = 20, 21, 31, 35, 40.$ 4 for $y = 28, 38, 48$ and 0 for all other y.
$d(20, v_y),\ 21 \le y \le 63$	2 for $y = 21, 31, 35, 40.$ 4 for $y = 28, 38, 48$ and 0 for all other y.

By explicit computation of the distance matrix, we obtain

$$WW(G) = \boxed{[1852]}.$$

Hence, the Hyper-Wiener index of the undirected zero-divisor graph derived from the upper triangular matrix ring $M_3(\mathbb{Z}_2)$ is 1852. □

Remark 3. *Let $A_{un}(M_n(\mathbb{Z}_m))$ denote the undirected zero-divisor graph of the upper-triangular matrix ring $M_n(\mathbb{Z}_m)$, and let*

$$WW(A_{un}(M_n(\mathbb{Z}_m)))$$

be its Hyper–Wiener index. The computed value for the smallest nontrivial case,

$$WW(A_{un}(M_3(\mathbb{Z}_2))) = 1852,$$

serves as a baseline. More generally, the Hyper–Wiener index is monotone increasing in both the matrix size n and the modulus m. In particular, for integers $n_1 < n_2$ and $m_1 < m_2$, we have the chain of inequalities

$$WW(A_{un}(M_{n_1}(\mathbb{Z}_{m_1}))) \le WW(A_{un}(M_{n_1}(\mathbb{Z}_{m_2}))) \le WW(A_{un}(M_{n_2}(\mathbb{Z}_{m_2}))).$$

Therefore,

$$WW(A_{un}(M_3(\mathbb{Z}_2))) = 1852 \le WW(A_{un}(M_n(\mathbb{Z}_m))) \qquad \text{for all } n \ge 3,\ m \ge 2.$$

Sketch of justification

(1) *Vertex growth.* The number of upper-triangular entries is $\frac{n(n+1)}{2}$. As m or n increases the ambient ring has strictly more elements and therefore (generically) more zero-divisor matrices; hence the vertex set of the zero-divisor graph does not decrease and in practice grows rapidly with n and m.
(2) *Edge / distance effect.* Increasing the number of vertices (and allowable nonzero zero-divisor patterns) increases the number of unordered vertex pairs and typically increases pairwise distances or yields additional longer paths; both effects increase the sums that define the Hyper–Wiener index (which aggregates distances and squared distances over all unordered pairs).
(3) *Monotonicity conclusion.* Combining the vertex increase with the nondecreasing contribution of pairwise distances yields the chain of inequalities above; hence the $M_3(\mathbb{Z}_2)$ value 1852 is a lower bound for the family $M_n(\mathbb{Z}_m)$ with larger n or m.

Asymptotic comment. From a computational perspective, this rapid growth in the number of vertices also affects the direct evaluation of the Hyper–Wiener index. If V denotes the number of vertices of the zero-divisor graph, then the index involves the distances between $\binom{V}{2} = O(V^2)$ unordered pairs of vertices. Since the number of upper-triangular matrices is of order $m^{\frac{n(n+1)}{2}}$, the corresponding pairwise computation can grow on the scale of $O(m^{n(n+1)})$. Thus, as n and/or m increase, the number of vertex pairs increases very rapidly, making direct computation of the Hyper–Wiener index increasingly demanding. This highlights the importance of deriving structural distance distributions and closed-form expressions for larger classes of upper-triangular matrix rings.

As the parameters n and m increase, the number of upper triangular matrices, and consequently the potential size of the corresponding zero-divisor graph, increases rapidly. Since the Hyper–Wiener index depends on the shortest-path distances between vertex pairs, this growth suggests that the index may increase for larger values of n and m. However, vertex growth alone does not establish monotonicity, as the distance structure may also change with the parameters. Therefore, we regard the observed growth as a computational observation rather than a general monotonicity result.

4. Discussion

When the coefficient ring is replaced by a commutative ring containing nonzero zero-divisors, the structure of the resulting zero-divisor graph becomes more involved. In particular, the presence of zero-divisors in the coefficient ring introduces additional possibilities for zero products among upper triangular matrices, thereby affecting the adjacency relationships between vertices. Moreover, as the size of the coefficient ring increases, the number of upper triangular matrices, and hence the number of vertices in the corresponding zero-divisor graph, also increases. This growth in the vertex set further contributes to the structural complexity of the graph. Consequently, determining the shortest-path distances requires considering a larger variety of algebraic cases and vertex pairs, which may lead to a more complicated distance distribution. Since the Wiener and Hyper-Wiener indices depend on the distances between vertex pairs, these additional algebraic possibilities and the increased number of vertices can make their computation more challenging. In contrast, working over the finite field $\mathbb{Z}_2$ provides a simpler algebraic setting with a relatively small and explicitly manageable vertex set, allowing the distance distribution and the corresponding indices to be determined systematically.

The computational analysis was performed by explicitly constructing the zero-divisor graphs of $M_3(\mathbb{Z}_2)$ and determining the adjacency and shortest-path distances between their vertices. The resulting distance tables were then used to evaluate the Hyper–Wiener index by applying its defining formula to all unordered pairs of vertices. Thus, the reported values are obtained directly from the constructed graph and its distance distribution.

5. Conclusions

In this study, we computed the Hyper-Wiener index of directed and undirected graphs derived from the upper triangular matrix $M_3(\mathbb{Z}_2)$ zero-divisor graph. Based on the zero-divisor graphs of an upper triangular matrix $M_3(\mathbb{Z}_2)$, we have found that the directed and undirected graphs have different Hyper-Wiener indices. For the zero-divisor graphs associated with the upper triangular matrix ring $M_3(\mathbb{Z}_2)$, the Hyper–Wiener indices are explicitly computed for both the directed and undirected cases. In particular, for the undirected zero-divisor graph, we obtain

$$WW(A_{un}(M_3(\mathbb{Z}_2))) = 1852,$$

while for the directed zero-divisor graph, we obtain

$$WW(A_D(M_3(\mathbb{Z}_2))) = 11258.$$

These results show that the directed and undirected zero-divisor graphs have different distance distributions, which consequently lead to different Hyper–Wiener indices.

In the future, we would expand our study to include Hyper-Wiener indices of directed and undirected graphs derived from the zero-divisor graph of the higher order matrix rings and also find some important topological indices such as Randic (Connectivity) Index, the Harary Index, and the First and Second Zagreb Indices of directed and undirected graphs derived from the zero-divisor graph of the upper triangular matrix $M_3(\mathbb{Z}_2)$.

Author contributions

Sahil Sharma: Conceptualization, Validation, Visualization; Omaima Al shanquiti: Formal analysis, Validation; Vijay Kumar Bhat: Conceptualization, Writing–original draft, supervision, Validation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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