AIMS Mathematics, 11(11): 36065–36079.
DOI: 10.3934/math.20261396
Received: 28 July 2026
Revised: 11 October 2026
Accepted: 22 October 2026
Published: 03 November 2026

https://www.aimspress.com/journal/Math

Research article

Convexity of the free boundary for two-dimensional Réthy flows

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Abstract: This paper investigates the convexity of the free boundaries for two-dimensional symmetric Réthy flows. Building on the known existence theory, we consider a steady, incompressible, irrotational flow past an obstacle. Under the assumptions that the upper nozzle wall is horizontal and that the obstacle profile satisfies suitable convexity conditions, we establish a relation between streamline curvature and the normal derivative of the flow speed. Using the subharmonicity of the squared speed and the maximum principle, we prove that the speed in the fluid interior is strictly less than its constant value on the free boundaries. Finally, Hopf's boundary-point lemma and the curvature–speed relation imply that the upper free-boundary graph is strictly convex and that the lower free-boundary graph is strictly concave.

Keywords: Réthy flow; free boundary; convexity; maximum principle; Hopf lemma
Mathematics Subject Classification: 35Q31, 35J25, 76B10, 35Q35

1. Introduction and main results

1.1. Introduction

In 1868, Helmholtz [1] studied flows with free streamlines. Hodograph transformations and conformal mapping theory [2, 3] subsequently became fundamental tools for studying two-dimensional, steady, irrotational jet and cavity flows. Alt and Caffarelli [4] established existence and regularity results for a variational free-boundary problem. Alt, Caffarelli, and Friedman subsequently developed variational approaches to jet and cavity flows [5–8]. Li [9] investigated two-dimensional, compressible, subsonic jet flows with general vorticity governed by the full Euler system. That work established a variational formulation for the stream function and proved the existence and uniqueness of smooth, subsonic jet-flow solutions using the variational method of Alt, Caffarelli, and Friedman. It also characterized the far-field behavior of the flow and established the existence of a critical upstream pressure. Building on the well-posedness results established by Li [9], Wang [10] proved that when the nozzle is concave toward the fluid, the free boundary is strictly

convex toward the fluid, and the horizontal velocity remains positive throughout the fluid domain.

The Réthy flow problem describes a jet issuing from a nozzle and passing an obstacle, with a cavity forming behind the obstacle. In the symmetric half-domain, the fluid region has two unknown free boundaries: an upper jet boundary and a lower cavity boundary. Their locations are determined together with the velocity and pressure by the governing equations and boundary conditions. In 2017, Cheng, Du, and Xiang [11] established the existence of two-dimensional symmetric Réthy flows and proved uniqueness for fixed admissible parameters. They subsequently studied the corresponding axisymmetric problem with two free boundaries in 2020 [12]. Beyond existence and uniqueness, the geometry of the free boundaries has been investigated in this and related flow configurations. Zhang established convexity results for axisymmetric rotational cavity flows [13] and axisymmetric Réthy flows [14]. In the planar setting, Cheng and Pei [15] proved the monotonicity and convexity of the free boundary for incompressible rotational cavity flows. For compressible subsonic flows, Wang obtained convexity results for impinging jets [16] and cavities [17]. Wang [18] also showed that for two-dimensional, compressible, subsonic jets with vorticity, a nozzle wall that is concave toward the fluid produces a free boundary that is strictly convex toward the fluid. For two-dimensional incompressible flows with gravity and vorticity, Zhang [19] derived a pressure–curvature identity and applied it to obtain a lower curvature bound for a class of flows with constant vorticity.

The present paper studies the free-boundary geometry of the planar symmetric Réthy flows constructed in [11]. The proof adapts the maximum-principle and curvature arguments used in [10, 13] and related works. In the present incompressible, irrotational setting, Bernoulli's law allows this strategy to be expressed in terms of the speed. The Réthy configuration contains both an upper jet boundary and a lower cavity boundary, and the boundary-extremum argument must also account for the obstacle vertex. Under the geometric assumptions stated below, we establish strict curvature inequalities for both free boundaries, thereby supplementing the existing theory for these flows with geometric information.

By symmetry, we consider only the upper half-plane. The upper nozzle wall is given by $N_1 : y = f_1(x)$, where

$$f_1(x) > 0 \quad \text{for } x \in (-\infty, a], \qquad f_1(a) = h_1, \qquad \lim_{x\to-\infty} f_1(x) = H. \tag{1.1}$$

The upper obstacle boundary is given by $N_2 : y = f_2(x)$, where

$$f_2(x) > 0 \quad \text{for } x \in (b, 0], \qquad f_2(b) = 0, \qquad f_2(0) = h_2. \tag{1.2}$$

Here, $H, h_1, h_2 > 0$, and $f_1 \in C^{2,\alpha}((-\infty, a])$ and $f_2 \in C^{2,\alpha}([b, 0])$ with $0 < \alpha < 1$ are nondecreasing. Let $A_1 = (a, h_1)$ and $A_2 = (0, h_2)$ denote the separation points of the two free boundaries, and let $B = (b, 0)$ denote the obstacle vertex. We assume that $a < b < 0$ and write

$$I := \{(x, 0) : x \le b\}$$

for the portion of the symmetry axis bounding the fluid domain (see Figure 1).

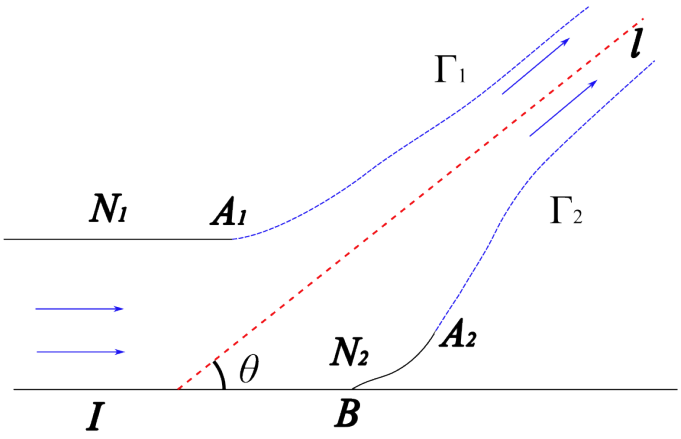


Figure 1. Schematic configuration of a Réthy flow.

We consider a two-dimensional symmetric, steady, incompressible, irrotational, and inviscid flow issuing from the nozzle. With the density normalized to one, the governing Euler equations are

$$\begin{cases} u_x + v_y = 0, \\ uu_x + vu_y + p_x = 0, \\ uv_x + vv_y + p_y = 0, \end{cases} \tag{1.3}$$

where $(x, y) \in \mathbb{R}^2$, and (u, v) and p denote the velocity field and pressure, respectively. The irrotationality condition is

$$v_x - u_y = 0. \tag{1.4}$$

The nozzle wall and obstacle boundary are impermeable, and symmetry gives the same zero-normal-velocity condition on I. Thus,

$$(u, v) \cdot \nu = 0 \quad \text{on } N_1 \cup N_2 \cup I, \tag{1.5}$$

where ν denotes the unit outward normal to the fluid domain at regular boundary points. Along Γ_1 and Γ_2, the velocity is tangent to the boundary so that $(u, v) \cdot \nu = 0$. The pressure there is prescribed by the surrounding atmosphere:

$$p = p_{\text{atm}} \quad \text{on } \Gamma_1 \cup \Gamma_2.$$

Define the speed by $q := \sqrt{u^2 + v^2}$. Equations (1.3)–(1.4) imply that $p + q^2/2$ is constant throughout the fluid domain. Consequently, for some $\lambda > 0$,

$$q = \lambda \quad \text{on } \Gamma_1 \cup \Gamma_2. \tag{1.6}$$

The downstream configuration is required to approach a jet between two parallel straight lines with direction $(\cos\theta, \sin\theta)$, where $0 < \theta < \pi/2$. We prescribe a positive inlet mass flux m_0. Incompressibility and (1.5) make the flux independent of the upstream section:

$$\int_0^{f_1(x_0)} u(x_0, y)\, dy = m_0 \qquad \text{for every } x_0 < a.$$

Réthy flow problem. For the given nozzle, obstacle, and flux m_0, determine the fields (u, v, p); the free boundaries Γ_1, Γ_2; and the parameters λ and θ. The fields must satisfy (1.3) and (1.4) in the fluid region Ω_0 bounded by N_1, N_2, I, Γ_1, and Γ_2, together with the boundary and flux conditions and the downstream behavior specified above. The two free boundaries must also attach continuously to the fixed boundaries at A_1 and A_2, respectively, with matching tangents. Thus, λ, θ, and the free-boundary profiles are determined jointly. This is the planar symmetric configuration studied in [11].

1.2. Existing results for two-dimensional Réthy flows

Theorem 1.1 (see [11])**.** *Assume that the nozzle wall N_1 and obstacle boundary N_2 satisfy* (1.1)*,* (1.2)*, and the regularity and monotonicity assumptions stated above. Given the inlet mass flux $m_0 > 0$, there exists a solution $(u, v, p, \Gamma_1, \Gamma_2)$ to the Réthy flow problem with the following properties.*

(1) The velocity and pressure satisfy

$$(u, v, p) \in \left(C^{1,\alpha}(\Omega_0) \cap C(\overline{\Omega_0})\right)^3.$$

They solve the Euler equations (1.3) *and satisfy the irrotationality condition* (1.4) *in Ω_0, where Ω_0 is the fluid domain bounded by N_1, N_2, I, Γ_1, and Γ_2. They also satisfy the boundary conditions* (1.5) *and* (1.6)*.*

(2) There exists a pair of parameters (λ, θ) with $\lambda > 0$ and $\theta \in (0, \pi/2)$ such that

$$\Gamma_1 = \{(x, k_{1,\lambda,\theta}(x)) : x > a\},$$
$$\Gamma_2 = \{(x, k_{2,\lambda,\theta}(x)) : x > 0\},$$

where

$$k_{1,\lambda,\theta} \in C^1([a, +\infty)), \qquad k_{2,\lambda,\theta} \in C^1([0, +\infty)).$$

Moreover, $k_{2,\lambda,\theta}(x) < k_{1,\lambda,\theta}(x)$ for $x \geq 0$. The continuous fit conditions are

$$k_{1,\lambda,\theta}(a) = h_1, \qquad k_{2,\lambda,\theta}(0) = h_2, \tag{1.7}$$

and the smooth fit conditions are

$$k'_{1,\lambda,\theta}(a + 0) = f'_1(a - 0), \qquad k'_{2,\lambda,\theta}(0 + 0) = f'_2(0 - 0). \tag{1.8}$$

There exists a constant β such that as $x \to +\infty$,

$$k_{1,\lambda,\theta}(x) = x \tan\theta + \beta + \frac{m_0}{2\lambda \cos\theta} + o(1),$$
$$k_{2,\lambda,\theta}(x) = x \tan\theta + \beta - \frac{m_0}{2\lambda \cos\theta} + o(1),$$

where

$$\beta \in \left[h_2 - \frac{m_0}{2\lambda \cos\theta}, h_1 - a \tan\theta + \frac{m_0}{2\lambda \cos\theta}\right].$$

In addition,

$$k'_{i,\lambda,\theta}(x) \to \tan\theta \quad \text{as } x \to +\infty, \qquad i = 1, 2.$$

(3) The velocity components satisfy

$$u > 0 \quad \text{in } \overline{\Omega_0} \setminus \{B\}, \qquad v > 0 \quad \text{in } \Omega_0.$$

There are no stagnation points in $\overline{\Omega_0} \setminus \{B\}$. *If* $h_1 < h_2$, *and* N_2 *lies below the line segment* $\overline{A_1A_2}$, *then*

$$\theta_0 := \arctan \frac{h_2 - h_1}{-a} < \theta < \frac{\pi}{2}.$$

(4) As $x \to -\infty$,

$$(u(x,y), v(x,y), p(x,y)) \longrightarrow \left(\frac{m_0}{H}, 0, p_{\text{atm}} + \frac{\lambda^2}{2} - \frac{m_0^2}{2H^2}\right),$$

and

$$\nabla u \to 0, \qquad \nabla v \to 0, \qquad \nabla p \to 0$$

uniformly for y *in any compact subset of* $(0, H)$. *Set* $d := m_0/(2\lambda \cos\theta)$. *As* $x \to +\infty$,

$$(u, v, p)(x, x\tan\theta + \beta + t) \longrightarrow (\lambda\cos\theta, \lambda\sin\theta, p_{\text{atm}})$$

uniformly for t *in any compact subset of* $(-d, d)$. *The gradients* ∇u, ∇v, *and* ∇p *converge to zero in the same sense.*

The following uniqueness statements are proved in [11, Theorem 1.6].

Theorem 1.2 (Uniqueness of the Réthy flow problem)**.** *(1) For any fixed pair of parameters* (λ, θ) *for which the continuous fit and smooth fit conditions* (1.7) *and* (1.8) *are satisfied, the corresponding Réthy flow described in Theorem 1.1 is unique.*

(2) Suppose that two parameter pairs (λ, θ_1) *and* (λ, θ_2) *with the same* λ *satisfy the continuous fit and smooth fit conditions* (1.7) *and* (1.8). *Then,* $\theta_1 = \theta_2$.

By the continuity equation in (1.3), we introduce the stream function ψ, defined by

$$u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x}.$$

The irrotationality condition (1.4) then implies that

$$\Delta\psi = 0 \quad \text{in } \Omega_0.$$

We normalize ψ so that

$$\psi = 0 \quad \text{on } I \cup N_2 \cup \Gamma_2, \qquad \psi = m_0 \quad \text{on } N_1 \cup \Gamma_1.$$

Because $q = |\nabla\psi|$, the free-boundary condition (1.6) becomes

$$|\nabla\psi| = \lambda \quad \text{on } \Gamma_1 \cup \Gamma_2.$$

1.3. Main results

We now study the geometry of the free boundaries for the flows described in Theorem 1.1. We assume that the upper nozzle wall is horizontal:

$$N_1 := \{(x, H) : x \in (-\infty, a]\} \tag{1.9}$$

so that $h_1 = H$ and that the obstacle profile satisfies

$$f_2''(x) \geq 0 \quad \text{for } x \in (b, 0), \qquad f_2'(b) > 0. \tag{1.10}$$

For simplicity, we write $k_i = k_{i,\lambda,\theta}$, $i = 1, 2$, suppressing the dependence on the parameters. The free-boundary regularity established in [11, Lemma 7.14] implies that both profiles are locally analytic away from the separation points.

Theorem 1.3 (Convexity of the free boundaries of two-dimensional symmetric Réthy flows)**.** *Assume that N_1 and N_2 satisfy the geometric, regularity, and monotonicity assumptions stated above, including* (1.1)*,* (1.2)*,* (1.9)*, and* (1.10)*. Then, the free-boundary profiles k_1 and k_2 of any Réthy flow supplied by Theorem 1.1 satisfy*

$$k_1''(x) > 0 \quad \textit{for } x \in (a, +\infty)$$

and

$$k_2''(x) < 0 \quad \textit{for } x \in (0, +\infty).$$

Remark 1.4. *For a graph $y = k(x)$, we use the upward-pointing unit normal*

$$n = \frac{(-k'(x), 1)}{\sqrt{1 + k'(x)^2}}$$

and the signed curvature

$$\kappa = -\frac{k''(x)}{(1 + k'(x)^2)^{3/2}}.$$

Thus, $\kappa > 0$ corresponds to $k'' < 0$, and $\kappa < 0$ corresponds to $k'' > 0$. The fluid lies below Γ_1 and above Γ_2. Hence, if ν_i is the outward unit normal on Γ_i, then $\nu_1 = n_1, \nu_2 = -n_2$.

Section 2 develops the curvature–speed identity and the estimate $q < \lambda$ in Ω_0, with the obstacle vertex treated as part of the speed estimate. Section 3 combines these results with Hopf's boundary point lemma to prove Theorem 1.3.

2. Preliminaries

The curvature–speed identity below is classical; see [7, p. 144]. We derive it using the streamlined calculation in [15, Lemma 3.3], with the normal and curvature convention of Remark 1.4. The proof also makes explicit that the identity holds on regular fixed-wall arcs as well as on the free boundaries.

Lemma 2.1. *Let S be a C^2-smooth arc of a streamline of the irrotational flow with $u > 0$ on S. Assume that the velocity field and the pressure are C^1 up to S. Set*

$$q = \sqrt{u^2 + v^2} = |\nabla\psi|, \qquad n = \frac{\nabla\psi}{|\nabla\psi|}.$$

With the curvature convention in Remark 1.4, we have

$$\frac{\partial q}{\partial n} + \kappa q = 0 \qquad \text{on } S. \tag{2.1}$$

In particular, this identity applies to regular arcs of the free boundaries and the impermeable walls.

Proof. Because $u = \psi_y > 0$ on S the streamline can locally be represented as $y = s(x)$, where $s \in C^2$, and

$$\psi(x, s(x)) = m$$

for some constant m. Differentiating this identity gives

$$\psi_x + s'\psi_y = 0.$$

Using $u = \psi_y$ and $v = -\psi_x$, we obtain

$$v = s'u, \qquad q^2 = (1 + (s')^2)u^2 \qquad \text{on } S.$$

Consequently,

$$n = \frac{(-s', 1)}{\sqrt{1 + (s')^2}}, \qquad \kappa = -\frac{s''}{(1 + (s')^2)^{3/2}},$$

where the second identity follows from Remark 1.4.

Differentiating $v(x, s(x)) = s'(x)u(x, s(x))$ with respect to x, we find that

$$v_x + s'v_y = s'(u_x + s'u_y) + s''u \qquad \text{on } S.$$

All the velocity components and their derivatives in the following calculation are evaluated at $(x, s(x))$. By the Euler equations (1.3),

$$p_x = -(uu_x + vu_y), \qquad p_y = -(uv_x + vv_y).$$

It follows that

$$\begin{aligned}
\frac{\partial p}{\partial n} &= \frac{-s'p_x + p_y}{\sqrt{1 + (s')^2}} \\
&= \frac{s'(uu_x + vu_y) - (uv_x + vv_y)}{\sqrt{1 + (s')^2}} \\
&= \frac{u}{\sqrt{1 + (s')^2}} \Big[s'(u_x + s'u_y) - (v_x + s'v_y) \Big] \\
&= -\frac{s''u^2}{\sqrt{1 + (s')^2}} \\
&= -\frac{s''}{(1 + (s')^2)^{3/2}} q^2 \\
&= \kappa q^2.
\end{aligned}$$

On the other hand, the irrotationality condition $v_x = u_y$ together with the Euler equations yields

$$\nabla p = -\nabla\left(\frac{u^2+v^2}{2}\right) = -q\nabla q.$$

For boundary streamlines, this identity extends to S by the assumed regularity. Therefore,

$$-q\frac{\partial q}{\partial n} = \frac{\partial p}{\partial n} = \kappa q^2.$$

Because $q \geq u > 0$ on S, division by q gives

$$\frac{\partial q}{\partial n} + \kappa q = 0,$$

which completes the proof. □

We next establish the subharmonicity of the squared speed, which will be used in the proof of Lemma 2.3.

Lemma 2.2. *Let $q = |\nabla\psi|$. Then,*

$$\Delta(q^2) = 2|D^2\psi|^2 \geq 0 \qquad \text{in } \Omega_0,$$

where

$$|D^2\psi|^2 = \psi_{xx}^2 + 2\psi_{xy}^2 + \psi_{yy}^2.$$

Consequently, q^2 is subharmonic in Ω_0.

Proof. Because ψ is harmonic in Ω_0, it is smooth in the interior of Ω_0. Using $q^2 = \psi_x^2 + \psi_y^2$, we compute

$$\Delta(q^2) = 2\left(\psi_{xx}^2 + 2\psi_{xy}^2 + \psi_{yy}^2\right) + 2\psi_x\Delta\psi_x + 2\psi_y\Delta\psi_y.$$

Moreover, $\Delta\psi = 0$ implies that

$$\Delta\psi_x = \partial_x(\Delta\psi) = 0, \qquad \Delta\psi_y = \partial_y(\Delta\psi) = 0.$$

Therefore,

$$\Delta(q^2) = 2\left(\psi_{xx}^2 + 2\psi_{xy}^2 + \psi_{yy}^2\right) = 2|D^2\psi|^2 \geq 0 \qquad \text{in } \Omega_0.$$

This proves the lemma. □

Lemma 2.3. *Under the assumptions of Theorem 1.3, the fluid speed $q = |\nabla\psi|$ satisfies*

$$q < \lambda \quad \text{in } \Omega_0, \qquad q = \lambda \quad \text{on } \Gamma_1 \cup \Gamma_2.$$

In particular,

$$\sup_{\Omega_0} q = \lambda.$$

Proof. We follow the boundary-extremum argument in [13, Lemma 3.3], applying the maximum principle to q^2. At regular boundary points, ν denotes the unit outward normal to the fluid domain, whereas $n = \nabla\psi/q$ is the normal used in Lemma 2.1.

By Lemma 2.2,

$$\Delta(q^2) = 2|D^2\psi|^2 \geq 0 \qquad \text{in } \Omega_0.$$

We first verify that $q(B) = 0$. By Theorem 1.1, Item 1.1, the velocity extends continuously to B. The boundary condition on I gives $v = 0$ there, whereas impermeability on N_2 gives

$$v(x, f_2(x)) = f_2'(x)u(x, f_2(x)) \qquad \text{for } b < x < 0.$$

Taking limits toward B along the two fixed boundary arcs yields

$$v(B) = 0, \qquad v(B) = f_2'(b)u(B).$$

Because $f_2'(b) > 0$ by (1.10), it follows that $u(B) = v(B) = 0$, and hence, $q(B) = 0$. This stagnation property is also noted in [11, Remark 1.4]. Because $q = \lambda > 0$ on the free boundaries, q^2 is nonconstant.

We next record the far-field behavior needed below. The upstream convergence in [11, Proposition 6.2] extends to the horizontal boundaries by odd reflection of ψ across I and of $\psi - m_0$ across N_1, followed by interior estimates for harmonic functions. In the downstream, the asymptotic flatness established in [11, Proposition 6.1] together with local free-boundary regularity and elliptic boundary estimates gives uniform $C^{1,\eta}$ estimates for ψ in translated boundary neighborhoods for some $\eta \in (0, 1)$. In particular,

$$|q(X) - \lambda| \leq C \ \operatorname{dist}(X, \Gamma_1 \cup \Gamma_2)^\eta$$

for points sufficiently close to the free boundaries and sufficiently far downstream, with C independent of X. Combining this estimate with the interior convergence in [11, Proposition 6.2], we obtain

$$\lim_{x\to-\infty} \sup_{0\leq y\leq H} \left| q(x, y) - \frac{m_0}{H} \right| = 0$$

and

$$\lim_{x\to+\infty} \sup_{k_2(x)\leq y\leq k_1(x)} |q(x, y) - \lambda| = 0.$$

Consequently, q is bounded in $\overline{\Omega_0}$.

Suppose, to the contrary, that

$$M := \sup_{\overline{\Omega_0}} q > \lambda,$$

and choose a sequence $P_j \in \overline{\Omega_0}$ such that $q(P_j) \to M$. The downstream limit excludes a maximizing sequence escaping downstream. Furthermore, continuity of the velocity and the free-boundary condition imply

$$q(A_1) = q(A_2) = \lambda.$$

Thus, neither the free boundaries nor the points A_1, A_2, and B can be a limit point of such a maximizing sequence.

If the sequence has a finite limit point P, then $q(P) = M$ by continuity. The strong maximum principle excludes $P \in \Omega_0$. It therefore remains to consider the following four cases.

Case 1. The maximizing sequence escapes upstream.

The upstream limit gives

$$M = \frac{m_0}{H} > \lambda. \tag{2.2}$$

On the other hand, passing to the limit in the comparison estimate of [11, Lemma 2.2] yields

$$\psi(x, y) \le \Psi(y) := \frac{m_0}{H} y \qquad \text{in } \Omega_0. \tag{2.3}$$

Set $w = \Psi - \psi$. Then, $w \ge 0$ in Ω_0, and $w(A_1) = 0$.

Because N_1 is horizontal, the smooth-fit condition and continuity of the velocity at $A_1 = (a, H)$ give

$$u(A_1) = \lambda, \qquad v(A_1) = 0.$$

For sufficiently small $t > 0$, the point $(a, H - t)$ lies in Ω_0. Therefore,

$$\begin{aligned} 0 \le w(a, H - t) &= \Psi(H - t) - \psi(a, H - t) \\ &= \left(\lambda - \frac{m_0}{H}\right) t + o(t). \end{aligned}$$

Dividing by t and letting $t \to 0^+$, we obtain

$$\frac{m_0}{H} \le \lambda,$$

which contradicts (2.2).

Case 2. The maximum is attained at $P \in N_1 \setminus \{A_1\}$.

Local boundary regularity for the harmonic function ψ gives $\psi \in C^{2,\alpha}$ up to N_1 near P. Because $\nu = (0, 1)$ on N_1, Hopf's boundary point lemma applied to q^2 yields

$$\partial_y(q^2)(P) > 0.$$

Using $q(P) = M > 0$, we obtain

$$\partial_y q(P) = \frac{\partial_y(q^2)(P)}{2q(P)} > 0. \tag{2.4}$$

However, $\psi(x, H) = m_0$ implies that

$$\psi_x(P) = 0, \qquad \psi_{xx}(P) = 0.$$

Because $\Delta\psi = 0$ extends to this regular boundary point, we also have $\psi_{yy}(P) = 0$. Differentiating $q = (\psi_x^2 + \psi_y^2)^{1/2}$ from the fluid side, we find that

$$\partial_y q(P) = \frac{\psi_x \psi_{xy} + \psi_y \psi_{yy}}{q}(P) = 0,$$

contradicting (2.4).

Case 3. The maximum is attained at $P \in N_2 \setminus \{B, A_2\}$.

Because N_2 is $C^{2,\alpha}$, and $\psi = 0$ on N_2, local boundary regularity gives $\psi \in C^{2,\alpha}$ up to N_2 near P. Hopf's boundary point lemma applied to q^2 yields

$$\frac{\partial(q^2)}{\partial \nu}(P) > 0.$$

Consequently,

$$\frac{\partial q}{\partial \nu}(P) = \frac{1}{2q(P)} \frac{\partial(q^2)}{\partial \nu}(P) > 0. \tag{2.5}$$

Write $P = (x_0, f_2(x_0))$. Because $\psi(x, f_2(x)) = 0$, and $u = \psi_y > 0$ on this boundary arc,

$$n = \frac{\nabla \psi}{q} = \frac{(-f_2'(x_0), 1)}{\sqrt{1 + (f_2'(x_0))^2}}$$

points into the fluid domain. Thus, the outward normal is

$$\nu = -n = \frac{(f_2'(x_0), -1)}{\sqrt{1 + (f_2'(x_0))^2}}.$$

With the curvature convention in Remark 1.4 and the assumption $f_2'' \geq 0$, we have

$$\kappa(P) = -\frac{f_2''(x_0)}{(1 + (f_2'(x_0))^2)^{3/2}} \leq 0.$$

The impermeability condition makes N_2 a streamline. Hence, Lemma 2.1 applies and gives

$$\frac{\partial q}{\partial n}(P) = -\kappa(P) q(P).$$

Because $\nu = -n$, it follows that

$$\frac{\partial q}{\partial \nu}(P) = -\frac{\partial q}{\partial n}(P) = \kappa(P) q(P) \leq 0,$$

which contradicts (2.5).

Case 4. The maximum is attained at $P \in I \setminus \{B\}$.

The outward normal on I is $\nu = (0, -1)$. By Hopf's boundary point lemma,

$$-\partial_y(q^2)(P) > 0.$$

Because $q(P) = M > 0$, we obtain

$$\partial_y q(P) < 0. \tag{2.6}$$

On the other hand, $\psi(x, 0) = 0$ implies that

$$\psi_x(P) = 0, \qquad \psi_{xx}(P) = 0.$$

Local boundary regularity and $\Delta\psi = 0$ then give $\psi_{yy}(P) = 0$. Consequently,

$$\partial_y q(P) = \frac{\psi_x\psi_{xy} + \psi_y\psi_{yy}}{q}(P) = 0,$$

contradicting (2.6).

All four cases lead to contradictions. Therefore,

$$q \le \lambda \qquad \text{in } \overline{\Omega_0}.$$

Because q^2 is nonconstant and subharmonic, the strong maximum principle further yields

$$q < \lambda \qquad \text{in } \Omega_0.$$

Together with $q = \lambda$ on $\Gamma_1 \cup \Gamma_2$ and continuity up to the free boundaries, this gives

$$\sup_{\Omega_0} q = \lambda.$$

The same boundary arguments as in Cases 2–4 exclude $q = \lambda$ on the regular fixed boundary arcs, and $q(B) = 0$. Thus, the maximum on $\overline{\Omega_0}$ is attained only on the free boundaries and their separation points A_1 and A_2. □

3. Proof of the main results

Proof of Theorem 1.3. By Lemma 2.3,

$$q^2 < \lambda^2 \quad \text{in } \Omega_0, \qquad q^2 = \lambda^2 \quad \text{on } \Gamma_1 \cup \Gamma_2.$$

Moreover, Lemma 2.2 gives

$$\Delta(q^2) \ge 0 \qquad \text{in } \Omega_0.$$

Fix an arbitrary point P on either free boundary, away from the separation points A_1 and A_2. Because the free boundary is $C^{2,\alpha}$ near P, and ψ is constant on it, local elliptic boundary regularity gives $\psi \in C^{2,\alpha}$ up to the free boundary near P. In particular, q^2 is continuously differentiable up to the boundary. The $C^{2,\alpha}$ regularity of the boundary also ensures the interior sphere condition at P.

Let ν_i denote the unit outward normal to Ω_0 on Γ_i, $i = 1, 2$. Hopf's boundary point lemma applied to q^2 yields

$$\frac{\partial(q^2)}{\partial\nu_i}(P) > 0 \qquad \text{for } P \in \Gamma_i, \quad i = 1, 2.$$

Because $q(P) = \lambda > 0$, it follows that

$$\frac{\partial q}{\partial\nu_i}(P) = \frac{1}{2\lambda}\frac{\partial(q^2)}{\partial\nu_i}(P) > 0.$$

We now relate the outward normals to the normal used in Lemma 2.1. Because ψ is constant on $\Gamma_i = \{y = k_i(x)\}$, we have

$$\psi_x + k_i'(x)\psi_y = 0 \qquad \text{on } \Gamma_i.$$

Using $u = \psi_y > 0$, we obtain

$$n_i := \frac{\nabla\psi}{q} = \frac{(-k_i'(x), 1)}{\sqrt{1 + (k_i'(x))^2}} \qquad \text{on } \Gamma_i.$$

The fluid lies below Γ_1 and above Γ_2. Consequently,

$$\nu_1 = n_1, \qquad \nu_2 = -n_2.$$

On the lower free boundary Γ_2, we therefore have

$$\frac{\partial q}{\partial n_2} = -\frac{\partial q}{\partial \nu_2} < 0. \tag{3.1}$$

By (3.1), Lemma 2.1, and $q = \lambda$ on Γ_2,

$$\kappa_2 = -\frac{1}{\lambda}\frac{\partial q}{\partial n_2} > 0. \tag{3.2}$$

Here, κ_2 denotes the curvature of Γ_2 with the convention in Remark 1.4:

$$\kappa_2 = -\frac{k_2''(x)}{(1 + (k_2'(x))^2)^{3/2}}.$$

Combining this identity with (3.2), we obtain

$$k_2''(x) < 0 \qquad \text{for } x \in (0, +\infty).$$

On the upper free boundary Γ_1, the identity $n_1 = \nu_1$ gives

$$\frac{\partial q}{\partial n_1} = \frac{\partial q}{\partial \nu_1} > 0.$$

Applying Lemma 2.1 again, we obtain

$$\kappa_1 = -\frac{1}{\lambda}\frac{\partial q}{\partial n_1} < 0.$$

Because

$$\kappa_1 = -\frac{k_1''(x)}{(1 + (k_1'(x))^2)^{3/2}},$$

we conclude that

$$k_1''(x) > 0 \qquad \text{for } x \in (a, +\infty).$$

This completes the proof. □

Together with the downstream asymptotics in Theorem 1.1, the strict curvature inequalities imply that the upper free-boundary slope k_1' increases to $\tan\theta$, whereas the lower free-boundary slope k_2' decreases to the same limit. Moreover, Bernoulli's law and Lemma 2.3 give

$$p - p_{\text{atm}} = \frac{\lambda^2 - q^2}{2} > 0 \qquad \text{in } \Omega_0.$$

Thus, the pressure throughout the fluid interior is strictly higher than the atmospheric pressure prescribed on both free boundaries.

4. Conclusions

This paper establishes the convexity of the free boundaries for two-dimensional symmetric Réthy flows with a horizontal upper nozzle wall and an obstacle satisfying the stated geometric assumptions. The proof combines a strict interior speed estimate, a curvature–speed identity, and Hopf's boundary point lemma. The upper free-boundary profile is strictly convex, whereas the lower profile is strictly concave. The analysis is restricted to steady, incompressible, and irrotational flows under the stated geometric assumptions. Further work could investigate analogous geometric properties of free boundaries in flows with nonhorizontal nozzle walls or nonzero vorticity.

Use of generative-AI tools declaration

The author used Doubao for English translation and language editing during the preparation of this manuscript. The tool was used only to improve grammar, clarity, and readability in the abstract and introduction. All mathematical arguments, results, and proofs were independently developed and verified by the author. The author reviewed and revised all AI-assisted text and takes full responsibility for the content of the manuscript.

Conflict of interest

The author declares no conflict of interest.

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