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Research article

Aczél–Alsina aggregation operators for bipolar (p, q)-quasirung orthopair fuzzy information with application to MCDM

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Abstract: This study extends the Aczél–Alsina operational and aggregation framework for (p, q)-quasirung orthopair fuzzy information to a four-component bipolar setting. Aczél–Alsina operational laws are formulated for bipolar (p, q)-quasirung orthopair fuzzy numbers, and the corresponding weighted averaging and weighted geometric aggregation operators are developed. The resulting operators are shown to preserve the bipolar (p, q)-quasirung orthopair fuzzy structure and satisfy the required aggregation properties. In addition, complement duality between the weighted averaging and geometric operators is established. The effect of the Aczél–Alsina parameter is characterized theoretically, showing the opposite responses of the positive- and negative-pole score components and demonstrating that monotonicity of the overall pole-weighted score does not follow in general. An explicit pole importance parameter α is incorporated into the ranking mechanism, and pole agreement, pole conflict, and the corresponding preference-switching threshold are characterized analytically. The proposed mechanisms are integrated into a multicriteria decision-making procedure and illustrated through a sustainable supplier selection problem. Numerical analyses examine the effects of the bipolar structure, criterion weights, and the parameters p, q, λ, and α. The leading alternative remains unchanged across the examined p, q, λ, and criterion weight scenarios, whereas changes in α can alter the leading alternative when the negative pole is heavily discounted. The complete ranking of the lower-ranked alternatives remains sensitive to the aggregation operator and parameter settings.

Keywords: bipolar fuzzy sets; (p, q)-quasirung orthopair fuzzy sets; Aczél–Alsina aggregation operators; multicriteria decision-making; pole-sensitive ranking
Mathematics Subject Classification: 03B52, 03E72

1. Introduction

Decision-making (DM) under realistic conditions is commonly associated with information that is uncertain, partial, imprecise, or expressed from different and sometimes opposing perspectives. In practical decision environments, experts may not evaluate an alternative solely in terms of the extent to which it satisfies a particular criterion. They may also provide evidence indicating dissatisfaction, opposition, or an unfavorable assessment. Moreover, the strength of positive and negative assessments may not follow the same pattern. These characteristics arise naturally in a wide range of applications, including sustainable supplier evaluation, medical diagnosis, environmental management, risk analysis, resource planning, and other multicriteria decision-making (MCDM) problems. Such situations motivate the development of mathematical models capable of preserving the richness of the information supplied by decision-makers while maintaining a tractable structure for subsequent analysis.

1.1. Motivation for conducting the research

The motivation for this study arises from the need to represent decision information that involves two distinct poles simultaneously: A positive pole reflecting favorable evidence and a negative pole reflecting opposing evidence. Conventional orthopair models generally use a common exponent for the membership and non membership degrees and therefore do not provide the asymmetric control offered by independent p and q parameters. Conventional bipolar models, on the other hand, distinguish positive and negative poles but do not, in the considered setting, provide an operational framework that combines this polarity with independent (p, q) orthopair constraints under the Aczél–Alsina (AA) operational mechanism. Three observations motivate the present study. First, existing bipolar orthopair frameworks do not combine the two poles with independent (p, q) exponents under a single closed operational framework. Second, the studies that develop AA aggregation operators for (p, q)-quasirung orthopair information treat the information as unipolar and therefore do not preserve the semantic distinction between favorable and unfavorable evidence. Third, although bipolar aggregation operators have been investigated in other environments, their interaction with the asymmetric (p, q)-orthopair structure has not been established. These observations motivate the development of a dedicated formulation.

1.2. Practical and methodological aims of the study

The present study pursues two complementary aims. From a practical perspective, it aims to provide a DM framework in which favorable and unfavorable assessments can be represented and aggregated without losing their distinct semantic meaning. The framework should support a complete MCDM procedure from information representation to final ranking, and should allow the decision-maker to control the relative importance of the two poles explicitly. From a methodological perspective, the study aims to develop a mathematical structure that (i) extends orthopair representations by allowing independent exponents for the two orthopair components, (ii) preserves the joint admissibility of both poles under AA operations, and (iii) supports a complete MCDM procedure from information representation to final ranking.

1.3. Main contributions

The main contributions of this study can be summarized as follows.

(1) We extend the AA operational and aggregation framework to bipolar (p, q)-quasirung orthopair fuzzy information, enabling the information of the positive and negative poles to be aggregated simultaneously under asymmetric membership and nonmembership exponents.

(2) We establish bipolar-specific properties of the resulting AA aggregation operators, including complement duality between the weighted averaging and geometric operators and the opposite responses of the positive and negative poles' score components to the AA parameter.

(3) We introduce an explicit pole importance parameter α into the ranking mechanism and characterize pole agreement, pole conflict, and the corresponding preference-switching threshold, thereby allowing the relative influence of positive and negative poles' evidence on the final ranking to be analyzed explicitly.

(4) We integrate the proposed bipolar AA aggregation and pole-sensitive ranking mechanisms into an MCDM procedure and evaluate their decision behavior through bipolar ablation, criterion weight sensitivity, and parameter sensitivity analyses, distinguishing the stability of the leading alternative from changes in the complete ranking.

1.4. Organization of the paper

The remainder of this paper is organized as follows. Section 2 provides a critical review of the relevant literature, focusing on bipolar and orthopair fuzzy models and their associated aggregation mechanisms. Section 3 recalls the mathematical preliminaries used throughout the study. Section 4 develops the proposed bipolar (p, q)-quasirung orthopair fuzzy set (B(p, q)-QOFS) framework and its operational laws. Section 5 introduces the two aggregation operators and establishes their algebraic properties. Section 6 formulates the MCDM procedure. Section 7 presents the numerical example and the sensitivity analyses. Section 8 provides the comparative analysis, discussion, limitations, and directions for future research. Section 9 concludes the paper.

The notation and abbreviations used throughout the proposed framework are collected in Table 1. These symbols are adopted consistently in the subsequent definitions, propositions, operational laws, aggregation operators, and MCDM procedure.

Table 1. Notation.

Symbol	Description
$\mathbb{X}$	Nonempty universal set (universe of discourse)
O	A (p, q)-quasirung orthopair fuzzy set (QOFS)
$\mathcal{G}$	A B(p, q)-QOFS
ξ	An element of the universal set $\mathbb{X}$
Ξ	A bipolar (p, q)-quasirung orthopair number (B(p, q)-QOFN)
$\aleph^{+}_{\mathcal{G}}(\xi)$	Positive membership degree (PMD) of element ξ in $\mathcal{G}$, where $\aleph^{+}_{\mathcal{G}}(\xi) \in [0, 1]$
$\mho^{+}_{\mathcal{G}}(\xi)$	Positive non-membership degree (PNMD) of element ξ in $\mathcal{G}$, where $\mho^{+}_{\mathcal{G}}(\xi) \in [0, 1]$
$\aleph^{-}_{\mathcal{G}}(\xi)$	Negative membership degree (NMD) of element ξ in $\mathcal{G}$, where $\aleph^{-}_{\mathcal{G}}(\xi) \in [-1, 0]$

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$\mho^{-}_{\mathcal{G}}(\xi)$	Negative non-membership degree (NNMD) of element ξ in $\mathcal{G}$, where $\mho^{-}_{\mathcal{G}}(\xi) \in [-1, 0]$
$\mathfrak{p}$	Exponent for the PMD ($\mathfrak{p} \geq 1$)
$\mathfrak{q}$	Exponent for the PNMD ($\mathfrak{q} \geq 1$)
$\mathfrak{f}$	Least common multiple of $\mathfrak{p}$ and $\mathfrak{q}$, i.e., $\mathfrak{f} = \text{lcm}(\mathfrak{p}, \mathfrak{q})$
$\pi^{+}_{\mathcal{G}}(\xi)$	Positive indeterminacy degree of element ξ in $\mathcal{G}$
$\pi^{-}_{\mathcal{G}}(\xi)$	Negative indeterminacy degree of element ξ in $\mathcal{G}$
$\Xi_1 \oplus_{AA} \Xi_2$	AA addition (algebraic sum) of two B(p, q)-QOFNs
$\Xi_1 \otimes_{AA} \Xi_2$	AA multiplication (algebraic product) of two B(p, q)-QOFNs
$\tau \cdot_{AA} \Xi$	AA scalar multiplication of a B(p, q)-QOFN by $\tau > 0$.
Ξ^{τ}	AA power operation of a B(p, q)-QOFN by $\tau > 0$
λ	Parameter of the AA t-norm and t-conorm ($\lambda > 0$)
T^{λ}_{AA}	AA t-norm
S^{λ}_{AA}	AA t-conorm
$P(\Xi)$	Positive pole preference component of the B(p, q)-QOFN Ξ
$N(\Xi)$	Negative pole preference component of the B(p, q)-QOFN Ξ
α	Pole importance parameter ($\alpha \in [0, 1]$)
$s_{\alpha}(\Xi)$	Pole weighted score function of the B(p, q)-QOFN Ξ
$\Theta_{\alpha}(\Xi)$	Normalized pole-weighted score function of the B(p, q)-QOFN Ξ ($0 \leq \Theta_{\alpha}(\Xi) \leq 1$)
$\Upsilon(\Xi)$	Accuracy function of the B(p, q)-QOFN Ξ ($0 \leq \Upsilon(\Xi) \leq 1$)
$\leq_C$	Componentwise order relation for B(p, q)-QOFNs
$\Xi^{\mathfrak{c}}$	Complement of the B(p, q)-QOFN Ξ
$\prec$	Inferior ranking relation for B(p, q)-QOFNs
$\succ$	Superior ranking relation for B(p, q)-QOFNs
$\simeq$	Indifference (equal ranking) relation for B(p, q)-QOFNs
$R_{(p,q)}$	Admissible region defined by $R_{(p,q)} = \{(\mathfrak{a}, \mathfrak{b}) \in [0, 1]^2 : \mathfrak{a}^{\mathfrak{p}} + \mathfrak{b}^{\mathfrak{q}} \leq 1\}$
F	Set of alternatives in the MCDM problem, $F = \{F_1, \ldots, F_{\mathfrak{m}}\}$
$\varkappa$	Set of criteria in the MCDM problem, $\varkappa = \{\varkappa_1, \ldots, \varkappa_{\mathfrak{n}}\}$
$\boldsymbol{\omega}$	Weight vector of the criteria, $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_{\mathfrak{n}})^T$
Ξ_{ij}	B(p, q)-QOFN representing the evaluation of alternative $F_{\mathfrak{i}}$ under criterion $\varkappa_{\mathfrak{j}}$
$\widetilde{W}_{ij}$	Normalized B(p, q)-QOFN evaluation (benefit-type or cost-type)
$\mathcal{D}$	B(p, q)-QOF decision matrix of size $\mathfrak{m} \times \mathfrak{n}$
B(p, q)-QOFAAWA	Bipolar (p, q)-quasirung orthopair fuzzy AA weighted averaging (B(p, q)-QOFAAWA) operator
B(p, q)-QOFAAWG	Bipolar (p, q)-quasirung orthopair fuzzy AA weighted geometric (B(p, q)-QOFAAWG) operator

2. Literature review

This section provides a critical review of the literature relevant to the present study, with particular emphasis on bipolar and orthopair fuzzy models and their associated aggregation mechanisms.

2.1. Development of orthopair fuzzy representations

The development of fuzzy set theory provided one of the fundamental mathematical approaches for dealing with imprecise and vague information. In the classical fuzzy set framework, the degree associated with an element is represented by a membership value belonging to the unit interval [1]. Although this formulation is effective for modeling gradual belonging, many decision problems require a richer description in which the evidence supporting and opposing an assessment is represented simultaneously. Intuitionistic fuzzy sets (IFSs) extended the classical framework by associating an element with both a membership degree and a nonmembership degree [2]. The resulting representation allows a decision-maker to distinguish between evidence in favor of an assessment and evidence against it. Subsequent research sought to enlarge the feasible domain available for these two degrees. In this direction, Pythagorean fuzzy sets introduced a less restrictive orthopair condition than that of IFSs [3,4], and the q-rung orthopair fuzzy set (Q-ROFS) subsequently generalized this idea by applying a parameterized admissibility condition [5].

The Q-ROFS framework has consequently attracted considerable attention in aggregation and MCDM research. Various aggregation operators and DM procedures have been developed to process q-rung orthopair information. Bonferroni mean and related aggregation mechanisms were investigated by [6, 7]. Heronian mean operators and interval-valued extensions were considered by [8, 9]. Additional contributions by [10, 11] and by [12, 13] further enlarged the representation domain. Fermatean fuzzy sets constitute another important development in this family, together with the corresponding aggregation mechanisms designed for decision analysis under more flexible uncertainty conditions [14, 15]. These developments illustrate a continuing research direction in which the admissible region of fuzzy evaluations is progressively generalized. A further extension of this line of research was proposed through the (p, q)-quasirung orthopair fuzzy set ((p, q)-QOFS) introduced by Seikh and Mandal [16]. Unlike orthopair models in which the same exponent controls the membership and nonmembership components, the (p, q)-QOFS formulation assigns two independent parameters to these components. This additional degree of freedom permits the two types of assessment to be treated differently and is therefore useful when favorable and unfavorable pieces of evidence exhibit unequal characteristics. Subsequent studies have explored extensions and aggregation mechanisms for (p, q)-QOFS information, including Hamacher-based aggregation operators [17], linguistic versions [18], and circular (p, q)-QOFS structures [19].

2.2. Bipolar fuzzy representations

Despite the increased flexibility offered by orthopair representations, conventional orthopair models generally organize information around a single evaluation direction. In practical settings, however, an assessment may contain two semantically different poles. One pole can represent information supporting the evaluated object, whereas the other can represent information opposing or negatively characterizing it. Bipolar fuzzy sets (BFSs) were introduced to address this type of situation by enabling positive and negative information to be represented within a common fuzzy framework [20–22]. The bipolar concept has subsequently been combined with a variety of fuzzy and uncertainty theories. Research has considered bipolar soft sets and bipolar neutrosophic structures [23–25]. Bipolar IFSs (BIFSs) and bipolar Pythagorean fuzzy sets (BPyFSs) have also been investigated [26–28]. More recent studies have developed complex bipolar fuzzy models [29],

bipolar (n, m)-rung orthopair fuzzy sets [30], and bipolar intuitionistic DM models [31]. Additional contributions include bipolar Fermatean fuzzy Hamacher approaches [32], similarity measures for interval-valued bipolar q-rung orthopair fuzzy sets [33], and bipolar complex q-rung orthopair fuzzy aggregation operators [34]. Nevertheless, the combination of bipolarity with the asymmetric (p, q)-orthopair structure requires a dedicated formulation because the two modeling mechanisms address different aspects of uncertain information.

2.3. AA aggregation mechanisms

In parallel with developments in fuzzy information representations, considerable attention has been devoted to the design of flexible aggregation mechanisms. Aggregation is a central component of DM because individual assessments must eventually be transformed into collective information before alternatives can be compared. Among the available parametric constructions, the AA t-norm and t-conorm provide a parameter-controlled family of operations that can be adapted to different aggregation behaviors [35–37]. Existing studies have developed AA-based aggregation operators for IFSs [38] and picture fuzzy sets [39]. Pythagorean fuzzy and Fermatean fuzzy extensions have also been investigated [40–42]. In the q-rung orthopair setting, AA operators have been developed by [43–45]. More recently, AA operators have been extended to (p, q)-quasirung orthopair fuzzy information [46, 47] and to bipolar fuzzy numbers [48]. However, the simultaneous integration of AA operations with bipolar and asymmetric (p, q)-orthopair information has not been established.

2.4. Research gap

From the perspective of the present study, the relevant literature establishes complementary but separate foundations. Ibrahim [30] developed the bipolar (n, m)-rung orthopair fuzzy framework, which represents the positive and negative poles information using two independent exponents and provides associated aggregation operators. Separately, AA operational and aggregation mechanisms have been established for unipolar (p, q)-quasirung orthopair fuzzy information [46, 47]. However, in the reviewed literature, these developments do not provide the corresponding four-component bipolar (p, q)-quasirung AA operational and aggregation framework in which the positive and negative poles' information is handled simultaneously under asymmetric membership and nonmembership exponents. Moreover, the reviewed frameworks do not provide an explicit pole importance parameter α for controlling the relative contribution of the positive and negative poles' evidence in the ranking process. Accordingly, the present study addresses this gap by extending the AA operational and aggregation framework to bipolar (p, q)-quasirung orthopair fuzzy information and coupling it with a pole-sensitive ranking mechanism that permits the influence of the two poles on the final ranking to be analyzed explicitly.

3. Preliminaries

In this section, we present the basic concepts and definitions that underpin the proposed framework. Unless otherwise specified, $\mathbb{X}$ denotes a nonempty universal set.

Definition 1. *Seikh and Mandal [16] introduced the (p, q)-QOFS as a further generalization that employs two independent exponents p and q for the membership and nonmembership degrees,*

respectively. A (p, q)-QOFS O on $\mathbb{X}$ is defined by

$$O = \{(\xi, \aleph_O(\xi), \mho_O(\xi)) \mid \xi \in \mathbb{X}\},$$

where $\aleph_O, \mho_O : \mathbb{X} \to [0, 1]$ satisfy

$$0 \leq (\aleph_O(\xi))^{\mathfrak{p}} + (\mho_O(\xi))^{\mathfrak{q}} \leq 1 \quad \text{for all } \xi \in \mathbb{X},$$

with $\mathfrak{p}, \mathfrak{q} \in \mathbb{N}$, $\mathfrak{p}, \mathfrak{q} \geq 1$. The indeterminacy degree of ξ is given by

$$\pi_O(\xi) = \sqrt[\operatorname{lcm}(\mathfrak{p},\mathfrak{q})]{1 - (\aleph_O(\xi))^{\mathfrak{p}} - (\mho_O(\xi))^{\mathfrak{q}}}.$$

The (p, q)-QOFS framework is unipolar: The membership degree $\aleph_O$ and the nonmembership degree $\mho_O$ describe the evaluation of an object within a single polarity. Thus, it does not explicitly represent separate positive and negative information poles.

Definition 2. *Following Aczél and Alsina (see [35]) and subsequent developments by Klement et al. [36, 37], the* AA *t-norm and t-conorm are recalled below. For a parameter $\lambda > 0$, the* AA *t-norm $T^{\lambda}_{AA} : [0, 1] \times [0, 1] \to [0, 1]$ is defined for $\mathfrak{u}, \mathfrak{v} \in [0, 1]$ by*

$$T^{\lambda}_{AA}(\mathfrak{u}, \mathfrak{v}) = \begin{cases} \min(\mathfrak{u}, \mathfrak{v}), & \lambda \to \infty, \\ e^{-\sqrt[\lambda]{(-\ln \mathfrak{u})^{\lambda} + (-\ln \mathfrak{v})^{\lambda}}}, & 0 < \lambda < \infty,\ \lambda \neq 1, \\ \mathfrak{u}\mathfrak{v}, & \lambda = 1. \end{cases}$$

The corresponding AA *t-conorm S^{λ}_{AA}: $[0, 1] \times [0, 1] \to [0, 1]$ is given for $\mathfrak{u}, \mathfrak{v} \in [0, 1]$ by*

$$S^{\lambda}_{AA}(\mathfrak{u}, \mathfrak{v}) = \begin{cases} \max(\mathfrak{u}, \mathfrak{v}), & \lambda \to \infty, \\ 1 - e^{-\sqrt[\lambda]{(-\ln(1-\mathfrak{u}))^{\lambda} + (-\ln(1-\mathfrak{v}))^{\lambda}}}, & 0 < \lambda < \infty,\ \lambda \neq 1, \\ \mathfrak{u} + \mathfrak{v} - \mathfrak{u}\mathfrak{v}, & \lambda = 1. \end{cases}$$

The parameter λ controls the aggregation behavior of the AA operators. When $\lambda = 1$, the t-norm and t-conorm reduce to the algebraic product and algebraic sum, respectively. As $\lambda \to \infty$, they converge to the minimum and maximum operators. For intermediate values of λ, the operators provide a continuous transition between these regimes, allowing the aggregation behavior to be adjusted according to the DM context.

Definition 3. *In the bipolar framework introduced by Zhang [20, 21] and further elaborated by Lee et al. [22], a BFS $\mathcal{E}$ on $\mathbb{X}$ is defined by*

$$\mathcal{E} = \{(\xi, \aleph^{+}_{\mathcal{E}}(\xi), \aleph^{-}_{\mathcal{E}}(\xi)) \mid \xi \in \mathbb{X}\},$$

where the positive membership function $\aleph^{+}_{\mathcal{E}} : \mathbb{X} \to [0, 1]$ and the negative membership function $\aleph^{-}_{\mathcal{E}} : \mathbb{X} \to [-1, 0]$ are independently specified for each $\xi \in \mathbb{X}$. The value $\aleph^{+}_{\mathcal{E}}(\xi)$ measures the strength of positive evidence, while $\aleph^{-}_{\mathcal{E}}(\xi)$ represents the degree associated with the negative pole (counter-property); the two poles are semantically independent. In the original BFS formulation, the negative pole is represented by a single membership function $\aleph^{-}_{\mathcal{E}}(\xi) \in [-1, 0]$, whose magnitude measures the strength of the opposing evidence. No distinction is made between different types of unfavorable information.

The proposed B(p, q)-QOFS framework, developed in Section 4, extends the BFS structure by introducing a second component in the negative pole and by combining both poles with the asymmetric (p, q)-orthopair admissibility conditions. The resulting representation preserves the semantic distinction between favorable and unfavorable information while allowing the two poles to be governed by independent exponents.

4. Proposed framework and operational laws (B(p, q)-QOFS)

This section develops the proposed B(p, q)-QOFS framework, establishes its fundamental properties, develops the associated ranking functions, and formulates the corresponding AA operational laws.

To provide a clear overview of the methodological development, Figure 1 illustrates how the relevant foundations are integrated into the proposed bipolar (p, q)-quasirung AA framework, followed by the aggregation, pole-sensitive ranking, and MCDM stages.

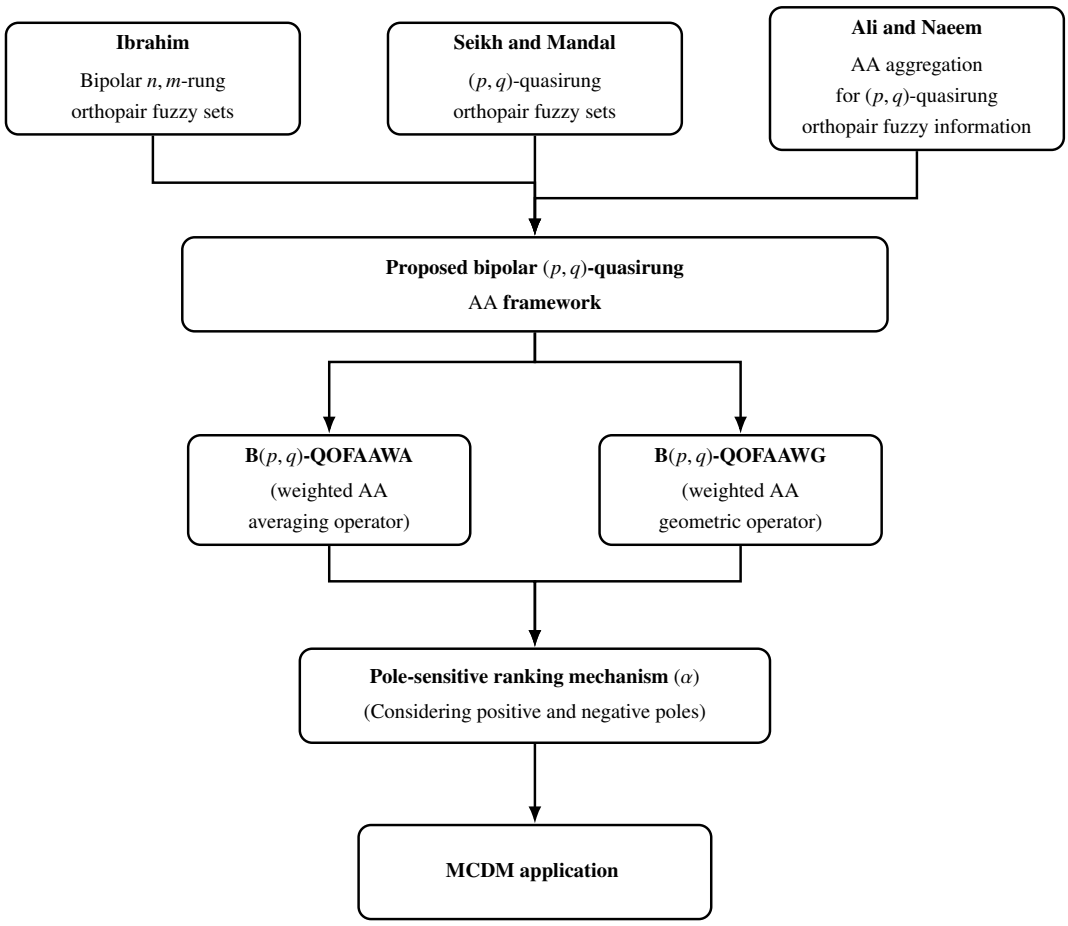


Figure 1. Conceptual structure of the proposed bipolar (p, q)-quasirung AA framework and its MCDM application.

As illustrated in Figure 1, the proposed model is not intended to replace the individual theoretical components but to establish a mathematical connection among them. The (p, q) structure supplies

independent control over the two orthopair degrees, bipolarity introduces separate positive and negative poles, and the AA mechanism determines how such information can be operationally combined. Their integration consequently produces a DM framework capable of retaining richer evaluation information throughout the aggregation process. The resulting methodology is applied to sustainable supplier selection, where alternatives may be assessed positively and negatively with respect to economic, environmental, operational, and other sustainability-related considerations. Such a setting provides a suitable illustration of the proposed framework because supplier evaluations can involve both supporting and opposing evidence and may exhibit unequal degrees of membership and nonmembership. The application therefore demonstrates the complete computational pathway from B (p, q)-QOF information representation to aggregation and the final alternative ranking.

4.1. Definition and basic properties

Definition 4. *Let $\mathbb{X}$ be a nonempty universal set, and let $\mathfrak{p}, \mathfrak{q} \in \mathbb{N}$ with $\mathfrak{p}, \mathfrak{q} \geq 1$. A $B(p, q)$-QOFS $\mathcal{G}$ on $\mathbb{X}$ is defined by*

$$\mathcal{G} = \left\{ \left\langle \xi, (\aleph_{\mathcal{G}}^{+}(\xi), \mho_{\mathcal{G}}^{+}(\xi)), (\aleph_{\mathcal{G}}^{-}(\xi), \mho_{\mathcal{G}}^{-}(\xi)) \right\rangle \mid \xi \in \mathbb{X} \right\},$$

where

$$\aleph_{\mathcal{G}}^{+}, \mho_{\mathcal{G}}^{+} : \mathbb{X} \to [0, 1], \qquad \aleph_{\mathcal{G}}^{-}, \mho_{\mathcal{G}}^{-} : \mathbb{X} \to [-1, 0],$$

and, for every $\xi \in \mathbb{X}$,

$$\left(\aleph_{\mathcal{G}}^{+}(\xi)\right)^{\mathfrak{p}} + \left(\mho_{\mathcal{G}}^{+}(\xi)\right)^{\mathfrak{q}} \leq 1$$

and

$$\left|\aleph_{\mathcal{G}}^{-}(\xi)\right|^{\mathfrak{p}} + \left|\mho_{\mathcal{G}}^{-}(\xi)\right|^{\mathfrak{q}} \leq 1.$$

Here, $(\aleph_{\mathcal{G}}^{+}(\xi), \mho_{\mathcal{G}}^{+}(\xi))$ represents the PMD *and* PNMD*, whereas $(\aleph_{\mathcal{G}}^{-}(\xi), \mho_{\mathcal{G}}^{-}(\xi))$ represents the corresponding* NMD *and* NNMD*. The absolute values in the latter constraint account for the negative components being in $[-1, 0]$.*

The two components of the negative pole describe the evaluation of the corresponding counter-property. The negative membership degree $\aleph_{\mathcal{G}}^{-}(\xi)$ represents the strength of evidence supporting the counter-property, whereas the negative nonmembership degree $\mho_{\mathcal{G}}^{-}(\xi)$ represents the degree of non-support for that counter-property. Thus, a larger $|\aleph_{\mathcal{G}}^{-}(\xi)|$ indicates stronger unfavorable evidence, while a larger $|\mho_{\mathcal{G}}^{-}(\xi)|$ indicates weaker unfavorable evidence.

Let

$$\mathfrak{f} = \operatorname{lcm}(\mathfrak{p}, \mathfrak{q}).$$

The positive and negative indeterminacy degrees are, respectively, defined by

$$\pi_{\mathcal{G}}^{+}(\xi) = \sqrt[\mathfrak{f}]{1 - \left(\aleph_{\mathcal{G}}^{+}(\xi)\right)^{\mathfrak{p}} - \left(\mho_{\mathcal{G}}^{+}(\xi)\right)^{\mathfrak{q}}}$$

and

$$\pi^{-}_{\mathcal{G}}(\xi) = -\sqrt[\mathfrak{f}]{1 - \left|\aleph^{-}_{\mathcal{G}}(\xi)\right|^{\mathfrak{p}} - \left|\mho^{-}_{\mathcal{G}}(\xi)\right|^{\mathfrak{q}}}.$$

The negative sign in $\pi^{-}_{\mathcal{G}}(\xi)$ preserves the negative polarity, so that $\pi^{-}_{\mathcal{G}}(\xi) \in [-1, 0]$.

For simplicity, a value

$$\Xi = \langle(\aleph^{+}, \mho^{+}), (\aleph^{-}, \mho^{-})\rangle$$

satisfying the above admissibility conditions is called a $B(p, q)$-quasirung orthopair fuzzy number ($B(p, q)$-QOFN).

Definition 5. *(Componentwise order) Let*

$$\Xi_1 = \langle(\aleph^{+}_1, \mho^{+}_1), (\aleph^{-}_1, \mho^{-}_1)\rangle$$

and

$$\Xi_2 = \langle(\aleph^{+}_2, \mho^{+}_2), (\aleph^{-}_2, \mho^{-}_2)\rangle$$

be two $B(p, q)$-QOFNs.

We say that Ξ_1 is less than or equal to Ξ_2 under the componentwise order, denoted $\Xi_1 \leq_C \Xi_2$, if the following four conditions hold:

$$(\aleph^{+}_1)^{\mathfrak{p}} \leq (\aleph^{+}_2)^{\mathfrak{p}}, \tag{4.1}$$

$$(\mho^{+}_1)^{\mathfrak{q}} \geq (\mho^{+}_2)^{\mathfrak{q}}, \tag{4.2}$$

$$|\aleph^{-}_1|^{\mathfrak{p}} \leq |\aleph^{-}_2|^{\mathfrak{p}}, \tag{4.3}$$

$$|\mho^{-}_1|^{\mathfrak{q}} \geq |\mho^{-}_2|^{\mathfrak{q}}. \tag{4.4}$$

Proposition 1. *The relation $\leq_C$ defined in Definition 5 is a partial order on the set of all $B(p, q)$-QOFNs.*

Proof. Let

$$\Xi_i = \langle(\aleph^{+}_i, \mho^{+}_i), (\aleph^{-}_i, \mho^{-}_i)\rangle, \qquad i = 1, 2, 3,$$

be three B(p, q)-QOFNs. We prove that $\leq_C$ is reflexive, antisymmetric, and transitive.

First, for every B(p, q)-QOFN Ξ_1, we have

$$(\aleph^{+}_1)^{\mathfrak{p}} \leq (\aleph^{+}_1)^{\mathfrak{p}}, \qquad (\mho^{+}_1)^{\mathfrak{q}} \geq (\mho^{+}_1)^{\mathfrak{q}}$$

and

$$|\aleph^{-}_1|^{\mathfrak{p}} \leq |\aleph^{-}_1|^{\mathfrak{p}}, \qquad |\mho^{-}_1|^{\mathfrak{q}} \geq |\mho^{-}_1|^{\mathfrak{q}}.$$

Hence, $\Xi_1 \leq_C \Xi_1$; therefore, $\leq_C$ is reflexive.

Next, suppose that

$$\Xi_1 \leq_C \Xi_2 \qquad \text{and} \qquad \Xi_2 \leq_C \Xi_1.$$

Then

$$(\aleph_1^+)^{\mathfrak{p}} = (\aleph_2^+)^{\mathfrak{p}}, \qquad (\mho_1^+)^{\mathfrak{q}} = (\mho_2^+)^{\mathfrak{q}}$$

and

$$|\aleph_1^-|^{\mathfrak{p}} = |\aleph_2^-|^{\mathfrak{p}}, \qquad |\mho_1^-|^{\mathfrak{q}} = |\mho_2^-|^{\mathfrak{q}}.$$

Since the functions $t \mapsto t^{\mathfrak{p}}$ and $t \mapsto t^{\mathfrak{q}}$ are strictly increasing on $[0, 1]$, it follows that

$$\aleph_1^+ = \aleph_2^+, \qquad \mho_1^+ = \mho_2^+$$

and

$$|\aleph_1^-| = |\aleph_2^-|, \qquad |\mho_1^-| = |\mho_2^-|.$$

Because $\aleph_i^-, \mho_i^- \in [-1, 0]$, the equality of their absolute values implies

$$\aleph_1^- = \aleph_2^-, \qquad \mho_1^- = \mho_2^-.$$

Therefore, $\Xi_1 = \Xi_2$, so $\leq_C$ is antisymmetric.

Finally, suppose that

$$\Xi_1 \leq_C \Xi_2 \qquad \text{and} \qquad \Xi_2 \leq_C \Xi_3.$$

Then

$$\begin{aligned}(\aleph_1^+)^{\mathfrak{p}} &\leq (\aleph_2^+)^{\mathfrak{p}} \leq (\aleph_3^+)^{\mathfrak{p}},\\ (\mho_1^+)^{\mathfrak{q}} &\geq (\mho_2^+)^{\mathfrak{q}} \geq (\mho_3^+)^{\mathfrak{q}},\\ |\aleph_1^-|^{\mathfrak{p}} &\leq |\aleph_2^-|^{\mathfrak{p}} \leq |\aleph_3^-|^{\mathfrak{p}}\end{aligned}$$

and

$$|\mho_1^-|^{\mathfrak{q}} \geq |\mho_2^-|^{\mathfrak{q}} \geq |\mho_3^-|^{\mathfrak{q}}.$$

Thus

$$(\aleph_1^+)^{\mathfrak{p}} \leq (\aleph_3^+)^{\mathfrak{p}}, \qquad (\mho_1^+)^{\mathfrak{q}} \geq (\mho_3^+)^{\mathfrak{q}}$$

and

$$|\aleph_1^-|^{\mathfrak{p}} \leq |\aleph_3^-|^{\mathfrak{p}}, \qquad |\mho_1^-|^{\mathfrak{q}} \geq |\mho_3^-|^{\mathfrak{q}}.$$

Therefore

$$\Xi_1 \leq_C \Xi_3,$$

and hence $\leq_C$ is transitive.

Consequently, $\leq_C$ is a partial order on the set of all B(p, q)-QOFNs. □

Remark 1. *The relation $\leq_C$ is a componentwise partial order defined by the four inequalities in Definition 5. It should be distinguished from the preference ranking induced by the score function introduced in Section 4.5. In particular, the componentwise order and the score-based ranking need not induce the same comparison between two $B(p,q)$-QOFNs.*

Example 1. *Consider two $B(p,q)$-QOFNs describing two suppliers under a criterion for reliable delivery. Suppose that one supplier has stronger positive membership but also stronger negative membership, while the other has weaker positive membership but weaker negative membership. Then one supplier is preferable with respect to the positive pole, whereas the other is preferable with respect to the negative pole. Because neither of the four componentwise inequalities is satisfied simultaneously, the two $B(p,q)$-QOFNs are incomparable under $\leq_C$. This situation reflects a genuine decision trade-off rather than a failure of the ordering relation, and it motivates the use of the pole-weighted score function s_α introduced in Section 4.5.*

Remark 2. *When there is no ambiguity, we write $\Xi = \langle(\aleph^+, \mho^+), (\aleph^-, \mho^-)\rangle$ instead of writing all component functions at a fixed element $\xi \in \mathbb{X}$.*

Remark 3. *For a given bipolar orthopair fuzzy number, the parameters $\mathfrak{p}$ and $\mathfrak{q}$ should be chosen so that both the positive and negative admissibility conditions are satisfied, namely $(\aleph^+)^{\mathfrak{p}} + (\mho^+)^{\mathfrak{q}} \leq 1$ and $|\aleph^-|^{\mathfrak{p}} + |\mho^-|^{\mathfrak{q}} \leq 1$. Since all powered components are taken from the interval $[0,1]$, increasing $\mathfrak{p}$ or $\mathfrak{q}$ generally enlarges the admissible region. Hence, for a fixed bipolar orthopair, more than one admissible pair $(\mathfrak{p},\mathfrak{q})$ may exist.*

4.2. Relation with existing models

The proposed B(p,q)-QOFS encompasses several established fuzzy models as special instances. This demonstrates that the proposed structure is not isolated from the existing literature, but rather extends known orthopair fuzzy models by combining bipolar information with asymmetric (p,q)-quasirung constraints.

For brevity, BIFS, BPyFS, BFFS, and Bq-ROFS denote bipolar intuitionistic fuzzy set, bipolar Pythagorean fuzzy set, bipolar Fermatean fuzzy set, and bipolar q-rung orthopair fuzzy set, respectively.

Corollary 1. *A $B(p,q)$-QOFS reduces to*

(1) A BIFS by taking $\mathfrak{p} = \mathfrak{q} = 1$ [26, 31].
(2) A BPyFS by taking $\mathfrak{p} = \mathfrak{q} = 2$ [27, 28].
(3) A BFFS by taking $\mathfrak{p} = \mathfrak{q} = 3$ [32].
(4) A Bq-ROFS by taking $\mathfrak{p} = \mathfrak{q}$ [49].
(5) An ordinary (p,q)-QOFS by taking the negative pole to be null, that is $\aleph^- = \mho^- = 0$, and then omitting the null negative pole [16, 46, 47].

Proof. If $\mathfrak{p} = \mathfrak{q} = 1$, then the admissibility conditions become $\aleph^+ + \mho^+ \leq 1$ and $|\aleph^-| + |\mho^-| \leq 1$. Thus, the proposed model reduces to a BIFS.

If $\mathfrak{p} = \mathfrak{q} = 2$, the conditions become $(\aleph^+)^2 + (\mho^+)^2 \leq 1$ and $|\aleph^-|^2 + |\mho^-|^2 \leq 1$. Thus, the proposed model reduces to a BPyFS.

If $\mathfrak{p} = \mathfrak{q} = 3$, the model becomes a BFFS.

When $\mathfrak{p} = \mathfrak{q}$, the exponent that controls the PMD and PNMD is the same, and the model reduces to a Bq-ROFS.

If $\aleph^- = \mho^- = 0$, then the negative pole carries no information. After omitting this null negative pole, the proposed model reduces to the ordinary (p, q)-QOFS.

Therefore, the proposed B(p, q)-QOFS generalizes several existing fuzzy structures. □

The proposed B(p, q)-QOFS should be distinguished from the bipolar (n, m)-rung orthopair fuzzy framework reported in the literature [30]. Both frameworks use independent exponents for the two components within each pole; therefore, replacing (n, m) by (p, q) alone does not constitute the main structural contribution of this study. The distinction of the present framework lies in two aspects. First, the proposed AA operational laws are formulated so that the joint admissibility of the positive and negative poles is preserved under every operation. Second, the ranking mechanism treats the two poles separately and allows their relative importance to be explicitly controlled by the pole importance parameter α, as discussed in Section 4.5. These features provide an operational and ranking framework that is not obtained by a direct replacement of (n, m) with (p, q) in the existing bipolar orthopair representation.

4.3. Inclusion property of B(p, q)-QOFSs

For any $\mathfrak{a} \in [0, 1]$ and the natural numbers $r > s$, we have $\mathfrak{a}^r \leq \mathfrak{a}^s$. The inequality is strict whenever $0 < \mathfrak{a} < 1$. This observation leads to the following inclusion property.

Theorem 1. *If $\mathcal{G}$ is a B(p_1, q_1)-QOFS, then for every $p_2 \geq p_1$ and $q_2 \geq q_1$, $\mathcal{G}$ is also a B(p_2, q_2)-QOFS.*

Proof. Since $\mathcal{G}$ is a B(p_1, q_1)-QOFS, we have $\aleph^+_{\mathcal{G}}(\xi)^{p_1} + \mho^+_{\mathcal{G}}(\xi)^{q_1} \leq 1$ for every $\xi \in \mathbb{X}$.

Since $\aleph^+_{\mathcal{G}}(\xi)^p, \mho^+_{\mathcal{G}}(\xi)^q \in [0, 1]$, and since $p_2 \geq p_1$ and $q_2 \geq q_1$, it follows that $\aleph^+_{\mathcal{G}}(\xi)^{p_2} \leq \aleph^+_{\mathcal{G}}(\xi)^{p_1}$ and $\mho^+_{\mathcal{G}}(\xi)^{q_2} \leq \mho^+_{\mathcal{G}}(\xi)^{q_1}$. Therefore, $\aleph^+_{\mathcal{G}}(\xi)^{p_2} + \mho^+_{\mathcal{G}}(\xi)^{q_2} \leq 1$.

The same argument applies to the negative pole because $|\aleph^-_{\mathcal{G}}(\xi)|^p, |\mho^-_{\mathcal{G}}(\xi)|^q \in [0, 1]$.

Hence, $|\aleph^-_{\mathcal{G}}(\xi)|^{p_2} + |\mho^-_{\mathcal{G}}(\xi)|^{q_2} \leq 1$. Thus, $\mathcal{G}$ is also a B(p_2, q_2)-QOFS. □

4.4. Comparison of admissible regions

In this subsection, we compare the admissible region of the proposed B(p, q)-QOFS with that of Bq-ROFSs. The comparison is understood componentwise for the positive pole and for the absolute values of the negative pole.

Let $R_{(p,q)} = \{(\mathfrak{a}, \mathfrak{b}) \in [0, 1]^2 \colon \mathfrak{a}^{\mathfrak{p}} + \mathfrak{b}^{\mathfrak{q}} \leq 1\}$. The bipolar admissible region of the B(p, q)-QOFS is governed by $R_{(p,q)}$ for the positive pole and by the same region for the absolute values of the negative pole.

Theorem 2. *If $p_1 < q_1$, then $R_{(p_1,p_1)} \subseteq R_{(p_1,q_1)} \subseteq R_{(q_1,q_1)}$.*

Equivalently, the admissible region of the Bq-ROFS with $q = p_1$ is contained in the admissible region of B(p_1, q_1)-QOFS, and the admissible region of B(p_1, q_1)-QOFS is contained in the admissible region of the Bq-ROFS with $q = q_1$.

Proof. Since $p_1 < q_1$, for every $\mathfrak{a}, \mathfrak{b} \in [0, 1]$, we have $\mathfrak{a}^{q_1} \leq \mathfrak{a}^{p_1}$ and $\mathfrak{b}^{q_1} \leq \mathfrak{b}^{p_1}$. If $\mathfrak{a}^{p_1} + \mathfrak{b}^{p_1} \leq 1$, then $\mathfrak{a}^{p_1} + \mathfrak{b}^{q_1} \leq \mathfrak{a}^{p_1} + \mathfrak{b}^{p_1} \leq 1$. Thus, $R_{(p_1,p_1)} \subseteq R_{(p_1,q_1)}$. Moreover, if $\mathfrak{a}^{p_1} + \mathfrak{b}^{q_1} \leq 1$, since $\mathfrak{a}^{q_1} \leq \mathfrak{a}^{p_1}$, then $\mathfrak{a}^{q_1} + \mathfrak{b}^{q_1} \leq \mathfrak{a}^{p_1} + \mathfrak{b}^{q_1} \leq 1$. Thus, $R_{(p_1,q_1)} \subseteq R_{(q_1,q_1)}$.

The same reasoning applies to the negative pole using the absolute values of the negative components. Hence, the proof is complete. □

Example 2. *Let*

$$\Xi = \langle (0.81, 0.61), (-0.70, -0.50) \rangle$$

*be a B(*2, 3*)-QOFN.*

For the positive pole, we have

$$(0.81)^2 + (0.61)^3 = 0.6561 + 0.226981 = 0.883081 < 1.$$

However, we have

$$(0.81)^2 + (0.61)^2 = 0.6561 + 0.3721 = 1.0282 > 1.$$

Thus, the positive orthopair is admissible for $(\mathfrak{p}, \mathfrak{q}) = (2, 3)$*; however, it is not admissible under the symmetric q-rung case with* $q = 2$*. For the negative pole, we have*

$$|-0.70|^2 + |-0.50|^3 = 0.49 + 0.125 = 0.615 < 1.$$

Hence, the bipolar value Ξ *is admissible under the B(*2, 3*)-QOFS framework. This example illustrates the additional flexibility obtained by using two independent exponents* $\mathfrak{p}$ *and* $\mathfrak{q}$*.*

4.5. Ranking of B(p, q)*-QOFNs*

To compare B(p, q)-QOFNs, we define the bipolar score and accuracy functions based on [16]. The normalized score function serves as the primary ranking index, while the accuracy function is used to break ties.

Definition 6. *(The positive and negative poles' preference components) The positive and negative poles' preference components are defined by the following expressions:*

$$P(\Xi) = (\aleph^+)^{\mathfrak{p}} - (\mho^+)^{\mathfrak{q}}$$

and

$$N(\Xi) = |\mho^-|^{\mathfrak{q}} - |\aleph^-|^{\mathfrak{p}}.$$

Proposition 2. *For any B*(p, q)*-QOFN* Ξ*, the preference components satisfy*

$$-1 \leq P(\Xi) \leq 1, \qquad -1 \leq N(\Xi) \leq 1.$$

Proof. By the positive pole admissibility condition, we have

$$(\aleph^+)^{\mathfrak{p}} + (\mho^+)^{\mathfrak{q}} \leq 1.$$

Since both powered components are non-negative, this implies

$$-1 \leq (\aleph^+)^{\mathfrak{p}} - (\mho^+)^{\mathfrak{q}} \leq 1.$$

Hence, $-1 \leq P(\Xi) \leq 1$. The same argument applied to the negative pole gives $-1 \leq N(\Xi) \leq 1$. □

Definition 7. *(Pole-weighted score function) Let $\alpha \in [0,1]$ be a pole importance parameter. The pole-weighted score function is defined by*

$$s_\alpha(\Xi) = \alpha P(\Xi) + (1-\alpha)N(\Xi).$$

The parameter α represents the relative importance assigned to the positive pole, whereas $1-\alpha$ represents the relative importance assigned to the negative pole. This parameter is independent of the criterion weights used in the subsequent aggregation procedure.

Proposition 3. *For any $B(p,q)$-QOFN Ξ and any $\alpha \in [0,1]$, we have*

$$-1 \le s_\alpha(\Xi) \le 1.$$

Proof. By Proposition 2, both $P(\Xi)$ and $N(\Xi)$ belong to $[-1,1]$. Since $\alpha \in [0,1]$, the convex combination $\alpha P(\Xi) + (1-\alpha)N(\Xi)$ also belongs to $[-1,1]$. Hence, $-1 \le s_\alpha(\Xi) \le 1$. □

Definition 8. *(Normalized pole-weighted score function) For applications requiring a score value in $[0,1]$, the normalized pole-weighted score function is defined by*

$$\Theta_\alpha(\Xi) = \frac{1 + s_\alpha(\Xi)}{2}.$$

Proposition 4. *For any $B(p,q)$-QOFN Ξ and any $\alpha \in [0,1]$, we have*

$$0 \le \Theta_\alpha(\Xi) \le 1.$$

Proof. By Proposition 3, we have $-1 \le s_\alpha(\Xi) \le 1$. Adding 1 and dividing by 2 gives $0 \le \Theta_\alpha(\Xi) \le 1$. □

Proposition 5. *(Monotonicity with respect to the negative pole) Let Ξ be a $B(p,q)$-QOFN and let $\alpha \in [0,1)$. If the magnitude of the negative membership component $|\aleph^-|$ increases within the admissible region while the other components remain fixed, then $s_\alpha(\Xi)$ strictly decreases. If the magnitude of the negative nonmembership component $|\mho^-|$ increases within the admissible region while the other components remain fixed, then $s_\alpha(\Xi)$ strictly increases.*

Proof. Consider an increase in $|\aleph^-|$ within the admissible region while $\aleph^+$, $\mho^+$, and $\mho^-$ remain fixed. Since $|\aleph^-| \in [0,1]$ and $\mathfrak{p} \ge 1$, the term $-|\aleph^-|^{\mathfrak{p}}$ in $N(\Xi)$ strictly decreases. Since $1-\alpha > 0$, the weighted contribution $(1-\alpha)N(\Xi)$ strictly decreases. The positive pole's contribution $\alpha P(\Xi)$ is unchanged. Hence, $s_\alpha(\Xi)$ strictly decreases.

Now consider an increase in $|\mho^-|$ within the admissible region while $\aleph^+$, $\mho^+$, and $\aleph^-$ remain fixed. Since $|\mho^-| \in [0,1]$ and $\mathfrak{q} \ge 1$, the term $|\mho^-|^{\mathfrak{q}}$ in $N(\Xi)$ strictly increases. Since $1-\alpha > 0$, the weighted contribution $(1-\alpha)N(\Xi)$ strictly increases. The positive pole's contribution is unchanged. Hence, $s_\alpha(\Xi)$ strictly increases. □

Definition 9. *(Accuracy function) Let $\Xi = \langle(\aleph^+, \mho^+), (\aleph^-, \mho^-)\rangle$ be a $B(p,q)$-QOFN. The accuracy function Υ is defined by*

$$\Upsilon(\Xi) = \frac{1}{2}\left[(\aleph^+)^{\mathfrak{p}} + (\mho^+)^{\mathfrak{q}} + |\aleph^-|^{\mathfrak{p}} + |\mho^-|^{\mathfrak{q}}\right].$$

Proposition 6. *For any $B(p,q)$-QOFN Ξ, the accuracy function satisfies*

$$0 \leq \Upsilon(\Xi) \leq 1.$$

Proof. By the admissibility conditions, we have

$$0 \leq (\aleph^+)^p + (\mho^+)^q \leq 1$$

and

$$0 \leq |\aleph^-|^p + |\mho^-|^q \leq 1.$$

Therefore

$$0 \leq (\aleph^+)^p + (\mho^+)^q + |\aleph^-|^p + |\mho^-|^q \leq 2.$$

Multiplying by $\frac{1}{2}$, we get

$$0 \leq \Upsilon(\Xi) \leq 1.$$

This completes the proof. □

Definition 10. *(Comparison rule) Let Ξ_1 and Ξ_2 be two $B(p,q)$-QOFNs, and let $\alpha \in [0,1]$. Then we have the following.*

- *If $\Theta_\alpha(\Xi_1) < \Theta_\alpha(\Xi_2)$, then $\Xi_1 \prec \Xi_2$.*
- *If $\Theta_\alpha(\Xi_1) > \Theta_\alpha(\Xi_2)$, then $\Xi_1 \succ \Xi_2$.*
- *If $\Theta_\alpha(\Xi_1) = \Theta_\alpha(\Xi_2)$ and $\Upsilon(\Xi_1) < \Upsilon(\Xi_2)$, then $\Xi_1 \prec \Xi_2$.*
- *If $\Theta_\alpha(\Xi_1) = \Theta_\alpha(\Xi_2)$ and $\Upsilon(\Xi_1) > \Upsilon(\Xi_2)$, then $\Xi_1 \succ \Xi_2$.*
- *If $\Theta_\alpha(\Xi_1) = \Theta_\alpha(\Xi_2)$ and $\Upsilon(\Xi_1) = \Upsilon(\Xi_2)$, then $\Xi_1 \simeq \Xi_2$.*

Pole agreement, pole conflict, and preference switching. The decomposition of the pole-weighted score into the two pole components P and N allows the ranking between any two B(p,q)-QOFNs to be analyzed in terms of the behavior of each pole separately. This analysis is useful when the two poles provide different or even opposing indications.

Definition 11. *(Pole agreement and pole conflict) Let Ξ_1, Ξ_2 be two $B(p,q)$-QOFNs, and define*

$$\Delta P_{12} = P(\Xi_1) - P(\Xi_2), \qquad \Delta N_{12} = N(\Xi_1) - N(\Xi_2).$$

If $\Delta P_{12} \cdot \Delta N_{12} > 0$, the two poles are in agreement with respect to the preference between Ξ_1 and Ξ_2. If $\Delta P_{12} \cdot \Delta N_{12} < 0$, the two poles are in conflict.

When the two poles are in agreement, the sign of $s_\alpha(\Xi_1)$ and $s_\alpha(\Xi_2)$ is independent of α. When the two poles are in conflict, the preference between the two numbers depends on α, and there is a unique value α^* at which the two numbers are equally preferred.

Theorem 3. *(Preference-switching threshold) Let Ξ_1, Ξ_2 be two $B(p,q)$-QOFNs whose poles are in conflict, that is, $\Delta P_{12} \cdot \Delta N_{12} < 0$. Then the pole-weighted score functions satisfy $s_\alpha(\Xi_1) = s_\alpha(\Xi_2)$ if and only if*

$$\alpha = \alpha^*_{12} = \frac{-\Delta N_{12}}{\Delta P_{12} - \Delta N_{12}}.$$

At $\alpha = \alpha^*_{12}$, the two $B(p,q)$-QOFNs have equal pole-weighted scores. As α crosses α^*_{12}, the sign of $s_\alpha(\Xi_1) - s_\alpha(\Xi_2)$ changes, and hence the preference between the two numbers is reversed.

Proof. Since $s_\alpha(\Xi) = \alpha P(\Xi) + (1-\alpha)N(\Xi)$, we have

$$s_\alpha(\Xi_1) - s_\alpha(\Xi_2) = \alpha \Delta P_{12} + (1-\alpha)\Delta N_{12}.$$

Setting this expression equal to zero gives

$$\alpha \Delta P_{12} + (1-\alpha)\Delta N_{12} = 0.$$

Rearranging, we have

$$\alpha(\Delta P_{12} - \Delta N_{12}) = -\Delta N_{12}.$$

Since the two poles are in conflict, $\Delta P_{12} - \Delta N_{12} \neq 0$, so

$$\alpha^*_{12} = \frac{-\Delta N_{12}}{\Delta P_{12} - \Delta N_{12}}.$$

Because $\Delta P_{12} \cdot \Delta N_{12} < 0$, this value lies strictly between 0 and 1, so it is a valid pole importance parameter. □

Example 3. *Consider a hypothetical decision between two treatments, A and B. The positive pole represents therapeutic effectiveness, whereas the negative pole represents the occurrence of adverse effects. Treatment A has stronger therapeutic effectiveness but is associated with a less favorable adverse effect profile, while Treatment B has lower therapeutic effectiveness but a more favorable adverse effect profile. Thus, the decision involves a conflict between the treatment's effectiveness and adverse effects. Let*

$$\Xi_A = \langle (0.92, 0.30), (-0.78, -0.25) \rangle, \qquad \Xi_B = \langle (0.76, 0.42), (-0.40, -0.62) \rangle$$

be two $B(3,2)$-QOFNs. Their pole-specific preference components are

$$P(\Xi_A) = 0.688688 > P(\Xi_B) = 0.262576,$$

showing that the positive pole favors Treatment A. In contrast,

$$N(\Xi_A) = -0.412052 < N(\Xi_B) = 0.320400,$$

so the negative pole favors Treatment B. Therefore, the two poles lead to opposite preferences. At equal pole importance, $\alpha = 0.5$, we have

$$\Theta_{0.5}(\Xi_A) = 0.569159, \qquad \Theta_{0.5}(\Xi_B) = 0.645744,$$

and hence Treatment B is preferred. According to Theorem 3, the preference-switching threshold is

$$\alpha^* = 0.632207.$$

Consequently, Treatment B is preferred when $\alpha < 0.632207$*, whereas Treatment A is preferred when* $\alpha > 0.632207$*; the two treatments are equally preferred at* $\alpha = 0.632207$*. This example demonstrates the practical role of* α*: When the two poles provide conflicting evidence, it explicitly controls their relative importance and identifies the point at which the preferred decision changes.*

4.6. Complement of a B(p, q)-QOFN

Definition 12. *Let*

$$\Xi = \langle(\aleph^+, \mho^+), (\aleph^-, \mho^-)\rangle$$

be a B(p, q)-QOFN. The complement of Ξ*, denoted* $\Xi^{\mathfrak{c}}$*, is defined as*

$$\Xi^{\mathfrak{c}} = \left\langle\left((\mho^+)^{\mathfrak{q}/\mathfrak{p}}, (\aleph^+)^{\mathfrak{p}/\mathfrak{q}}\right), \left(-|\mho^-|^{\mathfrak{q}/\mathfrak{p}}, -|\aleph^-|^{\mathfrak{p}/\mathfrak{q}}\right)\right\rangle. \tag{4.5}$$

The exponent transformations $\mathfrak{q}/\mathfrak{p}$ *and* $\mathfrak{p}/\mathfrak{q}$ *are required because the membership and nonmembership components within each pole are governed by the distinct exponents* $\mathfrak{p}$ *and* $\mathfrak{q}$*. They ensure that, after complementation, the powered components reproduce the original admissibility sums and therefore preserve the B(p, q)-QOFN domain.*

Proposition 7. *Let* Ξ *be a B(p, q)-QOFN. Then the complement operation satisfies the following properties:*

(1) $\Xi^{\mathfrak{c}}$ *is a B(p, q)-QOFN;*
(2) $(\Xi^{\mathfrak{c}})^{\mathfrak{c}} = \Xi$*;*
(3) $P(\Xi^{\mathfrak{c}}) = -P(\Xi)$ *and* $N(\Xi^{\mathfrak{c}}) = -N(\Xi)$*;*
(4) $s_\alpha(\Xi^{\mathfrak{c}}) = -s_\alpha(\Xi)$ *and* $\Theta_\alpha(\Xi^{\mathfrak{c}}) = 1 - \Theta_\alpha(\Xi)$*;*
(5) $\Upsilon(\Xi^{\mathfrak{c}}) = \Upsilon(\Xi)$*;*
(6) If $\Xi_1 \leq_C \Xi_2$*, then*

$$\Xi_2^{\mathfrak{c}} \leq_C \Xi_1^{\mathfrak{c}}.$$

Proof. Let

$$\Xi = \langle(\aleph^+, \mho^+), (\aleph^-, \mho^-)\rangle$$

be a B(p, q)-QOFN. Hence

$$(\aleph^+)^{\mathfrak{p}} + (\mho^+)^{\mathfrak{q}} \leq 1$$

and

$$|\aleph^-|^{\mathfrak{p}} + |\mho^-|^{\mathfrak{q}} \leq 1.$$

For the positive pole of Ξ^c, we have

$$\left((\mho^+)^{q/p}\right)^p + \left((\aleph^+)^{p/q}\right)^q = (\mho^+)^q + (\aleph^+)^p \leq 1.$$

For the negative pole, we have

$$\left|-|\mho^-|^{q/p}\right|^p + \left|-|\aleph^-|^{p/q}\right|^q = |\mho^-|^q + |\aleph^-|^p \leq 1.$$

Therefore, Ξ^c is a B(p, q)-QOFN.

Next, applying the complement operation twice gives

$$(\Xi^c)^c = \left\langle \left(\left((\aleph^+)^{p/q}\right)^{q/p}, \left((\mho^+)^{q/p}\right)^{p/q} \right), \left(-\left||\aleph^-|^{p/q}\right|^{q/p}, -\left||\mho^-|^{q/p}\right|^{p/q} \right) \right\rangle.$$

Since

$$\left((\aleph^+)^{p/q}\right)^{q/p} = \aleph^+, \qquad \left((\mho^+)^{q/p}\right)^{p/q} = \mho^+,$$

and similarly

$$-\left(|\aleph^-|^{p/q}\right)^{q/p} = -|\aleph^-| = \aleph^-,$$
$$-\left(|\mho^-|^{q/p}\right)^{p/q} = -|\mho^-| = \mho^-,$$

we obtain

$$(\Xi^c)^c = \Xi.$$

Using the definitions of P and N, we have

$$P(\Xi^c) = \left((\mho^+)^{q/p}\right)^p - \left((\aleph^+)^{p/q}\right)^q = (\mho^+)^q - (\aleph^+)^p = -P(\Xi).$$

Similarly

$$N(\Xi^c) = \left||\aleph^-|^{p/q}\right|^q - \left||\mho^-|^{q/p}\right|^p = |\aleph^-|^p - |\mho^-|^q = -N(\Xi).$$

Consequently

$$s_\alpha(\Xi^c) = \alpha P(\Xi^c) + (1-\alpha)N(\Xi^c) = -\alpha P(\Xi) - (1-\alpha)N(\Xi) = -s_\alpha(\Xi).$$

Therefore

$$\Theta_\alpha(\Xi^c) = \frac{1 + s_\alpha(\Xi^c)}{2} = \frac{1 - s_\alpha(\Xi)}{2} = 1 - \Theta_\alpha(\Xi).$$

Moreover

$$\Upsilon(\Xi^c) = \frac{1}{2}\left[(\mho^+)^q + (\aleph^+)^p + |\aleph^-|^p + |\mho^-|^q\right] = \Upsilon(\Xi).$$

Finally, suppose that

$$\Xi_1 \leq_C \Xi_2.$$

Then

$$(\aleph_1^+)^{\mathfrak{p}} \leq (\aleph_2^+)^{\mathfrak{p}}, \qquad (\mho_1^+)^{\mathfrak{q}} \geq (\mho_2^+)^{\mathfrak{q}}$$

and

$$|\aleph_1^-|^{\mathfrak{p}} \leq |\aleph_2^-|^{\mathfrak{p}}, \qquad |\mho_1^-|^{\mathfrak{q}} \geq |\mho_2^-|^{\mathfrak{q}}.$$

After applying the complement operation, these inequalities become

$$(\mho_2^+)^{\mathfrak{q}} \leq (\mho_1^+)^{\mathfrak{q}}, \qquad (\aleph_2^+)^{\mathfrak{p}} \geq (\aleph_1^+)^{\mathfrak{p}}$$

and

$$|\mho_2^-|^{\mathfrak{q}} \leq |\mho_1^-|^{\mathfrak{q}}, \qquad |\aleph_2^-|^{\mathfrak{p}} \geq |\aleph_1^-|^{\mathfrak{p}}.$$

Hence

$$\Xi_2^{\mathfrak{c}} \leq_C \Xi_1^{\mathfrak{c}}.$$

This completes the proof. □

4.7. AA *operational laws for B*(p, q)*-QOFNs*

In this section, we define AA operational laws for B(p, q)-QOFNs. The construction is based on the AA t-norm and t-conorm introduced in [35–37] and on subsequent AA aggregation developments (see [38, 44, 46]). The (p, q)-quasirung orthopair setting is adopted from [47]. These operations are introduced on the powered components to preserve the (p, q)-quasirung admissibility conditions.

Let $\lambda > 0$. For any $\mathfrak{a}, \mathfrak{b} \in [0, 1]$, the AA t-norm T_{AA}^{λ} is defined by

$$T_{AA}^{\lambda}(\mathfrak{a}, \mathfrak{b}) = e^{-\sqrt[\lambda]{(-\ln \mathfrak{a})^{\lambda}+(-\ln \mathfrak{b})^{\lambda}}}.$$

The AA t-conorm S_{AA}^{λ} is defined by

$$S_{AA}^{\lambda}(\mathfrak{a}, \mathfrak{b}) = 1 - e^{-\sqrt[\lambda]{(-\ln(1-\mathfrak{a}))^{\lambda}+(-\ln(1-\mathfrak{b}))^{\lambda}}}.$$

The logarithmic forms of T_{AA}^{λ} and S_{AA}^{λ} are used for arguments in $(0, 1)$. At the boundary points 0 and 1, the operations are understood through their endpoint values or, equivalently, through the limiting values of the corresponding AA t-norm and t-conorm [35–37].

Moreover, the AA t-conorm is the De Morgan dual of the AA t-norm, that is

$$S_{AA}^{\lambda}(\mathfrak{a}, \mathfrak{b}) = 1 - T_{AA}^{\lambda}(1 - \mathfrak{a}, 1 - \mathfrak{b}).$$

Definition 13. *Let*

$$\Xi_1 = \langle(\aleph_1^+, \mho_1^+), (\aleph_1^-, \mho_1^-)\rangle$$

and

$$\Xi_2 = \langle(\aleph_2^+, \mho_2^+), (\aleph_2^-, \mho_2^-)\rangle$$

be two $B(p,q)$-QOFNs. Then the AA *operational laws on $B(p,q)$-QOFNs are defined as follows.*

(i) AA ***addition:*** *The* AA *addition of Ξ_1 and Ξ_2 is given by*

$$\Xi_1 \oplus_{AA} \Xi_2 = \langle(\aleph_\oplus^+, \mho_\oplus^+), (\aleph_\oplus^-, \mho_\oplus^-)\rangle,$$

where

$$\aleph_\oplus^+ = \sqrt[\mathfrak{p}]{S_{AA}^{\lambda}\left((\aleph_1^+)^{\mathfrak{p}}, (\aleph_2^+)^{\mathfrak{p}}\right)}, \tag{4.6}$$

$$\mho_\oplus^+ = \sqrt[\mathfrak{q}]{T_{AA}^{\lambda}\left((\mho_1^+)^{\mathfrak{q}}, (\mho_2^+)^{\mathfrak{q}}\right)}, \tag{4.7}$$

$$\aleph_\oplus^- = -\sqrt[\mathfrak{p}]{S_{AA}^{\lambda}\left(|\aleph_1^-|^{\mathfrak{p}}, |\aleph_2^-|^{\mathfrak{p}}\right)}, \tag{4.8}$$

$$\mho_\oplus^- = -\sqrt[\mathfrak{q}]{T_{AA}^{\lambda}\left(|\mho_1^-|^{\mathfrak{q}}, |\mho_2^-|^{\mathfrak{q}}\right)}. \tag{4.9}$$

(ii) AA ***multiplication:*** *The* AA *multiplication of Ξ_1 and Ξ_2 is given by*

$$\Xi_1 \otimes_{AA} \Xi_2 = \langle(\aleph_\otimes^+, \mho_\otimes^+), (\aleph_\otimes^-, \mho_\otimes^-)\rangle,$$

where

$$\aleph_\otimes^+ = \sqrt[\mathfrak{p}]{T_{AA}^{\lambda}\left((\aleph_1^+)^{\mathfrak{p}}, (\aleph_2^+)^{\mathfrak{p}}\right)}, \tag{4.10}$$

$$\mho_\otimes^+ = \sqrt[\mathfrak{q}]{S_{AA}^{\lambda}\left((\mho_1^+)^{\mathfrak{q}}, (\mho_2^+)^{\mathfrak{q}}\right)}, \tag{4.11}$$

$$\aleph_\otimes^- = -\sqrt[\mathfrak{p}]{T_{AA}^{\lambda}\left(|\aleph_1^-|^{\mathfrak{p}}, |\aleph_2^-|^{\mathfrak{p}}\right)}, \tag{4.12}$$

$$\mho_\otimes^- = -\sqrt[\mathfrak{q}]{S_{AA}^{\lambda}\left(|\mho_1^-|^{\mathfrak{q}}, |\mho_2^-|^{\mathfrak{q}}\right)}. \tag{4.13}$$

(iii) AA ***scalar multiplication:*** *For $\tau > 0$, the* AA *scalar multiplication of Ξ_1 is defined by*

$$\tau \cdot_{AA} \Xi_1 = \langle(\aleph_\tau^+, \mho_\tau^+), (\aleph_\tau^-, \mho_\tau^-)\rangle,$$

where

$$\aleph_\tau^+ = \sqrt[\mathfrak{p}]{1 - \left(1 - (\aleph_1^+)^{\mathfrak{p}}\right)^{\tau^{1/\lambda}}}, \tag{4.14}$$

$$\mho_\tau^+ = \sqrt[\mathfrak{q}]{\left((\mho_1^+)^{\mathfrak{q}}\right)^{\tau^{1/\lambda}}}, \tag{4.15}$$

$$\aleph_\tau^- = -\sqrt[\mathfrak{p}]{1 - \left(1 - |\aleph_1^-|^{\mathfrak{p}}\right)^{\tau^{1/\lambda}}}, \tag{4.16}$$

$$\mho_\tau^- = -\sqrt[\mathfrak{q}]{\left(|\mho_1^-|^{\mathfrak{q}}\right)^{\tau^{1/\lambda}}}. \tag{4.17}$$

(iv) AA ***power operation:*** *For $\tau > 0$, the* AA *power operation of Ξ_1 is defined by*

$$\Xi_1^{\tau} = \langle (\aleph_{\mathrm{Pow}}^{+}, \mathfrak{U}_{\mathrm{Pow}}^{+}), (\aleph_{\mathrm{Pow}}^{-}, \mathfrak{U}_{\mathrm{Pow}}^{-}) \rangle,$$

where the subscript Pow *denotes the power operation, and*

$$\aleph_{\mathrm{Pow}}^{+} = \sqrt[\mathfrak{p}]{\left((\aleph_1^{+})^{\mathfrak{p}}\right)^{\tau^{1/\lambda}}}, \tag{4.18}$$

$$\mathfrak{U}_{\mathrm{Pow}}^{+} = \sqrt[\mathfrak{q}]{1 - \left(1 - (\mathfrak{U}_1^{+})^{\mathfrak{q}}\right)^{\tau^{1/\lambda}}}, \tag{4.19}$$

$$\aleph_{\mathrm{Pow}}^{-} = -\sqrt[\mathfrak{p}]{\left(|\aleph_1^{-}|^{\mathfrak{p}}\right)^{\tau^{1/\lambda}}}, \tag{4.20}$$

$$\mathfrak{U}_{\mathrm{Pow}}^{-} = -\sqrt[\mathfrak{q}]{1 - \left(1 - |\mathfrak{U}_1^{-}|^{\mathfrak{q}}\right)^{\tau^{1/\lambda}}}. \tag{4.21}$$

Remark 4. *In the addition operation, the membership components are combined through the* AA *t-conorm S_{AA}^{λ}, whereas the nonmembership components are processed using the* AA *t-norm T_{AA}^{λ}. The opposite assignment is adopted for multiplication. For the negative pole components, the corresponding operations are applied to $|\aleph^{-}|$ and $|\mathfrak{U}^{-}|$, and the resulting values are subsequently restored to their negative form. The absolute magnitudes represent the strengths of the negative membership and negative nonmembership degrees on* $[0, 1]$*, consistent with existing bipolar orthopair formulations [30]. Restoring the negative sign after the corresponding root operation ensures that the aggregated components remain in the negative pole* $[-1, 0]$*. At $\lambda = 1$, the* AA *t-norm and t-conorm become, respectively, the algebraic product and algebraic sum. Hence, under this parameter setting, the proposed $B(p, q)$-QOFN operations specialize to the corresponding algebraic operations [35–37] and to the* AA *aggregation setting in [38].*

4.8. Closure properties

In this subsection, we establish that the AA addition, multiplication, scalar multiplication, and power operations defined in Definition 13 preserve the B(p, q)-QOFN structure.

Lemma 1. *Let T_{AA}^{λ} and S_{AA}^{λ} be the* AA *t-norm and t-conorm, respectively. For any $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \in [0, 1]$, if*

$$\mathfrak{a} + \mathfrak{b} \leq 1, \qquad \mathfrak{c} + \mathfrak{d} \leq 1,$$

then

$$S_{AA}^{\lambda}(\mathfrak{a}, \mathfrak{c}) + T_{AA}^{\lambda}(\mathfrak{b}, \mathfrak{d}) \leq 1$$

and

$$T_{AA}^{\lambda}(\mathfrak{a}, \mathfrak{c}) + S_{AA}^{\lambda}(\mathfrak{b}, \mathfrak{d}) \leq 1.$$

Proof. Since $\mathfrak{a} + \mathfrak{b} \leq 1 \quad \text{and} \quad \mathfrak{c} + \mathfrak{d} \leq 1$, we have

$$\mathfrak{b} \leq 1 - \mathfrak{a} \quad \text{and} \quad \mathfrak{d} \leq 1 - \mathfrak{c}.$$

By the monotonicity of T_{AA}^{λ} in each argument, we have

$$T_{AA}^{\lambda}(\mathfrak{b}, \mathfrak{d}) \leq T_{AA}^{\lambda}(1 - \mathfrak{a}, 1 - \mathfrak{c}).$$

Using the De Morgan duality, we have

$$S_{AA}^{\lambda}(\mathfrak{a}, \mathfrak{c}) = 1 - T_{AA}^{\lambda}(1 - \mathfrak{a}, 1 - \mathfrak{c}),$$

we obtain

$$T_{AA}^{\lambda}(\mathfrak{b}, \mathfrak{d}) \leq 1 - S_{AA}^{\lambda}(\mathfrak{a}, \mathfrak{c}).$$

Hence

$$S_{AA}^{\lambda}(\mathfrak{a}, \mathfrak{c}) + T_{AA}^{\lambda}(\mathfrak{b}, \mathfrak{d}) \leq 1.$$

Similarly, from

$$\mathfrak{a} \leq 1 - \mathfrak{b} \quad \text{and} \quad \mathfrak{c} \leq 1 - \mathfrak{d},$$

by the monotonicity of T_{AA}^{λ}, we have

$$T_{AA}^{\lambda}(\mathfrak{a}, \mathfrak{c}) \leq T_{AA}^{\lambda}(1 - \mathfrak{b}, 1 - \mathfrak{d}).$$

Since

$$S_{AA}^{\lambda}(\mathfrak{b}, \mathfrak{d}) = 1 - T_{AA}^{\lambda}(1 - \mathfrak{b}, 1 - \mathfrak{d}),$$

it follows that

$$T_{AA}^{\lambda}(\mathfrak{a}, \mathfrak{c}) \leq 1 - S_{AA}^{\lambda}(\mathfrak{b}, \mathfrak{d}).$$

Therefore

$$T_{AA}^{\lambda}(\mathfrak{a}, \mathfrak{c}) + S_{AA}^{\lambda}(\mathfrak{b}, \mathfrak{d}) \leq 1.$$

This completes the proof. □

Theorem 4. *Let Ξ_1 and Ξ_2 be two $B(p, q)$-QOFNs and let $\tau > 0$. Then*

$$\Xi_1 \oplus_{AA} \Xi_2, \qquad \Xi_1 \otimes_{AA} \Xi_2, \qquad \tau \cdot_{AA} \Xi_1, \qquad \Xi_1^{\tau}$$

are also $B(p, q)$-QOFNs.

Proof. (i) Addition.

First, consider the AA addition. Put

$$\vartheta_i = (\aleph_i^{+})^{\mathfrak{p}}, \qquad \varsigma_i = (\mho_i^{+})^{\mathfrak{q}}, \quad i = 1, 2.$$

Since Ξ_i is a B(p,q)-QOFN, we have

$$\vartheta_i + \varsigma_i \leq 1, \quad i = 1, 2.$$

By Definition 13, the positive components of $\Xi_1 \oplus_{AA} \Xi_2$ are

$$\aleph_{\oplus}^{+} = \sqrt[\mathfrak{p}]{S_{AA}^{\lambda}(\vartheta_1, \vartheta_2)}, \qquad \mho_{\oplus}^{+} = \sqrt[\mathfrak{q}]{T_{AA}^{\lambda}(\varsigma_1, \varsigma_2)}.$$

Since S_{AA}^{λ} and T_{AA}^{λ} take values in $[0, 1]$, we have

$$\aleph_{\oplus}^{+}, \mho_{\oplus}^{+} \in [0, 1].$$

Moreover

$$(\aleph_{\oplus}^{+})^{\mathfrak{p}} + (\mho_{\oplus}^{+})^{\mathfrak{q}} = S_{AA}^{\lambda}(\vartheta_1, \vartheta_2) + T_{AA}^{\lambda}(\varsigma_1, \varsigma_2).$$

By Lemma 1, we have

$$S_{AA}^{\lambda}(\vartheta_1, \vartheta_2) + T_{AA}^{\lambda}(\varsigma_1, \varsigma_2) \leq 1.$$

Hence

$$(\aleph_{\oplus}^{+})^{\mathfrak{p}} + (\mho_{\oplus}^{+})^{\mathfrak{q}} \leq 1.$$

For the negative pole, set

$$\overline{\vartheta}_i = |\aleph_i^{-}|^{\mathfrak{p}}, \qquad \overline{\varsigma}_i = |\mho_i^{-}|^{\mathfrak{q}}, \qquad i = 1, 2.$$

Since Ξ_i is a B(p,q)-QOFN

$$\overline{\vartheta}_i + \overline{\varsigma}_i \leq 1, \quad i = 1, 2.$$

By Definition 13

$$\aleph_{\oplus}^{-} = -\sqrt[\mathfrak{p}]{S_{AA}^{\lambda}(\overline{\vartheta}_1, \overline{\vartheta}_2)}, \qquad \mho_{\oplus}^{-} = -\sqrt[\mathfrak{q}]{T_{AA}^{\lambda}(\overline{\varsigma}_1, \overline{\varsigma}_2)}.$$

Hence

$$\aleph_{\oplus}^{-}, \mho_{\oplus}^{-} \in [-1, 0].$$

Moreover

$$|\aleph_{\oplus}^{-}|^{\mathfrak{p}} + |\mho_{\oplus}^{-}|^{\mathfrak{q}} = S_{AA}^{\lambda}(\overline{\vartheta}_1, \overline{\vartheta}_2) + T_{AA}^{\lambda}(\overline{\varsigma}_1, \overline{\varsigma}_2).$$

Using Lemma 1 again, we get

$$|\aleph_{\oplus}^{-}|^{\mathfrak{p}} + |\mho_{\oplus}^{-}|^{\mathfrak{q}} \leq 1.$$

Thus, $\Xi_1 \oplus_{AA} \Xi_2$ is a B(p,q)-QOFN.

(ii) Multiplication.

Second, consider the AA multiplication. Using the same setting

$$\vartheta_i = (\aleph_i^+)^{\mathfrak{p}}, \qquad \varsigma_i = (\mho_i^+)^{\mathfrak{q}}, \quad i = 1, 2,$$

we have

$$\vartheta_i + \varsigma_i \leq 1, \quad i = 1, 2.$$

By Definition 13, the positive components of $\Xi_1 \otimes_{AA} \Xi_2$ are

$$\aleph_{\otimes}^+ = \sqrt[\mathfrak{p}]{T_{AA}^{\lambda}(\vartheta_1, \vartheta_2)}, \qquad \mho_{\otimes}^+ = \sqrt[\mathfrak{q}]{S_{AA}^{\lambda}(\varsigma_1, \varsigma_2)}.$$

Thus

$$\aleph_{\otimes}^+, \mho_{\otimes}^+ \in [0, 1].$$

Moreover

$$(\aleph_{\otimes}^+)^{\mathfrak{p}} + (\mho_{\otimes}^+)^{\mathfrak{q}} = T_{AA}^{\lambda}(\vartheta_1, \vartheta_2) + S_{AA}^{\lambda}(\varsigma_1, \varsigma_2).$$

By Lemma 1, we have

$$T_{AA}^{\lambda}(\vartheta_1, \vartheta_2) + S_{AA}^{\lambda}(\varsigma_1, \varsigma_2) \leq 1.$$

Thus

$$(\aleph_{\otimes}^+)^{\mathfrak{p}} + (\mho_{\otimes}^+)^{\mathfrak{q}} \leq 1.$$

For the negative pole, let

$$\overline{\vartheta}_i = |\aleph_i^-|^{\mathfrak{p}}, \qquad \overline{\varsigma}_i = |\mho_i^-|^{\mathfrak{q}}, \quad i = 1, 2.$$

Then

$$\overline{\vartheta}_i + \overline{\varsigma}_i \leq 1, \quad i = 1, 2.$$

By Definition 13, we have

$$\aleph_{\otimes}^- = -\sqrt[\mathfrak{p}]{T_{AA}^{\lambda}(\overline{\vartheta}_1, \overline{\vartheta}_2)}, \qquad \mho_{\otimes}^- = -\sqrt[\mathfrak{q}]{S_{AA}^{\lambda}(\overline{\varsigma}_1, \overline{\varsigma}_2)}.$$

Hence

$$\aleph_{\otimes}^-, \mho_{\otimes}^- \in [-1, 0].$$

Moreover

$$|\aleph_{\otimes}^-|^{\mathfrak{p}} + |\mho_{\otimes}^-|^{\mathfrak{q}} = T_{AA}^{\lambda}(\overline{\vartheta}_1, \overline{\vartheta}_2) + S_{AA}^{\lambda}(\overline{\varsigma}_1, \overline{\varsigma}_2).$$

By Lemma 1, we have

$$|\aleph_{\otimes}^-|^{\mathfrak{p}} + |\mho_{\otimes}^-|^{\mathfrak{q}} \leq 1.$$

Thus, $\Xi_1 \otimes_{AA} \Xi_2$ is a B(p, q)-QOFN.

(iii) Scalar multiplication ($\tau \cdot_{AA} \Xi_1$).

Third, consider the AA scalar multiplication. Set

$$\vartheta = (\aleph_1^+)^{\mathfrak{p}}, \qquad \varsigma = (\mho_1^+)^{\mathfrak{q}}, \qquad \rho = \tau^{1/\lambda}.$$

Since $\tau > 0$ and $\lambda > 0$, we have $\rho > 0$. In addition

$$\vartheta + \varsigma \leq 1.$$

By Definition 13, we have

$$\aleph_\tau^+ = \sqrt[\mathfrak{p}]{1 - (1 - \vartheta)^\rho}, \qquad \mho_\tau^+ = \sqrt[\mathfrak{q}]{\varsigma^\rho}.$$

Since $\vartheta, \varsigma \in [0, 1]$, it follows that

$$\aleph_\tau^+, \mho_\tau^+ \in [0, 1].$$

In addition

$$(\aleph_\tau^+)^{\mathfrak{p}} + (\mho_\tau^+)^{\mathfrak{q}} = 1 - (1 - \vartheta)^\rho + \varsigma^\rho.$$

Since $\vartheta + \varsigma \leq 1$, we have $\varsigma \leq 1 - \vartheta$. The function $x \mapsto x^\rho$ is increasing on $[0, 1]$. Hence

$$\varsigma^\rho \leq (1 - \vartheta)^\rho.$$

So

$$1 - (1 - \vartheta)^\rho + \varsigma^\rho \leq 1 - (1 - \vartheta)^\rho + (1 - \vartheta)^\rho = 1.$$

Thus

$$(\aleph_\tau^+)^{\mathfrak{p}} + (\mho_\tau^+)^{\mathfrak{q}} \leq 1.$$

For the negative pole, set

$$\overline{\vartheta} = |\aleph_1^-|^{\mathfrak{p}}, \qquad \overline{\varsigma} = |\mho_1^-|^{\mathfrak{q}}.$$

Since

$$\overline{\vartheta} + \overline{\varsigma} \leq 1.$$

Definition 13 gives

$$\aleph_\tau^- = -\sqrt[\mathfrak{p}]{1 - (1 - \overline{\vartheta})^\rho}, \qquad \mho_\tau^- = -\sqrt[\mathfrak{q}]{(\overline{\varsigma})^\rho}.$$

Thus, we have

$$\aleph_\tau^-, \mho_\tau^- \in [-1, 0].$$

Moreover,

$$|\aleph_\tau^-|^{\mathfrak{p}} + |\mho_\tau^-|^{\mathfrak{q}} = 1 - (1 - \overline{\vartheta})^\rho + (\overline{\varsigma})^\rho.$$

Since $\overline{\varsigma} \leq (1 - \overline{\vartheta})$, we have

$$\overline{\varsigma}^\rho \leq (1 - \overline{\vartheta})^\rho.$$

Therefore

$$|\aleph_\tau^-|^{\mathfrak{p}} + |\mho_\tau^-|^{\mathfrak{q}} = 1 - (1 - \overline{\vartheta})^\rho + (\overline{\varsigma})^\rho \leq 1 - (1 - \overline{\vartheta})^\rho + (1 - \overline{\vartheta})^\rho = 1.$$

In this case

$$|\aleph_\tau^-|^{\mathfrak{p}} + |\mho_\tau^-|^{\mathfrak{q}} \leq 1.$$

Thus, $\tau \cdot_{AA} \Xi_1$ is a B(p, q)-QOFN.

(iv) Power operation (Ξ_1^τ).

Fourth, consider the AA power operation. Let

$$\vartheta = (\aleph_1^+)^{\mathfrak{p}}, \qquad \varsigma = (\mho_1^+)^{\mathfrak{q}}, \qquad \rho = \tau^{1/\lambda}.$$

Then $\rho > 0$ and $\vartheta + \varsigma \leq 1$.
By Definition 13, we have

$$\aleph_{Pow}^+ = \sqrt[\mathfrak{p}]{\vartheta^\rho}, \qquad \mho_{Pow}^+ = \sqrt[\mathfrak{q}]{1 - (1 - \varsigma)^\rho}.$$

Since

$$\aleph_{Pow}^+, \mho_{Pow}^+ \in [0, 1].$$

Moreover,

$$(\aleph_{Pow}^+)^{\mathfrak{p}} + (\mho_{Pow}^+)^{\mathfrak{q}} = \vartheta^\rho + 1 - (1 - \varsigma)^\rho.$$

Since $\vartheta + \varsigma \leq 1$, we have $\vartheta \leq 1 - \varsigma$. Since $x \mapsto x^\rho$ is increasing on $[0, 1]$

$$\vartheta^\rho \leq (1 - \varsigma)^\rho.$$

Therefore

$$(\aleph_{Pow}^+)^{\mathfrak{p}} + (\mho_{Pow}^+)^{\mathfrak{q}} = \vartheta^\rho + 1 - (1 - \varsigma)^\rho \leq (1 - \varsigma)^\rho + 1 - (1 - \varsigma)^\rho = 1.$$

Hence

$$(\aleph_{Pow}^+)^{\mathfrak{p}} + (\mho_{Pow}^+)^{\mathfrak{q}} \leq 1.$$

For the negative pole, set

$$\overline{\vartheta} = |\aleph_1^-|^{\mathfrak{p}}, \qquad \overline{\varsigma} = |\mho_1^-|^{\mathfrak{q}}, \qquad \rho = \tau^{1/\lambda}.$$

Since

$$\overline{\vartheta} + \overline{\varsigma} \leq 1.$$

Definition 13 gives

$$\aleph_{Pow}^{-} = -\sqrt[\mathfrak{p}]{(\overline{\vartheta})^{\rho}}, \qquad \mho_{Pow}^{-} = -\sqrt[\mathfrak{q}]{1-(1-\overline{\varsigma})^{\rho}}.$$

Thus

$$\aleph_{Pow}^{-}, \mho_{Pow}^{-} \in [-1, 0].$$

In addition

$$|\aleph_{Pow}^{-}|^{\mathfrak{p}} + |\mho_{Pow}^{-}|^{\mathfrak{q}} = (\overline{\vartheta})^{\rho} + 1 - (1-\overline{\varsigma})^{\rho}.$$

Since $\overline{\vartheta} + \overline{\varsigma} \leq 1$, we have $\overline{\vartheta} \leq 1 - \overline{\varsigma}$. Since $x \mapsto x^{\rho}$ is increasing on $[0, 1]$, we get

$$\overline{\vartheta}^{\rho} \leq (1-\overline{\varsigma})^{\rho}.$$

In this case

$$|\aleph_{Pow}^{-}|^{\mathfrak{p}} + |\mho_{Pow}^{-}|^{\mathfrak{q}} = (\overline{\vartheta})^{\rho} + 1 - (1-\overline{\varsigma})^{\rho} \leq (1-\overline{\varsigma})^{\rho} + 1 - (1-\overline{\varsigma})^{\rho} = 1.$$

Thus

$$|\aleph_{Pow}^{-}|^{\mathfrak{p}} + |\mho_{Pow}^{-}|^{\mathfrak{q}} \leq 1.$$

Hence, Ξ_1^{τ} is a B(p, q)-QOFN.

Therefore, all four AA operational laws are closed in the class of B(p, q)-QOFNs. □

4.9. Algebraic properties

Theorem 5. *Let*

$$\begin{aligned}
\Xi_1 &= \langle (\aleph_1^{+}, \mho_1^{+}), (\aleph_1^{-}, \mho_1^{-}) \rangle, \\
\Xi_2 &= \langle (\aleph_2^{+}, \mho_2^{+}), (\aleph_2^{-}, \mho_2^{-}) \rangle, \\
\Xi_3 &= \langle (\aleph_3^{+}, \mho_3^{+}), (\aleph_3^{-}, \mho_3^{-}) \rangle,
\end{aligned}$$

be three B(p, q)-QOFNs. Then the following properties hold:

(i) $\Xi_1 \oplus_{AA} \Xi_2 = \Xi_2 \oplus_{AA} \Xi_1$;
(ii) $\Xi_1 \otimes_{AA} \Xi_2 = \Xi_2 \otimes_{AA} \Xi_1$;
(iii) $(\Xi_1 \oplus_{AA} \Xi_2) \oplus_{AA} \Xi_3 = \Xi_1 \oplus_{AA} (\Xi_2 \oplus_{AA} \Xi_3)$;
(iv) $(\Xi_1 \otimes_{AA} \Xi_2) \otimes_{AA} \Xi_3 = \Xi_1 \otimes_{AA} (\Xi_2 \otimes_{AA} \Xi_3)$.

Proof. (i) First, we prove $\Xi_1 \oplus_{AA} \Xi_2 = \Xi_2 \oplus_{AA} \Xi_1$.

By Definition 13, we have

$$\Xi_1 \oplus_{AA} \Xi_2 = \langle(\aleph_{\oplus}^{+}, \mho_{\oplus}^{+}), (\aleph_{\oplus}^{-}, \mho_{\oplus}^{-})\rangle,$$

where

$$\begin{aligned}
\aleph_{\oplus}^{+} &= \sqrt[\mathfrak{p}]{S_{AA}^{\lambda}\left((\aleph_1^{+})^{\mathfrak{p}}, (\aleph_2^{+})^{\mathfrak{p}}\right)},\\
\mho_{\oplus}^{+} &= \sqrt[\mathfrak{q}]{T_{AA}^{\lambda}\left((\mho_1^{+})^{\mathfrak{q}}, (\mho_2^{+})^{\mathfrak{q}}\right)},\\
\aleph_{\oplus}^{-} &= -\sqrt[\mathfrak{p}]{S_{AA}^{\lambda}\left(|\aleph_1^{-}|^{\mathfrak{p}}, |\aleph_2^{-}|^{\mathfrak{p}}\right)},\\
\mho_{\oplus}^{-} &= -\sqrt[\mathfrak{q}]{T_{AA}^{\lambda}\left(|\mho_1^{-}|^{\mathfrak{q}}, |\mho_2^{-}|^{\mathfrak{q}}\right)}.
\end{aligned}$$

Since $S_{AA}^{\lambda}(x, y) = S_{AA}^{\lambda}(y, x)$ and $T_{AA}^{\lambda}(x, y) = T_{AA}^{\lambda}(y, x)$ for all $x, y \in [0, 1]$, we obtain

$$\begin{aligned}
S_{AA}^{\lambda}\left((\aleph_1^{+})^{\mathfrak{p}}, (\aleph_2^{+})^{\mathfrak{p}}\right) &= S_{AA}^{\lambda}\left((\aleph_2^{+})^{\mathfrak{p}}, (\aleph_1^{+})^{\mathfrak{p}}\right),\\
T_{AA}^{\lambda}\left((\mho_1^{+})^{\mathfrak{q}}, (\mho_2^{+})^{\mathfrak{q}}\right) &= T_{AA}^{\lambda}\left((\mho_2^{+})^{\mathfrak{q}}, (\mho_1^{+})^{\mathfrak{q}}\right),\\
S_{AA}^{\lambda}\left(|\aleph_1^{-}|^{\mathfrak{p}}, |\aleph_2^{-}|^{\mathfrak{p}}\right) &= S_{AA}^{\lambda}\left(|\aleph_2^{-}|^{\mathfrak{p}}, |\aleph_1^{-}|^{\mathfrak{p}}\right),\\
T_{AA}^{\lambda}\left(|\mho_1^{-}|^{\mathfrak{q}}, |\mho_2^{-}|^{\mathfrak{q}}\right) &= T_{AA}^{\lambda}\left(|\mho_2^{-}|^{\mathfrak{q}}, |\mho_1^{-}|^{\mathfrak{q}}\right).
\end{aligned}$$

Hence, each component of $\Xi_1 \oplus_{AA} \Xi_2$ is equal to the corresponding component of $\Xi_2 \oplus_{AA} \Xi_1$. Therefore, we have

$$\Xi_1 \oplus_{AA} \Xi_2 = \Xi_2 \oplus_{AA} \Xi_1.$$

(ii) We prove $\Xi_1 \otimes_{AA} \Xi_2 = \Xi_2 \otimes_{AA} \Xi_1$.

By Definition 13, we have

$$\Xi_1 \otimes_{AA} \Xi_2 = \langle(\aleph_{\otimes}^{+}, \mho_{\otimes}^{+}), (\aleph_{\otimes}^{-}, \mho_{\otimes}^{-})\rangle,$$

where

$$\begin{aligned}
\aleph_{\otimes}^{+} &= \sqrt[\mathfrak{p}]{T_{AA}^{\lambda}\left((\aleph_1^{+})^{\mathfrak{p}}, (\aleph_2^{+})^{\mathfrak{p}}\right)},\\
\mho_{\otimes}^{+} &= \sqrt[\mathfrak{q}]{S_{AA}^{\lambda}\left((\mho_1^{+})^{\mathfrak{q}}, (\mho_2^{+})^{\mathfrak{q}}\right)},\\
\aleph_{\otimes}^{-} &= -\sqrt[\mathfrak{p}]{T_{AA}^{\lambda}\left(|\aleph_1^{-}|^{\mathfrak{p}}, |\aleph_2^{-}|^{\mathfrak{p}}\right)},\\
\mho_{\otimes}^{-} &= -\sqrt[\mathfrak{q}]{S_{AA}^{\lambda}\left(|\mho_1^{-}|^{\mathfrak{q}}, |\mho_2^{-}|^{\mathfrak{q}}\right)}.
\end{aligned}$$

Since $T_{AA}^{\lambda}(x, y) = T_{AA}^{\lambda}(y, x)$ and $S_{AA}^{\lambda}(x, y) = S_{AA}^{\lambda}(y, x)$, we get

$$\begin{aligned}
T_{AA}^{\lambda}\left((\aleph_1^{+})^{\mathfrak{p}}, (\aleph_2^{+})^{\mathfrak{p}}\right) &= T_{AA}^{\lambda}\left((\aleph_2^{+})^{\mathfrak{p}}, (\aleph_1^{+})^{\mathfrak{p}}\right),\\
S_{AA}^{\lambda}\left((\mho_1^{+})^{\mathfrak{q}}, (\mho_2^{+})^{\mathfrak{q}}\right) &= S_{AA}^{\lambda}\left((\mho_2^{+})^{\mathfrak{q}}, (\mho_1^{+})^{\mathfrak{q}}\right),
\end{aligned}$$

$$T^{\lambda}_{AA}\left(|\aleph_1^-|^{\mathfrak{p}}, |\aleph_2^-|^{\mathfrak{p}}\right) = T^{\lambda}_{AA}\left(|\aleph_2^-|^{\mathfrak{p}}, |\aleph_1^-|^{\mathfrak{p}}\right),$$
$$S^{\lambda}_{AA}\left(|\mho_1^-|^{\mathfrak{q}}, |\mho_2^-|^{\mathfrak{q}}\right) = S^{\lambda}_{AA}\left(|\mho_2^-|^{\mathfrak{q}}, |\mho_1^-|^{\mathfrak{q}}\right).$$

Thus, each component of $\Xi_1 \otimes_{AA} \Xi_2$ is equal to the corresponding component of $\Xi_2 \otimes_{AA} \Xi_1$. Therefore, we have

$$\Xi_1 \otimes_{AA} \Xi_2 = \Xi_2 \otimes_{AA} \Xi_1.$$

(iii) We prove $(\Xi_1 \oplus_{AA} \Xi_2) \oplus_{AA} \Xi_3 = \Xi_1 \oplus_{AA} (\Xi_2 \oplus_{AA} \Xi_3)$.

The powered PMD of $(\Xi_1 \oplus_{AA} \Xi_2) \oplus_{AA} \Xi_3$ is

$$S^{\lambda}_{AA}\left(S^{\lambda}_{AA}\left((\aleph_1^+)^{\mathfrak{p}}, (\aleph_2^+)^{\mathfrak{p}}\right), (\aleph_3^+)^{\mathfrak{p}}\right).$$

Using the associativity of S^{λ}_{AA}, we have

$$S^{\lambda}_{AA}\left(S^{\lambda}_{AA}\left((\aleph_1^+)^{\mathfrak{p}}, (\aleph_2^+)^{\mathfrak{p}}\right), (\aleph_3^+)^{\mathfrak{p}}\right) = S^{\lambda}_{AA}\left((\aleph_1^+)^{\mathfrak{p}}, S^{\lambda}_{AA}\left((\aleph_2^+)^{\mathfrak{p}}, (\aleph_3^+)^{\mathfrak{p}}\right)\right).$$

This is the powered PMD of $\Xi_1 \oplus_{AA} (\Xi_2 \oplus_{AA} \Xi_3)$. Hence, the PMD components of the two sides are equal.

The powered PNMD of $(\Xi_1 \oplus_{AA} \Xi_2) \oplus_{AA} \Xi_3$ is

$$T^{\lambda}_{AA}\left(T^{\lambda}_{AA}\left((\mho_1^+)^{\mathfrak{q}}, (\mho_2^+)^{\mathfrak{q}}\right), (\mho_3^+)^{\mathfrak{q}}\right).$$

Using the associativity of T^{λ}_{AA}, we have

$$T^{\lambda}_{AA}\left(T^{\lambda}_{AA}\left((\mho_1^+)^{\mathfrak{q}}, (\mho_2^+)^{\mathfrak{q}}\right), (\mho_3^+)^{\mathfrak{q}}\right) = T^{\lambda}_{AA}\left((\mho_1^+)^{\mathfrak{q}}, T^{\lambda}_{AA}\left((\mho_2^+)^{\mathfrak{q}}, (\mho_3^+)^{\mathfrak{q}}\right)\right).$$

This is the powered PNMD of $\Xi_1 \oplus_{AA} (\Xi_2 \oplus_{AA} \Xi_3)$. Hence, the PNMD components of the two sides are equal.

The powered absolute value of the NMD of $(\Xi_1 \oplus_{AA} \Xi_2) \oplus_{AA} \Xi_3$ is

$$S^{\lambda}_{AA}\left(S^{\lambda}_{AA}\left(|\aleph_1^-|^{\mathfrak{p}}, |\aleph_2^-|^{\mathfrak{p}}\right), |\aleph_3^-|^{\mathfrak{p}}\right).$$

Using the associativity of S^{λ}_{AA}, we obtain

$$S^{\lambda}_{AA}\left(S^{\lambda}_{AA}\left(|\aleph_1^-|^{\mathfrak{p}}, |\aleph_2^-|^{\mathfrak{p}}\right), |\aleph_3^-|^{\mathfrak{p}}\right) = S^{\lambda}_{AA}\left(|\aleph_1^-|^{\mathfrak{p}}, S^{\lambda}_{AA}\left(|\aleph_2^-|^{\mathfrak{p}}, |\aleph_3^-|^{\mathfrak{p}}\right)\right).$$

This is the powered absolute value of the NMD of $\Xi_1 \oplus_{AA} (\Xi_2 \oplus_{AA} \Xi_3)$. Since the negative sign is restored after taking the $\mathfrak{p}$-th root on both sides, the NMD components are equal.

The powered absolute value of the NNMD of $(\Xi_1 \oplus_{AA} \Xi_2) \oplus_{AA} \Xi_3$ is

$$T^{\lambda}_{AA}\left(T^{\lambda}_{AA}\left(|\mho_1^-|^{\mathfrak{q}}, |\mho_2^-|^{\mathfrak{q}}\right), |\mho_3^-|^{\mathfrak{q}}\right).$$

Using the associativity of T^{λ}_{AA}, we obtain

$$T^{\lambda}_{AA}\left(T^{\lambda}_{AA}\left(|\mho_1^-|^{\mathfrak{q}}, |\mho_2^-|^{\mathfrak{q}}\right), |\mho_3^-|^{\mathfrak{q}}\right) = T^{\lambda}_{AA}\left(|\mho_1^-|^{\mathfrak{q}}, T^{\lambda}_{AA}\left(|\mho_2^-|^{\mathfrak{q}}, |\mho_3^-|^{\mathfrak{q}}\right)\right).$$

This is the powered absolute value of the NNMD of $\Xi_1 \oplus_{AA} (\Xi_2 \oplus_{AA} \Xi_3)$. Since the negative sign is restored after taking the $\mathfrak{q}$-th root on both sides, the NNMD components are equal.

Therefore,

$$(\Xi_1 \oplus_{AA} \Xi_2) \oplus_{AA} \Xi_3 = \Xi_1 \oplus_{AA} (\Xi_2 \oplus_{AA} \Xi_3).$$

(iv) We prove $(\Xi_1 \otimes_{AA} \Xi_2) \otimes_{AA} \Xi_3 = \Xi_1 \otimes_{AA} (\Xi_2 \otimes_{AA} \Xi_3)$.

The powered PMD of $(\Xi_1 \otimes_{AA} \Xi_2) \otimes_{AA} \Xi_3$ is

$$T_{AA}^{\lambda}\left(T_{AA}^{\lambda}\left((\aleph_1^+)^{\mathfrak{p}}, (\aleph_2^+)^{\mathfrak{p}}\right), (\aleph_3^+)^{\mathfrak{p}}\right).$$

Using the associativity of T_{AA}^{λ}, we have

$$T_{AA}^{\lambda}\left(T_{AA}^{\lambda}\left((\aleph_1^+)^{\mathfrak{p}}, (\aleph_2^+)^{\mathfrak{p}}\right), (\aleph_3^+)^{\mathfrak{p}}\right) = T_{AA}^{\lambda}\left((\aleph_1^+)^{\mathfrak{p}}, T_{AA}^{\lambda}\left((\aleph_2^+)^{\mathfrak{p}}, (\aleph_3^+)^{\mathfrak{p}}\right)\right).$$

This is the powered PMD of $\Xi_1 \otimes_{AA} (\Xi_2 \otimes_{AA} \Xi_3)$. Hence, the PMD components of the two sides are equal.

The powered PNMD of $(\Xi_1 \otimes_{AA} \Xi_2) \otimes_{AA} \Xi_3$ is

$$S_{AA}^{\lambda}\left(S_{AA}^{\lambda}\left((\mho_1^+)^{\mathfrak{q}}, (\mho_2^+)^{\mathfrak{q}}\right), (\mho_3^+)^{\mathfrak{q}}\right).$$

Using the associativity of S_{AA}^{λ}, we have

$$S_{AA}^{\lambda}\left(S_{AA}^{\lambda}\left((\mho_1^+)^{\mathfrak{q}}, (\mho_2^+)^{\mathfrak{q}}\right), (\mho_3^+)^{\mathfrak{q}}\right) = S_{AA}^{\lambda}\left((\mho_1^+)^{\mathfrak{q}}, S_{AA}^{\lambda}\left((\mho_2^+)^{\mathfrak{q}}, (\mho_3^+)^{\mathfrak{q}}\right)\right).$$

This is the powered PNMD of $\Xi_1 \otimes_{AA} (\Xi_2 \otimes_{AA} \Xi_3)$. Hence, the PNMD components of the two sides are equal.

The powered absolute value of the NMD of $(\Xi_1 \otimes_{AA} \Xi_2) \otimes_{AA} \Xi_3$ is

$$T_{AA}^{\lambda}\left(T_{AA}^{\lambda}\left(|\aleph_1^-|^{\mathfrak{p}}, |\aleph_2^-|^{\mathfrak{p}}\right), |\aleph_3^-|^{\mathfrak{p}}\right).$$

Using the associativity of T_{AA}^{λ}, we obtain

$$T_{AA}^{\lambda}\left(T_{AA}^{\lambda}\left(|\aleph_1^-|^{\mathfrak{p}}, |\aleph_2^-|^{\mathfrak{p}}\right), |\aleph_3^-|^{\mathfrak{p}}\right) = T_{AA}^{\lambda}\left(|\aleph_1^-|^{\mathfrak{p}}, T_{AA}^{\lambda}\left(|\aleph_2^-|^{\mathfrak{p}}, |\aleph_3^-|^{\mathfrak{p}}\right)\right).$$

This is the powered absolute value of the NMD of $\Xi_1 \otimes_{AA} (\Xi_2 \otimes_{AA} \Xi_3)$. Since the negative sign is restored after taking the $\mathfrak{p}$-th root on both sides, the NMD components are equal.

The powered absolute value of the NNMD of $(\Xi_1 \otimes_{AA} \Xi_2) \otimes_{AA} \Xi_3$ is

$$S_{AA}^{\lambda}\left(S_{AA}^{\lambda}\left(|\mho_1^-|^{\mathfrak{q}}, |\mho_2^-|^{\mathfrak{q}}\right), |\mho_3^-|^{\mathfrak{q}}\right).$$

Using the associativity of S_{AA}^{λ}, we obtain

$$S_{AA}^{\lambda}\left(S_{AA}^{\lambda}\left(|\mho_1^-|^{\mathfrak{q}}, |\mho_2^-|^{\mathfrak{q}}\right), |\mho_3^-|^{\mathfrak{q}}\right) = S_{AA}^{\lambda}\left(|\mho_1^-|^{\mathfrak{q}}, S_{AA}^{\lambda}\left(|\mho_2^-|^{\mathfrak{q}}, |\mho_3^-|^{\mathfrak{q}}\right)\right).$$

This is the powered absolute value of the NNMD of $\Xi_1 \otimes_{AA} (\Xi_2 \otimes_{AA} \Xi_3)$. Since the negative sign is restored after taking the $\mathfrak{q}$-th root on both sides, the NNMD components are equal.

Therefore, we have

$$(\Xi_1 \otimes_{AA} \Xi_2) \otimes_{AA} \Xi_3 = \Xi_1 \otimes_{AA} (\Xi_2 \otimes_{AA} \Xi_3).$$

This completes the proof. □

Theorem 6. *Let Ξ, Ξ_1, Ξ_2 be $B(p,q)$-QOFNs, and let $\tau, \tau_1, \tau_2 > 0$. Then the following hold:*

(i) $\tau \cdot_{A_A} (\Xi_1 \oplus_{A_A} \Xi_2) = (\tau \cdot_{A_A} \Xi_1) \oplus_{A_A} (\tau \cdot_{A_A} \Xi_2)$;
(ii) $(\tau_1 \cdot_{A_A} \Xi) \oplus_{A_A} (\tau_2 \cdot_{A_A} \Xi) = (\tau_1 + \tau_2) \cdot_{A_A} \Xi$;
(iii) $(\Xi_1 \otimes_{A_A} \Xi_2)^\tau = \Xi_1^\tau \otimes_{A_A} \Xi_2^\tau$;
(iv) $\Xi^{\tau_1} \otimes_{A_A} \Xi^{\tau_2} = \Xi^{\tau_1+\tau_2}$.

Proof. For $r > 0$, define

$$\Phi_r(x) = 1 - (1 - x)^r, \qquad \Psi_r(x) = x^r, \qquad x \in [0, 1].$$

The additive generator forms of $S^\lambda_{A_A}$ and $T^\lambda_{A_A}$ give the following identities:

$$\Phi_r(S^\lambda_{A_A}(x, y)) = S^\lambda_{A_A}(\Phi_r(x), \Phi_r(y)), \tag{4.22}$$
$$\Psi_r(T^\lambda_{A_A}(x, y)) = T^\lambda_{A_A}(\Psi_r(x), \Psi_r(y)). \tag{4.23}$$

Indeed, $-\ln(1 - \Phi_r(x)) = r[-\ln(1 - x)]$ and $-\ln(\Psi_r(x)) = r[-\ln x]$; substitution into the definitions of $S^\lambda_{A_A}$ and $T^\lambda_{A_A}$ proves (4.22) and (4.23).

For $r_1, r_2 > 0$, set

$$R = (r_1^\lambda + r_2^\lambda)^{1/\lambda}.$$

The same generator calculation yields

$$S^\lambda_{A_A}(\Phi_{r_1}(x), \Phi_{r_2}(x)) = \Phi_R(x), \tag{4.24}$$
$$T^\lambda_{A_A}(\Psi_{r_1}(x), \Psi_{r_2}(x)) = \Psi_R(x). \tag{4.25}$$

All four identities extend to the boundary points by continuity.

We write the powered components of a B(p, q)-QOFN using the notation introduced in the preceding definition, namely $\vartheta = (\aleph^+)^{\mathfrak{p}}$, $\varsigma = (\mho^+)^{\mathfrak{q}}$, $\overline{\vartheta} = |\aleph^-|^{\mathfrak{p}}$, and $\overline{\varsigma} = |\mho^-|^{\mathfrak{q}}$.

Scalar multiplication applies the functions (Φ_r, Ψ_r) to the powered positive and negative components, where $r = \tau^{1/\lambda}$, while addition combines the two coordinates by $(S^\lambda_{A_A}, T^\lambda_{A_A})$. Therefore, the identities in (4.22) and (4.23) applied to both poles prove (i).

For (ii), take $r_i = \tau_i^{1/\lambda}$. Then

$$R = (r_1^\lambda + r_2^\lambda)^{1/\lambda} = (\tau_1 + \tau_2)^{1/\lambda},$$

so the identities in (4.24) and (4.25) give the powered components of $(\tau_1 + \tau_2) \cdot_{A_A} \Xi$.

The power operation applies (Ψ_r, Φ_r), whereas multiplication combines the powered components by $(T^\lambda_{A_A}, S^\lambda_{A_A})$. Hence, the same identities, in the reversed coordinate order, prove (iii). Finally, the identities in (4.24) and (4.25), with $r_i = \tau_i^{1/\lambda}$, prove (iv). In every case, the equalities also hold for the powered absolute values of the negative components; taking the appropriate roots and restoring their negative signs completes the four componentwise equalities. □

4.10. Computational complexity and representation gains

At the level of the number of componentwise operations, the proposed B(p, q)-QOFS framework has the same asymptotic order as standard bipolar orthopair aggregation. For an MCDM problem with

m alternatives and n criteria, the aggregation stage requires $O(mn)$ componentwise evaluations when p, q, and λ are treated as fixed parameters.

The introduction of the independent exponents p and q enlarges the admissible representation domain without changing this asymptotic order. In particular, when $p, q \geq 3$, the admissible region contains those of the BIFS, BPyFS, and BFFS frameworks; the containment is strict whenever at least one of the exponents exceeds 3. This provides additional representational flexibility for MCDM problems in which both bipolarity and asymmetric orthopair information are required.

5. AA **aggregation operators**

In this section, we define two AA aggregation operators for B(p, q)-QOFNs. The development is motivated by AA aggregation operators for intuitionistic and picture fuzzy environments (see the works [38,39,43]). Related contributions in Pythagorean and Fermatean fuzzy settings have also been considered [40,41,46]. More recent developments for (p, q)-quasirung orthopair information are taken into account [45,47]. The closed forms and principal properties of these operators are then established.

5.1. B(p, q)-QOFAAWA operator

Definition 14. *Let $\Xi_j = \langle(\aleph_j^+, \mho_j^+), (\aleph_j^-, \mho_j^-)\rangle$, $j = 1, 2, \ldots, n$, be a collection of B(p, q)-QOFNs. Let $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be a weight vector such that $\omega_j > 0$ and $\sum_{j=1}^n \omega_j = 1$.*

The bipolar (p, q)-quasirung orthopair fuzzy Aczél–Alsina weighted averaging (B(p, q)-QOFAAWA) operator is defined by

$$B(p,q)\text{-}QOFAAWA(\Xi_1, \Xi_2, \ldots, \Xi_n) = \bigoplus_{j=1}^{n} (\omega_j \cdot_{AA} \Xi_j).$$

Theorem 7. *Let $\Xi_j = \langle(\aleph_j^+, \mho_j^+), (\aleph_j^-, \mho_j^-)\rangle$, $j = 1, 2, \ldots, n$, be a collection of B(p, q)-QOFNs with weight vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_n)^T$, where $\omega_j > 0$ and $\sum_{j=1}^n \omega_j = 1$. Set*

$$\mathfrak{a}_j = (\aleph_j^+)^{\mathfrak{p}}, \qquad \mathfrak{b}_j = (\mho_j^+)^{\mathfrak{q}}, \qquad \overline{\mathfrak{a}}_j = |\aleph_j^-|^{\mathfrak{p}}, \qquad \overline{\mathfrak{b}}_j = |\mho_j^-|^{\mathfrak{q}}.$$

Then

$$B(p,q)\text{-}QOFAAWA(\Xi_1, \Xi_2, \ldots, \Xi_n) = \langle(\aleph_A^+, \mho_A^+), (\aleph_A^-, \mho_A^-)\rangle,$$

where

$$\aleph_A^+ = \sqrt[\mathfrak{p}]{1 - e^{-\sqrt[\lambda]{\sum_{j=1}^n \omega_j (-\ln(1-\mathfrak{a}_j))^\lambda}}}, \tag{5.1}$$

$$\mho_A^+ = \sqrt[\mathfrak{q}]{e^{-\sqrt[\lambda]{\sum_{j=1}^n \omega_j (-\ln(\mathfrak{b}_j))^\lambda}}}, \tag{5.2}$$

$$\aleph_A^- = -\sqrt[\mathfrak{p}]{1 - e^{-\sqrt[\lambda]{\sum_{j=1}^n \omega_j (-\ln(1-\overline{\mathfrak{a}}_j))^\lambda}}}, \tag{5.3}$$

$$\mho_A^- = -\sqrt[\mathfrak{q}]{e^{-\sqrt[\lambda]{\sum_{j=1}^n \omega_j (-\ln(\overline{\mathfrak{b}}_j))^\lambda}}}. \tag{5.4}$$

Proof. For each $j = 1, 2, \ldots, n$, set $\rho_j = \sqrt[\lambda]{\omega_j}$. Then $\rho_j^\lambda = \omega_j$.

Since the AA t-norm and t-conorm are associative, their repeated application can be expressed in the corresponding n-ary form. By Definition 13, the powered PMD component of $\omega_j \cdot_{AA} \Xi_j$ is $1-(1-\mathfrak{a}_j)^{\rho_j}$. Hence, the powered PMD component of $\bigoplus_{j=1}^{n}(\omega_j \cdot_{AA} \Xi_j)$ is obtained by applying the AA t-conorm to the values

$$1-(1-\mathfrak{a}_1)^{\rho_1}, 1-(1-\mathfrak{a}_2)^{\rho_2}, \ldots, 1-(1-\mathfrak{a}_n)^{\rho_n}.$$

Thus

$$S_{AA}^{\lambda}\left(1-(1-\mathfrak{a}_1)^{\rho_1}, \ldots, 1-(1-\mathfrak{a}_n)^{\rho_n}\right) = 1-e^{-\sqrt[\lambda]{\sum_{j=1}^{n}(-\ln(1-(1-(1-\mathfrak{a}_j)^{\rho_j})))^{\lambda}}}.$$

Since

$$1-(1-(1-\mathfrak{a}_j)^{\rho_j}) = (1-\mathfrak{a}_j)^{\rho_j},$$

we have

$$-\ln((1-\mathfrak{a}_j)^{\rho_j}) = \rho_j(-\ln(1-\mathfrak{a}_j)).$$

Therefore

$$(-\ln((1-\mathfrak{a}_j)^{\rho_j}))^{\lambda} = \rho_j^{\lambda}(-\ln(1-\mathfrak{a}_j))^{\lambda} = \omega_j(-\ln(1-\mathfrak{a}_j))^{\lambda}.$$

Hence

$$S_{AA}^{\lambda}\left(1-(1-\mathfrak{a}_1)^{\rho_1}, \ldots, 1-(1-\mathfrak{a}_n)^{\rho_n}\right) = 1-e^{-\sqrt[\lambda]{\sum_{j=1}^{n}\omega_j(-\ln(1-\mathfrak{a}_j))^{\lambda}}}.$$

After taking the $\mathfrak{p}$-th root, we obtain the PMD component.

$$\aleph_A^{+} = \sqrt[\mathfrak{p}]{1-e^{-\sqrt[\lambda]{\sum_{j=1}^{n}\omega_j(-\ln(1-\mathfrak{a}_j))^{\lambda}}}}.$$

Next, the powered PNMD component of $\omega_j \cdot_{AA} \Xi_j$ is $\mathfrak{b}_j^{\rho_j}$. Hence, the powered PNMD component of $\bigoplus_{j=1}^{n}(\omega_j \cdot_{AA} \Xi_j)$ is obtained by applying the AA t-norm to the values

$$\mathfrak{b}_1^{\rho_1}, \mathfrak{b}_2^{\rho_2}, \ldots, \mathfrak{b}_n^{\rho_n}.$$

Thus

$$T_{AA}^{\lambda}(\mathfrak{b}_1^{\rho_1}, \mathfrak{b}_2^{\rho_2}, \ldots, \mathfrak{b}_n^{\rho_n}) = e^{-\sqrt[\lambda]{\sum_{j=1}^{n}(-\ln(\mathfrak{b}_j^{\rho_j}))^{\lambda}}}.$$

Since

$$-\ln(\mathfrak{b}_j^{\rho_j}) = \rho_j(-\ln(\mathfrak{b}_j)),$$

we have

$$(-\ln(\mathfrak{b}_j^{\rho_j}))^{\lambda} = \rho_j^{\lambda}(-\ln(\mathfrak{b}_j))^{\lambda} = \omega_j(-\ln(\mathfrak{b}_j))^{\lambda}.$$

Therefore

$$T_{AA}^{\lambda}(\mathfrak{b}_1^{\rho_1}, \mathfrak{b}_2^{\rho_2}, \ldots, \mathfrak{b}_n^{\rho_n}) = e^{-\sqrt[\lambda]{\sum_{j=1}^{n} \omega_j(-\ln(\mathfrak{b}_j))^{\lambda}}}.$$

After taking the $\mathfrak{q}$-th root, we obtain the PNMD component

$$\mho_A^{+} = \sqrt[\mathfrak{q}]{e^{-\sqrt[\lambda]{\sum_{j=1}^{n} \omega_j(-\ln(\mathfrak{b}_j))^{\lambda}}}}.$$

For the NMD, the operations are applied to the powered absolute values, and then the negative sign is restored.

The powered absolute value of the NMD of $\boldsymbol{\omega}_j \cdot_{AA} \Xi_j$ is

$$1 - (1 - \overline{\mathfrak{a}}_j)^{\rho_j}.$$

Hence, the powered absolute value of the NMD of $\bigoplus_{j=1}^{n}(\boldsymbol{\omega}_j \cdot_{AA} \Xi_j)$ is obtained by applying the AA t-conorm to the values

$$1 - (1 - \overline{\mathfrak{a}}_1)^{\rho_1}, 1 - (1 - \overline{\mathfrak{a}}_2)^{\rho_2}, \ldots, 1 - (1 - \overline{\mathfrak{a}}_n)^{\rho_n}.$$

Thus

$$S_{AA}^{\lambda}\left(1 - (1 - \overline{\mathfrak{a}}_1)^{\rho_1}, \ldots, 1 - (1 - \overline{\mathfrak{a}}_n)^{\rho_n}\right) = 1 - e^{-\sqrt[\lambda]{\sum_{j=1}^{n}(-\ln(1-(1-(1-\overline{\mathfrak{a}}_j)^{\rho_j})))^{\lambda}}}.$$

Since

$$1 - (1 - (1 - \overline{\mathfrak{a}}_j)^{\rho_j}) = (1 - \overline{\mathfrak{a}}_j)^{\rho_j},$$

we have

$$-\ln((1 - \overline{\mathfrak{a}}_j)^{\rho_j}) = \rho_j(-\ln(1 - \overline{\mathfrak{a}}_j)).$$

Therefore

$$(-\ln((1 - \overline{\mathfrak{a}}_j)^{\rho_j}))^{\lambda} = \rho_j^{\lambda}(-\ln(1 - \overline{\mathfrak{a}}_j))^{\lambda} = \boldsymbol{\omega}_j(-\ln(1 - \overline{\mathfrak{a}}_j))^{\lambda}.$$

Hence

$$S_{AA}^{\lambda}\left(1 - (1 - \overline{\mathfrak{a}}_1)^{\rho_1}, \ldots, 1 - (1 - \overline{\mathfrak{a}}_n)^{\rho_n}\right) = 1 - e^{-\sqrt[\lambda]{\sum_{j=1}^{n} \omega_j(-\ln(1-\overline{\mathfrak{a}}_j))^{\lambda}}}.$$

After taking the $\mathfrak{p}$-th root and restoring the negative sign, we obtain the NMD

$$\aleph_A^{-} = -\sqrt[\mathfrak{p}]{1 - e^{-\sqrt[\lambda]{\sum_{j=1}^{n} \omega_j(-\ln(1-\overline{\mathfrak{a}}_j))^{\lambda}}}}.$$

Finally, the powered absolute value of the NNMD of $\boldsymbol{\omega}_j \cdot_{AA} \Xi_j$ is $\overline{\mathfrak{b}}_j^{\rho_j}$. Hence, the powered absolute value of the NNMD of $\bigoplus_{j=1}^{n}(\boldsymbol{\omega}_j \cdot_{AA} \Xi_j)$ is obtained by applying the AA t-norm to the values

$$\overline{\mathfrak{b}}_1^{\rho_1}, \overline{\mathfrak{b}}_2^{\rho_2}, \ldots, \overline{\mathfrak{b}}_n^{\rho_n}.$$

Thus

$$T_{AA}^{\lambda}(\overline{\mathfrak{b}}_1^{\rho_1}, \overline{\mathfrak{b}}_2^{\rho_2}, \ldots, \overline{\mathfrak{b}}_n^{\rho_n}) = e^{-\sqrt[\lambda]{\sum_{j=1}^{n}(-\ln(\overline{\mathfrak{b}}_j^{\rho_j}))^{\lambda}}}.$$

Since

$$-\ln(\overline{\mathfrak{b}}_j^{\rho_j}) = \rho_j(-\ln(\overline{\mathfrak{b}}_j)),$$

we have

$$(-\ln(\overline{\mathfrak{b}}_j^{\rho_j}))^{\lambda} = \rho_j^{\lambda}(-\ln(\overline{\mathfrak{b}}_j))^{\lambda} = \boldsymbol{\omega}_j(-\ln(\overline{\mathfrak{b}}_j))^{\lambda}.$$

Therefore

$$T_{AA}^{\lambda}(\overline{\mathfrak{b}}_1^{\rho_1}, \overline{\mathfrak{b}}_2^{\rho_2}, \ldots, \overline{\mathfrak{b}}_n^{\rho_n}) = e^{-\sqrt[\lambda]{\sum_{j=1}^{n}\boldsymbol{\omega}_j(-\ln(\overline{\mathfrak{b}}_j))^{\lambda}}}.$$

After taking the $\mathfrak{q}$-th root and restoring the negative sign, we obtain the NNMD

$$\mho_A^{-} = -\sqrt[\mathfrak{q}]{e^{-\sqrt[\lambda]{\sum_{j=1}^{n}\boldsymbol{\omega}_j(-\ln(\overline{\mathfrak{b}}_j))^{\lambda}}}}.$$

Therefore, the stated closed form of the B(p, q)-QOFAAWA operator is obtained. The boundary cases are understood under the convention stated in Section 4.7.

This completes the proof. □

Theorem 8. *The value obtained by the B(p, q)-QOFAAWA operator is a B(p, q)-QOFN.*

Proof. Let the intermediate values be defined by the scalar multiplication operation

$$\upsilon_j = \boldsymbol{\omega}_j \cdot_{AA} \Xi_j, \qquad j = 1, 2, \ldots, n.$$

By the closure of scalar multiplication established in the previous section, each intermediate value is a B(p, q)-QOFN. The proof then proceeds by induction on the number of criteria.

Now define $A_1 = \upsilon_1$, and for $k = 2, 3, \ldots, n$, define

$$A_k = A_{k-1} \oplus_{AA} \upsilon_k.$$

Since $A_1 = \upsilon_1$ is a B(p, q)-QOFN, the result is true for $k = 1$.

Assume that the previous partial aggregation belongs to the same class of B(p, q)-QOFNs. Since the next intermediate value also belongs to this class, the closure of the addition operation ensures that the new partial aggregation belongs to the same class as well.

By induction, A_n is a B(p, q)-QOFN. However

$$A_n = \bigoplus_{j=1}^{n}(\boldsymbol{\omega}_j \cdot_{AA} \Xi_j) = \text{B}(p, q)\text{-QOFAAWA}(\Xi_1, \Xi_2, \ldots, \Xi_n).$$

Hence, the B(p, q)-QOFAAWA operator produces a B(p, q)-QOFN.

This completes the proof. □

5.2. $B(p,q)$-QOFAAWG operator

Definition 15. *Let*

$$\Xi_j = \langle(\aleph_j^+, \mho_j^+), (\aleph_j^-, \mho_j^-)\rangle, \quad j = 1, 2, \ldots, n,$$

be a collection of $B(p,q)$-QOFNs. Let $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be a weight vector such that $\omega_j > 0$ and $\sum_{j=1}^n \omega_j = 1$. The bipolar (p,q)-quasirung orthopair fuzzy Aczél–Alsina weighted geometric ($B(p,q)$-QOFAAWG) operator is defined by

$$B(p,q)\text{-}QOFAAWG(\Xi_1, \Xi_2, \ldots, \Xi_n) = \bigotimes_{j=1}^{n} \Xi_j^{\omega_j}.$$

Theorem 9. *Let $\Xi_j = \langle(\aleph_j^+, \mho_j^+), (\aleph_j^-, \mho_j^-)\rangle$, $j = 1, 2, \ldots, n$, be a collection of $B(p,q)$-QOFNs with the weight vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_n)^T$, where $\omega_j > 0$ and $\sum_{j=1}^n \omega_j = 1$. Set*

$$\mathfrak{a}_j = (\aleph_j^+)^{\mathfrak{p}}, \qquad \mathfrak{b}_j = (\mho_j^+)^{\mathfrak{q}}, \qquad \overline{\mathfrak{a}}_j = |\aleph_j^-|^{\mathfrak{p}}, \qquad \overline{\mathfrak{b}}_j = |\mho_j^-|^{\mathfrak{q}}.$$

Then

$$B(p,q)\text{-}QOFAAWG(\Xi_1, \Xi_2, \ldots, \Xi_n) = \langle(\aleph_G^+, \mho_G^+), (\aleph_G^-, \mho_G^-)\rangle,$$

where

$$\aleph_G^+ = \sqrt[\mathfrak{p}]{e^{-\sqrt[\lambda]{\sum_{j=1}^n \omega_j(-\ln(\mathfrak{a}_j))^\lambda}}}, \tag{5.5}$$

$$\mho_G^+ = \sqrt[\mathfrak{q}]{1 - e^{-\sqrt[\lambda]{\sum_{j=1}^n \omega_j(-\ln(1-\mathfrak{b}_j))^\lambda}}}, \tag{5.6}$$

$$\aleph_G^- = -\sqrt[\mathfrak{p}]{e^{-\sqrt[\lambda]{\sum_{j=1}^n \omega_j(-\ln(\overline{\mathfrak{a}}_j))^\lambda}}}, \tag{5.7}$$

$$\mho_G^- = -\sqrt[\mathfrak{q}]{1 - e^{-\sqrt[\lambda]{\sum_{j=1}^n \omega_j(-\ln(1-\overline{\mathfrak{b}}_j))^\lambda}}}. \tag{5.8}$$

Proof. For each $j = 1, 2, \ldots, n$, set $\rho_j = \sqrt[\lambda]{\omega_j}$. Then $\rho_j^\lambda = \omega_j$.

Since the AA t-norm and t-conorm are associative, their repeated application can be expressed in the corresponding n-ary form. By Definition 13, the powered PMD component of $\Xi_j^{\omega_j}$ is $\mathfrak{a}_j^{\rho_j}$.

Hence, the powered PMD component of $\bigotimes_{j=1}^n \Xi_j^{\omega_j}$ is obtained by applying the AA t-norm to the values

$$\mathfrak{a}_1^{\rho_1}, \mathfrak{a}_2^{\rho_2}, \ldots, \mathfrak{a}_n^{\rho_n}.$$

Thus

$$T_{AA}^\lambda(\mathfrak{a}_1^{\rho_1}, \ldots, \mathfrak{a}_n^{\rho_n}) = e^{-\sqrt[\lambda]{\sum_{j=1}^n (-\ln(\mathfrak{a}_j^{\rho_j}))^\lambda}}.$$

Since

$$-\ln(\mathfrak{a}_j^{\rho_j}) = \rho_j(-\ln(\mathfrak{a}_j)),$$

we have

$$(-\ln(\mathfrak{a}_j^{\rho_j}))^{\lambda} = \rho_j^{\lambda}(-\ln(\mathfrak{a}_j))^{\lambda} = \boldsymbol{\omega}_j(-\ln(\mathfrak{a}_j))^{\lambda}.$$

Hence

$$T_{AA}^{\lambda}(\mathfrak{a}_1^{\rho_1}, \ldots, \mathfrak{a}_n^{\rho_n}) = e^{-\sqrt[\lambda]{\sum_{j=1}^{n} \boldsymbol{\omega}_j(-\ln(\mathfrak{a}_j))^{\lambda}}}.$$

After taking the $\mathfrak{p}$-th root, we obtain the PMD component.

$$\aleph_G^{+} = \sqrt[\mathfrak{p}]{e^{-\sqrt[\lambda]{\sum_{j=1}^{n} \boldsymbol{\omega}_j(-\ln(\mathfrak{a}_j))^{\lambda}}}}.$$

Next, the powered PNMD component of $\Xi_j^{\boldsymbol{\omega}_j}$ is

$$1 - (1 - \mathfrak{b}_j)^{\rho_j}.$$

Hence, the powered PNMD component of $\bigotimes_{j=1}^{n} \Xi_j^{\boldsymbol{\omega}_j}$ is obtained by applying the AA t-conorm to the values

$$1 - (1 - \mathfrak{b}_1)^{\rho_1}, 1 - (1 - \mathfrak{b}_2)^{\rho_2}, \ldots, 1 - (1 - \mathfrak{b}_n)^{\rho_n}.$$

Thus

$$S_{AA}^{\lambda}\left(1 - (1 - \mathfrak{b}_1)^{\rho_1}, \ldots, 1 - (1 - \mathfrak{b}_n)^{\rho_n}\right) = 1 - e^{-\sqrt[\lambda]{\sum_{j=1}^{n}(-\ln(1-(1-(1-\mathfrak{b}_j)^{\rho_j})))^{\lambda}}}.$$

Since

$$-\ln(1 - (1 - (1 - \mathfrak{b}_j)^{\rho_j})) = -\ln((1 - \mathfrak{b}_j)^{\rho_j}),$$

we have

$$(-\ln((1 - \mathfrak{b}_j)^{\rho_j}))^{\lambda} = \rho_j^{\lambda}(-\ln(1 - \mathfrak{b}_j))^{\lambda} = \boldsymbol{\omega}_j(-\ln(1 - \mathfrak{b}_j))^{\lambda}.$$

Therefore

$$S_{AA}^{\lambda}\left(1 - (1 - \mathfrak{b}_1)^{\rho_1}, \ldots, 1 - (1 - \mathfrak{b}_n)^{\rho_n}\right) = 1 - e^{-\sqrt[\lambda]{\sum_{j=1}^{n} \boldsymbol{\omega}_j(-\ln(1-\mathfrak{b}_j))^{\lambda}}}.$$

After taking the $\mathfrak{q}$-th root, we obtain the PNMD component

$$\mho_G^{+} = \sqrt[\mathfrak{q}]{1 - e^{-\sqrt[\lambda]{\sum_{j=1}^{n} \boldsymbol{\omega}_j(-\ln(1-\mathfrak{b}_j))^{\lambda}}}}.$$

For the NMD, the operations are applied to the powered absolute values, and then the negative sign is restored.

The powered absolute value of the NMD of $\Xi_j^{\boldsymbol{\omega}_j}$ is $\overline{\mathfrak{a}}_j^{\rho_j}$.

Hence, the powered absolute value of the NMD of $\bigotimes_{j=1}^{n} \Xi_j^{\boldsymbol{\omega}_j}$ is obtained by applying the AA t-norm to the values

$$\overline{\mathfrak{a}}_1^{\rho_1}, \overline{\mathfrak{a}}_2^{\rho_2}, \ldots, \overline{\mathfrak{a}}_n^{\rho_n}.$$

Thus

$$T_{AA}^{\lambda}(\overline{\mathfrak{a}}_1^{\rho_1}, \ldots, \overline{\mathfrak{a}}_n^{\rho_n}) = e^{-\sqrt[\lambda]{\sum_{j=1}^{n}(-\ln(\overline{\mathfrak{a}}_j^{\rho_j}))^{\lambda}}}.$$

Since

$$-\ln(\overline{\mathfrak{a}}_j^{\rho_j}) = \rho_j(-\ln(\overline{\mathfrak{a}}_j)),$$

we have

$$(-\ln(\overline{\mathfrak{a}}_j^{\rho_j}))^{\lambda} = \rho_j^{\lambda}(-\ln(\overline{\mathfrak{a}}_j))^{\lambda} = \boldsymbol{\omega}_j(-\ln(\overline{\mathfrak{a}}_j))^{\lambda}.$$

Hence

$$T_{AA}^{\lambda}(\overline{\mathfrak{a}}_1^{\rho_1}, \ldots, \overline{\mathfrak{a}}_n^{\rho_n}) = e^{-\sqrt[\lambda]{\sum_{j=1}^{n}\boldsymbol{\omega}_j(-\ln(\overline{\mathfrak{a}}_j))^{\lambda}}}.$$

After taking the $\mathfrak{p}$-th root and restoring the negative sign, we obtain the NMD

$$\aleph_G^{-} = -\sqrt[\mathfrak{p}]{e^{-\sqrt[\lambda]{\sum_{j=1}^{n}\boldsymbol{\omega}_j(-\ln(\overline{\mathfrak{a}}_j))^{\lambda}}}}.$$

Next, the powered absolute value of the NNMD of $\Xi_j^{\boldsymbol{\omega}_j}$ is

$$1 - (1 - \overline{\mathfrak{b}}_j)^{\rho_j}.$$

Hence, the powered absolute value of the NNMD of $\bigotimes_{j=1}^{n}\Xi_j^{\boldsymbol{\omega}_j}$ is obtained by applying the AA t-conorm to the values

$$1 - (1 - \overline{\mathfrak{b}}_1)^{\rho_1}, 1 - (1 - \overline{\mathfrak{b}}_2)^{\rho_2}, \ldots, 1 - (1 - \overline{\mathfrak{b}}_n)^{\rho_n}.$$

Thus

$$S_{AA}^{\lambda}\left(1 - (1 - \overline{\mathfrak{b}}_1)^{\rho_1}, \ldots, 1 - (1 - \overline{\mathfrak{b}}_n)^{\rho_n}\right) = 1 - e^{-\sqrt[\lambda]{\sum_{j=1}^{n}(-\ln(1-(1-(1-\overline{\mathfrak{b}}_j)^{\rho_j}))^{\lambda}}}.$$

Since

$$-\ln(1 - (1 - (1 - \overline{\mathfrak{b}}_j)^{\rho_j})) = -\ln((1 - \overline{\mathfrak{b}}_j)^{\rho_j}),$$

we have

$$(-\ln((1 - \overline{\mathfrak{b}}_j)^{\rho_j}))^{\lambda} = \rho_j^{\lambda}(-\ln(1 - \overline{\mathfrak{b}}_j))^{\lambda} = \boldsymbol{\omega}_j(-\ln(1 - \overline{\mathfrak{b}}_j))^{\lambda}.$$

Therefore

$$S_{AA}^{\lambda}\left(1 - (1 - \overline{\mathfrak{b}}_1)^{\rho_1}, \ldots, 1 - (1 - \overline{\mathfrak{b}}_n)^{\rho_n}\right) = 1 - e^{-\sqrt[\lambda]{\sum_{j=1}^{n}\boldsymbol{\omega}_j(-\ln(1-\overline{\mathfrak{b}}_j))^{\lambda}}}.$$

After taking the $\mathfrak{q}$-th root and restoring the negative sign, we obtain the NNMD

$$\mho_G^{-} = -\sqrt[\mathfrak{q}]{1 - e^{-\sqrt[\lambda]{\sum_{j=1}^{n}\boldsymbol{\omega}_j(-\ln(1-\overline{\mathfrak{b}}_j))^{\lambda}}}}.$$

Therefore, the stated closed form of the B(p, q)-QOFAAWG operator is obtained. The boundary cases are understood under the convention stated in Section 4.7.

This completes the proof. □

Theorem 10. *The value obtained by the B(p, q)-QOFAAWG operator is a B(p, q)-QOFN.*

Proof. Let the intermediate values be defined by the power operation.

$$\upsilon_j = \Xi_j^{\omega_j}, \qquad j = 1, 2, \ldots, n.$$

By the closure of the power operation established in the previous section, each intermediate value belongs to the same class of B(p, q)-QOFNs. The proof then proceeds by induction on the number of criteria.

Now define $G_1 = \upsilon_1$ and for $k = 2, 3, \ldots, n$, define

$$G_k = G_{k-1} \otimes_{AA} \upsilon_k.$$

Since $G_1 = \upsilon_1$ is a B(p, q)-QOFN, the result is true for $k = 1$.

Assume that the previous partial aggregation belongs to the same class of B(p, q)-QOFNs. Since the next intermediate value also belongs to this class, the closure of the multiplication operation ensures that the new partial aggregation belongs to the same class as well.

By induction, G_n is a B(p, q)-QOFN. However

$$G_n = \bigotimes_{j=1}^{n} (\Xi_j^{\omega_j}) = \text{B}(p, q)\text{-QOFAAWG}(\Xi_1, \Xi_2, \ldots, \Xi_n).$$

Hence, the B(p, q)-QOFAAWG operator produces a B(p, q)-QOFN.

This completes the proof. □

5.3. Properties of the proposed aggregation operators

Theorem 11. *Let $\Xi_j = \langle (\aleph_j^+, \mho_j^+), (\aleph_j^-, \mho_j^-) \rangle$, $j = 1, 2, \ldots, n$, be a collection of B(p, q)-QOFNs. Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be a weight vector such that $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1$. Then the B$(p, q)$-QOFAAWA and B$(p, q)$-QOFAAWG operators satisfy the following properties:*

(1) Idempotency;
(2) Monotonicity;
(3) Boundedness.

Proof. We prove the three properties separately. Throughout the proof, $\lambda > 0$, $\omega_j > 0$, and $\sum_{j=1}^{n} \omega_j = 1$. Expressions involving the endpoint values 0 and 1 are understood through the continuous extension of the corresponding AA expressions.

(i) Idempotency: Suppose that

$$\Xi_1 = \Xi_2 = \cdots = \Xi_n = \Xi = \langle (\aleph^+, \mho^+), (\aleph^-, \mho^-) \rangle .$$

For the B(p, q)-QOFAAWA operator, the positive membership powered component becomes

$$1 - \exp\left\{ - \left(\sum_{j=1}^{n} \omega_j [-\ln(1 - (\aleph^+)^p)]^\lambda \right)^{1/\lambda} \right\}.$$

Since $\sum_{j=1}^{n} \omega_j = 1$, we have

$$1 - \exp\left\{-\left(\sum_{j=1}^{n} \omega_j[-\ln(1-(\aleph^+)^p)]^\lambda\right)^{1/\lambda}\right\} = 1 - \exp\left\{-\left([-\ln(1-(\aleph^+)^p)]^\lambda \sum_{j=1}^{n} \omega_j\right)^{1/\lambda}\right\}$$
$$= 1 - \exp\{\ln(1-(\aleph^+)^p)\} = (\aleph^+)^p.$$

Similarly, the positive nonmembership powered component satisfies

$$\exp\left\{-\left(\sum_{j=1}^{n} \omega_j[-\ln((\mho^+)^q)]^\lambda\right)^{1/\lambda}\right\} = \exp\{\ln((\mho^+)^q)\} = (\mho^+)^q.$$

The same substitutions give $|\aleph^-|^p$ and $|\mho^-|^q$ for the two negative pole powered magnitudes. After taking the corresponding p-th and q-th roots and restoring the negative signs of the negative pole components, we obtain

$$\mathrm{B}(p,q)\text{-QOFAAWA}(\Xi,\ldots,\Xi) = \Xi.$$

Direct substitution into the closed form of the B(p,q)-QOFAAWG operator yields the same result

$$\mathrm{B}(p,q)\text{-QOFAAWG}(\Xi,\ldots,\Xi) = \Xi.$$

Hence, both operators are idempotent.

(ii) Monotonicity: Suppose that

$$\Xi_j \leq_C Z_j, \quad j = 1,\ldots,n,$$

where

$$\Xi_j = \left\langle (\aleph_j^+, \mho_j^+), (\aleph_j^-, \mho_j^-)\right\rangle$$

and

$$Z_j = \left\langle (\aleph_j'^+, \mho_j'^+), (\aleph_j'^-, \mho_j'^-)\right\rangle.$$

By Definition 5, we have

$$(\aleph_j^+)^p \leq (\aleph_j'^+)^p, \quad (\mho_j^+)^q \geq (\mho_j'^+)^q,$$
$$|\aleph_j^-|^p \leq |\aleph_j'^-|^p, \quad |\mho_j^-|^q \geq |\mho_j'^-|^q.$$

For the B(p,q)-QOFAAWA operator, since $x \mapsto -\ln(1-x)$ is increasing on $[0,1)$, we have

$$-\ln\left(1-(\aleph_j^+)^p\right) \leq -\ln\left(1-(\aleph_j'^+)^p\right).$$

Hence

$$1 - \exp\left\{-\left(\sum_{j=1}^{n} \omega_j[-\ln(1-(\aleph_j^+)^p)]^\lambda\right)^{1/\lambda}\right\} \leq 1 - \exp\left\{-\left(\sum_{j=1}^{n} \omega_j[-\ln(1-(\aleph_j'^+)^p)]^\lambda\right)^{1/\lambda}\right\}.$$

Since $x \mapsto -\ln x$ is decreasing on $(0, 1]$

$$-\ln((\mho_j^+)^q) \leq -\ln((\mho_j'^+)^q),$$

and therefore

$$\exp\left\{-\left(\sum_{j=1}^{n} \omega_j[-\ln((\mho_j^+)^q)]^\lambda\right)^{1/\lambda}\right\} \geq \exp\left\{-\left(\sum_{j=1}^{n} \omega_j[-\ln((\mho_j'^+)^q)]^\lambda\right)^{1/\lambda}\right\}.$$

Similarly, from

$$|\aleph_j^-|^p \leq |\aleph_j'^-|^p$$

and the monotonicity of $x \mapsto -\ln(1-x)$, it follows that

$$1-\exp\left\{-\left(\sum_{j=1}^{n} \omega_j[-\ln(1-|\aleph_j^-|^p)]^\lambda\right)^{1/\lambda}\right\} \leq 1-\exp\left\{-\left(\sum_{j=1}^{n} \omega_j[-\ln(1-|\aleph_j'^-|^p)]^\lambda\right)^{1/\lambda}\right\}.$$

Furthermore, since

$$|\mho_j^-|^q \geq |\mho_j'^-|^q$$

and $x \mapsto -\ln x$ is decreasing, we have

$$-\ln(|\mho_j^-|^q) \leq -\ln(|\mho_j'^-|^q),$$

which gives

$$\exp\left\{-\left(\sum_{j=1}^{n} \omega_j[-\ln(|\mho_j^-|^q)]^\lambda\right)^{1/\lambda}\right\} \geq \exp\left\{-\left(\sum_{j=1}^{n} \omega_j[-\ln(|\mho_j'^-|^q)]^\lambda\right)^{1/\lambda}\right\}.$$

Thus, the four powered components satisfy the componentwise inequalities required by Definition 5. Therefore, we have

$$\mathrm{B}(p,q)\text{-QOFAAWA}(\Xi_1,\ldots,\Xi_n) \leq_C \mathrm{B}(p,q)\text{-QOFAAWA}(Z_1,\ldots,Z_n).$$

For the $\mathrm{B}(p,q)$-QOFAAWG operator, using the same monotonicity arguments, we obtain

$$\exp\left\{-\left(\sum_{j=1}^{n} \omega_j[-\ln((\aleph_j^+)^p)]^\lambda\right)^{1/\lambda}\right\} \leq \exp\left\{-\left(\sum_{j=1}^{n} \omega_j[-\ln((\aleph_j'^+)^p)]^\lambda\right)^{1/\lambda}\right\},$$

$$1-\exp\left\{-\left(\sum_{j=1}^{n} \omega_j[-\ln(1-(\mho_j^+)^q)]^\lambda\right)^{1/\lambda}\right\} \geq 1-\exp\left\{-\left(\sum_{j=1}^{n} \omega_j[-\ln(1-(\mho_j'^+)^q)]^\lambda\right)^{1/\lambda}\right\},$$

$$\exp\left\{-\left(\sum_{j=1}^{n} \omega_j[-\ln(|\aleph_j^-|^p)]^\lambda\right)^{1/\lambda}\right\} \leq \exp\left\{-\left(\sum_{j=1}^{n} \omega_j[-\ln(|\aleph_j'^-|^p)]^\lambda\right)^{1/\lambda}\right\}$$

and

$$1-\exp\left\{-\left(\sum_{j=1}^{n}\omega_j[-\ln(1-|\mho_j^-|^q)]^\lambda\right)^{1/\lambda}\right\}\geq 1-\exp\left\{-\left(\sum_{j=1}^{n}\omega_j[-\ln(1-|\mho_j'^-|^q)]^\lambda\right)^{1/\lambda}\right\}.$$

Hence

$$\mathrm{B}(p,q)\text{-QOFAAWG}(\Xi_1,\ldots,\Xi_n)\leq_C \mathrm{B}(p,q)\text{-QOFAAWG}(Z_1,\ldots,Z_n).$$

Therefore, both operators are monotone with respect to $\leq_C$.

(iii) Boundedness: Define

$$\Xi_L=\left\langle\left([\min_j(\aleph_j^+)^p]^{1/p},[\max_j(\mho_j^+)^q]^{1/q}\right),\left(-[\min_j|\aleph_j^-|^p]^{1/p},-[\max_j|\mho_j^-|^q]^{1/q}\right)\right\rangle$$

and

$$\Xi_U=\left\langle\left([\max_j(\aleph_j^+)^p]^{1/p},[\min_j(\mho_j^+)^q]^{1/q}\right),\left(-[\max_j|\aleph_j^-|^p]^{1/p},-[\min_j|\mho_j^-|^q]^{1/q}\right)\right\rangle.$$

We first verify that Ξ_L and Ξ_U satisfy the B(p,q)-QOF admissibility conditions. Let k_1 be such that

$$(\mho_{k_1}^+)^q=\max_j(\mho_j^+)^q.$$

Then

$$\min_j(\aleph_j^+)^p+\max_j(\mho_j^+)^q\leq(\aleph_{k_1}^+)^p+(\mho_{k_1}^+)^q\leq 1.$$

Similarly, if k_2 satisfies

$$(\aleph_{k_2}^+)^p=\max_j(\aleph_j^+)^p,$$

then

$$\max_j(\aleph_j^+)^p+\min_j(\mho_j^+)^q\leq(\aleph_{k_2}^+)^p+(\mho_{k_2}^+)^q\leq 1.$$

For the negative pole, let k_3 satisfy

$$|\mho_{k_3}^-|^q=\max_j|\mho_j^-|^q.$$

Then

$$\min_j|\aleph_j^-|^p+\max_j|\mho_j^-|^q\leq|\aleph_{k_3}^-|^p+|\mho_{k_3}^-|^q\leq 1.$$

Finally, if k_4 satisfies

$$|\aleph_{k_4}^-|^p=\max_j|\aleph_j^-|^p,$$

then

$$\max_j |\aleph_j^-|^p + \min_j |\mho_j^-|^q \leq |\aleph_{k_4}^-|^p + |\mho_{k_4}^-|^q \leq 1.$$

Therefore, both Ξ_L and Ξ_U are admissible B(p,q)-QOFNs.
For every j, we have

$$\begin{aligned}
\min_i (\aleph_i^+)^p \leq (\aleph_j^+)^p \leq \max_i (\aleph_i^+)^p,\\
\max_i (\mho_i^+)^q \geq (\mho_j^+)^q \geq \min_i (\mho_i^+)^q,\\
\min_i |\aleph_i^-|^p \leq |\aleph_j^-|^p \leq \max_i |\aleph_i^-|^p
\end{aligned}$$

and

$$\max_i |\mho_i^-|^q \geq |\mho_j^-|^q \geq \min_i |\mho_i^-|^q.$$

Hence

$$\Xi_L \leq_C \Xi_j \leq_C \Xi_U.$$

For the B(p,q)-QOFAAWA operator, we have

$$\begin{aligned}
\min_i (\aleph_i^+)^p \leq 1 - \exp\left\{-\left(\sum_{j=1}^n \omega_j[-\ln(1-(\aleph_j^+)^p)]^\lambda\right)^{1/\lambda}\right\} \leq \max_i (\aleph_i^+)^p,\\
\min_i (\mho_i^+)^q \leq \exp\left\{-\left(\sum_{j=1}^n \omega_j[-\ln((\mho_j^+)^q)]^\lambda\right)^{1/\lambda}\right\} \leq \max_i (\mho_i^+)^q,\\
\min_i |\aleph_i^-|^p \leq 1 - \exp\left\{-\left(\sum_{j=1}^n \omega_j[-\ln(1-|\aleph_j^-|^p)]^\lambda\right)^{1/\lambda}\right\} \leq \max_i |\aleph_i^-|^p
\end{aligned}$$

and

$$\min_i |\mho_i^-|^q \leq \exp\left\{-\left(\sum_{j=1}^n \omega_j[-\ln(|\mho_j^-|^q)]^\lambda\right)^{1/\lambda}\right\} \leq \max_i |\mho_i^-|^q.$$

After taking the corresponding p-th and q-th roots and restoring the negative signs of the negative pole's components, these inequalities give

$$\Xi_L \leq_C \text{B}(p,q)\text{-QOFAAWA}(\Xi_1, \ldots, \Xi_n) \leq_C \Xi_U.$$

For the B(p,q)-QOFAAWG operator

$$\min_i (\aleph_i^+)^p \leq \exp\left\{-\left(\sum_{j=1}^n \omega_j[-\ln((\aleph_j^+)^p)]^\lambda\right)^{1/\lambda}\right\} \leq \max_i (\aleph_i^+)^p,$$

$$\min_i(\mathfrak{V}_i^+)^q \le 1 - \exp\left\{-\left(\sum_{j=1}^n \omega_j[-\ln(1-(\mathfrak{V}_j^+)^q)]^\lambda\right)^{1/\lambda}\right\} \le \max_i(\mathfrak{V}_i^+)^q,$$

$$\min_i |\aleph_i^-|^p \le \exp\left\{-\left(\sum_{j=1}^n \omega_j[-\ln(|\aleph_j^-|^p)]^\lambda\right)^{1/\lambda}\right\} \le \max_i |\aleph_i^-|^p$$

and

$$\min_i |\mathfrak{V}_i^-|^q \le 1 - \exp\left\{-\left(\sum_{j=1}^n \omega_j[-\ln(1-|\mathfrak{V}_j^-|^q)]^\lambda\right)^{1/\lambda}\right\} \le \max_i |\mathfrak{V}_i^-|^q.$$

After taking the appropriate roots and restoring the signs of the negative-pole components,

$$\Xi_L \le_C \mathrm{B}(p,q)\text{-QOFAAWG}(\Xi_1,\ldots,\Xi_n) \le_C \Xi_U.$$

Therefore, both the B(p,q)-QOFAAWA and B(p,q)-QOFAAWG operators satisfy idempotency, monotonicity, and boundedness.

This completes the proof. □

Theorem 12. *(Response of the pole contributions to the* AA *parameter) Let* $0 < \lambda_1 \le \lambda_2$*, with the input B*$(p,q)$*-QOFNs and the weight vector being fixed. Denote the positive and negative poles' contributions for the aggregated B*(p,q)*-QOFN by*

$$P_\lambda = (\aleph_\lambda^+)^p - (\mathfrak{V}_\lambda^+)^q, \qquad N_\lambda = |\mathfrak{V}_\lambda^-|^q - |\aleph_\lambda^-|^p.$$

For the B(p,q)*-QOFAAWA operator, we have*

$$P_{\lambda_1} \le P_{\lambda_2}, \qquad N_{\lambda_1} \ge N_{\lambda_2},$$

whereas for the B(p,q)*-QOFAAWG operator, we have*

$$P_{\lambda_1} \ge P_{\lambda_2}, \qquad N_{\lambda_1} \le N_{\lambda_2}.$$

For either operator, let

$$\Delta P = P_{\lambda_2} - P_{\lambda_1}, \qquad \Delta N = N_{\lambda_2} - N_{\lambda_1}.$$

Then

$$s_\alpha(\lambda_2) - s_\alpha(\lambda_1) = \alpha\Delta P + (1-\alpha)\Delta N.$$

Whenever ΔP *and* ΔN *are nonzero and have opposite signs, there is a unique*

$$\alpha^* = \frac{-\Delta N}{\Delta P - \Delta N} \in (0,1)$$

such that

$$s_{\alpha^*}(\lambda_2) = s_{\alpha^*}(\lambda_1).$$

Proof. From the closed form of the B(p, q)-QOFAAWA operator in Theorem 7, the dependence on λ occurs through weighted power means of the form

$$\left(\sum_{j=1}^{n} \omega_j x_j^{\lambda}\right)^{1/\lambda},$$

which are nondecreasing in λ. Since $1 - e^{-x}$ is increasing and e^{-x} is decreasing, for $0 < \lambda_1 \leq \lambda_2$, the four powered components of the AAWA aggregate satisfy

$$(\aleph_{\lambda_1}^{+})^p \leq (\aleph_{\lambda_2}^{+})^p, \qquad (\mho_{\lambda_1}^{+})^q \geq (\mho_{\lambda_2}^{+})^q$$

and

$$|\aleph_{\lambda_1}^{-}|^p \leq |\aleph_{\lambda_2}^{-}|^p, \qquad |\mho_{\lambda_1}^{-}|^q \geq |\mho_{\lambda_2}^{-}|^q.$$

Therefore, using

$$P_\lambda = (\aleph_\lambda^{+})^p - (\mho_\lambda^{+})^q, \qquad N_\lambda = |\mho_\lambda^{-}|^q - |\aleph_\lambda^{-}|^p,$$

we obtain

$$P_{\lambda_1} \leq P_{\lambda_2}, \qquad N_{\lambda_1} \geq N_{\lambda_2}.$$

For the B(p, q)-QOFAAWG operator, Theorem 9 reverses the four componentwise responses to λ. Hence

$$P_{\lambda_1} \geq P_{\lambda_2}, \qquad N_{\lambda_1} \leq N_{\lambda_2}.$$

Thus, for either operator, increasing λ changes the positive and negative poles' contributions in the opposite directions. Let

$$\Delta P = P_{\lambda_2} - P_{\lambda_1}, \qquad \Delta N = N_{\lambda_2} - N_{\lambda_1}.$$

Since

$$s_\alpha(\lambda) = \alpha P_\lambda + (1 - \alpha) N_\lambda,$$

the corresponding change in score is

$$s_\alpha(\lambda_2) - s_\alpha(\lambda_1) = \alpha \Delta P + (1 - \alpha)\Delta N.$$

If ΔP and ΔN are nonzero and have opposite signs, the value of α for which the change in score is zero is obtained from

$$\alpha \Delta P + (1 - \alpha)\Delta N = 0.$$

Therefore

$$\alpha(\Delta P - \Delta N) = -\Delta N,$$

and hence

$$\alpha^* = \frac{-\Delta N}{\Delta P - \Delta N}.$$

Because ΔP and ΔN have opposite signs,

$$0 < \alpha^* < 1,$$

and, consequently , we have

$$s_{\alpha^*}(\lambda_2) = s_{\alpha^*}(\lambda_1).$$

Endpoint cases are understood by continuous extension. □

Theorem 13. *(Complement duality of the $B(p,q)$-QOFAAWA and $B(p,q)$-QOFAAWG operators) Let Ξ_j, $j = 1, 2, \ldots, n$, be $B(p,q)$-QOFNs with the weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$, where $\omega_j > 0$ and*

$$\sum_{j=1}^{n} \omega_j = 1.$$

Then, we have

$$[\mathrm{B}(p,q)\text{-}QOFAAWA(\Xi_1, \Xi_2, \ldots, \Xi_n)]^c = \mathrm{B}(p,q)\text{-}QOFAAWG(\Xi_1^c, \Xi_2^c, \ldots, \Xi_n^c)$$

and

$$[\mathrm{B}(p,q)\text{-}QOFAAWG(\Xi_1, \Xi_2, \ldots, \Xi_n)]^c = \mathrm{B}(p,q)\text{-}QOFAAWA(\Xi_1^c, \Xi_2^c, \ldots, \Xi_n^c).$$

Proof. Let

$$\Xi_j = \left\langle (\aleph_j^+, \mho_j^+), (\aleph_j^-, \mho_j^-) \right\rangle.$$

By Definition 12, we have

$$\Xi_j^c = \left\langle \left((\mho_j^+)^{q/p}, (\aleph_j^+)^{p/q} \right), \left(-|\mho_j^-|^{q/p}, -|\aleph_j^-|^{p/q} \right) \right\rangle.$$

Hence, the four powered components of Ξ_j^c are $(\mho_j^+)^q$, $(\aleph_j^+)^p$, $|\mho_j^-|^q$, $|\aleph_j^-|^p$. Let

$$\mathrm{B}(p,q)\text{-QOFAAWA}(\Xi_1, \ldots, \Xi_n) = \left\langle (\aleph_A^+, \mho_A^+), (\aleph_A^-, \mho_A^-) \right\rangle.$$

Using the closed forms of the $\mathrm{B}(p,q)$-QOFAAWA and $\mathrm{B}(p,q)$-QOFAAWG operators in Theorems 7 and 9, respectively, and substituting the powered components of Ξ_j^c, the four components of the $\mathrm{B}(p,q)$-QOFAAWG aggregate of the complemented inputs satisfy

$$\begin{aligned}
\aleph_G^+(\Xi_1^c, \ldots, \Xi_n^c) &= (\mho_A^+)^{q/p}, \\
\mho_G^+(\Xi_1^c, \ldots, \Xi_n^c) &= (\aleph_A^+)^{p/q}, \\
\aleph_G^-(\Xi_1^c, \ldots, \Xi_n^c) &= -|\mho_A^-|^{q/p},
\end{aligned}$$

and

$$\mho_G^-(\Xi_1^c, \ldots, \Xi_n^c) = -|\aleph_A^-|^{p/q}.$$

Therefore

$$\mathrm{B}(p,q)\text{-QOFAAWG}(\Xi_1^c, \ldots, \Xi_n^c) = \left\langle \left((\mho_A^+)^{q/p}, (\aleph_A^+)^{p/q}\right), \left(-|\mho_A^-|^{q/p}, -|\aleph_A^-|^{p/q}\right)\right\rangle,$$

which, by Definition 12, is exactly

$$\left[\mathrm{B}(p,q)\text{-QOFAAWA}(\Xi_1, \ldots, \Xi_n)\right]^c.$$

Thus

$$\left[\mathrm{B}(p,q)\text{-QOFAAWA}(\Xi_1, \ldots, \Xi_n)\right]^c = \mathrm{B}(p,q)\text{-QOFAAWG}(\Xi_1^c, \ldots, \Xi_n^c).$$

The second identity follows by applying the first identity to the complemented inputs and using the involutive property $(\Xi^c)^c = \Xi$ established in Proposition 7. Hence, we have

$$\left[\mathrm{B}(p,q)\text{-QOFAAWG}(\Xi_1, \ldots, \Xi_n)\right]^c = \mathrm{B}(p,q)\text{-QOFAAWA}(\Xi_1^c, \ldots, \Xi_n^c).$$

This completes the proof. □

6. MCDM **method**

This section formulates an MCDM procedure that uses the proposed B(p,q)-QOF AA aggregation operators for the synthesis and evaluation of decision information. The procedure draws on general MCDM methodologies [9, 13, 16]. It also builds on AA-based decision frameworks (see [38, 44, 46]) and on the (p,q)-quasirung orthopair setting [47]. The resulting procedure combines the aggregated B(p,q)-QOF assessments with the score and accuracy functions defined in Section 4.5 to obtain an ordering of the alternatives.

Let $F = \{F_1, F_2, \ldots, F_{\mathfrak{m}}\}$ be a finite set of alternatives, and let $\varkappa = \{\varkappa_1, \varkappa_2, \ldots, \varkappa_{\mathfrak{n}}\}$ be a finite set of criteria. Let $\boldsymbol{\omega} = (\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots, \boldsymbol{\omega}_{\mathfrak{n}})^T$ be the weight vector of the criteria, where $\boldsymbol{\omega}_j > 0$ and

$$\sum_{j=1}^{\mathfrak{n}} \boldsymbol{\omega}_j = 1.$$

Assume that the evaluation of the alternative $F_{\mathfrak{i}}$ with respect to the criterion $\varkappa_{\mathfrak{i}}$ is given by a B(p,q)-QOFN

$$\Xi_{ij} = \langle (\aleph_{ij}^+, \mho_{ij}^+), (\aleph_{ij}^-, \mho_{ij}^-) \rangle,$$

where $0 \le \aleph_{ij}^+, \mho_{ij}^+ \le 1$, $-1 \le \aleph_{ij}^-, \mho_{ij}^- \le 0$, and

$$(\aleph_{ij}^+)^{\mathfrak{p}} + (\mho_{ij}^+)^{\mathfrak{q}} \le 1, \qquad |\aleph_{ij}^-|^{\mathfrak{p}} + |\mho_{ij}^-|^{\mathfrak{q}} \le 1.$$

Then the B(p,q)-QOF decision matrix is written as $\mathcal{D} = (\Xi_{ij})_{\mathfrak{m}\times\mathfrak{n}}$.

The proposed MCDM procedure is described as follows.

Step 1: Construct the decision matrix

For each alternative $F_{\mathfrak{i}}$ and each criterion $\varkappa_{\mathfrak{j}}$, collect the evaluation value

$$\Xi_{ij} = \langle(\aleph_{ij}^{+}, \mho_{ij}^{+}), (\aleph_{ij}^{-}, \mho_{ij}^{-})\rangle.$$

Then we construct the decision matrix $\mathcal{D} = (\Xi_{ij})_{\mathfrak{m}\times\mathfrak{n}}$.

Step 2: Normalize the decision matrix

In MCDM problems, the criteria may be benefit-type or cost-type. To obtain a unified benefit-type decision matrix, each cost-type criterion is transformed by using the complement operation, while each benefit-type criterion remains unchanged

$$\widetilde{W}_{ij} = \begin{cases} \Xi_{ij}, & \text{if } \varkappa_{\mathfrak{j}} \text{ is a benefit-type criterion,} \\ \Xi_{ij}^{\mathfrak{c}}, & \text{if } \varkappa_{\mathfrak{j}} \text{ is a cost-type criterion.} \end{cases}$$

For $\Xi_{ij} = \langle(\aleph_{ij}^{+}, \mho_{ij}^{+}), (\aleph_{ij}^{-}, \mho_{ij}^{-})\rangle$, the complement is given by

$$\Xi_{ij}^{\mathfrak{c}} = \left\langle \left((\mho_{ij}^{+})^{\frac{\mathfrak{q}}{\mathfrak{p}}}, (\aleph_{ij}^{+})^{\frac{\mathfrak{p}}{\mathfrak{q}}}\right), \left(-|\mho_{ij}^{-}|^{\frac{\mathfrak{q}}{\mathfrak{p}}}, -|\aleph_{ij}^{-}|^{\frac{\mathfrak{p}}{\mathfrak{q}}}\right)\right\rangle.$$

Hence, the normalized decision matrix is written as $\widetilde{\mathcal{D}} = (\widetilde{W}_{ij})_{\mathfrak{m}\times\mathfrak{n}}$.

Step 3: Aggregate the criterion values of each alternative

For each alternative $F_{\mathfrak{i}}$, aggregate the normalized values $\widetilde{W}_{i1}, \widetilde{W}_{i2}, \ldots, \widetilde{W}_{in}$ by using one of the proposed AA aggregation operators.

Using the B(p, q)-QOFAAWA operator, the aggregated value of $F_{\mathfrak{i}}$ is

$$\Xi_i^A = \text{B}(p, q)\text{-QOFAAWA}(\widetilde{W}_{i1}, \widetilde{W}_{i2}, \ldots, \widetilde{W}_{in}),$$

where

$$\Xi_i^A = \bigoplus_{j=1}^{\mathfrak{n}} (\omega_j \cdot_{AA} \widetilde{W}_{ij}).$$

Using the B(p, q)-QOFAAWG operator, the aggregated value of $F_{\mathfrak{i}}$ is

$$\Xi_i^G = \text{B}(p, q)\text{-QOFAAWG}(\widetilde{W}_{i1}, \widetilde{W}_{i2}, \ldots, \widetilde{W}_{in}),$$

where

$$\Xi_i^G = \bigotimes_{j=1}^{\mathfrak{n}} (\widetilde{W}_{ij})^{\omega_j}.$$

By Theorems 8 and 10, both Ξ_i^A and Ξ_i^G are B(p, q)-QOFNs. Hence, the aggregated value remains in the proposed B(p, q)-QOF environment.

Step 4: Compute the normalized score value

Let the pole importance parameter be chosen so that it lies in the unit interval, and let the aggregated value obtained by either aggregation operator be denoted as

$$\Xi_i = \langle (\aleph_i^+, \mho_i^+), (\aleph_i^-, \mho_i^-) \rangle.$$

The pole-weighted score of this aggregated value is given by

$$s_\alpha(\Xi_i) = \alpha \left[(\aleph_i^+)^{\mathfrak{p}} - (\mho_i^+)^{\mathfrak{q}} \right] + (1 - \alpha) \left[|\mho_i^-|^{\mathfrak{q}} - |\aleph_i^-|^{\mathfrak{p}} \right],$$

where $\alpha \in [0, 1]$ denotes the pole importance parameter. The corresponding normalized score is then computed as

$$\Theta_\alpha(\Xi_i) = \frac{1 + s_\alpha(\Xi_i)}{2}.$$

The alternative with the larger normalized score is preferred.

Step 5: Compute the accuracy value if needed

If two alternatives have the same score value, the accuracy function is used to break the tie. The accuracy value of Ξ_i is given by

$$\Upsilon(\Xi_i) = \frac{1}{2} \left[(\aleph_i^+)^{\mathfrak{p}} + (\mho_i^+)^{\mathfrak{q}} + |\aleph_i^-|^{\mathfrak{p}} + |\mho_i^-|^{\mathfrak{q}} \right].$$

The alternative with the larger accuracy value is preferred when the score values are equal.

Step 6: Rank the alternatives

For a fixed pole importance parameter, the comparison rule is given as follows:

- If $\Theta_\alpha(\Xi_i) > \Theta_\alpha(\Xi_k)$, then $F_{\mathfrak{i}} \succ F_{\mathfrak{k}}$.
- If $\Theta_\alpha(\Xi_i) = \Theta_\alpha(\Xi_k)$ and $\Upsilon(\Xi_i) > \Upsilon(\Xi_k)$, then $F_{\mathfrak{i}} \succ F_{\mathfrak{k}}$.
- If $\Theta_\alpha(\Xi_i) = \Theta_\alpha(\Xi_k)$ and $\Upsilon(\Xi_i) = \Upsilon(\Xi_k)$, then $F_{\mathfrak{i}} \simeq F_{\mathfrak{k}}$.

The alternatives are ordered from highest to lowest according to their normalized pole-weighted score values Θ_α. In the event of a tie in the score, the corresponding accuracy values are considered as the secondary ranking criterion. Accordingly, the alternative achieving the highest position in the resulting order is identified as the preferred choice.

The pole importance parameter α controls the relative contribution of the two poles to the final ranking. When the two poles agree, the resulting ordering is independent of α. When the two poles conflict, the preference between any two alternatives depends on α, and a unique threshold value at which the preference switches can be determined as described in the preceding section. This property is investigated further in the numerical study.

6.1. Selection principle for the aggregation operator

The B(p, q)-QOFAAWA and B(p, q)-QOFAAWG operators are not regarded as competitors in which one is universally superior to the other. They reflect two different aggregation behaviors. The B(p, q)-QOFAAWA operator uses a compensatory mechanism, in which relatively strong performance on some criteria can offset weaker performance on others. The B(p, q)-QOFAAWG operator uses a less compensatory mechanism, in which the aggregated value is more strongly influenced by the lowest criterion-level assessments. The choice between the two operators should therefore be guided by the nature of the criteria and by whether trade-offs among them are acceptable. When the decision problem permits trade-offs, the averaging operator is appropriate; when balanced performance across criteria is required, the geometric operator is preferable.

6.2. Determination of criterion weights

The criterion weights used in the numerical study are assumed to be provided by the decision-maker and are normalized so that they sum to one. In practice, these weights may be obtained by any of the established weighting procedures available in the literature. A dedicated extension of the proposed framework for automatic weight elicitation, including a B(p, q)-QOF best–worst method or an entropy-based weighting scheme, is identified as a direction for future research.

6.3. Algorithm

The proposed algorithm can be summarized as follows. The procedure is formally presented in Algorithm 6, and its logical flow is visually illustrated in Figure 2.

Algorithm 1: Proposed B(p, q)-QOFAAWA/B(p, q)-QOFAAWG-based MCDM procedure.

Input : The alternative set $F = \{F_1, F_2, \ldots, F_{\mathfrak{m}}\}$
The criterion set $\varkappa = \{\varkappa_1, \varkappa_2, \ldots, \varkappa_{\mathfrak{n}}\}$
The criterion-weight vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_{\mathfrak{n}})^T$
The B(p, q)-QOF decision matrix $\mathcal{D} = (\Xi_{ij})_{\mathfrak{m}\times\mathfrak{n}}$
The parameters $\mathfrak{p}, \mathfrak{q} \in \mathbb{N}$, with $\mathfrak{p}, \mathfrak{q} \geq 1$, and $\lambda > 0$
The criterion-type vector $C = (c_1, c_2, \ldots, c_{\mathfrak{n}})$, where $c_j \in \{\text{benefit-type}, \text{cost-type}\}$
The aggregation operator $O \in \{\text{B}(p, q)\text{-QOFAAWA}, \text{B}(p, q)\text{-QOFAAWG}\}$
The pole importance parameter $\alpha \in [0, 1]$.

1 **Step 1.** Form the B(p, q)-QOF decision matrix.
2 **Step 2.** Transform the matrix to a normalized form whenever cost-type criteria are present.
3 **Step 3.** Apply the selected aggregation operator to the normalized criterion assessments for each alternative.
4 **Step 4.** Determine the pole-weighted score and the normalized score of each aggregated B(p, q)-QOF assessment using the pole importance parameter α.
5 **Step 5.** Evaluate the corresponding accuracy value whenever it is required to resolve equal score values.
6 **Step 6.** Rank the alternatives according to their normalized pole-weighted scores, using the accuracy values to resolve ties when necessary.

Output: Identify the highest-ranked alternative as the preferred choice.

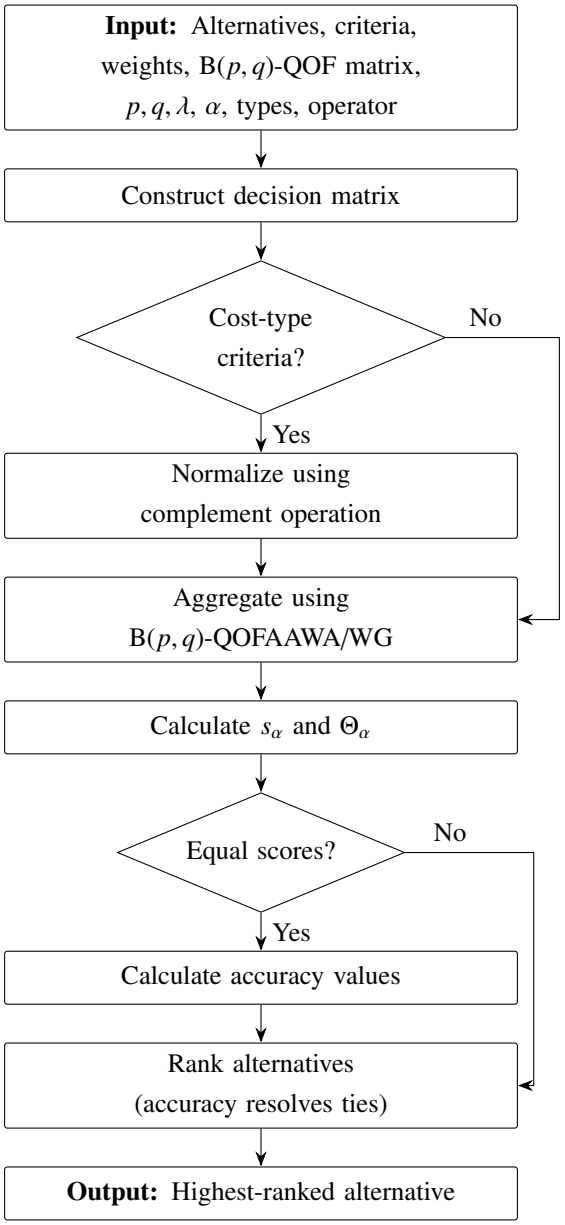


Figure 2. Flowchart of the proposed B(p, q)-QOF-based MCDM procedure.

7. Numerical example

This section provides a numerical illustration of the proposed B(p, q)-QOF-based MCDM methodology. The purpose of this example is to show how the proposed AA aggregation operators can be implemented to combine the evaluations of several criteria and subsequently determine the relative preference of the candidate alternatives. The analysis also examines the stability of the obtained ranking when the parameters of the proposed model are varied.

7.1. Problem description

Consider a hypothetical smart-material manufacturing company that needs to select a suitable supplier for an upcoming production project. Four candidate suppliers are considered, namely

$$F = \{F_1, F_2, F_3, F_4\}.$$

Their performances are assessed according to the following four criteria:

$$\varkappa = \{\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4\},$$

where $\varkappa_1$ represents environmental compliance, $\varkappa_2$ denotes the operational cost, $\varkappa_3$ describes delivery resilience, and $\varkappa_4$ reflects the level of technical cooperation.

The first, third, and fourth criteria are benefit-type criteria, whereas the second criterion is a cost-type criteria. The importance assigned to the four criteria is represented by the weight vector

$$\boldsymbol{\omega} = (0.29, 0.18, 0.31, 0.22)^T$$

with

$$\sum_{j=1}^{4} \boldsymbol{\omega}_j = 1.$$

The criterion weights used in this numerical study are assumed to be provided by the decision-maker and are normalized so that they sum to one. A dedicated sensitivity analysis is conducted in Section 7.8 to examine the dependence of the final ranking on these weights. For the main calculation, the model parameters are selected as

$$\mathfrak{p} = 3, \qquad \mathfrak{q} = 4, \qquad \lambda = 2.$$

For the baseline analysis, the pole importance parameter is set to $\alpha = 0.5$, assigning equal weight to the positive and negative poles in the ranking score. The sensitivity analysis in Section 7.5 examines how alternative values of α affect the resulting ranking.

7.2. Initial decision matrix and cost-criterion normalization

The assessments provided for the four suppliers are expressed as B(p, q)-QOFNs. The evaluations corresponding to the first two criteria are given in Table 2, while the evaluations for the remaining two criteria are reported in Table 3.

Table 2. Initial evaluations for $\varkappa_1$ and $\varkappa_2$.

Alternative	$\varkappa_1$	$\varkappa_2$
F_1	$\langle(0.76, 0.42), (-0.64, -0.34)\rangle$	$\langle(0.55, 0.65), (-0.44, -0.60)\rangle$
F_2	$\langle(0.66, 0.53), (-0.55, -0.45)\rangle$	$\langle(0.48, 0.70), (-0.39, -0.66)\rangle$
F_3	$\langle(0.82, 0.34), (-0.72, -0.28)\rangle$	$\langle(0.39, 0.73), (-0.34, -0.67)\rangle$
F_4	$\langle(0.70, 0.48), (-0.60, -0.40)\rangle$	$\langle(0.58, 0.61), (-0.46, -0.56)\rangle$

Table 3. Initial evaluations for $\varkappa_3$ and $\varkappa_4$.

Alternative	$\varkappa_3$	$\varkappa_4$
F_1	$\langle((0.69, 0.47), (-0.59, -0.39)\rangle$	$\langle((0.73, 0.43), (-0.62, -0.36)\rangle$
F_2	$\langle((0.74, 0.41), (-0.63, -0.35)\rangle$	$\langle((0.61, 0.56), (-0.50, -0.48)\rangle$
F_3	$\langle((0.78, 0.37), (-0.68, -0.30)\rangle$	$\langle((0.80, 0.35), (-0.70, -0.29)\rangle$
F_4	$\langle((0.65, 0.52), (-0.55, -0.44)\rangle$	$\langle((0.76, 0.39), (-0.66, -0.33)\rangle$

The second criterion represents operational cost and is therefore a cost-oriented criterion. To place all criteria on a common preference scale, its evaluations are transformed into the corresponding benefit-oriented form using the complement operation defined in Section 4.6 for the B(p, q)-QOF framework. The resulting normalized decision matrix is shown in Table 4.

Table 4. Normalized B(p, q)-QOF decision matrix.

Alternative	$\varkappa_1$	$\varkappa_2$	$\varkappa_3$	$\varkappa_4$
F_1	$\langle((0.7600, 0.4200), (-0.6400, -0.3400)\rangle$	$\langle((0.5631, 0.6387), (-0.5061, -0.5402)\rangle$	$\langle((0.6900, 0.4700), (-0.5900, -0.3900)\rangle$	$\langle((0.7300, 0.4300), (-0.6200, -0.3600)\rangle$
F_2	$\langle((0.6600, 0.5300), (-0.5500, -0.4500)\rangle$	$\langle((0.6215, 0.5767), (-0.5746, -0.4935)\rangle$	$\langle((0.7400, 0.4100), (-0.6300, -0.3500)\rangle$	$\langle((0.6100, 0.5600), (-0.5000, -0.4800)\rangle$
F_3	$\langle((0.8200, 0.3400), (-0.7200, -0.2800)\rangle$	$\langle((0.6573, 0.4935), (-0.5863, -0.4453)\rangle$	$\langle((0.7800, 0.3700), (-0.6800, -0.3000)\rangle$	$\langle((0.8000, 0.3500), (-0.7000, -0.2900)\rangle$
F_4	$\langle((0.7000, 0.4800), (-0.6000, -0.4000)\rangle$	$\langle((0.5173, 0.6646), (-0.4616, -0.5586)\rangle$	$\langle((0.6500, 0.5200), (-0.5500, -0.4400)\rangle$	$\langle((0.7600, 0.3900), (-0.6600, -0.3300)\rangle$

The normalized matrix preserves both poles of every evaluation. For the cost-oriented criterion $\varkappa_2$, the complement operation interchanges the powered membership and nonmembership components according to the exponents specified in Section 4.6, yielding benefit-oriented evaluations within the admissible B(p, q)-QOF domain.

7.3. Aggregation and ranking results

The normalized evaluations are next combined according to the criterion weights. First, the proposed B(p, q)-QOFAAWA operator is applied to each alternative. This produces one aggregated B(p, q)-QOFN for each supplier, as reported in Table 5.

Table 5. Aggregated B(p, q)-QOFNs obtained using the B(p, q)-QOFAAWA operator.

Alternative	Aggregated value
F_1	$\langle((0.7147, 0.4645), (-0.6054, -0.3852)\rangle$
F_2	$\langle((0.6817, 0.4959), (-0.5801, -0.4243)\rangle$
F_3	$\langle((0.7873, 0.3723), (-0.6881, -0.3097)\rangle$
F_4	$\langle((0.6912, 0.4883), (-0.5929, -0.4126)\rangle$

For comparison, the same normalized decision information is processed using the B(p, q)-QOFAAWG operator. The resulting aggregated evaluations are listed in Table 6.

Table 6. Aggregated B(p, q)-QOFNs obtained using the B(p, q)-QOFAAWG operator.

Alternative	Aggregated value
F_1	$\langle(0.6827, 0.5349), (-0.5903, -0.4487)\rangle$
F_2	$\langle(0.6600, 0.5327), (-0.5626, -0.4547)\rangle$
F_3	$\langle(0.7629, 0.4139), (-0.6731, -0.3666)\rangle$
F_4	$\langle(0.6473, 0.5632), (-0.5620, -0.4714)\rangle$

Using the pole importance parameter $\alpha = 0.5$ the pole-weighted score and the normalized score of each aggregated B(p, q)-QOFN are computed as described in Section 6. The resulting values, together with the corresponding accuracy values, are reported in Table 7.

Table 7. Normalized score and accuracy values under the baseline setting, computed with $\alpha = 0.5$.

Alternative	Operator	Normalized score Θ_α	Accuracy Υ
F_1	B(p, q)-QOFAAWA	0.5297	0.3278
F_1	B(p, q)-QOFAAWG	0.5178	0.3232
F_2	B(p, q)-QOFAAWA	0.5234	0.3025
F_2	B(p, q)-QOFAAWG	0.5179	0.2944
F_3	B(p, q)-QOFAAWA	0.5380	0.4211
F_3	B(p, q)-QOFAAWG	0.5320	0.3982
F_4	B(p, q)-QOFAAWA	0.5235	0.3122
F_4	B(p, q)-QOFAAWG	0.5106	0.2994

A visual comparison of the normalized score values produced by the two operators is provided in Figure 3.

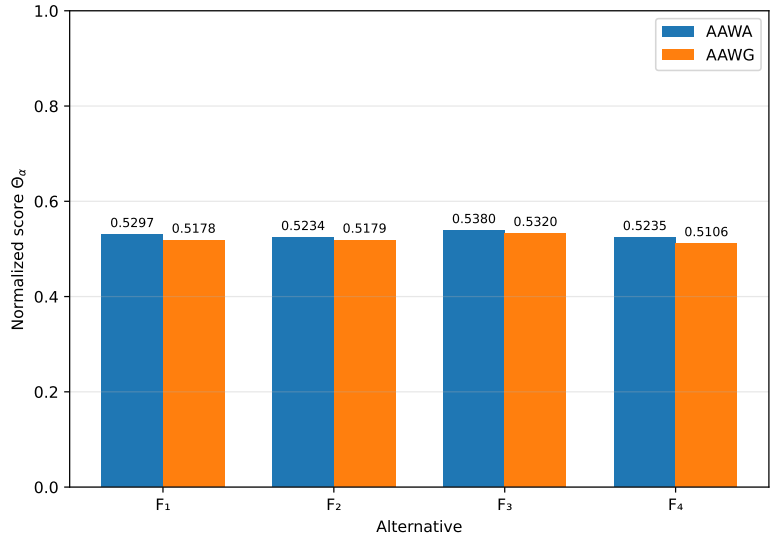


Figure 3. Comparison of the normalized score values produced by the B(p, q)-QOFAAWA and B(p, q)-QOFAAWG operators under the baseline setting.

Using the normalized score as the primary ranking measure, with the accuracy value providing additional information when necessary, the resulting orders of the four alternatives are summarized in Table 8.

Table 8. Ranking results under the baseline setting.

Operator	Rank	Alternative	Normalized score Θ_α	Accuracy Υ
B(p,q)-QOFAAWA	1	F_3	0.5380	0.4211
B(p,q)-QOFAAWA	2	F_1	0.5297	0.3278
B(p,q)-QOFAAWA	3	F_4	0.5235	0.3122
B(p,q)-QOFAAWA	4	F_2	0.5234	0.3025
B(p,q)-QOFAAWG	1	F_3	0.5320	0.3982
B(p,q)-QOFAAWG	2	F_2	0.5179	0.2944
B(p,q)-QOFAAWG	3	F_1	0.5178	0.3232
B(p,q)-QOFAAWG	4	F_4	0.5106	0.2994

The ranking generated by the B(p,q)-QOFAAWA operator is

$$F_3 \succ F_1 \succ F_4 \succ F_2,$$

whereas the B(p,q)-QOFAAWG operator produces

$$F_3 \succ F_2 \succ F_1 \succ F_4.$$

Both aggregation mechanisms identify F_3 as the preferred supplier. The two operators, however, do not produce identical orderings of the remaining alternatives.

7.4. Interpretation of the two aggregation operators

The B(p,q)-QOFAAWA operator provides a more compensatory aggregation mechanism, allowing stronger assessments on some criteria to offset weaker assessments on others. In contrast, the B(p,q)-QOFAAWG operator provides a less compensatory mechanism, so that relatively weak criterion-level assessments can have a stronger influence on the aggregated evaluation. The baseline results are consistent with this interpretation. Both operators place F_3 in the first position. Under B(p,q)-QOFAAWA, the second position is occupied by F_1, whereas under B(p,q)-QOFAAWG, the second position is occupied by F_2. This difference is attributable to the distinct AA aggregation mechanisms used by the two operators. Neither operator is universally superior to the other. The choice between them should be guided by the decision context and by whether trade-offs among criteria are acceptable.

7.5. Sensitivity to the pole importance parameter

The pole importance parameter α controls the relative contribution of the two poles to the ranking score. The baseline analysis used the symmetric value $\alpha = 0.5$. To examine the influence of this parameter, the aggregation and ranking calculations are repeated for several alternative values of α. The resulting rankings are reported in Table 9.

Table 9. Ranking results under alternative values of the pole importance parameter α.

α	B(p,q)-QOFAAWA ranking	B(p,q)-QOFAAWG ranking
0.1	$F_2 \succ F_4 \succ F_1 \succ F_3$	$F_4 \succ F_2 \succ F_1 \succ F_3$
0.3	$F_2 \succ F_4 \succ F_1 \succ F_3$	$F_2 \succ F_4 \succ F_1 \succ F_3$
0.5	$F_3 \succ F_1 \succ F_4 \succ F_2$	$F_3 \succ F_2 \succ F_1 \succ F_4$
0.7	$F_3 \succ F_1 \succ F_4 \succ F_2$	$F_3 \succ F_1 \succ F_2 \succ F_4$
0.9	$F_3 \succ F_1 \succ F_4 \succ F_2$	$F_3 \succ F_1 \succ F_2 \succ F_4$

The results show that the pole importance parameter has a substantive effect on the resulting ranking. For $\alpha \leq 0.3$, the positive poles contribution is heavily discounted, and the leading alternative changes. For $\alpha \geq 0.5$, the leading alternative is F_3 under both operators. This behavior confirms that α is not a purely formal parameter: It encodes a genuine trade-off between the positive and negative poles' evidence. The corresponding ranking sensitivity to α is illustrated in Figure 4.

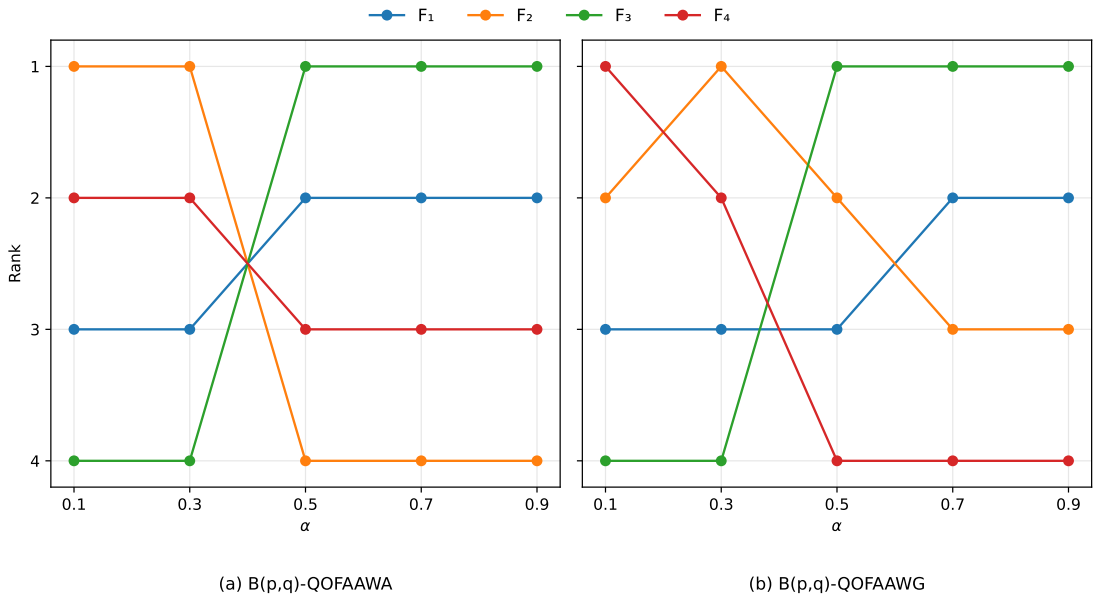


Figure 4. Sensitivity of rankings to the pole importance parameter α: (a) B(p,q)-QOFAAWA; (b) B(p,q)-QOFAAWG.

7.6. Sensitivity to the AA *parameter*

The parameter λ controls the operational behavior of the AA norm and conorm. To examine whether changes in this parameter can alter the final decision, its value is varied over a predefined range. The corresponding ranking results are summarized in Table 10.

For the numerical data considered here, the leading alternative remains F_3 across all tested values of λ. However, the complete ranking is not invariant. Under B(p,q)-QOFAAWA, the positions of F_2 and F_4 change when λ varies, whereas under B(p,q)-QOFAAWG, the relative positions of F_1 and F_2 change between the lower and higher tested values of λ.

Table 10. Ranking results for different values of λ.

λ	B(p,q)-QOFAAWA ranking	B(p,q)-QOFAAWG ranking
0.5	$F_3 \succ F_1 \succ F_2 \succ F_4$	$F_3 \succ F_1 \succ F_2 \succ F_4$
1.0	$F_3 \succ F_1 \succ F_2 \succ F_4$	$F_3 \succ F_1 \succ F_2 \succ F_4$
2.0	$F_3 \succ F_1 \succ F_4 \succ F_2$	$F_3 \succ F_2 \succ F_1 \succ F_4$
5.0	$F_3 \succ F_1 \succ F_2 \succ F_4$	$F_3 \succ F_2 \succ F_1 \succ F_4$
10.0	$F_3 \succ F_1 \succ F_2 \succ F_4$	$F_3 \succ F_2 \succ F_1 \succ F_4$

These results indicate that λ has a limited effect on the leading decision but can influence the relative ordering of lower-ranked alternatives. The ranking behavior for different values of λ is depicted in Figure 5, and the variation of the normalized score values with respect to λ is shown in Figure 6.

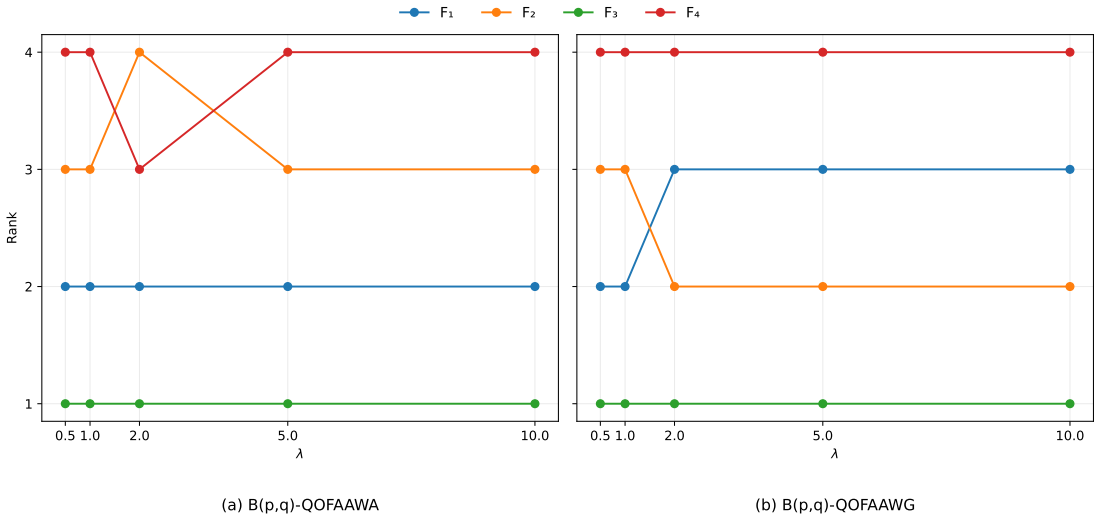


Figure 5. Ranking behavior for different values of the AA parameter λ.

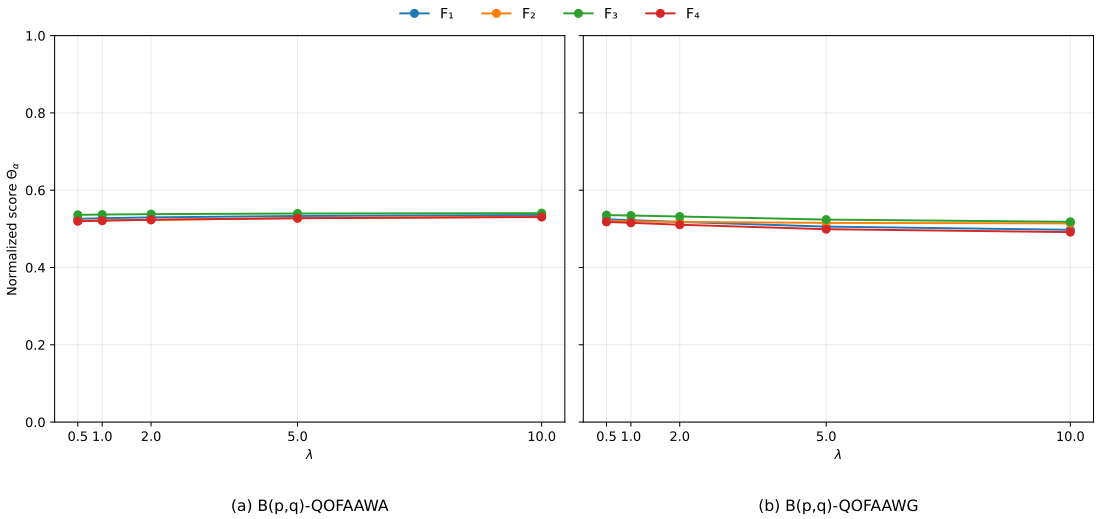


Figure 6. Variation of normalized score values with respect to λ.

7.7. Sensitivity to the parameters p and q

The proposed framework contains two independent exponents, p and q, which govern the admissible orthopair structure. The influence of these two parameters on the final ranking is examined separately.

First, the parameter p is varied from 1 to 6, while $q = 4$ and $\lambda = 2$ are held fixed. The ranking results for these values of p are presented in Table 11.

Table 11. Ranking sensitivity to p for $q = 4$ and $\lambda = 2$.

p	B(p, q)-QOFAAWA ranking	B(p, q)-QOFAAWG ranking
1	$F_3 \succ F_1 \succ F_2 \succ F_4$	$F_3 \succ F_2 \succ F_1 \succ F_4$
2	$F_3 \succ F_1 \succ F_2 \succ F_4$	$F_3 \succ F_2 \succ F_1 \succ F_4$
3	$F_3 \succ F_1 \succ F_4 \succ F_2$	$F_3 \succ F_2 \succ F_1 \succ F_4$
4	$F_3 \succ F_1 \succ F_4 \succ F_2$	$F_3 \succ F_1 \succ F_2 \succ F_4$
5	$F_3 \succ F_1 \succ F_4 \succ F_2$	$F_3 \succ F_1 \succ F_2 \succ F_4$
6	$F_3 \succ F_1 \succ F_4 \succ F_2$	$F_3 \succ F_1 \succ F_4 \succ F_2$

The results show that F_3 remains the leading alternative throughout the tested range of p. Under B(p, q)-QOFAAWA, the ranking is

$$F_3 \succ F_1 \succ F_2 \succ F_4$$

for $p = 1, 2$, whereas it becomes

$$F_3 \succ F_1 \succ F_4 \succ F_2$$

for $p = 3, 4, 5, 6$. Thus, the transition in the lower ordering occurs when p increases from 2 to 3.

Under B(p, q)-QOFAAWG, the ordering evolves from

$$F_3 \succ F_2 \succ F_1 \succ F_4$$

for $p = 1, 2, 3$ to

$$F_3 \succ F_1 \succ F_2 \succ F_4$$

for $p = 4, 5$. At $p = 6$, the ranking becomes

$$F_3 \succ F_1 \succ F_4 \succ F_2.$$

The corresponding ranking behavior is illustrated in Figure 7.

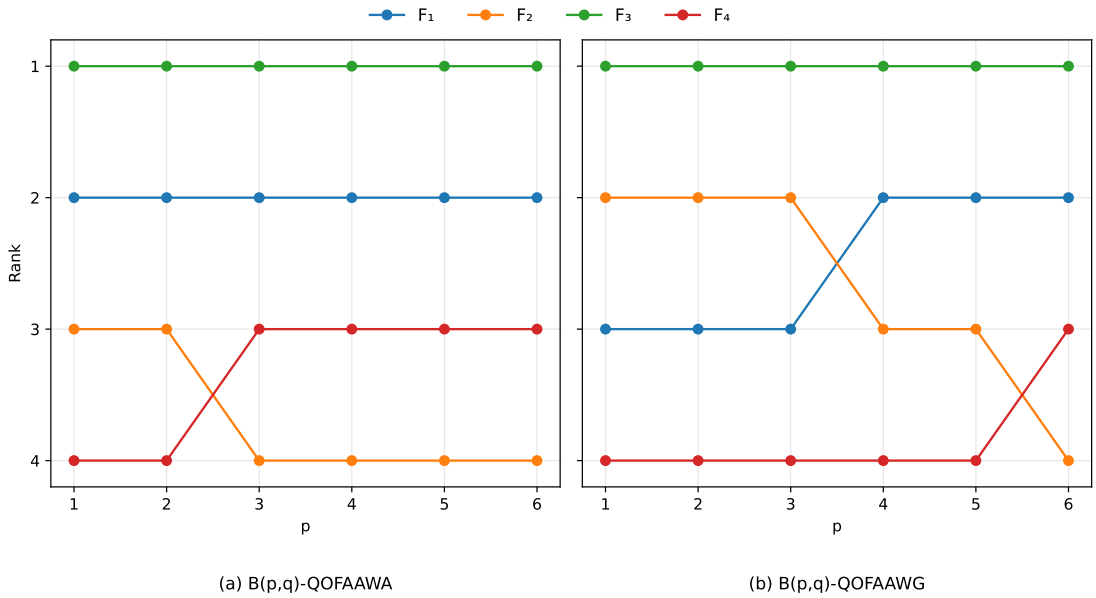


Figure 7. Ranking behavior for different values of p with $q = 4$ and $\lambda = 2$: (a) B(p, q)-QOFAAWA; (b) B(p, q)-QOFAAWG.

Second, the parameter q is varied from 1 to 6 while $p = 3$ and $\lambda = 2$ are held fixed. The corresponding rankings are reported in Table 12.

Table 12. Ranking sensitivity to q for $p = 3$ and $\lambda = 2$.

q	B(p, q)-QOFAAWA ranking	B(p, q)-QOFAAWG ranking
1	$F_3 \succ F_1 \succ F_2 \succ F_4$	$F_3 \succ F_1 \succ F_2 \succ F_4$
2	$F_3 \succ F_1 \succ F_4 \succ F_2$	$F_3 \succ F_1 \succ F_2 \succ F_4$
3	$F_3 \succ F_1 \succ F_4 \succ F_2$	$F_3 \succ F_1 \succ F_2 \succ F_4$
4	$F_3 \succ F_1 \succ F_4 \succ F_2$	$F_3 \succ F_2 \succ F_1 \succ F_4$
5	$F_3 \succ F_1 \succ F_2 \succ F_4$	$F_3 \succ F_2 \succ F_1 \succ F_4$
6	$F_3 \succ F_1 \succ F_2 \succ F_4$	$F_3 \succ F_2 \succ F_1 \succ F_4$

The leading alternative remains F_3 throughout the tested range of q. Under B(p, q)-QOFAAWA, the ordering of the lower-ranked alternatives changes between $q = 1$ and $q = 2$; remains unchanged for $q = 2, 3, 4$; and changes again at $q = 5$, remaining stable through $q = 6$. Under B(p, q)-QOFAAWG, the ordering is stable for $q \leq 3$ and then changes at $q = 4$. Overall, the two exponents p and q have a measurable influence on the middle ordering, whereas the leading alternative remains unchanged throughout the tested configurations. The ranking behavior for different values of q is depicted in Figure 8.

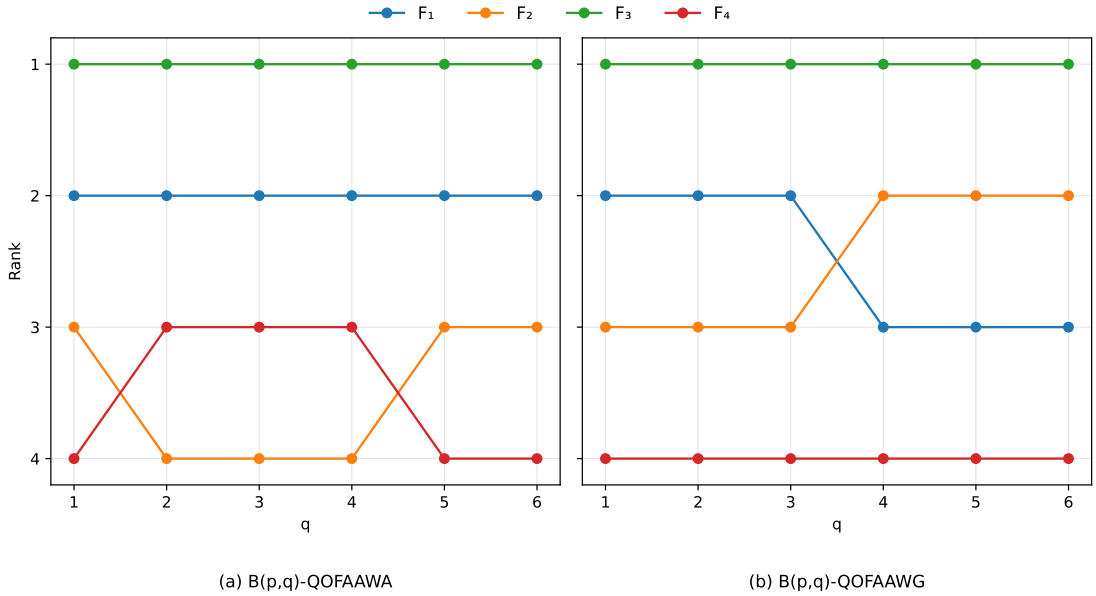


Figure 8. Ranking behavior for different values of q with $p = 3$ and $\lambda = 2$: (a) B(p, q)-QOFAAWA; (b) B(p, q)-QOFAAWG.

7.8. Sensitivity to criterion weights

The criterion weights used in the baseline analysis are assumed to be provided by the decision-maker. To examine the dependence of the final ranking on these weights, several alternative weighting scenarios are considered. The ranking results under these scenarios are reported in Table 13.

Table 13. Ranking under alternative criterion weight scenarios.

Scenario	ω	B(p, q)-QOFAAWA ranking	B(p, q)-QOFAAWG ranking
Baseline	$(0.29, 0.18, 0.31, 0.22)^T$	$F_3 \succ F_1 \succ F_4 \succ F_2$	$F_3 \succ F_2 \succ F_1 \succ F_4$
Equal weights	$(0.25, 0.25, 0.25, 0.25)^T$	$F_3 \succ F_1 \succ F_4 \succ F_2$	$F_3 \succ F_2 \succ F_1 \succ F_4$
Emphasis on $\varkappa_1$	$(0.40, 0.20, 0.20, 0.20)^T$	$F_3 \succ F_1 \succ F_4 \succ F_2$	$F_3 \succ F_1 \succ F_2 \succ F_4$
Emphasis on $\varkappa_2$	$(0.20, 0.40, 0.20, 0.20)^T$	$F_3 \succ F_1 \succ F_4 \succ F_2$	$F_3 \succ F_2 \succ F_1 \succ F_4$
Emphasis on $\varkappa_3$	$(0.20, 0.20, 0.40, 0.20)^T$	$F_3 \succ F_1 \succ F_2 \succ F_4$	$F_3 \succ F_2 \succ F_1 \succ F_4$
Emphasis on $\varkappa_4$	$(0.20, 0.20, 0.20, 0.40)^T$	$F_3 \succ F_1 \succ F_4 \succ F_2$	$F_3 \succ F_1 \succ F_2 \succ F_4$

The criterion weight sensitivity analysis shows that F_3 remains the top-ranked alternative under all six weighting scenarios for both aggregation operators. This indicates that the leading decision is stable with respect to the tested redistributions of criterion importance, although the relative ordering of the remaining alternatives changes in several scenarios.

7.9. Negative pole ablation analysis

To examine the contribution of the bipolar structure to the final ranking, an ablation analysis is conducted. The negative pole components of every evaluation are removed, so that only the positive pole contributes to the aggregation. The comparison between the full bipolar model and the positive-pole-only model is reported in Table 14.

Table 14. Comparison between the full bipolar model and the positive-pole-only model.

Operator	Alternative	Full model Θ_α	Rank	Positive-only Θ_α	Rank
B(p,q)-QOFAAWA	F_1	0.5297	2	0.5796	2
B(p,q)-QOFAAWA	F_2	0.5234	4	0.5641	4
B(p,q)-QOFAAWA	F_3	0.5380	1	0.6172	1
B(p,q)-QOFAAWA	F_4	0.5235	3	0.5683	3
B(p,q)-QOFAAWG	F_1	0.5178	3	0.5591	2
B(p,q)-QOFAAWG	F_2	0.5179	2	0.5518	3
B(p,q)-QOFAAWG	F_3	0.5320	1	0.6037	1
B(p,q)-QOFAAWG	F_4	0.5106	4	0.5426	4

The ablation analysis provides evidence that the negative pole can materially affect the decision outcome, although its effect is operator-dependent. Under B(p,q)-QOFAAWA, removing the pole's information does not change the ranking in this example. In contrast, under B(p,q)-QOFAAWG, the second and third positions are interchanged. Thus, the negative pole is not redundant in this numerical example: Its influence becomes visible under the B(p,q)-QOFAAWG aggregation mechanism, even though the B(p,q)-QOFAAWA ranking remains unchanged. This result demonstrates that the contribution of bipolar information can depend on both the aggregation mechanism and the structure of the decision data.

8. Comparative analysis and discussion

This section evaluates the proposed B(p,q)-QOFS framework from two complementary perspectives. First, its representational capability is examined in relation to established bipolar and orthopair models. Second, its numerical behavior is discussed in light of the ablation and sensitivity analyses conducted in Section 7.

8.1. Structural comparison with existing models

The proposed B(p,q)-QOFS framework combines several features that, in existing bipolar and orthopair models, are typically addressed separately. To make this comparison explicit, Table 15 summarizes the key structural characteristics of the most closely related models.

Table 15. Structural comparison between the proposed B(p,q)-QOFS framework and the related bipolar and orthopair models.

Model	Bipolar polarity	Independent exponents	AA operations	Pole-sensitive ranking	Pole conflict switching analysis
Li et al. [49]	Yes	No (single q)	No	No	No
Ibrahim [30]	Yes	Yes (n,m)	Not reported	Not reported	Not reported
Ali and Naeem [46]	No	Yes (p,q)	Yes	No	No
Ali et al. [47]	No	Yes (p,q)	Yes	No	No
Zararsiz [48]	Yes	No (no p,q)	Yes	No	No
Proposed B(p,q)-QOFS	Yes	Yes (p,q)	Yes	Yes (α)	Yes

The comparison highlights three aspects that distinguish the proposed framework from the existing models considered here. First, the proposed framework combines independent orthopair exponents p and q with an explicit bipolar polarity. Among the models considered here, the bipolar orthopair models either use a single orthopair exponent, as in the bipolar q-ROFS case, or are based on a different exponent pair (n, m) whose interaction with AA operations is not reported in the corresponding study. Second, the proposed AA operational laws are formulated so that the joint admissibility of the two poles is preserved. To the best of our examination of the considered literature, such a joint admissibility property under AA operations has not been explicitly established for the bipolar orthopair models listed in Table 15. Third, the ranking mechanism allows the decision-maker to control the relative importance of the two poles through the parameter α. This feature is not reported in the compared models.

8.2. Numerical evidence from ablation and sensitivity analyses

A direct numerical comparison with the models listed in Table 15 would require transforming the bipolar data of Section 7 into representations that do not preserve the same information content. In particular, applying a unipolar model to the bipolar data would discard the negative pole, whereas applying the bipolar q-ROFS model would require reducing the independent exponents to a common exponent. Such transformations would not yield a fair comparison. Instead, the numerical contribution of the bipolar structure and of the free parameters is examined directly through the ablation and sensitivity analyses conducted in Section 7. The negative pole ablation in Section 7.9 shows that removing the negative pole's information changes the ranking under the geometric operator, demonstrating that the bipolar structure is not redundant in the numerical example considered here. The sensitivity analyses in Sections 7.5 to 7.8 show that the leading alternative remains F_3 across a range of values of λ, p, and q and across alternative criterion weight scenarios, whereas the complete ranking responds to these parameters and to the pole importance parameter α.

8.3. Theoretical and practical discussion

The results of Sections 7 and 8 support four observations about the behavior of the proposed framework.

(1) The two aggregation operators, B(p, q)-QOFAAWA and B(p, q)-QOFAAWG, are not competitors in which one is universally superior. They reflect two different aggregation mechanisms, one more compensatory and one less compensatory, and the resulting orderings differ even when the leading alternative is the same.
(2) The parameters p, q, and λ preserve the leading alternative in the numerical example considered here, but they can influence the ordering of the lower-ranked alternatives. This distinction between the stability of the leading decision and the stability of the complete ranking is important for decision support.
(3) The pole importance parameter α has a substantive effect on the ranking. For sufficiently small values of α, the leading alternative changes, indicating that α encodes a genuine trade-off between the evidence of the positive pole and negative pole.
(4) The negative pole is not redundant in the numerical example: Removing it changes the ranking under the geometric aggregation mechanism, although the effect is operator-dependent.

Taken together, these observations indicate that the proposed framework is most useful in decision

settings where favorable and unfavorable pieces of evidence need to be retained simultaneously and where the relative importance of the two poles is a decision-relevant consideration.

8.4. Limitations and future directions

The present study has several limitations that should be acknowledged. First, the numerical example is based on synthetic data. While this choice ensures reproducibility, it does not constitute an empirical validation of the proposed framework. Future work should apply the framework to real-world decision problems. Second, the criterion weights used in the numerical example are assumed to be provided by the decision-maker. The proposed framework does not include a dedicated weighting mechanism. Extending the framework with a B(p, q)-QOF best–worst method or an entropy-based weighting scheme would provide a more complete decision-support tool. Third, the contribution of the bipolar structure is operator-dependent in the numerical example. Under the averaging operator, removing the negative pole does not change the ranking, whereas under the geometric operator, it does. Further empirical work is needed to characterize the conditions under which the bipolar structure is essential. Fourth, the complete ranking remains sensitive to the choice of aggregation operator and to the values of p, q, λ, and α. This sensitivity is not necessarily a limitation but it should be communicated clearly to decision-makers. Future research could address these limitations by applying the framework to real case studies, by developing dedicated weighting procedures, and by extending the framework to dynamic or group DM settings.

9. Conclusions

This study developed a B(p, q)-QOF framework for MCDM problems involving favorable and unfavorable assessments. The proposed formulation combines bipolar polarity with the asymmetric (p, q)-quasirung orthopair structure, representing both poles within a common mathematical setting with independent control through $\mathfrak{p}$ and $\mathfrak{q}$. Within this framework, AA operational rules were formulated for B(p, q)-QOFNs, leading to the B(p, q)-QOFAAWA and B(p, q)-QOFAAWG operators. Their principal properties, including closure, idempotency, monotonicity, and boundedness, were established. The resulting operators were then incorporated into an MCDM procedure with a pole-weighted score mechanism that allows the decision-maker to control the relative contribution of the positive and negative poles' evidence through the parameter α. The numerical study demonstrated the practical behavior of the proposed framework through a sustainable supplier selection example. Both aggregation operators identified F_3 as the preferred supplier under the baseline setting, although they produced different orderings for the remaining alternatives.

This result illustrates that the choice between averaging and geometric aggregation should depend on the decision context. The averaging operator is preferable when greater compensation among criteria is acceptable, whereas the geometric operator is preferable when relatively weak assessments should have a stronger influence on the aggregated evaluation. Neither operator is universally superior. The sensitivity analyses showed that the parameters $\mathfrak{p}$, $\mathfrak{q}$, and λ can affect the ordering of lower-ranked alternatives, while the leading alternative remained unchanged over the tested configurations. Similarly, the tested criterion weight scenarios preserved F_3 as the leading alternative, indicating that the stability of the leading decision should be distinguished from stability of the complete ranking. In contrast, the pole importance parameter α had a more substantive Therefor Sufficiently small

values of α changed the leading alternative. Thus, α should be selected according to the decision-maker's assessment of the relative importance of favorable and unfavorable evidence rather than treated as a purely technical parameter. The negative pole ablation analysis provided additional evidence for the role of bipolar information. Removing the negative pole did not change the ranking under the B(p, q)-QOFAAWA operator, whereas it changed the relative positions of F_1 and F_2 under the B(p, q)-QOFAAWG operator. Therefore, the contribution of bipolar information is operator and data-dependent. The framework is consequently most relevant to decision environments in which favorable and unfavorable evidence have distinct meanings and both should be retained during aggregation and ranking.

From a practical perspective, the proposed framework provides several adjustable modeling choices. The parameters $\mathfrak{p}$ and $\mathfrak{q}$ can be selected independently to reflect the desired flexibility of the membership and nonmembership components, while λ controls the operational behavior of the AA aggregation mechanism. The parameter α should reflect the relative importance assigned to the positive and negative poles' evidence. The choice between the averaging and geometric operators should likewise be based on whether compensatory trade-offs among criteria are desirable. These choices should preferably be supported by a sensitivity analysis rather than by assuming a universally optimal parameter configuration.

Author contributions

Shatha Abdulhadi Saleh Alharbi: conceptualization, methodology, investigation, and writing original draft; Kholood Alsager: supervision, conceptualization, writing review, and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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