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Research article

Convex interpolative t-simulation contractions in rectangular b-metric spaces with applications

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Abstract: This work develops the concept of convex interpolative t-simulation contractions within the framework of rectangular b-metric spaces. The proposed condition employs an $\mathcal{L}$-simulation function and a convex parameter to combine the ordinary distance between two points with an interpolative expression determined by their displacements under the mapping. Suitable hypotheses are formulated under which the Picard sequence converges to the sole fixed point of the underlying operator. A carefully constructed example illustrates the effectiveness of the new condition and shows that it applies even when certain familiar contractive requirements are not satisfied. Its relationships with several existing contraction principles are also examined through appropriate parameter choices. As an application, the obtained theorem is used to guarantee a uniquely determined continuous solution of a nonlinear Fredholm integral equation.

Keywords: convex interpolative contraction; t-simulation contraction; L-simulation function; rectangular b-metric space
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1. Introduction

Fixed point methods constitute an essential component of nonlinear analysis and provide effective tools for investigating numerous equations appearing throughout mathematics and allied disciplines. Among the earliest and most influential results in this area is the Banach contraction principle [5]. Besides guaranteeing that a contractive self-map admits exactly one fixed point, this principle yields a successive-approximation process for locating that point. Its broad applicability has encouraged

researchers to replace the classical contraction requirement with less restrictive conditions and to formulate analogous results in spaces equipped with nonstandard notions of distance.

Such investigations have produced several important extensions of ordinary metric spaces. Czerwik [8] developed the concept of a b-metric by permitting a constant factor in the triangular relation. In a different direction, Branciari [6] proposed a rectangular distance, where the standard three-point triangle condition is replaced by a four-point inequality containing two auxiliary elements. These two generalizations were subsequently brought together by George et al. [12], who introduced rectangular b-metric spaces and obtained basic fixed point results for contractive mappings in this environment. The resulting structure is broad enough to accommodate fixed point problems that may fall outside the scope of the conventional metric framework.

Alongside the development of generalized spaces, considerable effort has been devoted to new forms of contractive mappings. Classical alternatives to Banach's condition include the contractions of Kannan [16], Reich [23], and Chatterjea [7]. More recently, Khojasteh et al. [15] introduced simulation functions, providing an auxiliary-function approach to contractive conditions. Cho [9] developed fixed point results for L-contractions in generalized metric spaces. Aydi et al. [4] subsequently investigated L-simulation mappings in partial metric spaces. These developments motivate the use of suitable auxiliary functions to formulate contractive conditions in generalized metric settings.

Following Karapınar's work on interpolative Kannan-type contractions [17], Karapınar et al. [18] studied interpolative Reich–Rus–Ćirić-type contractions in partial metric spaces. Subsequent investigations extended the interpolative approach to other generalized metric frameworks. Aydi et al. [2] studied interpolative Ćirić–Reich–Rus-type contractions using the Branciari distance, whereas Debnath and de la Sen [10] investigated modified and interpolative Ćirić–Reich–Rus-type contractions in b-metric spaces. Mishra et al. [22] established fixed-point results for interpolative Reich–Rus–Ćirić and Hardy–Rogers contractions in quasi-partial b-metric spaces. Subsequently, Fulga [11] examined interpolative contractions involving rational expressions in b-metric spaces. Konwar and Debnath [20] introduced several families of interpolative F-contractions in b-metric spaces, including interpolative Kannan-type and extended interpolative Ćirić–Reich–Rus-type F-contractions. More recently, Wangwe and Kumar [24] developed interpolative ψ-Hardy–Rogers contractions in quasi-partial b-metric spaces. These contributions demonstrate that interpolative contractive conditions have been successfully studied in a variety of generalized metric frameworks. Nevertheless, their systematic treatment in extended b-metric spaces, where the coefficient in the generalized triangle inequality depends on the pair of points, remains comparatively limited and motivates the present investigation. Another related contribution is the work of Abodayeh et al. [1], who investigated hybrid contractions combining linear and nonlinear contractive expressions in Branciari and Branciari b-distance spaces. Their study provides a relevant context for combining different distance quantities within a single contractive condition. In the present work, we consider a convex combination of the direct distance and an interpolative displacement expression, governed by an L-simulation function, in the setting of rectangular b-metric spaces.

Motivated by these two approaches, we introduce in this paper a convex interpolative t-simulation contraction in rectangular b-metric spaces. The proposed condition is governed by the quantity

$$M(\varkappa, \varrho) = \lambda \mho(\varkappa, \varrho) + (1 - \lambda)[\mho(\varkappa, \mathcal{T}\varkappa)]^{\alpha}[\mho(\varrho, \mathcal{T}\varrho)]^{\beta}\mho(\varkappa, \varrho)^{1-\alpha-\beta},$$

where

$$0 \leq \lambda \leq 1, \qquad \alpha, \beta > 0, \qquad \alpha + \beta < 1.$$

The parameter λ produces a convex combination of the direct distance $\mathfrak{V}(\varkappa, \varrho)$ and an interpolative term involving the displacement of the points under the mapping. Thus, rather than selecting one distance quantity through a maximum-type expression, the proposed formulation allows both components to contribute simultaneously to the contractive mechanism.

The main objective of the paper is to establish an existence and uniqueness theorem for this new class of mappings in complete rectangular b-metric spaces by means of L-simulation functions. The convergence of the associated Picard sequence is also obtained. A nontrivial example is included to illustrate the introduced condition and to show that it may hold for a mapping which does not satisfy the classical Banach contraction condition. We further discuss concise connections of the proposed framework with Banach-type, interpolative, Kannan-type, and Reich-type contractive structures under appropriate parameter choices or estimates. Finally, the main theorem is applied to a nonlinear Fredholm integral equation to establish the existence and uniqueness of its continuous solution.

The paper is organized as follows. Section 2 collects the concepts required in the sequel. In Section 3, the convex interpolative t-simulation contraction is introduced, and the principal fixed point result is established together with illustrative examples. Some consequences and connections with existing contractive conditions are discussed thereafter. The final section is devoted to an application to a nonlinear Fredholm integral equation.

2. Materials and methods

This section assembles the principal definitions and supporting notions required in the subsequent analysis.

We first present the concept of a rectangular b-metric space, originally formulated by George et al. [12].

Definition 1. *Let $\mathfrak{X} \neq \emptyset$, and fix a real number $s \geq 1$. A function*

$$\mathfrak{V} : \mathfrak{X} \times \mathfrak{X} \longrightarrow [0, \infty)$$

is said to define a rectangular b-metric *with coefficient s whenever the following properties are satisfied for arbitrary $\varkappa, \varrho \in \mathfrak{X}$:*

(1) $\mathfrak{V}(\varkappa, \varrho) = 0 \iff \varkappa = \varrho$;
(2) $\mathfrak{V}(\varkappa, \varrho) = \mathfrak{V}(\varrho, \varkappa)$;
(3) *for every pair of distinct elements $u, v \in \mathfrak{X} \setminus \{\varkappa, \varrho\}$,*

$$\mathfrak{V}(\varkappa, \varrho) \leq s \left\{ \mathfrak{V}(\varkappa, u) + \mathfrak{V}(u, v) + \mathfrak{V}(v, \varrho) \right\}.$$

In this case, the ordered pair $(\mathfrak{X}, \mathfrak{V})$ is referred to as a rectangular b-metric space.

Remark 1. *An ordinary metric generates a rectangular b-metric with coefficient $s = 1$. Thus, the class of rectangular b-metric spaces encompasses both rectangular metric spaces and b-metric spaces. Nevertheless, the three-point triangle inequality associated with a usual metric is not, in general, valid for a rectangular b-metric.*

The following example shows that a rectangular b-metric need not be a b-metric. More precisely, we construct a rectangular metric, which is a rectangular b-metric with coefficient $s = 1$, but does not satisfy the b-metric inequality for any finite coefficient.

Example 1. *Let*

$$\mathfrak{X} = \{a, b\} \cup \{c_n : n \in \mathbb{N}\},$$

where all these elements are distinct. Define a symmetric function $\mathfrak{V} : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ *by*

$$\mathfrak{V}(x, x) = 0 \quad \textit{for every } x \in \mathfrak{X},$$

$$\mathfrak{V}(a, b) = 1, \qquad \mathfrak{V}(c_n, c_m) = 1 \quad \textit{whenever } n \neq m$$

and

$$\mathfrak{V}(a, c_n) = \mathfrak{V}(b, c_n) = \frac{1}{n} \quad \textit{for every } n \in \mathbb{N}.$$

We show that $\mathfrak{V}$ *is a rectangular metric, and hence a rectangular b-metric with coefficient* $s = 1$*, but is not a b-metric for any finite coefficient.*

Positivity and symmetry follow directly from the definition. To verify the rectangular inequality, put

$$A = \{a, b\}, \qquad B = \{c_n : n \in \mathbb{N}\}.$$

Let $x, y \in \mathfrak{X}$ *be distinct, and let* $u, v \in \mathfrak{X} \setminus \{x, y\}$ *be distinct.*

If x *and* y *belong to the same part, then the three-edge path* x, u, v, y *must contain an edge whose endpoints belong to the same part. Indeed, three successive changes of the part would place* x *and* y *in different parts. Since the four points are distinct, such an edge has distance* 1*. Therefore,*

$$\mathfrak{V}(x, y) = 1 \leq \mathfrak{V}(x, u) + \mathfrak{V}(u, v) + \mathfrak{V}(v, y).$$

If x *and* y *belong to different parts, symmetry allows us to assume that* $x \in A$ *and* $y = c_n \in B$*. If the path* x, u, v, y *contains an edge within one part, its total length is at least* $1 \geq 1/n$*. Otherwise, the path alternates between the two parts, so* $v \in A$ *and* $\mathfrak{V}(v, c_n) = 1/n$*. In either case,*

$$\mathfrak{V}(x, y) = \frac{1}{n} \leq \mathfrak{V}(x, u) + \mathfrak{V}(u, v) + \mathfrak{V}(v, y).$$

Thus, $(\mathfrak{X}, \mathfrak{V})$ *is a rectangular metric space.*

Suppose, however, that $\mathfrak{V}$ *were a b-metric with some finite coefficient* $K \geq 1$*. Then, for every* $n \in \mathbb{N}$*,*

$$1 = \mathfrak{V}(a, b) \leq K[\mathfrak{V}(a, c_n) + \mathfrak{V}(c_n, b)] = \frac{2K}{n}.$$

Choosing $n > 2K$ *gives a contradiction. Consequently,* $\mathfrak{V}$ *is not a b-metric for any finite coefficient.*

Definition 2. *Consider a rectangular b-metric space* $(\mathfrak{X}, \mathfrak{V})$ *and a sequence* $(\varkappa_n)_{n\in\mathbb{N}} \subseteq \mathfrak{X}$*.*

(i) *A point* $\varkappa \in \mathfrak{X}$ *is called a* limit *of* $(\varkappa_n)$ *whenever*

$$\mathfrak{V}(\varkappa_n, \varkappa) \longrightarrow 0 \qquad \textit{as } n \longrightarrow \infty.$$

In this situation, we write $\varkappa_n \to \varkappa$*.*

(ii) *The sequence* $(\varkappa_n)$ *is called* Cauchy *provided that*

$$\mho(\varkappa_n, \varkappa_m) \longrightarrow 0 \qquad \text{as } n, m \longrightarrow \infty.$$

(iii) *The space* $(\mathfrak{X}, \mho)$ *is termed* complete *if each Cauchy sequence of its elements possesses a limit belonging to* $\mathfrak{X}$.

We next recall the notion of orbital continuity.

Definition 3. *Let* $(\mathfrak{X}, \mho)$ *be a rectangular b-metric space, and consider a self-map* $\mathcal{T}$ *on* $\mathfrak{X}$. *The map* $\mathcal{T}$ *is called* orbitally continuous *if, for every* $\varkappa \in \mathfrak{X}$ *and every convergent subsequence* $(\mathcal{T}^{n_i}\varkappa)$ *of its orbit, one has*

$$\mathcal{T}^{n_i}\varkappa \longrightarrow z \quad \Longrightarrow \quad \mathcal{T}^{n_i+1}\varkappa \longrightarrow \mathcal{T}z.$$

Equivalently, applying $\mathcal{T}$ *preserves convergence along each convergent subsequence of an orbit.*

Motivated by the class of auxiliary functions introduced by Jleli and Samet [14] and its subsequent variants, we consider the following class Θ suitable for the present framework.

Definition 4. *Let* Θ *denote the family of functions*

$$\vartheta : [0, \infty) \longrightarrow [1, \infty)$$

satisfying the following conditions:

(Θ_1) ϑ *is continuous and strictly increasing;*
(Θ_2) $\vartheta(r) = 1 \quad \Longleftrightarrow \quad r = 0$;
(Θ_3) *for every sequence* $\{r_n\} \subset [0, \infty)$,

$$\vartheta(r_n) \longrightarrow 1 \quad \Longleftrightarrow \quad r_n \longrightarrow 0.$$

For instance,

$$\vartheta(r) = e^r, \qquad r \geq 0,$$

belongs to Θ.

We now recall the form of an L-simulation function used in the present work.

Definition 5. *[9] A mapping*

$$\zeta : [1, \infty) \times [1, \infty) \longrightarrow \mathbb{R}$$

is said to be an L-simulation function *if the following conditions are satisfied:*

(L_1)

$$\zeta(1, 1) = 1;$$

(L_2) *for all* $u, v > 1$,

$$\zeta(u, v) < \frac{v}{u};$$

(L_3) *if* $\{u_n\}$ *and* $\{v_n\}$ *are sequences in* $(1, \infty)$ *satisfying*

$$u_n \longrightarrow \ell, \qquad v_n \longrightarrow \ell, \qquad \ell > 1,$$

then

$$\limsup_{n\to\infty} \zeta(u_n, v_n) < 1.$$

The class of all L-simulation functions is denoted by $\mathcal{Z}_L$.

The following function provides a useful example.

Example 2. *Let $0 < \gamma < 1$ and define*

$$\zeta(u, v) = \frac{v^\gamma}{u}, \qquad u, v \geq 1.$$

Then

$$\zeta(1, 1) = 1.$$

Moreover, for $u, v > 1$,

$$\zeta(u, v) = \frac{v^\gamma}{u} < \frac{v}{u}.$$

If

$$u_n \to \ell, \qquad v_n \to \ell, \qquad \ell > 1,$$

then

$$\limsup_{n\to\infty} \zeta(u_n, v_n) = \ell^{\gamma-1} < 1.$$

Hence

$$\zeta \in \mathcal{Z}_L.$$

The following elementary inequality will be useful in the proof of the main result.

Lemma 1. *Let $a, b, c \geq 0$, and let $\alpha, \beta > 0$ satisfy*

$$\alpha + \beta < 1.$$

Then

$$a^\alpha b^\beta c^{1-\alpha-\beta} \leq \alpha a + \beta b + (1 - \alpha - \beta)c.$$

Proof. The assertion follows directly from the weighted arithmetic–geometric mean inequality. □

In particular, for $0 < \beta < 1$,

$$a^{1-\beta} b^\beta \leq (1 - \beta)a + \beta b.$$

We shall repeatedly employ these inequalities in estimating the successive distances generated by the Picard iteration

$$\varkappa_{n+1} = \mathcal{T}\varkappa_n.$$

3. Results

Let $\mathcal{Z}_L^*$ denote the subclass of all L-simulation functions $\zeta \in \mathcal{Z}_L$ satisfying

$$\zeta(u, 1) < 1 \qquad \text{for every } u > 1.$$

Definition 6. *Let $(\mathfrak{X}, \mathfrak{U})$ be a rectangular b-metric space with coefficient $s \geq 1$, and let $\mathcal{T} : \mathfrak{X} \to \mathfrak{X}$ be a self-mapping. Let $\vartheta \in \Theta$, ζ be an L-simulation function, and let*

$$0 < t < 1, \qquad 0 \leq \lambda \leq 1, \qquad \alpha, \beta > 0, \qquad \alpha + \beta < 1.$$

For $\varkappa, \varrho \in \mathfrak{X}$, define

$$M(\varkappa, \varrho) = \lambda \mathfrak{U}(\varkappa, \varrho) + (1 - \lambda)[\mathfrak{U}(\varkappa, \mathcal{T}\varkappa)]^{\alpha}[\mathfrak{U}(\varrho, \mathcal{T}\varrho)]^{\beta}\mathfrak{U}(\varkappa, \varrho)^{1-\alpha-\beta}.$$

Then $\mathcal{T}$ is said to be a convex interpolative *t*-simulation contraction *if*

$$\zeta(\vartheta(s\,\mathfrak{U}(\mathcal{T}\varkappa, \mathcal{T}\varrho)), \vartheta(t\,M(\varkappa, \varrho))) \geq 1$$

for all $\varkappa, \varrho \in \mathfrak{X}$ such that $\mathcal{T}\varkappa \neq \mathcal{T}\varrho$.

To demonstrate that the proposed contractive condition is nontrivial and genuinely extends the classical contractive framework, we present an example of a convex interpolative *t*-simulation contraction in a rectangular *b*-metric space for which the corresponding mapping is not a Banach contraction.

Example 3. *Let*

$$\mathfrak{X} = \{0, 1, 2, 3\},$$

and define $\mathfrak{U} : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ by

$$\mathfrak{U}(\varkappa, \varkappa) = 0$$

for every $\varkappa \in \mathfrak{X}$, and

$$\mathfrak{U}(0,1) = 22, \qquad \mathfrak{U}(0,2) = 4, \qquad \mathfrak{U}(0,3) = 20,$$
$$\mathfrak{U}(1,2) = 17, \qquad \mathfrak{U}(1,3) = 4, \qquad \mathfrak{U}(2,3) = 26,$$

together with symmetry

$$\mathfrak{U}(\varkappa, \varrho) = \mathfrak{U}(\varrho, \varkappa).$$

One can verify that $(\mathfrak{X}, \mathfrak{U})$ is a rectangular metric space, and hence a rectangular b-metric space with coefficient $s = 1$. Notice that it is not an ordinary metric space. Indeed,

$$\mathfrak{U}(0,1) = 22 > \mathfrak{U}(0,2) + \mathfrak{U}(2,1) = 4 + 17 = 21,$$

so the usual triangle inequality fails.

Define $\mathcal{T} : \mathfrak{X} \to \mathfrak{X}$ by

$$\mathcal{T}0 = 1, \qquad \mathcal{T}1 = 1, \qquad \mathcal{T}2 = 3, \qquad \mathcal{T}3 = 1.$$

Choose

$$\lambda = 0.3, \qquad \alpha = \beta = 0.2, \qquad t = 0.98.$$

Then

$$\alpha + \beta = 0.4 < 1.$$

Let

$$\vartheta(r) = e^{r},$$

and define

$$\zeta(u, v) = \frac{v^{0.98}}{u}, \qquad u, v \geq 1.$$

We first verify that ζ is an L-simulation function. Clearly,

$$\zeta(1, 1) = 1.$$

Also, for $u, v > 1$,

$$\zeta(u, v) = \frac{v^{0.98}}{u} < \frac{v}{u}.$$

Furthermore, if

$$u_n \to \ell, \qquad v_n \to \ell, \qquad \ell > 1,$$

then

$$\limsup_{n\to\infty} \zeta(u_n, v_n) = \ell^{-0.02} < 1.$$

Hence ζ is an L-simulation function.

For

$$M(\varkappa, \varrho) = \lambda \mho(\varkappa, \varrho) + (1 - \lambda)[\mho(\varkappa, \mathcal{T}\varkappa)]^{\alpha}[\mho(\varrho, \mathcal{T}\varrho)]^{\beta}\mho(\varkappa, \varrho)^{1-\alpha-\beta},$$

the only pairs for which $\mathcal{T}\varkappa \neq \mathcal{T}\varrho$ are

$$(0, 2),\ (2, 0),\ (1, 2),\ (2, 1),\ (2, 3),\ (3, 2).$$

For $(\varkappa, \varrho) = (0, 2)$, we have

$$\mho(\mathcal{T}0, \mathcal{T}2) = \mho(1, 3) = 4,$$

while

$$\mho(0, \mathcal{T}0) = 22, \qquad \mho(2, \mathcal{T}2) = 26, \qquad \mho(0, 2) = 4.$$

Therefore

$$\begin{aligned} M(0, 2) &= 0.3(4) + 0.7(22)^{0.2}(26)^{0.2}(4)^{0.6} \\ &\approx 6.9255. \end{aligned}$$

Hence

$$0.98 \times 0.98 \times M(0, 2) \approx 6.6512 > 4.$$

Thus

$$\begin{aligned} &\zeta(\vartheta(\mho(\mathcal{T}0, \mathcal{T}2)), \vartheta(tM(0, 2))) \\ &= \frac{e^{0.98\, tM(0,2)}}{e^{\mho(\mathcal{T}0, \mathcal{T}2)}} > 1. \end{aligned}$$

For $(\varkappa, \varrho) = (1, 2)$, since

$$\mho(1, \mathcal{T}1) = 0,$$

the interpolative term vanishes and

$$M(1, 2) = 0.3\, \mho(1, 2) = 0.3(17) = 5.1.$$

Hence

$$0.98 \times 0.98 \times 5.1 \approx 4.898 > 4 = \mho(\mathcal{T}1, \mathcal{T}2).$$

Therefore

$$\zeta(\vartheta(\mathfrak{U}(\mathcal{T}1,\mathcal{T}2)),\vartheta(tM(1,2)))>1.$$

For $(\varkappa,\varrho)=(2,3)$*, we have*

$$\mathfrak{U}(\mathcal{T}2,\mathcal{T}3)=\mathfrak{U}(3,1)=4$$

and

$$\mathfrak{U}(2,\mathcal{T}2)=26,\qquad \mathfrak{U}(3,\mathcal{T}3)=4,\qquad \mathfrak{U}(2,3)=26.$$

Thus

$$\begin{aligned}M(2,3)&=0.3(26)+0.7(26)^{0.2}(4)^{0.2}(26)^{0.6}\\&\approx 20.3167.\end{aligned}$$

Consequently,

$$0.98\times 0.98\times M(2,3)>4,$$

and hence

$$\zeta(\vartheta(\mathfrak{U}(\mathcal{T}2,\mathcal{T}3)),\vartheta(tM(2,3)))>1.$$

By symmetry, the same calculations hold for the reversed pairs $(2,0)$*,* $(2,1)$*, and* $(3,2)$*.*

Therefore,

$$\zeta(\vartheta(s\mathfrak{U}(\mathcal{T}\varkappa,\mathcal{T}\varrho)),\vartheta(tM(\varkappa,\varrho)))\geq 1$$

for every $\varkappa,\varrho\in\mathfrak{X}$ *with* $\mathcal{T}\varkappa\neq\mathcal{T}\varrho$*. Hence* $\mathcal{T}$ *is a convex interpolative t-simulation contraction.*

Moreover, $\mathcal{T}$ *is not a Banach contraction. Indeed, for the pair* $(0,2)$*,*

$$\mathfrak{U}(\mathcal{T}0,\mathcal{T}2)=\mathfrak{U}(1,3)=4=\mathfrak{U}(0,2).$$

Thus there cannot exist any $k\in[0,1)$ *satisfying*

$$\mathfrak{U}(\mathcal{T}\varkappa,\mathcal{T}\varrho)\leq k\,\mathfrak{U}(\varkappa,\varrho)$$

for all $\varkappa,\varrho\in\mathfrak{X}$*.*

Hence this example shows that the proposed convex interpolative t-simulation contraction applies to a mapping that is not a Banach contraction.

Lemma 2. *Let* $(\mathfrak{X},\mathfrak{U})$ *be a rectangular b-metric space with coefficient* $s\geq 1$*, and let* $\{\varkappa_n\}_{n\geq 0}$ *be a sequence of pairwise distinct points. Suppose that there exist constants* $H>0$ *and* $r\in(0,1)$ *with* $sr^2<1$ *such that*

$$\mathfrak{U}(\varkappa_n,\varkappa_{n+1})\leq Hr^n,\qquad \mathfrak{U}(\varkappa_n,\varkappa_{n+2})\leq Hr^n$$

for every $n\geq 0$*. Then* $\{\varkappa_n\}$ *is Cauchy. More precisely, for every* $m>n$*,*

$$\mathfrak{U}(\varkappa_n,\varkappa_m)\leq Hr^n\left(1+\frac{s(1+r)}{1-sr^2}\right).$$

Proof. Write

$$\delta_j=\mathfrak{U}(\varkappa_j,\varkappa_{j+1}),\qquad \rho_j=\mathfrak{U}(\varkappa_j,\varkappa_{j+2}).$$

The cases $m-n=1$ and $m-n=2$ follow directly from the hypotheses.

Suppose first that $m - n = 2\ell + 1$, where $\ell \geq 1$. Repeated application of the rectangular inequality, using $\varkappa_{n+2j+1}$ and $\varkappa_{n+2j+2}$ as intermediate points, gives

$$\mho(\varkappa_n, \varkappa_m) \leq \sum_{j=0}^{\ell-1} s^{j+1}(\delta_{n+2j} + \delta_{n+2j+1}) + s^{\ell}\delta_{n+2\ell}.$$

All these applications are admissible because the sequence consists of pairwise distinct points. Consequently,

$$\begin{aligned}\mho(\varkappa_n, \varkappa_m) &\leq sH(1+r)r^n \sum_{j=0}^{\ell-1}(sr^2)^j + Hr^n(sr^2)^{\ell} \\ &\leq Hr^n\left(\frac{s(1+r)}{1-sr^2} + 1\right).\end{aligned}$$

Suppose next that $m - n = 2\ell$, where $\ell \geq 2$. In this case, repeated application of the rectangular inequality terminates with the two-step distance ρ_{m-2}, yielding

$$\mho(\varkappa_n, \varkappa_m) \leq \sum_{j=0}^{\ell-2} s^{j+1}(\delta_{n+2j} + \delta_{n+2j+1}) + s^{\ell-1}\rho_{n+2\ell-2}.$$

Therefore,

$$\begin{aligned}\mho(\varkappa_n, \varkappa_m) &\leq sH(1+r)r^n \sum_{j=0}^{\ell-2}(sr^2)^j + Hr^n(sr^2)^{\ell-1} \\ &\leq Hr^n\left(\frac{s(1+r)}{1-sr^2} + 1\right).\end{aligned}$$

Thus, the stated bound holds for every $m > n$. Its right-hand side tends to zero as $n \to \infty$, independently of m. Hence the sequence is Cauchy. □

Theorem 1. *Let $(\mathfrak{X}, \mho)$ be a complete rectangular b-metric space whose coefficient is $s \geq 1$, and let $\mathcal{T} : \mathfrak{X} \to \mathfrak{X}$ be orbitally continuous. Assume that $\mathcal{T}$ satisfies the convex interpolative t-simulation contractive condition; that is,*

$$\zeta(\vartheta(s\,\mho(\mathcal{T}\varkappa, \mathcal{T}\varrho)), \vartheta(t\,M(\varkappa, \varrho))) \geq 1, \tag{3.1}$$

for all $\varkappa, \varrho \in \mathfrak{X}$ with $\mathcal{T}\varkappa \neq \mathcal{T}\varrho$, where

$$M(\varkappa, \varrho) = \lambda\mho(\varkappa, \varrho) + (1-\lambda)[\mho(\varkappa, \mathcal{T}\varkappa)]^{\alpha}[\mho(\varrho, \mathcal{T}\varrho)]^{\beta}\mho(\varkappa, \varrho)^{\,1-\alpha-\beta} \tag{3.2}$$

and

$$0 < t < 1, \qquad 0 \leq \lambda \leq 1, \qquad \alpha, \beta > 0, \qquad \alpha + \beta < 1,$$

where $\vartheta \in \Theta$ and ζ is an L-simulation function,

Then $\mathcal{T}$ has a unique fixed point $\varkappa^* \in \mathfrak{X}$. Moreover, for every $\varkappa_0 \in \mathfrak{X}$, the Picard sequence $\{\varkappa_n\}$ defined by

$$\varkappa_{n+1} = \mathcal{T}\varkappa_n, \qquad n \geq 0$$

converges to $\varkappa^*$.

Proof. Let $\varkappa_0 \in \mathfrak{X}$ be arbitrary, and define the Picard sequence $\{\varkappa_n\}$ by

$$\varkappa_{n+1} = \mathcal{T}\varkappa_n, \qquad n \geq 0.$$

If $\varkappa_n = \varkappa_{n+1}$ for some $n \geq 0$, then $\varkappa_n = \mathcal{T}\varkappa_n$, and hence $\varkappa_n$ is a fixed point of $\mathcal{T}$. Therefore, throughout the remaining part of the proof, we assume that

$$\varkappa_n \neq \varkappa_{n+1} \qquad \text{for every } n \geq 0.$$

Put

$$\delta_n = \mho(\varkappa_n, \varkappa_{n+1}), \qquad n \geq 0.$$

Since

$$\mathcal{T}\varkappa_n = \varkappa_{n+1} \quad \text{and} \quad \mathcal{T}\varkappa_{n+1} = \varkappa_{n+2},$$

the contractive condition gives

$$\zeta(\vartheta(s\delta_{n+1}), \vartheta(tM(\varkappa_n, \varkappa_{n+1}))) \geq 1.$$

By the defining property of an L-simulation function,

$$\zeta(u, v) < \frac{v}{u}, \qquad u, v > 1.$$

Consequently,

$$1 \leq \zeta(\vartheta(s\delta_{n+1}), \vartheta(tM(\varkappa_n, \varkappa_{n+1}))) < \frac{\vartheta(tM(\varkappa_n, \varkappa_{n+1}))}{\vartheta(s\delta_{n+1})}.$$

Hence

$$\vartheta(s\delta_{n+1}) < \vartheta(tM(\varkappa_n, \varkappa_{n+1})).$$

Since ϑ is strictly increasing,

$$s\delta_{n+1} < tM(\varkappa_n, \varkappa_{n+1}). \tag{3.3}$$

Now,

$$M(\varkappa_n, \varkappa_{n+1}) = \lambda\mho(\varkappa_n, \varkappa_{n+1}) + (1-\lambda)[\mho(\varkappa_n, \mathcal{T}\varkappa_n)]^{\alpha}[\mho(\varkappa_{n+1}, \mathcal{T}\varkappa_{n+1})]^{\beta}\mho(\varkappa_n, \varkappa_{n+1})^{1-\alpha-\beta}.$$

Using

$$\mho(\varkappa_n, \mathcal{T}\varkappa_n) = \delta_n, \qquad \mho(\varkappa_{n+1}, \mathcal{T}\varkappa_{n+1}) = \delta_{n+1},$$

we obtain

$$M(\varkappa_n, \varkappa_{n+1}) = \lambda\delta_n + (1-\lambda)\delta_n^{1-\beta}\delta_{n+1}^{\beta}.$$

Thus, from (3.3),

$$s\delta_{n+1} < t\left[\lambda\delta_n + (1-\lambda)\delta_n^{1-\beta}\delta_{n+1}^{\beta}\right]. \tag{3.4}$$

By the weighted Young inequality,

$$a^{1-\beta}b^{\beta} \leq (1-\beta)a + \beta b, \qquad a, b \geq 0,$$

we have

$$\delta_n^{1-\beta}\delta_{n+1}^{\beta} \leq (1-\beta)\delta_n + \beta\delta_{n+1}.$$

Therefore

$$s\delta_{n+1} < t\left[\lambda\delta_n + (1-\lambda)((1-\beta)\delta_n + \beta\delta_{n+1})\right].$$

Consequently,

$$[s - t\beta(1-\lambda)]\delta_{n+1} < t[1-\beta(1-\lambda)]\delta_n.$$

Hence

$$\delta_{n+1} < q\delta_n, \tag{3.5}$$

where

$$q = \frac{t[1-\beta(1-\lambda)]}{s - t\beta(1-\lambda)}.$$

We claim that

$$sq < 1.$$

Indeed,

$$sq < 1$$

is equivalent to

$$st[1-\beta(1-\lambda)] < s - t\beta(1-\lambda).$$

The difference between the right- and left-hand sides is

$$s(1-t) + t\beta(1-\lambda)(s-1) > 0,$$

because $0 < t < 1$ and $s \geq 1$. Thus

$$0 < q < \frac{1}{s} \leq 1.$$

Iterating (3.5), we obtain

$$\delta_n \leq q^n\delta_0, \qquad n \geq 0. \tag{3.6}$$

Consequently,

$$\lim_{n\to\infty} \delta_n = 0.$$

We next show that

$$\mho(\varkappa_n, \varkappa_{n+2}) \longrightarrow 0.$$

Set

$$\rho_n = \mho(\varkappa_n, \varkappa_{n+2}).$$

If $\rho_n = 0$ for some n, there is nothing to prove for that index. Otherwise, applying the contractive condition to $\varkappa_{n-1}$ and $\varkappa_{n+1}$ gives

$$s\rho_n < tM(\varkappa_{n-1}, \varkappa_{n+1}).$$

Now

$$M(\varkappa_{n-1}, \varkappa_{n+1}) = \lambda\rho_{n-1} + (1-\lambda)\delta_{n-1}^{\alpha}\delta_{n+1}^{\beta}\rho_{n-1}^{1-\alpha-\beta}.$$

By the weighted arithmetic–geometric mean inequality,

$$a^{\alpha}b^{\beta}c^{1-\alpha-\beta} \leq \alpha a + \beta b + (1-\alpha-\beta)c,$$

and therefore

$$s\rho_n < t\Big[A\rho_{n-1} + B\delta_{n-1} + C\delta_{n+1}\Big],$$

where

$$A = \lambda + (1-\lambda)(1-\alpha-\beta),$$
$$B = (1-\lambda)\alpha, \qquad C = (1-\lambda)\beta,$$

and

$$A + B + C = 1.$$

Thus

$$\rho_n \leq p\rho_{n-1} + Dq^{n-1}, \tag{3.7}$$

where

$$p = \frac{tA}{s} < \frac{1}{s}$$

and

$$D = \frac{t}{s}(B + Cq^2)\delta_0.$$

By iteration of (3.7),

$$\rho_n \leq p^n\rho_0 + D\sum_{j=1}^{n} p^{n-j}q^{j-1}.$$

To justify the convergence of the sum, put

$$r = \max\{p, q\}.$$

Since $0 < p, q < 1/s \leq 1$, we have $0 < r < 1$. For every $j \in \{1, \ldots, n\}$,

$$p^{n-j}q^{j-1} \leq r^{n-j}r^{j-1} = r^{n-1}.$$

Consequently,

$$0 \leq \sum_{j=1}^{n} p^{n-j}q^{j-1} \leq nr^{n-1} \longrightarrow 0 \qquad \text{as } n \to \infty.$$

Indeed, if $a_n = nr^{n-1}$, then

$$\frac{a_{n+1}}{a_n} = r\left(1 + \frac{1}{n}\right) \longrightarrow r < 1,$$

which implies $a_n \to 0$. Since D is a fixed nonnegative constant and $p^n\rho_0 \to 0$, it follows that

$$0 \leq \rho_n \leq p^n\rho_0 + Dnr^{n-1} \longrightarrow 0.$$

Therefore

$$\lim_{n\to\infty} \mho(\kappa_n, \kappa_{n+2}) = 0. \tag{3.8}$$

We now prove that $\{\varkappa_n\}$ is a Cauchy sequence. First observe that the orbit cannot contain a nontrivial repetition. Indeed, if $\varkappa_m = \varkappa_n$ for some $m > n$, then the corresponding successive distances would be periodic, whereas (3.6) yields

$$\delta_m \leq q^{m-n}\delta_n < \delta_n,$$

a contradiction. Hence, unless a fixed point has already occurred, the points of the Picard orbit are distinct.

Choose r such that

$$\max\{p, q\} < r < \frac{1}{s}.$$

Such a choice is possible because $p, q < 1/s$. From Eq (3.6), we have

$$\delta_n \leq \delta_0 q^n \leq \delta_0 r^n.$$

We next obtain a geometric bound for ρ_n. Set

$$C = \max\left\{\rho_0, \frac{D}{r-p}\right\}.$$

We claim that $\rho_n \leq Cr^n$ for every $n \geq 0$. The assertion holds for $n = 0$. If it holds for $n-1$, then Eq (3.7) gives

$$\begin{aligned}\rho_n &\leq p\rho_{n-1} + Dq^{n-1} \\ &\leq pCr^{n-1} + Dr^{n-1} \\ &\leq Cr^n,\end{aligned}$$

where the last inequality follows from $D \leq C(r-p)$. The claim therefore follows by induction.

Put

$$H = \max\{\delta_0, C\}.$$

Then

$$\delta_n \leq Hr^n, \qquad \rho_n \leq Hr^n.$$

Moreover, $r < 1/s$ implies $r < 1$ and $sr^2 < 1$. Since the Picard orbit consists of pairwise distinct points, Lemma 2 applies and gives

$$\mho(\varkappa_n, \varkappa_m) \leq Hr^n\left(1 + \frac{s(1+r)}{1-sr^2}\right) \qquad (m > n).$$

The bound tends to zero uniformly in $m > n$. Consequently, $\{\varkappa_n\}$ is a Cauchy sequence.

Since $(\mathfrak{X}, \mho)$ is complete, there exists $\varkappa^* \in \mathfrak{X}$ such that

$$\varkappa_n \longrightarrow \varkappa^*.$$

Since $\mathcal{T}$ is orbitally continuous and

$$\varkappa_{n+1} = \mathcal{T}\varkappa_n,$$

we have

$$\mathcal{T}\varkappa_n \longrightarrow \mathcal{T}\varkappa^*.$$

On the other hand,

$$\mathcal{T}\varkappa_n = \varkappa_{n+1} \longrightarrow \varkappa^*.$$

Therefore

$$\mathcal{T}\varkappa^* = \varkappa^*.$$

Hence $\varkappa^*$ is a fixed point of $\mathcal{T}$.

It remains to prove uniqueness. Suppose that $u, v \in \mathfrak{X}$ are two distinct fixed points of $\mathcal{T}$. Then

$$\mathcal{T}u = u, \qquad \mathcal{T}v = v$$

and hence

$$\mathfrak{V}(u, \mathcal{T}u) = \mathfrak{V}(v, \mathcal{T}v) = 0.$$

Since $\alpha, \beta > 0$, the interpolative term vanishes and

$$M(u, v) = \lambda \mathfrak{V}(u, v).$$

Applying the contractive condition, we obtain

$$1 \leq \zeta(\vartheta(s\mathfrak{V}(u, v)), \vartheta(t\lambda\mathfrak{V}(u, v))) < \frac{\vartheta(t\lambda\mathfrak{V}(u, v))}{\vartheta(s\mathfrak{V}(u, v))}.$$

Therefore

$$\vartheta(s\mathfrak{V}(u, v)) < \vartheta(t\lambda\mathfrak{V}(u, v)).$$

Since ϑ is strictly increasing,

$$s\mathfrak{V}(u, v) < t\lambda\mathfrak{V}(u, v).$$

As $\mathfrak{V}(u, v) > 0$, this implies

$$s < t\lambda,$$

which is impossible because

$$s \geq 1, \qquad 0 < t < 1, \qquad 0 \leq \lambda \leq 1.$$

Hence $u = v$.

Therefore, $\mathcal{T}$ has a unique fixed point in $\mathfrak{X}$, and the Picard iteration converges to it. □

The following example illustrates the applicability of Theorem 1 in a complete rectangular b-metric space.

Example 4. *Let* $\mathfrak{X} = \{0, 1, 10, 11\} \subset \mathbb{R}$*, and define* $\mathfrak{V}(\varkappa, \varrho) = |\varkappa - \varrho|^2, \quad \varkappa, \varrho \in \mathfrak{X}$.

For any admissible distinct points $\varkappa, \varrho, u, v \in \mathfrak{X}$*, the usual triangle inequality gives*

$$|\varkappa - \varrho| \leq |\varkappa - u| + |u - v| + |v - \varrho|.$$

Consequently,

$$\begin{aligned}
\mathfrak{V}(\varkappa, \varrho) &= |\varkappa - \varrho|^2 \\
&\leq (|\varkappa - u| + |u - v| + |v - \varrho|)^2 \\
&\leq 3\left(|\varkappa - u|^2 + |u - v|^2 + |v - \varrho|^2\right) \\
&= 3\left[\mathfrak{V}(\varkappa, u) + \mathfrak{V}(u, v) + \mathfrak{V}(v, \varrho)\right].
\end{aligned}$$

Hence $(\mathfrak{X}, \mathfrak{V})$ *is a rectangular b-metric space with an admissible coefficient* $s = 3$*. Since* $\mathfrak{X}$ *is finite,* $(\mathfrak{X}, \mathfrak{V})$ *is complete.*

Define $\mathcal{T} : \mathfrak{X} \to \mathfrak{X}$ *by*

$$\mathcal{T}(0) = 0, \qquad \mathcal{T}(1) = 0, \qquad \mathcal{T}(10) = 1, \qquad \mathcal{T}(11) = 0.$$

Choose

$$t = 0.9, \qquad \lambda = 0.2, \qquad \alpha = \beta = 0.4.$$

Clearly,

$$\alpha + \beta = 0.8 < 1.$$

Let

$$\vartheta(r) = e^r,$$

and define

$$\zeta(a, b) = \frac{b^{0.9}}{a}, \qquad a, b \geq 1.$$

It is immediate that

$$\zeta(1, 1) = 1,$$

and for $a, b > 1$*,*

$$\zeta(a, b) = \frac{b^{0.9}}{a} < \frac{b}{a}.$$

Moreover, if

$$a_n \to \ell, \qquad b_n \to \ell, \qquad \ell > 1,$$

then

$$\limsup_{n\to\infty} \zeta(a_n, b_n) = \ell^{-0.1} < 1.$$

Thus ζ *is an L-simulation function.*

For $\varkappa, \varrho \in \mathfrak{X}$*, put*

$$M(\varkappa, \varrho) = 0.2\,\mho(\varkappa, \varrho) + 0.8\,[\mho(\varkappa, \mathcal{T}\varkappa)]^{0.4}[\mho(\varrho, \mathcal{T}\varrho)]^{0.4}\mho(\varkappa, \varrho)^{0.2}.$$

Observe that $\mathcal{T}\varkappa \neq \mathcal{T}\varrho$ *only when one of the points is* 10 *and the other belongs to* $\{0, 1, 11\}$*. Therefore, it is sufficient to check the following three pairs.*

For $(\varkappa, \varrho) = (0, 10)$*, we have*

$$\mho(\mathcal{T}0, \mathcal{T}10) = \mho(0, 1) = 1$$

and

$$\mho(0, \mathcal{T}0) = 0.$$

Hence

$$M(0, 10) = 0.2\,\mho(0, 10) = 0.2(100) = 20.$$

Therefore

$$\begin{aligned}
&\zeta\left(\vartheta\left(3\mho(\mathcal{T}0, \mathcal{T}10)\right), \vartheta\left(0.9M(0, 10)\right)\right) \\
&= \frac{\left(e^{0.9M(0,10)}\right)^{0.9}}{e^{3\mho(T0,T10)}} \\
&= \exp\left(0.81(20) - 3\right) > 1.
\end{aligned}$$

Next, for $(\varkappa, \varrho) = (1, 10)$,

$$\mho(\mathcal{T}1, \mathcal{T}10) = \mho(0, 1) = 1,$$

while

$$\mho(1, \mathcal{T}1) = 1, \qquad \mho(10, \mathcal{T}10) = 81, \qquad \mho(1, 10) = 81.$$

Thus

$$\begin{aligned} M(1, 10) &= 0.2(81) + 0.8(1)^{0.4}(81)^{0.4}(81)^{0.2} \\ &= 16.2 + 0.8(81)^{0.6} \\ &\approx 27.3733. \end{aligned}$$

Consequently,

$$0.81M(1, 10) > 3 = \mho(\mathcal{T}1, \mathcal{T}10)\, s,$$

and hence

$$\zeta\left(\vartheta\left(3\mho(\mathcal{T}1, \mathcal{T}10)\right), \vartheta\left(0.9M(1, 10)\right)\right) > 1.$$

Finally, consider $(\varkappa, \varrho) = (10, 11)$. *We have*

$$\mho(\mathcal{T}10, \mathcal{T}11) = \mho(1, 0) = 1,$$

whereas

$$\mho(10, \mathcal{T}10) = 81, \qquad \mho(11, \mathcal{T}11) = 121, \qquad \mho(10, 11) = 1.$$

Therefore,

$$\begin{aligned} M(10, 11) &= 0.2 + 0.8(81)^{0.4}(121)^{0.4} \\ &\approx 31.7935. \end{aligned}$$

Thus

$$0.81M(10, 11) > 3,$$

and consequently

$$\zeta\left(\vartheta\left(3\mho(\mathcal{T}10, \mathcal{T}11)\right), \vartheta\left(0.9M(10, 11)\right)\right) > 1.$$

Since $\alpha = \beta$, *the same computations apply to the reversed pairs. Therefore,*

$$\zeta\left(\vartheta\left(s\mho(\mathcal{T}\varkappa, \mathcal{T}\varrho)\right), \vartheta\left(tM(\varkappa, \varrho)\right)\right) \geq 1$$

for all $\varkappa, \varrho \in \mathfrak{X}$ *with* $\mathcal{T}\varkappa \neq \mathcal{T}\varrho$.

Hence T *is a convex interpolative t-simulation contraction.*

Furthermore, the Picard iterations are

$$10 \longmapsto 1 \longmapsto 0 \longmapsto 0,$$

$$11 \longmapsto 0 \longmapsto 0,$$

and

$$1 \longmapsto 0 \longmapsto 0.$$

Hence, $\mathcal{T}$ *is orbitally continuous, and all the assumptions of Theorem 1 are fulfilled. Consequently,* $\mathcal{T}$ *admits a unique fixed point* $\varkappa^* = 0$.

We now show that the above mapping does not satisfy some familiar contractive conditions, thereby demonstrating the usefulness of the proposed convex interpolative t-simulation contraction.

First, $\mathcal{T}$ is not a Banach contraction. Indeed, for $\varkappa = 10$ and $\varrho = 11$, we have

$$\mho(10, 11) = |10 - 11|^2 = 1,$$

whereas

$$\mho(\mathcal{T}10, \mathcal{T}11) = \mho(1, 0) = 1.$$

Hence

$$\frac{\mho(\mathcal{T}10, \mathcal{T}11)}{\mho(10, 11)} = 1.$$

Consequently, no constant $k \in [0, 1)$ exists for which

$$\mho(\mathcal{T}\varkappa, \mathcal{T}\varrho) \leq k\, \mho(\varkappa, \varrho)$$

for all $\varkappa, \varrho \in \mathfrak{X}$.

Next, we show that the ordinary L-simulation contractive condition also fails. Recall that, for the chosen functions

$$\vartheta(r) = e^r, \qquad \zeta(a, b) = \frac{b^{0.9}}{a},$$

the ordinary L-simulation condition would require

$$\zeta\left(\vartheta(s\mho(\mathcal{T}\varkappa, \mathcal{T}\varrho)), \vartheta(\mho(\varkappa, \varrho))\right) \geq 1.$$

Taking again $\varkappa = 10$ and $\varrho = 11$, we obtain

$$s\mho(\mathcal{T}10, \mathcal{T}11) = 3$$

and

$$\mho(10, 11) = 1.$$

Therefore,

$$\begin{aligned}\zeta\left(\vartheta(s\mho(\mathcal{T}10, \mathcal{T}11)), \vartheta(\mho(10, 11))\right) &= \frac{\left(e^1\right)^{0.9}}{e^3} \\ &= e^{-2.1} < 1.\end{aligned}$$

Thus, $\mathcal{T}$ does not satisfy the ordinary L-simulation contractive condition.

We further observe that the pure interpolative t-simulation condition, corresponding to $\lambda = 0$, also fails. In this case, the interpolative quantity becomes

$$M_0(\varkappa, \varrho) = [\mho(\varkappa, \mathcal{T}\varkappa)]^{\alpha}[\mho(\varrho, \mathcal{T}\varrho)]^{\beta}\mho(\varkappa, \varrho)^{1-\alpha-\beta}.$$

Consider the pair $\varkappa = 0$ and $\varrho = 10$. Since

$$\mathcal{T}0 = 0$$

and

$$\mathcal{T}10 = 1,$$

we have

$$\mho(\mathcal{T}0, \mathcal{T}10) = \mho(0, 1) = 1.$$

However,

$$\mho(0, \mathcal{T}0) = \mho(0, 0) = 0.$$

Consequently,

$$M_0(0, 10) = [\mho(0, \mathcal{T}0)]^{0.4}[\mho(10, \mathcal{T}10)]^{0.4}\mho(0, 10)^{0.2} = 0.$$

Hence

$$\zeta\left(\vartheta(s\mho(\mathcal{T}0, \mathcal{T}10)), \vartheta(tM_0(0, 10))\right)$$
$$= \zeta\left(e^3, 1\right) = \frac{1}{e^3} < 1.$$

Therefore, $\mathcal{T}$ does not satisfy the pure interpolative t-simulation contraction either.

On the other hand, as verified above, for

$$\lambda = 0.2, \qquad t = 0.9, \qquad \alpha = \beta = 0.4,$$

the mapping $\mathcal{T}$ satisfies

$$\zeta\left(\vartheta(s\mho(\mathcal{T}\varkappa, \mathcal{T}\varrho)), \vartheta(tM(\varkappa, \varrho))\right) \geq 1$$

for every $\varkappa, \varrho \in \mathfrak{X}$ with $\mathcal{T}\varkappa \neq T\varrho$.

Thus, the mapping $\mathcal{T}$ satisfies the proposed convex interpolative t-simulation contraction although the Banach contraction, the ordinary L-simulation contraction, and the corresponding pure interpolative t-simulation contraction all fail. This demonstrates that the convex combination appearing in $M(\varkappa, \varrho)$ genuinely enlarges the class of mappings covered by the fixed point theorem.

This example demonstrates the genuine scope of the proposed framework: Although the mapping fails to satisfy the Banach, ordinary L-simulation, and pure interpolative t-simulation contractive conditions, it satisfies the convex interpolative t-simulation condition introduced herein.

4. Consequences and related results

We briefly indicate how the proposed convex interpolative *t*-simulation framework is related to some familiar contractive conditions. Recall that

$$M(\varkappa, \varrho) = \lambda\mho(\varkappa, \varrho) + (1 - \lambda)[\mho(\varkappa, \mathcal{T}\varkappa)]^{\alpha}[\mho(\varrho, \mathcal{T}\varrho)]^{\beta}\mho(\varkappa, \varrho)^{1-\alpha-\beta}.$$

Remark 2. *If $\lambda = 1$, then*

$$M(\varkappa, \varrho) = \mho(\varkappa, \varrho),$$

and the contractive condition reduces to

$$\zeta(\vartheta(s\mho(\mathcal{T}\varkappa, \mathcal{T}\varrho)), \vartheta(t\mho(\varkappa, \varrho))) \geq 1.$$

In particular, by taking

$$\vartheta(r) = e^r, \qquad \zeta(u, v) = \frac{v^{\gamma}}{u}, \quad 0 < \gamma < 1,$$

we obtain

$$\mathfrak{U}(\mathcal{T}\varkappa, T\varrho) \leq \frac{\gamma t}{s}\,\mathfrak{U}(\varkappa, \varrho).$$

Thus, the proposed condition contains a Banach-type contraction as a special case. For $s = 1$, this reduces to the corresponding contraction in the usual metric setting.

Remark 3. *If $\lambda = 0$, then*

$$M(\varkappa, \varrho) = [\mathfrak{U}(\varkappa, \mathcal{T}\varkappa)]^{\alpha}[\mathfrak{U}(\varrho, \mathcal{T}\varrho)]^{\beta}\mathfrak{U}(\varkappa, \varrho)^{1-\alpha-\beta},$$

and hence the proposed condition becomes a pure interpolative t-simulation contraction:

$$\zeta\Big(\vartheta(s\mathfrak{U}(\mathcal{T}\varkappa, \mathcal{T}\varrho)), \vartheta\Big(t[\mathfrak{U}(\varkappa, \mathcal{T}\varkappa)]^{\alpha}[\mathfrak{U}(\varrho, \mathcal{T}\varrho)]^{\beta}\mathfrak{U}(\varkappa, \varrho)^{1-\alpha-\beta}\Big)\Big) \geq 1.$$

Therefore, the parameter λ provides a continuous transition between the classical distance component and the purely interpolative component.

Remark 4. *The proposed condition is also naturally related to Kannan- and Reich-type contractions. Indeed, by the weighted arithmetic–geometric mean inequality,*

$$[\mathfrak{U}(\varkappa, \mathcal{T}\varkappa)]^{\alpha}[\mathfrak{U}(\varrho, \mathcal{T}\varrho)]^{\beta}\mathfrak{U}(\varkappa, \varrho)^{1-\alpha-\beta} \leq \alpha\mathfrak{U}(\varkappa, \mathcal{T}\varkappa) + \beta\mathfrak{U}(\varrho, \mathcal{T}\varrho) + (1-\alpha-\beta)\mathfrak{U}(\varkappa, \varrho).$$

Consequently,

$$M(\varkappa, \varrho) \leq A\,\mathfrak{U}(\varkappa, \varrho) + B\,\mathfrak{U}(\varkappa, \mathcal{T}\varkappa) + C\,\mathfrak{U}(\varrho, \mathcal{T}\varrho),$$

where

$$A = \lambda + (1-\lambda)(1-\alpha-\beta), \qquad B = (1-\lambda)\alpha, \qquad C = (1-\lambda)\beta,$$

and

$$A + B + C = 1.$$

Thus, the interpolative expression is controlled by an additive Reich-type quantity. The relationship with interpolative Kannan-type expressions should be understood at the level of the contractive form. Formally setting $\lambda = 0$ and $\alpha + \beta = 1$ gives

$$M(\varkappa, \varrho) = [\mathfrak{U}(\varkappa, \mathcal{T}\varkappa)]^{\alpha}[\mathfrak{U}(\varrho, \mathcal{T}\varrho)]^{1-\alpha}, \qquad 0 < \alpha < 1.$$

Here, the direct-distance factor is omitted in the boundary expression. In particular, for $\alpha = 1/2$,

$$M(\varkappa, \varrho) = \sqrt{\mathfrak{U}(\varkappa, \mathcal{T}\varkappa)\mathfrak{U}(\varrho, \mathcal{T}\varrho)} \leq \frac{1}{2}\left[\mathfrak{U}(\varkappa, \mathcal{T}\varkappa) + \mathfrak{U}(\varrho, \mathcal{T}\varrho)\right].$$

However, the boundary condition $\alpha+\beta = 1$ lies outside the hypotheses of Theorem 1, which requires $\alpha + \beta < 1$. Accordingly, this observation is not a corollary of that theorem and does not establish a fixed point result at the boundary. Such a result would require a separate justification.

Likewise, the weighted arithmetic–geometric mean estimate above shows that M is bounded by an additive Reich-type expression. This is an implication between contractive bounds, rather than an equivalence of the corresponding conditions.

Thus, the preceding observations describe connections with familiar contractive expressions. They do not assert that all classical Banach, Kannan, or Reich fixed point theorems are recovered as special cases of Theorem 1.

The above observations show that the proposed convex interpolative t-simulation condition provides a unified framework connecting Banach-type, interpolative, and certain Kannan–Reich-type contractive structures while retaining the additional flexibility provided by the convex parameter λ and the L-simulation function.

As an application to a nonlinear problem, Theorem 1 is now utilized to derive conditions guaranteeing that a nonlinear Fredholm integral equation admits precisely one solution.

5. Application to a nonlinear Fredholm integral equation

We now illustrate how Theorem 1 can be employed to analyze a nonlinear Fredholm integral equation. More precisely, consider

$$u(\varkappa) = g(\varkappa) + \int_0^1 K(\varkappa, r, u(r))\, dr, \qquad \varkappa \in [0,1], \tag{5.1}$$

where $g \in C([0,1], \mathbb{R})$ and

$$K : [0,1] \times [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$$

is continuous. Let

$$\mathfrak{X} = C([0,1], \mathbb{R}),$$

and equip $\mathfrak{X}$ with the distance

$$\mho(u,v) = \|u - v\|_\infty^2 = \left(\sup_{\kappa \in [0,1]} |u(\kappa) - v(\kappa)| \right)^2, \qquad u, v \in \mathfrak{X}.$$

We verify that $(\mathfrak{X}, \mho)$ is a complete rectangular b-metric space with an admissible coefficient $s = 3$.

Positivity and symmetry follow immediately from the properties of the supremum norm. For $u, v \in \mathfrak{X}$ and distinct $w, z \in \mathfrak{X} \setminus \{u, v\}$, the triangle inequality for the supremum norm gives

$$\|u - v\|_\infty \leq \|u - w\|_\infty + \|w - z\|_\infty + \|z - v\|_\infty.$$

Using $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$, we obtain

$$\mho(u,v) \leq 3[\mho(u,w) + \mho(w,z) + \mho(z,v)].$$

Thus, $\mho$ is a rectangular b-metric with coefficient $s = 3$.

Moreover, a sequence is Cauchy with respect to $\mho$ if and only if it is Cauchy in the supremum norm. Since $C([0,1], \mathbb{R})$ is complete in that norm, $(\mathfrak{X}, \mho)$ is complete.

Define the operator $\mathcal{T} : \mathfrak{X} \to \mathfrak{X}$ by

$$(\mathcal{T}u)(\varkappa) = g(\varkappa) + \int_0^1 K(\varkappa, r, u(r))\, dr. \tag{5.2}$$

For each $u \in \mathfrak{X}$, the continuity of g and K ensures that $\mathcal{T}u$ is continuous. Hence, $\mathcal{T}$ is a well-defined self-mapping of $\mathfrak{X}$. Solving the Fredholm integral equation is equivalent to finding a fixed point of $\mathcal{T}$.

For $u, v \in \mathfrak{X}$, put

$$M(u,v) = \lambda \mho(u,v) + (1 - \lambda)[\mho(u, \mathcal{T}u)]^\alpha [\mho(v, \mathcal{T}v)]^\beta \mho(u,v)^{1-\alpha-\beta},$$

where

$$0 < \lambda \leq 1, \qquad \alpha, \beta > 0, \qquad \alpha + \beta < 1.$$

Theorem 2. *Suppose that some constants $t, \gamma \in (0,1)$ exist such that*

$$|K(\varkappa, r, u(r)) - K(\varkappa, r, v(r))| \leq \sqrt{\frac{\gamma t}{3}}\ \sqrt{M(u,v)} \tag{5.3}$$

for all $u, v \in \mathfrak{X}$ and $\varkappa, r \in [0,1]$. Then (5.1) *admits exactly one solution in $C([0,1], \mathbb{R})$.*

Proof. Fix $u, v \in \mathfrak{X}$ and $\varkappa \in [0,1]$. From the definition of $\mathcal{T}$, it follows that

$$\begin{aligned} |(\mathcal{T}u)(\varkappa) - (\mathcal{T}v)(\varkappa)| &= \left| \int_0^1 \left[K(\varkappa, r, u(r)) - K(\varkappa, r, v(r)) \right] dr \right| \\ &\leq \int_0^1 |K(\varkappa, r, u(r)) - K(\varkappa, r, v(r))|\, dr. \end{aligned}$$

Applying (5.3) and using the fact that the integration interval has length one, we find

$$|(\mathcal{T}u)(\varkappa) - (\mathcal{T}v)(\varkappa)| \leq \sqrt{\frac{\gamma t}{3}}\, \sqrt{M(u,v)}.$$

Passing to the supremum with respect to $\varkappa$ and squaring both sides gives

$$\mathfrak{U}(\mathcal{T}u, \mathcal{T}v) \leq \frac{\gamma t}{3} M(u,v),$$

or, equivalently,

$$3\mathfrak{U}(\mathcal{T}u, \mathcal{T}v) \leq \gamma t M(u,v).$$

Consider the functions

$$\vartheta(r) = e^r \qquad \text{and} \qquad \zeta(a,b) = \frac{b^\gamma}{a}, \quad a, b \geq 1.$$

The function ζ is an L-simulation function, as established earlier. Therefore,

$$\begin{aligned} &\zeta\left(\vartheta(3\mathfrak{U}(\mathcal{T}u, \mathcal{T}v)), \vartheta(tM(u,v))\right) \\ &\quad = \frac{\left(e^{tM(u,v)}\right)^\gamma}{e^{3\mathfrak{U}(\mathcal{T}u, \mathcal{T}v)}} \\ &\quad = \exp\left(\gamma t M(u,v) - 3\mathfrak{U}(\mathcal{T}u, \mathcal{T}v)\right) \geq 1. \end{aligned}$$

Hence, $\mathcal{T}$ fulfills the convex interpolative t-simulation contractive requirement.

The continuity assumptions imposed on g and K ensure that $\mathcal{T}$ is continuous and, in particular, orbitally continuous. All conditions of Theorem 1 are therefore fulfilled. It follows that $\mathcal{T}$ possesses a unique fixed point $u^* \in \mathfrak{X}$. By the definition of $\mathcal{T}$, this fixed point is precisely the unique continuous solution of (5.1). □

Remark 5. *Condition (5.3) is an operator-dependent hypothesis: The quantity $M(u,v)$ contains the displacements $\mathfrak{U}(u, \mathcal{T}u)$ and $\mathfrak{U}(v, \mathcal{T}v)$. These quantities are well defined by the prescribed integral operator and do not require prior knowledge of a solution. Nevertheless, the condition may be difficult to verify directly.*

A more explicit sufficient condition is the existence of a constant $L \geq 0$ such that

$$|K(\varkappa, r, a) - K(\varkappa, r, b)| \leq L|a - b|$$

for all $\varkappa, r \in [0, 1]$ and $a, b \in \mathbb{R}$, where

$$3L^2 \leq \gamma t \lambda.$$

Here, $t, \gamma \in (0, 1)$ and $\lambda \in (0, 1]$ are the parameters appearing in Theorem 2.

Indeed, since

$$M(u, v) \geq \lambda \mathfrak{V}(u, v) = \lambda \|u - v\|_\infty^2,$$

we obtain

$$\begin{aligned} |K(\varkappa, r, u(r)) - K(\varkappa, r, v(r))| &\leq L|u(r) - v(r)| \\ &\leq L\|u - v\|_\infty \\ &\leq \sqrt{\frac{\gamma t \lambda}{3}}\, \|u - v\|_\infty \\ &\leq \sqrt{\frac{\gamma t}{3}}\, \sqrt{M(u, v)}. \end{aligned}$$

Thus, condition (5.3) *holds, and Theorem 2 applies.*

This Lipschitz condition is a sufficient criterion for verifying (5.3)*; it is not asserted to be equivalent to the operator-dependent hypothesis.*

Example 5. *Consider the nonlinear Fredholm integral equation*

$$u(\varkappa) = \sin \varkappa + \int_0^1 \frac{\tanh(u(r))}{10(1 + r^2)}\, dr, \qquad \varkappa \in [0, 1]. \tag{5.4}$$

Let

$$\mathfrak{X} = C([0, 1], \mathbb{R})$$

and

$$\mathfrak{V}(u, v) = \|u - v\|_\infty^2.$$

As shown above, $(\mathfrak{X}, \mathfrak{V})$ is a complete rectangular b-metric space with coefficient $s = 3$.

Define

$$(\mathcal{T}u)(\varkappa) = \sin \varkappa + \int_0^1 \frac{\tanh(u(r))}{10(1 + r^2)}\, dr.$$

Since

$$|\tanh a - \tanh b| \leq |a - b|,$$

consequently, the kernel satisfies

$$|K(\varkappa, r, a) - K(\varkappa, r, b)| \leq \frac{1}{10}|a - b|.$$

Hence, we may take $L = 1/10$. For the chosen parameters $t = \gamma = 0.9$ and $\lambda = 0.1$, we have

$$3L^2 = 0.03 < 0.081 = \gamma t \lambda.$$

Therefore, the explicit sufficient condition in Remark 5 is satisfied. We have

$$\begin{aligned}|(\mathcal{T}u)(\varkappa) - (\mathcal{T}v)(\varkappa)| &\leq \frac{1}{10}\int_0^1 \frac{|u(r) - v(r)|}{1 + r^2}\, dr \\ &\leq \frac{\|u - v\|_\infty}{10}\int_0^1 \frac{1}{1 + r^2}\, dr \\ &= \frac{\pi}{40}\|u - v\|_\infty.\end{aligned}$$

Therefore,

$$\mho(\mathcal{T}u, \mathcal{T}v) \leq \left(\frac{\pi}{40}\right)^2 \mho(u, v).$$

Consequently,

$$3\mho(\mathcal{T}u, \mathcal{T}v) \leq 3\left(\frac{\pi}{40}\right)^2 \mho(u, v).$$

Numerically,

$$3\left(\frac{\pi}{40}\right)^2 \approx 0.01851.$$

Choose

$$t = 0.9, \qquad \gamma = 0.9, \qquad \lambda = 0.1, \qquad \alpha = \beta = \frac{1}{4}.$$

Then

$$\gamma t \lambda = 0.081.$$

Since

$$0.01851 < 0.081$$

and

$$M(u, v) \geq \lambda \mho(u, v) = 0.1\mho(u, v),$$

we obtain

$$3\mho(\mathcal{T}u, \mathcal{T}v) < \gamma t M(u, v).$$

Thus,

$$\zeta\left(\vartheta(3\mho(\mathcal{T}u, \mathcal{T}v)), \vartheta(tM(u, v))\right) \geq 1,$$

where

$$\vartheta(r) = e^r, \qquad \zeta(a, b) = \frac{b^{0.9}}{a}.$$

Hence, $\mathcal{T}$ satisfies the convex interpolative t-simulation contractive condition. Consequently, Theorem 1 guarantees that the integral equation (5.4) *admits a unique continuous solution. Moreover, since the integral term is independent of $\varkappa$, the solution has the form*

$$u(\varkappa) = \sin\varkappa + C,$$

where C satisfies

$$C = \frac{1}{10}\int_0^1 \frac{\tanh(\sin r + C)}{1 + r^2}\, dr.$$

Thus, the constant C may be approximated through the iteration

$$C_{n+1} = \frac{1}{10}\int_0^1 \frac{\tanh(\sin r + C_n)}{1 + r^2}\, dr.$$

6. Conclusions

In this paper, we introduced the notion of a convex interpolative t-simulation contraction in the setting of rectangular b-metric spaces. An existence and uniqueness fixed point theorem was established, together with the convergence of the associated Picard iteration. A nontrivial example demonstrated that the proposed framework can cover mappings for which the Banach, ordinary L-simulation, and pure interpolative t-simulation conditions fail. Connections with some well-known contractive conditions were also discussed. Finally, the main result was applied to establish the existence and uniqueness of a solution of a nonlinear Fredholm integral equation.

The proposed framework may be further investigated in other generalized metric structures and for multivalued or common fixed point problems. It would also be interesting to develop weaker versions of the present contractive condition, study stability and data-dependence properties, and explore applications to broader classes of integral and differential equations. Related nonlinear partial differential equation models provide another possible direction for future investigation. Examples include the chemotaxis–haptotaxis model studied by Liu [21] and the haptotactic system modeling oncolytic virotherapy studied by Zheng and Ke [26]. Exploring whether suitable reformulations of such problems admit operators satisfying the contractive conditions considered here would require additional analysis and is beyond the scope of the present paper.

Author contributions

All authors have equal contribution and are agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflict of interest in this paper.

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