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Research article

Boundedness and stabilization in a two-species parabolic-elliptic-parabolic-elliptic chemotaxis system with two chemicals

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Abstract: In this work, we studied dynamic properties of classical solutions to the chemotaxis system in a two-dimensional smoothly bounded domain. By applying the Agmon-Douglis-Nirenberg L^p estimates on linear elliptic equations and the two-dimensional Gagliardo-Nirenberg interpolation inequality, we showed that $\frac{\chi\tilde{\chi}}{\xi\tilde{\xi}} \geq 1$, which had nothing to do with β and δ, was sufficient to ensure the boundedness of the classical solutions. Meanwhile, by constructing Lyapunov function and the super-solution of the elliptic equations and employing the elliptic maximum principle, it was proved that the assumption $\frac{\chi\tilde{\chi}}{\xi\tilde{\xi}} > \max\{\frac{\beta}{\delta}, \frac{\delta}{\beta}\}$ can guarantee the stabilization of the classical solutions. This study displayed the differences of both research methods and conclusions between the parabolic-elliptical-parabolic-elliptic system and fully parabolic system, and thus we extended the existing research ones.

Keywords: boundedness; stabilization; two species and chemicals; chemotaxis system; smoothly bounded domain
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1. Introduction

To understand chemotactic interaction in the presence of several chemicals, Liu et al. ([1, 2]) studied the following system,

$$\begin{cases} u_t = \Delta u + \chi\nabla\cdot(u\nabla z) - \xi\nabla\cdot(u\nabla v), & (x,t)\in\Omega\times(0,\infty),\\ \tau_1 v_t = \Delta v - \beta v + \alpha\omega, & (x,t)\in\Omega\times(0,\infty),\\ \omega_t = \Delta\omega + \bar{\chi}\nabla\cdot(\omega\nabla v) - \bar{\xi}\nabla\cdot(\omega\nabla z), & (x,t)\in\Omega\times(0,\infty),\\ \tau_2 z_t = \Delta z - \delta z + \gamma u, & (x,t)\in\Omega\times(0,\infty),\\ \frac{\partial u}{\partial\vec{\nu}} = \frac{\partial v}{\partial\vec{\nu}} = \frac{\partial\omega}{\partial\vec{\nu}} = \frac{\partial z}{\partial\vec{\nu}} = 0, & (x,t)\in\partial\Omega\times(0,\infty),\\ (u,\tau_1 v,\omega,\tau_2 z)(x,0) = (u_0(x),\tau_1 v_0(x),\omega_0(x),\tau_2 z_0(x)), & x\in\overline{\Omega}. \end{cases} \tag{1.1}$$

Here, all of the parameters $\chi,\xi,\bar{\chi},\bar{\xi},\alpha,\beta,\gamma,\delta$ are positive, $\tau_1,\tau_2\geq 0$, $\Omega\subset\mathbb{R}^2$ is a smoothly bounded domain, u and ω denote the densities of the two species, and the corresponding chemicals produced by them are z and v. The current system (1.1) can describe the following biological phenomena involving interactions between different species or cell types. For example, there is a typical mutualistic relationship between ants and aphids. Aphids secrete a syrup containing dew to attract ants, which use them as a food source. At the same time, by secreting chemicals that attract aphids, ants protect them and deter or disturb predators. Clownfish bring food scraps that attract anemones and provide them with nutrients. Meanwhile, the tentacles of anemones release pheromones that attract clownfish, thus reducing the likelihood of being predation [3]. Up to now, some important results are listed below in the two-dimensional smoothly bounded domain:

(R_1) When $\tau_1=\tau_2=1$, system (1.1) possesses a globally bounded classical solution if initial species mass $\|u_0\|_{L^1(\Omega)}+\|\omega_0\|_{L^1(\Omega)}$ is appropriately small [1]. Moreover, if $\frac{\chi\bar{\chi}}{\xi\bar{\xi}}>\max\{\frac{\beta}{\delta},\frac{\delta}{\beta}\}$, then the solution (u,v,ω,z) of system (1.1) is globally bounded and converges exponentially to constant equilibria([3]).

(R_2) When $\tau_1=\tau_2=0$, the system (1.1) still admits a globally bounded classical solution if $\alpha=\beta=\gamma=\delta=1$ and $\max\{\|u_0\|_{L^1(\Omega)},\|\omega_0\|_{L^1(\Omega)}\}$ is below some threshold; or $\alpha=\beta=\gamma=\delta=1$ as well as $\chi=\bar{\chi}=\xi=\bar{\xi}$. Meanwhile, there exist finite-time blow-up solutions if the second moment of the initial data is sufficiently small [2].

It is known that there are some differences in research methods between fully parabolic systems and parabolic-elliptic-parabolic-elliptic systems. Let's recall the system (1.1) with $\tau_1=\tau_2=1$ to derive the boundedness and stabilization of the classical solution in [3], we first construct the nonnegative Lyapunov function

$$\bar{E}(t)=\frac{\gamma}{\xi}\int_\Omega u\ln\frac{u}{\bar{u}}+\frac{\alpha}{\bar{\xi}}\int_\Omega\omega\ln\frac{\omega}{\bar{\omega}}+\frac{\tilde{\chi}}{2\bar{\xi}}\int_\Omega|\nabla v|^2+\frac{\chi}{2\xi}\int_\Omega|\nabla z|^2-\int_\Omega\nabla z\cdot\nabla v,$$

and auxiliary function

$$\begin{aligned}\bar{F}(t)=&\frac{\gamma}{\xi}\int_\Omega\frac{|\nabla u|^2}{u}+\frac{\alpha}{\bar{\xi}}\int_\Omega\frac{|\nabla\omega|^2}{\omega}+\frac{\tilde{\chi}}{\bar{\xi}}\int_\Omega|\Delta v|^2+\frac{\chi}{\xi}\int_\Omega|\Delta z|^2\\&-2\int_\Omega\Delta z\cdot\Delta v+\frac{\beta\tilde{\chi}}{\bar{\xi}}\int_\Omega|\nabla v|^2+\frac{\delta\chi}{\xi}\int_\Omega|\nabla z|^2-(\beta+\delta)\int_\Omega\nabla z\cdot\nabla v.\end{aligned}$$

Second, with the aid of the theory of positive definite matrices of quadratic forms, we obtained the key result:

$$\|(u\ln u,\omega\ln\omega)\|_{L^1(\Omega)}+\|(\nabla v,\nabla z)\|_{L^2(\Omega)}\leq c.$$

Once again, based on the above estimate, we obtained $\|(u,\omega)\|_{L^2(\Omega)} \le c$ and, further, $\|(u,\omega)\|_{L^\infty(\Omega)} + \|(v,z)\|_{W^{1,\infty}(\Omega)} \le c$ for some positive constant c. Finally, the assumption $\frac{\chi\bar{\chi}}{\xi\bar{\xi}} > \max\{\frac{\beta}{\delta}, \frac{\delta}{\beta}\}$ simultaneously ensures the boundedness and stabilization. However, for the system (1.1) with $\tau_1 = \tau_2 = 0$, both the second and the fourth equations are transformed into elliptic equations, and the nonnegative Lyapunov function $\bar{E}(t)$ and auxiliary function $\bar{F}(t)$ in [3] can't be applied again. Therefore, we need to recheck system (1.1) with $\tau_1 = \tau_2 = 0$.

Fortunately, motivated by [4, lemma 4.1], with the aid of the mass conservation of the species and the Agmon-Douglis-Nirenberg L^p estimates on linear elliptic equations, we will see that assumption $\frac{\chi\tilde{\chi}}{\xi\tilde{\xi}} \ge 1$, which is clearer and has nothing to do with β and δ, is sufficient to ensure the boundedness of classical solutions. Moreover, by establish the following Lyapunov function

$$E(t) = \frac{\gamma}{\xi}\int_\Omega u\ln\frac{u}{\bar{u}} + \frac{\alpha}{\tilde{\xi}}\int_\Omega \omega\ln\frac{\omega}{\bar{\omega}},$$

and auxiliary function

$$\begin{aligned} F(t) = {} & \frac{\gamma}{\xi}\int_\Omega \frac{|\nabla u|^2}{u} + \frac{\alpha}{\tilde{\xi}}\int_\Omega \frac{|\nabla\omega|^2}{\omega} + \frac{\tilde{\chi}}{\tilde{\xi}}\int_\Omega |\Delta v|^2 + \frac{\chi}{\xi}\int_\Omega |\Delta z|^2 \\ & - 2\int_\Omega \Delta z\cdot\Delta v + \frac{\beta\tilde{\chi}}{\tilde{\xi}}\int_\Omega |\nabla v|^2 + \frac{\delta\chi}{\xi}\int_\Omega |\nabla z|^2 + (\beta+\delta)\int_\Omega \nabla z\cdot\nabla v, \end{aligned}$$

we indicate that criterion $\frac{\chi\tilde{\chi}}{\xi\tilde{\xi}} > \max\{\frac{\beta}{\delta}, \frac{\delta}{\beta}\}$ can still ensure the stability of the classical solution.

The novelty of this work lies in the following: on one hand, we adopted a different method from that in [3, Theorem 1.1] to obtain the boundedness criterion, that is, $\frac{\chi\tilde{\chi}}{\xi\tilde{\xi}} \ge 1$. On another hand, by constructing the super-solution of the elliptic equations different from fully parabolic equations and employing the elliptic maximum principle, we obtain the stability criterion of the solutions, namely, $\frac{\chi\tilde{\chi}}{\xi\tilde{\xi}} > \max\{\frac{\beta}{\delta}, \frac{\delta}{\beta}\}$. Our main results are stated as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a smoothly bounded domain, positive initial data $(u_0, \omega_0) \in C^0(\bar{\Omega}) \times C^0(\bar{\Omega})$, and the parameters satisfy*

$$\frac{\chi\tilde{\chi}}{\xi\tilde{\xi}} \ge 1. \tag{1.2}$$

Then, system (1.1) *with $\tau_1 = \tau_2 = 0$ admits positive global classical solution $(u, v, \omega, z) \in [C^0(\bar{\Omega}\times[0,\infty);\mathbb{R}) \cap C^{2,1}(\bar{\Omega}\times(0,\infty);\mathbb{R})]^4$. Moreover, the solution (u, v, ω, z) is bounded in the sense that*

$$\|u\|_{L^\infty(\Omega)} + \|v\|_{W^{1,\infty}(\Omega)} + \|\omega\|_{L^\infty(\Omega)} + \|z\|_{W^{1,\infty}(\Omega)} \le C, \quad for\ all\ t \ge 0.$$

In addition, there exists $\sigma \in (0,1)$ such that for all $t > 1$,

$$\|u\|_{C^{\sigma,\frac{\sigma}{2}}(\bar{\Omega}\times[t,t+1])} + \|\omega\|_{C^{\sigma,\frac{\sigma}{2}}(\bar{\Omega}\times[t,t+1])} \le C,$$

where $C > 0$ is some constant independent of time t.

Theorem 1.1 tells us that strong repulsive effect (1.2) is beneficial to the global existence of solutions. The following let us recall system (1.1) with $\tau_1 = \tau_2 = 1$, where the assumption $\frac{\chi\bar{\chi}}{\xi\bar{\xi}} > \max\{\frac{\beta}{\delta}, \frac{\delta}{\beta}\} \geq 1$ in [3, Theorem 1.1(4)] warrants that it admits a globally bounded classical solution. However, for system (1.1) with $\tau_1 = \tau_2 = 0$, we will see that the assumption $\frac{\chi\bar{\chi}}{\xi\bar{\xi}} \geq 1$ is sufficient to ensure the global boundedness of the solution. Furthermore, as a piece of crucial evidence, the proof of Proposition 3.1 indeed shows the differences in research methods between fully parabolic systems and the parabolic-elliptic-parabolic-elliptic systems.

The following result provides the global asymptotic stability of the solution (u, v, ω, z).

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a smoothly bounded domain, positive initial data $(u_0, \omega_0) \in C^0(\bar{\Omega}) \times C^0(\bar{\Omega})$, and the parameters satisfy*

$$\frac{\chi\tilde{\chi}}{\xi\tilde{\xi}} > \max\{\frac{\beta}{\delta}, \frac{\delta}{\beta}\}. \tag{1.3}$$

Then, there exist positive constant c and $\tilde{\eta}$ such that the bounded solution (u, v, ω, z) of system (1.1) with $\tau_1 = \tau_2 = 0$ satisfies

$$\|u - \bar{u}_0\|_{L^\infty(\Omega)} + \|v - \frac{\alpha}{\beta}\bar{\omega}_0\|_{L^\infty(\Omega)} + \|\omega - \bar{\omega}_0\|_{L^\infty(\Omega)} + \|z - \frac{\gamma}{\delta}\bar{u}_0\|_{L^\infty(\Omega)} \leq ce^{-\tilde{\eta}t} \quad for\ t > 0,$$

where $\bar{u}_0 = \frac{1}{|\Omega|}\int_\Omega u$, $\bar{\omega}_0 = \frac{1}{|\Omega|}\int_\Omega \omega$, and $\tilde{\eta}$ depends only on $\Omega, \chi, \xi, \tilde{\chi}, \tilde{\xi}, \alpha, \beta, \gamma, \delta$.

Theorem 1.2 shows that strong repulsive effect (1.3) can still guarantee the global asymptotic stability of the solution to system (1.1), with $\tau_1 = \tau_2 = 0$ as that in [3, Theorem 1.2] for system (1.1) with $\tau_1 = \tau_2 = 1$.

Before giving proof of the main results in Sections 3 and 4, we first show the local existence of a classical solution to system (1.1) and provide some preliminary results.

2. Local existence of solutions and preliminaries

As far as the local existence of solutions is concerned, the proof in time is similar to that in [5], which is achieved by applying Banach's fixed point theorem.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a smoothly bounded domain and all the other parameters $\chi, \tilde{\chi}, \xi, \tilde{\xi}, \beta, \alpha, \delta, \gamma$ be positive constants. Then, for any positive initial data $(u_0, \omega_0) \in C^0(\bar{\Omega}) \times C^0(\bar{\Omega})$, there exist $T_{max} \in (0, \infty]$ and positive functions u, v, ω, z in $\bar{\Omega} \times (0, T_{max})$ determined by*

$$(u, v, \omega, z) \in [C^0(\bar{\Omega} \times [0, T_{max}); \mathbb{R}) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max}); \mathbb{R})]^4,$$

which solve system (1.1) with $\tau_1 = \tau_2 = 0$ in $\Omega \times (0, T_{max})$, and if $T_{max} < \infty$, then

$$\limsup_{t \nearrow T_{\max}} (\|u\|_{L^\infty(\Omega)} + \|v\|_{W^{1,\infty}(\Omega)} + \|\omega\|_{L^\infty(\Omega)} + \|z\|_{W^{1,\infty}(\Omega)}) = \infty.$$

The following result gives some fundamental properties of solutions to system (1.1) with $\tau_1 = \tau_2 = 0$.

Proposition 2.2. *Assume that the positive initial data* $(u_0, \omega_0) \in C^0(\bar{\Omega}) \times C^0(\bar{\Omega})$. *Then,*

$$\begin{cases} \|u(\cdot,t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)}, & \|v(\cdot,t)\|_{L^1(\Omega)} = \dfrac{\alpha}{\beta}\|\omega_0\|_{L^1(\Omega)}, \\ \|\omega(\cdot,t)\|_{L^1(\Omega)} = \|\omega_0\|_{L^1(\Omega)}, & \|z(\cdot,t)\|_{L^1(\Omega)} = \dfrac{\gamma}{\delta}\|u_0\|_{L^1(\Omega)}. \end{cases}$$

Proof. Integrating each equation in system (1.1) with $\tau_1 = \tau_2 = 0$ over Ω yields the desired results. □

The following lemma, which was achieved based on the widely known smoothing $L^p - L^q$ estimates of the Neumann heat semigroup or the standard $W^{2,p}$-elliptic theory, plays a crucial role in the process of $\|(v,z)\|_{W^{1,\infty}(\Omega)}$ from $\|(u,\omega)\|_{L^2(\Omega)}$. This is particularly evident in the proof of Theorem 1.1.

Lemma 2.3. *[6, Lemma 2.2] Let* $\Omega \subset \mathbb{R}^2$ *be a smoothly bounded domain, and*

$$\begin{cases} q \in [1, \dfrac{2p}{2-p}), & if\ p \in [1,2], \\ q \in [1,\infty], & if\ p \in (2,\infty). \end{cases}$$

Then, there exist $\bar{C}_1 = \bar{C}_1(p,q,\tau_1 v_0,\Omega) > 0$ *and* $\bar{C}_2 = \bar{C}_2(p,q,\tau_2 z_0,\Omega) > 0$ *such that the unique local-in-time classical solution* (u,v,ω,z) *of system* (1.1) *satisfies*

$$\begin{cases} \|v\|_{W^{1,q}(\Omega)} \le \bar{C}_1(1 + \sup\limits_{s\in(0,t)} \|\omega\|_{L^p(\Omega)}), & \forall t \in (0, T_{\max}), \\ \|z\|_{W^{1,q}(\Omega)} \le \bar{C}_2(1 + \sup\limits_{s\in(0,t)} \|u\|_{L^p(\Omega)}), & \forall t \in (0, T_{\max}). \end{cases}$$

3. Boundedness criterion and Lyapunov function

3.1. Boundedness criterion

Motivated by [4, lemma 4.1], with the aid of the mass conservation of the species and the Agmon-Douglis-Nirenberg L^p estimates on linear elliptic equations, we will first establish a boundedness criterion of the solution (u,ω) to system (1.1) with $\tau_1 = \tau_2 = 0$. In other words, if $\dfrac{\chi\tilde{\chi}}{\xi\tilde{\xi}} \ge 1$, then $\int_\Omega u^2 + \int_\Omega \omega^2 < C$ holds for some positive constant C.

Proposition 3.1. *Suppose that* $(u_0,\omega_0) \in C^0(\bar{\Omega}) \times C^0(\bar{\Omega})$ *and* (1.2) *hold. Then, the solution* (u,ω) *of system* (1.1) *with* $\tau_1 = \tau_2 = 0$ *satisfies*

$$\int_\Omega u^2 + \int_\Omega \omega^2 < C, \quad \forall t \in (0, T_{\max}), \tag{3.1}$$

where the positive constant C is independent of t.

Proof. For the convenience of the argument, we define the following parameters:

$$k_1 = \chi^{\frac{1}{2}}\tilde{\xi}^{\frac{3}{2}}\gamma^2,\ k_2 = \tilde{\chi}^{\frac{1}{2}}\xi^{\frac{3}{2}}\alpha^2,\ \text{both } \epsilon_1 \to 0^+ \text{ and } \epsilon_3 \to 0^+ \text{are small enough}, \epsilon_2 = \frac{\chi\gamma}{\alpha\xi},\ \epsilon_4 = \frac{\alpha\tilde{\chi}}{\tilde{\xi}\gamma}. \tag{3.2}$$

Multiplying the first equation of system (1.1) by u and integrating by parts over Ω produces

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega u^2+\int_\Omega|\nabla u|^2\\
&=\frac{\chi}{2}\int_\Omega u^2\Delta z-\frac{\xi}{2}\int_\Omega u^2\Delta v\\
&=\frac{\chi}{2}\int_\Omega u^2(\delta z-\gamma u)-\frac{\xi}{2}\int_\Omega u^2(\beta v-\alpha\omega)\\
&=\frac{\chi\delta}{2}\int_\Omega u^2 z-\frac{\chi\gamma}{2}\int_\Omega u^3-\frac{\chi\beta}{2}\int_\Omega u^2 v+\frac{\xi\alpha}{2}\int_\Omega u^2\omega,
\end{aligned}$$

and following the same argument as that for u gives

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\omega^2+\int_\Omega|\nabla\omega|^2=\frac{\tilde{\chi}\beta}{2}\int_\Omega\omega^2 v-\frac{\alpha\tilde{\chi}}{2}\int_\Omega\omega^3-\frac{\tilde{\xi}\delta}{2}\int_\Omega\omega^2 z+\frac{\tilde{\xi}\gamma}{2}\int_\Omega\omega^2 u.$$

They, along with $v\geq 0$ and $z\geq 0$, yield

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2+\int_\Omega|\nabla u|^2\leq\frac{\chi\delta}{2}\int_\Omega u^2 z-\frac{\chi\gamma}{2}\int_\Omega u^3+\frac{\xi\alpha}{2}\int_\Omega u^2\omega. \tag{3.3}$$

Following the same argument as that for u leads to

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\omega^2+\int_\Omega|\nabla\omega|^2\leq\frac{\tilde{\chi}\beta}{2}\int_\Omega\omega^2 v-\frac{\tilde{\chi}\alpha}{2}\int_\Omega\omega^3+\frac{\tilde{\xi}\gamma}{2}\int_\Omega\omega^2 u. \tag{3.4}$$

Let $k_1\times(3.3)+k_2\times(3.4)$, and we conclude

$$\begin{aligned}
&\frac{d}{dt}(k_1\int_\Omega u^2+k_2\int_\Omega\omega^2)+2k_1\int_\Omega|\nabla u|^2+2k_2\int_\Omega|\nabla\omega|^2\\
&\leq\chi\delta k_1\int_\Omega u^2 z-\chi\gamma k_1\int_\Omega u^3+\xi\alpha k_1\int_\Omega u^2\omega\\
&\quad+\tilde{\chi}\beta k_2\int_\Omega\omega^2 v-\tilde{\chi}\alpha k_2\int_\Omega\omega^3+\tilde{\xi}\gamma k_2\int_\Omega\omega^2 u.
\end{aligned} \tag{3.5}$$

Applying Young's inequality to integral terms $\int_\Omega u^2 z$, $\int_\Omega u^2\omega$, $\int_\Omega\omega^2 v$, and $\int_\Omega\omega^2 u$ on the righthand side of (3.5), we derive

$$\begin{aligned}
&\frac{d}{dt}(k_1\int_\Omega u^2+k_2\int_\Omega\omega^2)+2k_1\int_\Omega|\nabla u|^2+2k_2\int_\Omega|\nabla\omega|^2+k_1\int_\Omega u^2+k_2\int_\Omega\omega^2\\
&\leq(\frac{2\chi\delta k_1\epsilon_1}{3}+\frac{2\alpha\xi k_1\epsilon_2}{3}+\frac{\tilde{\xi}\gamma k_2}{3\epsilon_4^2}-\chi\gamma k_1)\int_\Omega u^3+\frac{\chi\delta k_1}{3\epsilon_1^2}\int_\Omega z^3+k_1\int_\Omega u^2\\
&\quad+(\frac{\alpha\xi k_1}{3\epsilon_2^2}+\frac{2\tilde{\chi}\beta k_2\epsilon_3}{3}+\frac{2\tilde{\xi}\gamma k_2\epsilon_4}{3}-\alpha\tilde{\chi}k_2)\int_\Omega\omega^3+\frac{\tilde{\chi}\beta k_2}{3\epsilon_3^2}\int_\Omega v^3+k_2\int_\Omega\omega^2,
\end{aligned} \tag{3.6}$$

where all ϵ_i $(i=1,\cdots,4)$ and k_j $(j=1,2)$ are defined in (3.2). In the coming part, noting that v solves

$$\begin{cases}
-\triangle v+\beta v=\alpha\omega, & (x,t)\in\Omega\times(0,T_{\max}),\\
\frac{\partial v}{\partial\vec{\nu}}=0, & (x,t)\in\partial\Omega\times(0,T_{\max}),
\end{cases} \tag{3.7}$$

and applying the Agmon-Douglis-Nirenberg L^p estimates ([7, 8]) on linear elliptic equations with the (zero) Neumann boundary condition just as [4, (4.5)–(4.8)], there exists some positive constant $c(\epsilon_3)$ such that

$$\int_\Omega v^3 \le 2\epsilon_3^3 \int_\Omega \omega^3 + c(\epsilon_3). \tag{3.8}$$

By the same procedure as that for v, there exists some positive constant $c(\epsilon_1)$ such that

$$\int_\Omega z^3 \le 2\epsilon_1^3 \int_\Omega u^3 + c(\epsilon_1). \tag{3.9}$$

Meanwhile, in order to remove integral term $\int_\Omega u^2$ and $\int_\Omega \omega^2$ on the righthand side of (3.6), we employ the well-known Gagliardo-Nirenberg interpolation inequality to deduce some positive constant c_1 fufilling

$$\|u\|_{L^2(\Omega)} \le c_1 \|\nabla u\|_{L^2(\Omega)}^{\frac{1}{2}} \|u\|_{L^1(\Omega)}^{\frac{1}{2}} + c_1 \|u\|_{L^1(\Omega)}. \tag{3.10}$$

Due to the boundedness of $\|u\|_{L^1(\Omega)}$, and applying Young's inequality to (3.10), there exists some positive constant c_2 such that

$$\|u\|_{L^2(\Omega)}^2 \le 2\|\nabla u\|_{L^2(\Omega)}^2 + c_2. \tag{3.11}$$

In precisely the same manner as that for u, there exists some positive constant c_3 such that

$$\|\omega\|_{L^2(\Omega)}^2 \le 2\|\nabla \omega\|_{L^2(\Omega)}^2 + c_3. \tag{3.12}$$

Substituting (3.8)-(3.9) and (3.11)-(3.12) into (3.6) results in

$$\begin{aligned}
&\frac{d}{dt}(k_1 \int_\Omega u^2 + k_2 \int_\Omega \omega^2) + 2k_1 \int_\Omega |\nabla u|^2 + 2k_2 \int_\Omega |\nabla \omega|^2 + k_1 \int_\Omega u^2 + k_2 \int_\Omega \omega^2 \\
\le\ & (\frac{4\chi\delta k_1 \epsilon_1}{3} + \frac{2\alpha\xi k_1 \epsilon_2}{3} + \frac{\tilde{\xi}\gamma k_2}{3\epsilon_4^2} - \chi\gamma k_1) \int_\Omega u^3 + 2k_1 \|\nabla u\|_{L^2(\Omega)}^2 \\
&+ (\frac{\alpha\xi k_1}{3\epsilon_2^2} + \frac{4\tilde{\chi}\beta k_2 \epsilon_3}{3} + \frac{2\tilde{\xi}\gamma k_2 \epsilon_4}{3} - \alpha\tilde{\chi} k_2) \int_\Omega \omega^3 + 2k_2 \|\nabla \omega\|_{L^2(\Omega)}^2 + c_4 \\
:=\ & A_1 \int_\Omega u^3 + A_2 \int_\Omega \omega^3 + 2k_1 \|\nabla u\|_{L^2(\Omega)}^2 + 2k_2 \|\nabla \omega\|_{L^2(\Omega)}^2 + c_4,
\end{aligned} \tag{3.13}$$

where c_4 is some positive constant. In order to obtain four positive parameters $\epsilon_i (i = 1, \cdots, 4)$ and two positive parameters $k_j (j = 1, 2)$ on the righthand side of (3.13) and boundedness criteria, we rewrite $A_i < 0$ $(i = 1, 2)$ from (3.13) as

$$\begin{cases}
\frac{4\chi\delta}{3}\epsilon_1 + \frac{2\alpha\xi}{3}\epsilon_2 + \frac{\tilde{\xi}\gamma}{3\epsilon_4^2}\frac{k_2}{k_1} < \chi\gamma, \\
\frac{\alpha\xi}{3\epsilon_2^2}\frac{k_1}{k_2} + \frac{4\tilde{\chi}\beta}{3}\epsilon_3 + \frac{2\tilde{\xi}\gamma}{3}\epsilon_4 < \alpha\tilde{\chi}.
\end{cases} \tag{3.14}$$

Let $\epsilon_1 \to 0^+$ and $\epsilon_3 \to 0^+$ be small enough in (3.14), then we get

$$\begin{cases}
\frac{2\alpha\xi}{3}\epsilon_2 + \frac{\tilde{\xi}\gamma}{3\epsilon_4^2}\frac{k_2}{k_1} \le \chi\gamma, \\
\frac{\alpha\xi}{3\epsilon_2^2}\frac{k_1}{k_2} + \frac{2\tilde{\xi}\gamma}{3}\epsilon_4 \le \alpha\tilde{\chi}.
\end{cases} \tag{3.15}$$

Multiplying the first and second inequalities by $\alpha\tilde{\chi}$ and $\chi\gamma$, respectively, we get

$$\begin{cases} \dfrac{2\alpha\xi\alpha\tilde{\chi}}{3}\epsilon_2 + \dfrac{\tilde{\xi}\gamma\alpha\tilde{\chi}}{3\epsilon_4^2}\dfrac{k_2}{k_1} \le \chi\gamma\alpha\tilde{\chi}, \\ \dfrac{\alpha\xi\chi\gamma}{3\epsilon_2^2}\dfrac{k_1}{k_2} + \dfrac{2\tilde{\xi}\gamma\chi\gamma}{3}\epsilon_4 \le \alpha\tilde{\chi}\chi\gamma. \end{cases} \tag{3.16}$$

Once again we easily verify

$$k_1 = \chi^{\frac{1}{2}}\tilde{\xi}^{\frac{3}{2}}\gamma^2,\ k_2 = \tilde{\chi}^{\frac{1}{2}}\xi^{\frac{3}{2}}\alpha^2, \epsilon_2 = \frac{\chi\gamma}{\alpha\xi},\ \ \epsilon_4 = \frac{\alpha\tilde{\chi}}{\tilde{\xi}\gamma},\ \ \frac{\chi\tilde{\chi}}{\xi\tilde{\xi}} \ge 1$$

guaranteeing $\dfrac{2\alpha\xi\alpha\tilde{\chi}}{3}\epsilon_2 = \dfrac{2\tilde{\xi}\gamma\chi\gamma}{3}\epsilon_4 = \dfrac{2}{3}\chi\gamma\alpha\tilde{\chi}$ and $\dfrac{\tilde{\xi}\gamma\alpha\tilde{\chi}}{3\epsilon_4^2}\dfrac{k_2}{k_1} = \dfrac{\alpha\xi\chi\gamma}{3\epsilon_2^2}\dfrac{k_1}{k_2} \le \dfrac{1}{3}\chi\gamma\alpha\tilde{\chi}$ in (3.16).

Up to now, we obtain that there exists some positive constant c_5 such that

$$\frac{d}{dt}(k_1\int_\Omega u^2 + k_2\int_\Omega \omega^2) + k_1\int_\Omega u^2 + k_2\int_\Omega \omega^2 \le c_5.$$

Thanks to the comparison principle of ordinary differential equations, (3.1) is clear. □

With the above Proposition 3.1 at hand, along with the existing Lemma 2.3 and extensibility criterion Proposition 2.1, we can immediately give the proof of Theorem 1.1.

Proof of Theorem 1.1. Since $\|u(\cdot,t)\|_{L^2(\Omega)} + \|\omega(\cdot,t)\|_{L^2(\Omega)} \le M$ for some positive constant M, applying Lemma 2.3 to the second and fourth equations of system 1.1 with $\tau_1 = \tau_2 = 0$, there exists positive constant c_6 such that

$$\|\nabla v\|_{L^4(\Omega)} + \|\nabla z\|_{L^4(\Omega)} \le c_6.$$

We denote $n = 2, r = 2n$, and follow the same argument as that for [9, (25)–(30)] to deduce

$$\|u\|_{L^\infty(\Omega)} + \|\omega\|_{L^\infty(\Omega)} \le c_7 \ \ for\ all\ t \in (0, T_{\max}). \tag{3.17}$$

In particular, we get

$$\|u\|_{L^3(\Omega)} + \|\omega\|_{L^3(\Omega)} \le c_8.$$

Applying Lemma 2.3 to the second and fourth equations of system (1.1) once again yields

$$\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|z(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le c_9 \ \ for\ all\ t \in (0, T_{\max}). \tag{3.18}$$

Due to (3.17) and (3.18), with the help of the extensibility criterion Proposition 2.1, we can deduce that $T_{\max}$ must be infinite. Apart from that, similar arguments as that for [9, Lemma 3.2] also show that there exists $\sigma \in (0,1)$, such that for all $t > 1$,

$$\|u\|_{C^{\sigma,\frac{\sigma}{2}}(\bar{\Omega}\times[t,t+1])} + \|\omega\|_{C^{\sigma,\frac{\sigma}{2}}(\bar{\Omega}\times[t,t+1])} \le c_{10}.$$

Herein $c_i (i = 7, \cdots, 10)$ are some positive constants. As mentioned above, we complete the proof. □

3.2. Lyapunov function

As is well known, just like many chemotaxis systems, Lyapunov function plays a crucial role in the study of the asymptotic stability of solutions. In this subsection, by constructing the following Lyapunov function

$$E(t) = \frac{\gamma}{\xi}\int_\Omega u\ln\frac{u}{\bar{u}} + \frac{\alpha}{\tilde{\xi}}\int_\Omega \omega\ln\frac{\omega}{\bar{\omega}} \tag{3.19}$$

and auxiliary function

$$\begin{aligned} F(t) = &\frac{\gamma}{\xi}\int_\Omega \frac{|\nabla u|^2}{u} + \frac{\alpha}{\tilde{\xi}}\int_\Omega \frac{|\nabla\omega|^2}{\omega} + \frac{\tilde{\chi}}{\tilde{\xi}}\int_\Omega |\Delta v|^2 + \frac{\chi}{\xi}\int_\Omega |\Delta z|^2 \\ &- 2\int_\Omega \Delta z\cdot\Delta v + \frac{\beta\tilde{\chi}}{\tilde{\xi}}\int_\Omega |\nabla v|^2 + \frac{\delta\chi}{\xi}\int_\Omega |\nabla z|^2 + (\beta+\delta)\int_\Omega \nabla z\cdot\nabla v, \end{aligned} \tag{3.20}$$

we can achieve the study on asymptotic stability of solutions (u, v, ω, z). More precisely, we list the following outline:

Proposition 3.2. *Let (u, v, ω, z) be the solution of system* (1.1) *with $\tau_1 = \tau_2 = 0$. Then, one can get*

$$\frac{d}{dt}E(t) + F(t) = 0, \tag{3.21}$$

where $E(t) \geq 0$ and $F(t)$ are defined as (3.19) *and* (3.20)*, respectively. Moreover, if $\frac{\chi\tilde{\chi}}{\xi\tilde{\xi}} > \max\{\frac{\beta}{\delta}, \frac{\delta}{\beta}\}$ holds, then $F(t) \geq 0$ for all $t > 0$.*

Proof. Multiplying $\ln\frac{u}{\bar{u}}$, $-\Delta v$, $\ln\frac{\omega}{\bar{\omega}}$, and $-\Delta z$ at both ends of the first, second, third, and fourth equations of system (1.1) with $\tau_1 = \tau_2 = 0$, respectively, one can get

$$\frac{d}{dt}\int_\Omega u\ln\frac{u}{\bar{u}} + \int_\Omega \frac{|\nabla u|^2}{u} = -\chi\int_\Omega \nabla u\cdot\nabla z + \xi\int_\Omega \nabla u\cdot\nabla v, \tag{3.22}$$

$$\int_\Omega |\Delta v|^2 + \beta\int_\Omega |\nabla v|^2 = \alpha\int_\Omega \nabla\omega\cdot\nabla v, \tag{3.23}$$

$$\frac{d}{dt}\int_\Omega \omega\ln\frac{\omega}{\bar{\omega}} + \int_\Omega \frac{|\nabla\omega|^2}{\omega} = -\tilde{\chi}\int_\Omega \nabla\omega\cdot\nabla v + \tilde{\xi}\int_\Omega \nabla\omega\cdot\nabla z, \tag{3.24}$$

$$\int_\Omega |\Delta z|^2 + \delta\int_\Omega |\nabla z|^2 = \gamma\int_\Omega \nabla u\cdot\nabla z. \tag{3.25}$$

Meanwhile, some series of calculations lead to

$$\begin{aligned} -\int_\Omega \Delta z\cdot\Delta v &= \beta\int_\Omega \nabla v\cdot\nabla z - \alpha\int_\Omega \nabla\omega\cdot\nabla z \\ &= \delta\int_\Omega \nabla z\cdot\nabla v - \gamma\int_\Omega \nabla u\cdot\nabla v. \end{aligned} \tag{3.26}$$

Let $\frac{\gamma}{\xi}\times(3.22)+\frac{\tilde{\chi}}{\tilde{\xi}}\times(3.23)+\frac{\alpha}{\tilde{\xi}}\times(3.24)+\frac{\chi}{\xi}\times(3.25)+2\times(3.26)$, and we obtain

$$\frac{d}{dt}E(t) + F(t) = 0, \tag{3.27}$$

In other words, (3.21) holds.

Next, based on an obvious fact $s \ln s \geq s-1, \forall s>0$, taking $s=\frac{u}{\bar{u}}$ results in $E(t) \geq 0$. To show that $F(t) \geq 0$ for all $t>0$, we rewrite $F(t)$ in (3.20) as

$$
\begin{aligned}
F(t)=&\frac{\gamma}{\xi} \int_{\Omega} \frac{|\nabla u|^2}{u}+\frac{\alpha}{\tilde{\xi}} \int_{\Omega} \frac{|\nabla \omega|^2}{\omega}+\int_{\Omega}\begin{bmatrix}\Delta v & \Delta z\end{bmatrix}\begin{bmatrix}\frac{\tilde{\chi}}{\tilde{\xi}} & -1 \\ -1 & \frac{\chi}{\xi}\end{bmatrix}\begin{bmatrix}\Delta v \\ \Delta z\end{bmatrix} \\
&+\int_{\Omega}\begin{bmatrix}\nabla v & \nabla z\end{bmatrix}\begin{bmatrix}\frac{\beta \tilde{\chi}}{\tilde{\xi}} & \frac{\beta+\delta}{2} \\ \frac{\beta+\delta}{2} & \frac{\delta \chi}{\xi}\end{bmatrix}\begin{bmatrix}\nabla v \\ \nabla z\end{bmatrix}.
\end{aligned}
$$

Performing an argument similar to that in [3, Lemma 3.2], one can obtain $F(t) \geq 0$ provided that $\frac{\chi \tilde{\chi}}{\xi \tilde{\xi}}>\max\{\frac{\beta}{\delta}, \frac{\delta}{\beta}\}$. As mentioned above, the proof is completed. □

4. Proof of Theorem 1.2

In this section, under the assumption (1.3), we give a key proposition on asymptotical behavior of solution to system (1.1) with $\tau_1=\tau_2=0$.

Proposition 4.1. *Suppose that* $(u_0, \omega_0) \in C^0(\bar{\Omega}) \times C^0(\bar{\Omega})$ *and* (1.3) *hold. Then, the solution* (u, v, ω, z) *of system* (1.1) *with* $\tau_1=\tau_2=0$ *fulfills*

$$
\|u \ln u\|_{L^1(\Omega)} \leq c, \|\omega \ln \omega\|_{L^1(\Omega)} \leq c, \int_{\Omega}|\nabla v|^2 \leq c, \int_{\Omega}|\nabla z|^2 \leq c \tag{4.1}
$$

and

$$
\int_0^t \int_{\Omega} \frac{|\nabla u|^2}{u} \leq c, \int_0^t \int_{\Omega} \frac{|\nabla \omega|^2}{\omega} \leq c. \tag{4.2}
$$

Here $c>0$ *is a constant independent of* t.

Proof. In fact, the boundedness of the first two L^1-norms in (4.1) and that of integral (4.2) can be proved as that in [3, Proposition 3]. Readers can easily verify them, and thus it will not be restated here. In addition, the boundedness of the last two integral terms in (4.1) are the direct consequence of Lemma 2.3. □

4.1. Convergence

In this subsection, we first obtain the convergence of $\|u-\bar{u}_0\|_{L^\infty(\Omega)}$ and $\|\omega-\bar{\omega}_0\|_{L^\infty(\Omega)}$ as well as $\|z-\frac{\gamma}{\delta}\bar{u}_0\|_{L^\infty(\Omega)}$ and $\|v-\frac{\alpha}{\beta}\bar{\omega}_0\|_{L^\infty(\Omega)}$ via using the argument of contradiction, constructing the super-solution of two elliptic systems and employing the elliptic maximum principle.

Proposition 4.2. *Suppose that* $(u_0, \omega_0) \in C^0(\bar{\Omega}) \times C^0(\bar{\Omega})$ *and* (1.3) *hold. Then, the solution* (u, v, ω, z) *of system* (1.1) *with* $\tau_1=\tau_2=0$ *satisfies that*

$$
\begin{aligned}
&\|u-\bar{u}_0\|_{L^\infty(\Omega)} \to 0, \quad \|\omega-\bar{\omega}_0\|_{L^\infty(\Omega)} \to 0, \\
&\|z-\frac{\gamma}{\delta}\bar{u}_0\|_{L^\infty(\Omega)} \to 0, \|v-\frac{\alpha}{\beta}\bar{\omega}_0\|_{L^\infty(\Omega)} \to 0 \ \ as\ t \to \infty,
\end{aligned}
$$

where $\bar{u}_0 = \frac{1}{|\Omega|}\int_\Omega u$ *and* $\bar{\omega}_0 = \frac{1}{|\Omega|}\int_\Omega \omega$.

Proof. By the argument of contradiction just like in [9, Lemma 4.4] and [10, Lemma 3.10], or (31)–(35) in [3, Propersition 5], we easily get

$$\|u - \bar{u}_0\|_{C^0(\Omega)} \to 0 \text{ and } \|u - \bar{u}_0\|_{L^\infty(\Omega)} \to 0, \text{ as } t \to \infty, \tag{4.3}$$

and

$$\|\omega - \bar{\omega}_0\|_{C^0(\Omega)} \to 0 \text{ and } \|\omega - \bar{\omega}_0\|_{L^\infty(\Omega)} \to 0, \text{ as } t \to \infty. \tag{4.4}$$

Throughout what follows, we will construct the super-solution of two new systems and employ the elliptic maximum principle to indicate

$$\|z - \frac{\gamma}{\delta}\bar{u}_0\|_{L^\infty(\Omega)} \to 0, \|v - \frac{\alpha}{\beta}\bar{\omega}_0\|_{L^\infty(\Omega)} \to 0 \text{ as } t \to \infty.$$

To be specific, let $\phi(x,t) = v(x,t) - \frac{\alpha}{\beta}\bar{\omega}_0$ and $\psi(x,t) = z(x,t) - \frac{\gamma}{\delta}\bar{u}_0$, then we rewrite the second and fourth equations of system (1.1) as

$$\begin{cases} -\Delta\phi + \beta\phi = \alpha(\omega - \bar{\omega}_0), & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial\phi}{\partial\nu} = 0, & (x,t) \in \Omega \times (0,\infty), \end{cases} \tag{4.5}$$

as well as

$$\begin{cases} -\Delta\psi + \delta\psi = \gamma(u - \bar{u}_0), & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial\psi}{\partial\nu} = 0, & (x,t) \in \Omega \times (0,\infty). \end{cases} \tag{4.6}$$

Applying the elliptic maximum principle ([4, (5.15)], [11]), we can obtain from (4.5) and (4.6)

$$\begin{cases} \|v(\cdot,t) - \frac{\alpha}{\beta}\bar{\omega}_0\|_{L^\infty(\Omega)} = \|\phi(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{\alpha}{\beta}\|\omega - \bar{\omega}_0\|_{L^\infty(\Omega)} \to 0, \\ \|z(\cdot,t) - \frac{\gamma}{\delta}\bar{u}_0\|_{L^\infty(\Omega)} = \|\psi(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{\gamma}{\delta}\|u - \bar{u}_0\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty. \end{cases} \tag{4.7}$$

As shown in (4.3)-(4.4) and (4.7) above, we complete the proof. □

4.2. Decay rate

In this subsection, by establishing differential control inequalities of the Lyapunov function $E(t)$, we show that the convergence rates of $\|u - \bar{u}_0\|_{L^1(\Omega)}$ and $\|\omega - \bar{\omega}_0\|_{L^1(\Omega)}$ are exponential. Then, we also get the estimate of $\|\nabla v\|_{L^p(\Omega)}$ and $\|\nabla z\|_{L^p(\Omega)}$ for any $p \in (1,\infty)$ by the Agmon-Douglis-Nirenberg L^p estimates on linear elliptic equations with the (zero) Neumann boundary condition.

Proposition 4.3. *Suppose that* $(u_0,\omega_0) \in C^0(\bar{\Omega}) \times C^0(\bar{\Omega})$ *and* (1.3) *hold. Then, the solution* (u,ω) *of system* (1.1) *with* $\tau_1 = \tau_2 = 0$ *satisfies*

$$\|u - \bar{u}_0\|_{L^1(\Omega)} \le Ce^{-\lambda t}, \quad \|\omega - \bar{\omega}_0\|_{L^1(\Omega)} \le Ce^{-\lambda t} \text{ for all } t > 0,$$

where $\bar{u}_0 = \frac{1}{|\Omega|}\int_\Omega u, \bar{\omega}_0 = \frac{1}{|\Omega|}\int_\Omega \omega$, *and both positive constants* C *and* λ *depend only on* $\Omega, \chi, \xi, \tilde{\chi}, \tilde{\xi}, \alpha, \beta, \gamma, \delta$.

Proof. Similar to [3, Proposition 6], we can easily obtain the proof. The details are not listed here. □

Proposition 4.4. *Suppose that* (u, v, ω, z) *is a globally bounded classical solution of system* (1.1) *with initial data* $(u_0, \omega_0) \in C^0(\bar{\Omega}) \times C^0(\bar{\Omega})$ *and* (1.3) *holding. Then, for any* $p \in (1, \infty)$*, the solution* (v, z) *of system* (1.1) *with* $\tau_1 = \tau_2 = 0$ *fulfills*

$$\|\nabla v\|_{L^p(\Omega)} \le Ce^{-\frac{t}{2\mu p}}, \quad \|\nabla z\|_{L^p(\Omega)} \le Ce^{-\frac{t}{2\mu p}} \text{ for all } t > 0,$$

where positive constants C *and* μ *depend only on* $\Omega, \chi, \xi, \tilde{\chi}, \tilde{\xi}, \alpha, \beta, \gamma, \delta$*, and the Poincaré constant, respectively.*

Proof. First, we make the following transformations

$$U^* = u - \bar{u}_0, V^* = v - \frac{\alpha}{\beta}\bar{\omega}_0, W^* = \omega - \bar{\omega}_0, Z^* = z - \frac{\gamma}{\delta}\bar{u}_0,$$

then system (1.1) with $\tau_1 = \tau_2 = 0$ can be converted into the following system

$$\begin{cases} U_t^* = \Delta U^* + \chi\nabla\cdot(U^*\nabla Z^*) - \xi\nabla\cdot(U^*\nabla V^*) + \chi\bar{u}_0\Delta Z^* - \xi\bar{u}_0\Delta V^*, & (x,t) \in \Omega\times(0,\infty), \\ -\Delta V^* + \beta V^* = \alpha W^*, & (x,t) \in \Omega\times(0,\infty), \\ W_t^* = \Delta W^* + \tilde{\chi}\nabla\cdot(W^*\nabla V^*) - \tilde{\xi}\nabla\cdot(W^*\nabla Z^*) + \tilde{\chi}\bar{\omega}_0\Delta V^* - \tilde{\xi}\bar{\omega}_0\Delta Z^*, & (x,t) \in \Omega\times(0,\infty), \\ -\Delta Z^* + \delta Z^* = \gamma U^*, & (x,t) \in \Omega\times(0,\infty). \end{cases} \tag{4.8}$$

Applying the Agmon-Douglis-Nirenberg L^p estimates ([7, 8]) on linear elliptic equations with the (zero) Neumann boundary condition leads to some positive constant c_{11} such that

$$\|\nabla v\|_{L^p(\Omega)} = \|\nabla V^*\|_{L^p(\Omega)} \le c_{11}\|V^*\|_{W^{2,p}(\Omega)} \le c_{11}\|W^*\|_{L^p(\Omega)},$$

which, along with the interpolation inequality $\|W^*\|_{L^p(\Omega)} \le \|W^*\|_{L^\infty(\Omega)}^{\frac{p-1}{p}}\|W^*\|_{L^1(\Omega)}^{\frac{1}{p}}$, gives

$$\|\nabla v\|_{L^p(\Omega)} = \|\nabla V^*\|_{L^p(\Omega)} \le c_{12}e^{-\frac{t}{2\mu p}}.$$

The similar argument as that for v leads to

$$\|\nabla z\|_{L^p(\Omega)} = \|\nabla Z^*\|_{L^p(\Omega)} \le c_{13}e^{-\frac{t}{2\mu p}}.$$

Here c_{12} and c_{13} are some positive constants. As summarized above, we get the desired results. □

With above preparation at hand, following the idea in [12, Proof of Theorem 1.2], we obtain the decay rate of $\|u - \bar{u}_0\|_{L^\infty(\Omega)}$ and $\|\omega - \bar{\omega}_0\|_{L^\infty(\Omega)}$.

4.3. Proof of Theorem 1.2

Proof of Theorem 1.2. First of all, representing U^* in (4.8) according to the variation-of-constants formula results in

$$\begin{aligned} U^* = e^{t\Delta}U_0^* &+ \chi\int_0^t e^{(t-s)\Delta}\nabla\cdot(U^*\nabla Z^*) - \xi\int_0^t e^{(t-s)\Delta}\nabla\cdot(U^*\nabla V^*) \\ &+ \chi\bar{u}_0\int_0^t e^{(t-s)\Delta}\Delta Z^* - \xi\bar{u}_0\int_0^t e^{(t-s)\Delta}\Delta V^*. \end{aligned}$$

Second, we get

$$\begin{aligned}\|U^*\|_{L^\infty(\Omega)} &\le \|e^{t\Delta}U_0^*\|_{L^\infty(\Omega)} + \chi\int_0^t \|e^{(t-s)\Delta}\nabla\cdot(U^*\nabla Z^*)\|_{L^\infty(\Omega)} + \xi\int_0^t \|e^{(t-s)\Delta}\nabla\cdot(U^*\nabla V^*)\|_{L^\infty(\Omega)}\\ &\quad + \chi\bar{u}_0\int_0^t \|e^{(t-s)\Delta}\nabla\cdot(\nabla Z^*)\|_{L^\infty(\Omega)} + \xi\bar{u}_0\int_0^t \|e^{(t-s)\Delta}\nabla\cdot(\nabla V^*)\|_{L^\infty(\Omega)}\\ &=: I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t), \text{ for all } \quad t > 0.\end{aligned}$$

Once again, following the idea as that in [12, Proof of Theorem 1.2], there exists some positive constant c_{14} such that

$$\|u - \bar{u}_0\|_{L^\infty(\Omega)} \le c_{14}e^{-\tilde{\eta}t} \quad for\, t > 0.$$

Then, following the same argument as that for U^*, there exists some positive constant c_{15} such that

$$\|\omega - \bar{\omega}_0\|_{L^\infty(\Omega)} \le c_{15}e^{-\tilde{\eta}t} \quad for\, t > 0.$$

Finally, the two inequalities in (4.7) imply, for some positive constant c_{16},

$$\|v - \frac{\alpha}{\beta}\bar{\omega}_0\|_{L^\infty(\Omega)} + \|z - \frac{\gamma}{\delta}\bar{u}_0\|_{L^\infty(\Omega)} \le c_{16}e^{-\tilde{\eta}t} \quad for\, t > 0.$$

As mentioned above, we complete the proof. □

5. Conclusions

In this work, we consider a two-species attraction-repulsion chemotaxis system with two chemicals. Motivated by [4, Lemma 4.1], we first show that the assumption $\frac{\chi\tilde{\chi}}{\xi\tilde{\xi}} \ge 12^{\frac{2}{3}} \approx 5.24$ ensures the boundedness of the classical solutions. It is worth noting that we should be very grateful to a friendly and extremely patient anonymous reviewer who provided a very valuable suggestion, that is, the modified boundedness threshold $\frac{\chi\tilde{\chi}}{\xi\tilde{\xi}} \ge \frac{16}{9}\sqrt[3]{16} \approx 4.48$ can be substantially improved to $\frac{\chi\tilde{\chi}}{\xi\tilde{\xi}} \ge 1$ by employing the conserved mass together with elliptic regularity to obtain uniform bounds for the chemical variables. By applying the Agmon-Douglis-Nirenberg L^p estimates on linear elliptic equations and selecting new parameters, we further strengthen the boundedness threshold from $\frac{\chi\tilde{\chi}}{\xi\tilde{\xi}} \ge \frac{2}{3}\sqrt[3]{16} \approx 1.68$ to $\frac{\chi\tilde{\chi}}{\xi\tilde{\xi}} \ge 1$, and ultimately achieve the results expected by the anonymous reviewer. Meanwhile, by constructing a Lyapunov function and the super-solution of the elliptic equations, the assumption $\frac{\chi\tilde{\chi}}{\xi\tilde{\xi}} > \max\{\frac{\beta}{\delta}, \frac{\delta}{\beta}\}$ guarantees the stabilization of the classical solution. As previously mentioned, in this work, the current research displays the differences of both research methods and conclusions between parabolic-elliptical-parabolic-elliptic systems and fully parabolic systems.

Author contributions

Aichao Liu carried out the majority of writing for the initial, revised and final manuscript; Junhong Xiong reviewed and polished each version of the manuscript; Lufang Mi conducted calculation verification for the manuscript and provided the fund support. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

No generative artificial intelligence (AI) tools were used in the writing of current work, all the contents were produced on the basis of literature reading and personal research.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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