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Research article

m-QMDS codes over mixed alphabets via orthogonal arrays

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Abstract: The construction of quantum error-correcting codes (QECCs) with good parameters is a hot topic in the area of quantum information and quantum computing. Quantum maximum distance separable (MDS) codes are optimal because the minimum distance cannot be improved for given length, dimension, and alphabet sizes. Quantum MDS codes over mixed alphabets are scarcely studied, and even the existence and construction of QECCs over mixed alphabets with a minimum distance of at least three remain an open question. In this paper, we define an m-QMDS code over mixed alphabets, which is a generalization of quantum MDS (QMDS) codes. We establish a relation between such codes and asymmetric orthogonal arrays with orthogonal partitions. This relation is used for the construction of m-QMDS codes with varying local dimension, length, and minimum distance. As applications of this method, numerous infinite families of m-QMDS codes over mixed alphabets can be constructed explicitly. The constructed codes not only have more flexibility in the choice of parameters than other models but also possess better error correction capability and encoding efficiency.

Keywords: quantum error-correcting code over mixed alphabets; orthogonal array; orthogonal partition; m-QMDS code over mixed alphabets; Hamming distance
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1. Introduction

The no-go theorems such as the no-cloning theorem [1–3], the no-broadcasting theorem [4], the no-deleting theorem [5], and the no-hiding theorem [6] are consequences of the linearity and the unitarity of quantum mechanics. One of the problems with the feasibility of quantum computation is the difficulty of eliminating errors caused by inaccuracy and decoherence. Because the classical error-correcting techniques based on redundancy or repetition codes seemed to contradict the quantum no-cloning theorem, classical error-correcting codes can not be used in quantum

computation. Quantum error-correcting codes will be necessary for preserving coherent states against noise and other unwanted interactions in quantum computation and communication [7]. Quantum systems are more fragile than classical systems. Errors are unavoidable when quantum information travels across a noisy channel [7–9]. The primary tool to deal with different types of quantum noises is quantum error-correcting codes (QECCs) [7, 8, 10, 11]. They are the backbone of quantum fault-tolerant protocols needed to reliably operate scalable quantum computers and play an important role in quantum information tasks, such as quantum key distribution [12] and entanglement purification [13–15]. Since their discovery, the construction of QECCs has come a long way. The construction of quantum MDS (QMDS) codes over a single alphabet has been exhaustively investigated in the literature [16–19]. It merits attention that, in addition to quantum MDS codes over a single alphabet, constructing non-MDS QECCs with optimal, best-known, or record-breaking parameters is also an interesting research direction that has attracted considerable attention in recent years [20–22]. However, we are often faced with a more complex situation where quantum information is encoded in quantum resources with different dimensions. In particular, we often use hybrid systems of different dimensions [23–25] to store, transmit, and process quantum information. Therefore, it is necessary to extend QECCs over a single alphabet to mixed alphabets [26].

For QECCs over a single alphabet, there are several general construction methods, such as the Calderbank–Shor–Steane (CSS)-type construction and Hermitian-type construction. For QECCs over mixed alphabets, however, the presence of different local dimensions makes an analogous general extension of these constructions extremely difficult. However, the quantum Singleton bound has been generalized to QECCs over mixed alphabets [26]. A QECC attaining the quantum Singleton bound is called a QMDS. Like QECCs over a single alphabet, quantum MDS codes over mixed alphabets are also optimal because the minimum distance cannot be improved for given length, dimension, and alphabet sizes.

However, the construction of QECCs over mixed alphabets is rarely studied. Other than some QECCs over mixed alphabets given in [17, 26–28], little is known about the existence of QECCs $((n, K, d+1))_{s_1 s_2 \cdots s_n}$. Especially, the existence and construction of QECCs over mixed alphabets with a minimum distance of at least three remain an open question [26]. It is well-known that constructing such codes is difficult, and quantum MDS codes with certain parameters do not exist; even the quantum MDS codes over mixed alphabets with $K = 1$ have not been found so far [17, 26–28]. It is therefore of interest to develop new approaches to construct QECCs with better performance, but this is challenging.

To address this problem, this paper develops a combinatorial construction framework based on asymmetric orthogonal arrays with orthogonal partitions for QECCs over mixed alphabets. We define an m-QMDS code over mixed alphabets, which is a generalization of quantum MDS codes. For a QECC $((n, K, d+1))_{s_1 s_2 \ldots s_n}$, when the parameters n, $s_1, s_2, \ldots, s_n$ are fixed, $d+1$ indicates the error correction capability of the code, and K implies the encoding efficiency of the code. The m-QMDS codes over mixed alphabets are a type of code whose performance lies between quantum MDS codes and ordinary QECCs over mixed alphabets. Their error correction capability and encoding efficiency are near to those of quantum MDS codes. Therefore, m-QMDS codes over mixed alphabets can partially address the challenge, especially when there are no quantum MDS codes with the same parameters. Therefore, constructing m-QMDS codes over mixed alphabets provides a promising approach to obtaining codes with parameters close to those of quantum MDS codes.

This paper establishes a relation between m-QMDS codes over mixed alphabets and asymmetric orthogonal arrays with orthogonal partitions. This relation is used for the construction of m-QMDS codes with varying local dimension, length, and minimum distance. As applications of this method, numerous infinite families of m-QMDS codes over mixed alphabets can be constructed explicitly, which make it possible to detect at most d errors and correct an arbitrary number of errors affecting up to $\frac{d}{2}$ qudits. These codes can deal with complicated and practical situations where encoded particles have different numbers of dimensions.

The remainder of this paper is organized as follows. Section 2 presents some notations, along with basic concepts of orthogonal arrays, the Hamming distance, difference schemes, orthogonal partitions of asymmetric orthogonal arrays, quantum error-correcting codes over mixed alphabets, the quantum Singleton bound, quantum MDS codes, near quantum MDS codes, and m-QMDS codes. Additionally, it includes several useful lemmas. In Section 3, we employ difference schemes, the expansive replacement method, and the multiplication of orthogonal arrays to construct numerous m-QMDS codes over mixed alphabets. Section 4 provides comparisons with existing QECCs. The conclusion is drawn in Section 5. In Appendix A, we provide some logical codewords of the QECC $((5, 9, 2))_{3^4 2^1}$. Tables are included in Appendix B.

2. Preliminaries

To present our results, we first make some preparations.

$\mathbb{C}^s$ is a Hilbert space. Let $\mathbb{Z}_s^n$ denote the n dimensional space over the ring $\mathbb{Z}_s = \{0, 1, \ldots, s-1\}$. $\mathbb{F}_s$ denotes a Galois field. $\mathbb{Z}_s = \mathbb{F}_s$ when s is a prime power. Let A^T be the transpose of the matrix A and $\mathbf{(s)} = (0, 1, \ldots, s-1)^T$. The symbols $\mathbf{0}_r$ and $\mathbf{1}_r$ represent the $r\times 1$ vector of 0s and 1s, respectively. $\prod$ denotes continued product. $C \subset \{1, 2, \ldots, n\}$ means that C is a subset of $\{1, 2, \ldots, n\}$ and $C \neq \{1, 2, \ldots, n\}$. $|C|$ denotes the number of elements in the set C.

$$\min\{a_1, a_2\} = \begin{cases} a_1, \text{ if } a_1 \leq a_2, \\ a_2, \text{ if } a_1 > a_2. \end{cases}$$

$$\max\{a_1, a_2\} = \begin{cases} a_2, \text{ if } a_1 \leq a_2, \\ a_1, \text{ if } a_1 > a_2. \end{cases}$$

If $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{u\times v}$ have entries from a finite field with binary operations (usually addition $+$ and multiplication $\cdot$), the Kronecker sum $A \oplus B$ is defined as $A \oplus B = (a_{ij} + B)_{mu\times nv}$, where $a_{ij} + B$ represents the $u \times v$ matrix with entries $a_{ij} + b_{rs}$ $(1 \leq r \leq u, 1 \leq s \leq v)$, and the Kronecker product $A \otimes B$ is defined as $A \otimes B = (a_{ij} \cdot B)_{mu\times nv}$.

Definition 1. *[29] An $r \times n$ matrix A, having k_i columns with s_i levels, $i = 1, \ldots, v$, $n = \sum_{i=1}^{v} k_i$, $s_i \neq s_j$, for $i \neq j$, is called an orthogonal array (OA) $OA(r, n, s_1^{k_1} s_2^{k_2} \cdots s_v^{k_v}, t)$ of strength t and size r if each $r \times t$ submatrix of A contains all of the possible row vectors with the same frequency. An orthogonal array with $s_1 = s_2 = \cdots = s_v = s$ is called symmetric; otherwise, it is called asymmetric. If $\sum_{i=1}^{v} k_i(s_i - 1) = r - 1$, the $\mathrm{OA}(r, n, s_1^{k_1} s_2^{k_2} \cdots s_v^{k_v}, 2)$ is saturated.*

As is well-known, an $OA(\prod_{i=1}^{k} s_i, k, s_1^1 s_2^1 \cdots s_k^1, k)$ can also be regarded as an OA of strength t for any $t \geq 1$ whether or not t exceeds k [30].

Definition 2. *[29] Let $S^l = \{(v_1, \ldots, v_l) \mid v_i \in \mathbb{Z}_{s_i}, i = 1, 2, \ldots, l\}$. The Hamming distance HD($\boldsymbol{u}$,$\boldsymbol{v}$) between two vectors $\boldsymbol{u}$= $(u_1, \ldots, u_l)$, $\boldsymbol{v}$= $(v_1, \ldots, v_l) \in S^l$ is defined as the number of positions in which they differ. The minimal distance of a matrix A, denoted by MD(A), is defined to be the minimal Hamming distance between its distinct rows. Let HD(A) denote all possible values of the Hamming distance between two distinct rows of A.*

Let $\mathcal{A}$ be an additive group of s elements. $\mathcal{A}^t$, $t \geq 1$ denotes the additive group of order s^t consisting of all t-tuples of entries from $\mathcal{A}$ with the usual vector addition as the binary operation. Let $\mathcal{A}_0^t = \{(x_1, x_2, \ldots, x_t) : x_1 = \cdots = x_t \in \mathcal{A}\}$. Then, $\mathcal{A}_0^t$ is a subgroup of $\mathcal{A}^t$ of order s, and its cosets will be denoted by $\mathcal{A}_i^t$, $i = 1, \ldots, s^{t-1} - 1$.

Definition 3. *[29] An $r \times c$ matrix D based on $\mathcal{A}$ is called a difference scheme of strength t if for every $r \times t$ submatrix, each set $\mathcal{A}_i^t$, $i = 0, 1, \ldots, s^{t-1} - 1$ is represented equally often when the rows of the submatrix are viewed as elements of $\mathcal{A}^t$. Such a matrix is denoted by $D_t(r, c, s)$. Especially, $D_2(r, c, s)$ is short for $D(r, c, s)$.*

Definition 4. *[31] Let A be an $OA(r, n, s_1^{k_1} s_2^{k_2} \cdots s_v^{k_v}, t)$. Suppose the rows of A can be partitioned into u submatrices $A_1, A_2, \ldots, A_u$ such that each A_i is an $OA(\frac{r}{u}, n, s_1^{k_1} s_2^{k_2} \cdots s_v^{k_v}, t_1)$ with $t_1 \geq 0$. Then, the set $\{A_1, A_2, \ldots, A_u\}$ is called an orthogonal partition of strength t_1 of A.*

Definition 5. *[27] A subspace of $\mathbb{C}^{s_1} \otimes \mathbb{C}^{s_2} \otimes \cdots \otimes \mathbb{C}^{s_n}$, Q is an $((n, K, d + 1))_{s_1, s_2, \ldots, s_n}$ QECC over mixed alphabets $\{s_1, s_2, \ldots, s_n\}$ if for any d parties, the reductions of all states in Q to the d parties are identical.*

The integers n, K, $d + 1$, s_1, s_2, …, s_n may be referred to as the parameters of the QECC. Here, n is the number of qudits or the length of the code, K is the dimension of the encoding state, $d + 1$ is the minimum distance, and $s_1, s_2, \ldots, s_n$ are the alphabet sizes, or s_i indicates the dimensions of the ith particle. s_i is also called the number of levels of the subsystem. A QECC over mixed alphabets is also called a QECC in heterogeneous systems. It is a code where the physical resources for encoding have different numbers of energy levels.

Definition 6. *[26, 27] An $((n, K, d + 1))_{s_1 s_2 \cdots s_n}$ QECC satisfies the quantum Singleton bound $K \leq \min\{\prod_{j \in C} s_j \mid C \subset \{1, 2, \ldots, n\}, |C| = n - 2d\}$ for $n \geq 2d + 1$ and $K \leq 1$ for $n = 2d$. Particularly, an $((n, K, d + 1))_s$ QECC has the quantum Singleton bound $K \leq s^{n-2d}$. A QECC attaining the quantum Singleton bound is called a quantum MDS.*

Definition 7. *[28] An $((n, K, d + 1))_{s_1 s_2 \cdots s_n}$ QECC is called a near quantum MDS code if $K = \min\{\prod_{j \in C} s_j \mid C \subset \{1, 2, \ldots, n\}, |C| = n - 2d\} - 1$ for $n \geq 2d + 1$.*

Definition 8. *Suppose that $s_1 \leq s_2 \leq \cdots \leq s_n$. An $((n, K, d + 1))_{s_1 s_2 \cdots s_n}$ QECC is called an m-QMDS code if $K = s_1 s_2 \cdots s_{n-2d} - m$ with $1 \leq m \leq s_3 \cdots s_{n-2d}(s_1 s_2 - 1)$ for $n \geq 2d + 3$, $1 \leq m \leq s_1 s_2 - 1$ for $n = 2d + 2$, or $1 \leq m \leq s_1 - 1$ for $n = 2d + 1$.*

The definition essentially characterizes codes with a bounded gap between their encoding dimension and the mixed-alphabet Singleton bound. Obviously, an m-QMDS code is a quantum MDS code if $m = 0$, and an m-QMDS code is a near quantum MDS code if $m = 1$ [28]. If a QECC is neither a quantum MDS code nor an m-QMDS code, then it is an ordinary QECC. When $n = 2d + 1$, and $n = 2d + 2$, all QECCs are either quantum MDS codes or m-QMDS codes. When $n \geq 2d + 3$, there exist

ordinary QECCs for example QECC $((7,1,3))_{64^1 8^3 2^3}$. The parameter m carries the following physical meaning: For fixed code length, minimum distances, and alphabet sizes, a smaller m corresponds to a higher encoding efficiency and yields a code closer to the quantum MDS case. The codes with different values of m form a hierarchical structure. For example, consider three codes: an m_1-QMDS code $((n, K_1, d+1))_{s_1 \cdots s_n}$, an m_2-QMDS code $((n, K_2, d+1))_{s_1 \cdots s_n}$ and an m_3-QMDS code $((n, K_3, d+1))_{s_1 \cdots s_n}$. If $m_1 < m_2 < m_3$, then $K_1 > K_2 > K_3$. Therefore, the m_1-QMDS code has the highest encoding efficiency among the three codes.

Lemma 1. *[29] A difference scheme $D(s, s, s)$ and a difference scheme $D(2s, 2s, s)$ exist for any prime power s.*

Lemma 2. *[32] (i) There is a difference matrix $D_3(s^2, 4, s)$ over $\mathbb{Z}_s$ for any $s \geq 2$.*
(ii) Let D be a difference scheme $D_t(r, c, s)$. Then, $D \oplus (\boldsymbol{s})$ is an $OA(rs, c, s, t)$.

As stated in [29], difference schemes are a simple but powerful tool for the construction of orthogonal arrays of strength two. The $D_3(r, c, s)$ also plays an important role in the construction of OAs in [32, 33]. Therefore, in this paper, we naturally apply this difference schemes to the construction of QECCs.

Lemma 3. *[34] If A is an $OA(s^t, n, s, t)$, then $MD(A) = n - t + 1$.*

Lemma 4. *[35] (Expansive replacement method) Suppose A is an orthogonal array of strength t with column 1 having s_1 levels and that B also is an orthogonal array of strength t with s_1 rows. After making a one-to-one mapping between the levels of column 1 in A and the rows of B, if each level of column 1 in A is replaced by the corresponding row from B, we can obtain an orthogonal array of strength t.*

Lemma 5. *[28] Assume that an $OA(r, n, s_1 s_2 \cdots s_n, t)$ exists with $MD = h$ and an orthogonal partition $\{A_1, \ldots, A_K\}$ of strength t'. Let $d + 1 = \min\{t' + 1, h\}$. Then, there exists an $((n, K, d+1))_{s_1 s_2 \cdots s_n}$ QECC.*

Lemma 6. *[36] Consider any two distinct rows of a saturated $OA(r, m_1 + m_2, s_1^{m_1} s_2^{m_2}, 2)$, A. For $i = 1, 2$, assume d_i is the Hamming distance between these two rows of the s_i-level columns. Then, $HD(A) = \{d_1 + d_2 : s_1 d_1 + s_2 d_2 = r, 0 \leq d_i \leq m_i, i = 1, 2\}$. Particularly, if $r = s_1^2$, then*

$$HD(A) = \{\frac{s_1^2 + (s_2 - s_1)(m_1 - 1)}{s_2}, \frac{s_1^2 + (s_2 - s_1)m_1}{s_2}\}.$$

Lemma 7. *[29] (i) If $s \geq 2$ is a prime power, then an $OA(s^t, s + 1, s, t)$ of index unity exists whenever $s \geq t - 1 \geq 0$. (ii) If $s = 2^m$, $m \geq 1$, and $t = 3$, then there exists an $OA(s^3, s + 2, s, t)$.*

Lemma 8. *Assume that A is an $OA(r_1, n, s_1 s_2 \cdots s_n, t)$ with $MD(A) = h_1$, and that B is an $OA(r_2, n, q_1 q_2 \cdots q_n, t)$ with $MD(B) = h_2$. Let $h = \min\{h_1, h_2\}$. Then, there exists an $OA(r_1 r_2, n, (s_1 q_1)(s_2 q_2) \cdots (s_n q_n), t)$ with $MD = h$.*

Proof. The proof is similar to Lemma 7 in [17]. □

Lemma 9. *[29] Taking the runs in an $OA(r, n, s, t)$ that begin with 0 (or any other particular symbol) and omitting the first column yields an $OA(r/s, n - 1, s, t - 1)$. If we assume that these are the initial runs, the process can be represented by the following diagram:*

$\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}$	$OA(r/s, n-1, s, t-1)$
$\begin{matrix} 1 \\ \vdots \\ 1 \end{matrix}$	$OA(r/s, n-1, s, t-1)$
$\vdots$	$\vdots$

3. New m-QMDS code from orthogonal arrays

In this section, we construct numerous infinite classes of m-QMDS codes over mixed alphabets using asymmetric orthogonal arrays. For given $s_1, \ldots, s_n$, n, and d, the coding efficiency of an m-QMDS code $((n, K, d+1))_{s_1 \cdots s_n}$ is close to that of quantum MDS codes. The smaller the value of m, the closer it is. Therefore, we can use an m-QMDS code to replace the quantum MDS code when it is unavailable or hard to construct. In all the literature, quantum MDS codes over mixed alphabets $((2d+1, K, d+1))_{s_1 \cdots s_{2d+1}}$ for $K > 1$ have not been found so far. Therefore, we have constructed infinite classes of m-QMDS codes $((2d+1, 1, d+1))_{s_1 \cdots s_{2d+1}}$ in Theorems 1–3. We also have constructed infinite classes of new m-QMDS codes $((2d+2, 1, d+1))_{s_1 \cdots s_{2d+1}}$ and ordinary QECCs in Theorems 1 and 2. By using Theorems 4 and 5, we have further constructed a large number of m-QMDS codes over mixed alphabets with dimension greater than 1. They are also flexible for length, minimum distance, mixed alphabets, and dimension. In Theorem 4 (ii), we construct numerous infinite classes of m-QMDS codes that include the existing infinite classes of near quantum MDS codes in [28] as special cases. Compared with the existing QECCs in [17, 26–28], the new m-QMDS codes we have constructed not only possess higher flexibility in terms of code length, dimension of the encoding state, and alphabet sizes, but also have better properties.

By first constructing an orthogonal array via a difference scheme and then applying the expansive replacement method for OAs, we have the following result.

Theorem 1. *There exist two* $(s-1)$*-QMDS codes* $((5, 1, 3))_{(s^2)^1 s^4}$ *and* $((6, 1, 3))_{(s^2)^1 s^3 s_1^1 s_2^1}$ *for* $s = s_1 s_2$. *In addition, an ordinary QECC* $((4+n, 1, 3))_{(s^2)^1 s^3 s_1^1 s_2^1 \cdots s_n^1}$ *exists for* $s = s_1 s_2 \cdots s_n$ *and* $n \geq 3$.

Proof. By Lemma 2 (i), there exists a difference scheme $D = D_3(s^2, 4, s)$ for $s \geq 2$. Then, $A = D \oplus \mathbf{(s)}$ is an $OA(s^3, 4, s, 3)$ with $MD(A) = 2$ from Lemmas 2 (ii) and 3. We construct an array $B = OA(s^3, 5, (s^2)^1 s^4, 2)$ as follows:

$$B = ((\mathbf{s^2}) \otimes \mathbf{1}_s, D \oplus \mathbf{(s)}) = \begin{pmatrix} 0 \otimes \mathbf{1}_s & d_1 \oplus \mathbf{(s)} \\ 1 \otimes \mathbf{1}_s & d_2 \oplus \mathbf{(s)} \\ \vdots & \vdots \\ (s^2-1) \otimes \mathbf{1}_s & d_{s^2} \oplus \mathbf{(s)} \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_{s^2} \end{pmatrix}$$

where d_i is the ith row of D for $1 \leq i \leq s^2$. Apparently, $MD(B_i) = 4$, $MD(B_{i_1}, B_{i_2}) = 1 + 2 = 3$ for $1 \leq i_1 < i_2 \leq s^2$ from $MD(A) = 2$. Then, $MD(B) = 3$. An $OA(s^3, 4+n, (s^2)^1 s^3 s_1^1 s_2^1 \cdots s_n^1, 2)$

with $MD = 3$ for $s = s_1 s_2 \cdots s_n$ and $s \geq 4$ can be obtained after an s-level column of B is replaced by an $OA(s, n, s_1^1 s_2^1 \cdots s_n^1, n)$ from Lemma 4. Therefore, a QECC $((4+n, 1, 3))_{(s^2)^1 s^3 s_1^1 s_2^1 \cdots s_n^1}$ exists for $s = s_1 s_2 \cdots s_n$ from Lemma 5. Obviously, there exist two $(s-1)$-QMDS codes $((5, 1, 3))_{(s^2)^1 s^4}$ and $((6, 1, 3))_{(s^2)^1 s^3 s_1^1 s_2^1}$ from Definition 8. □

For example, we can obtain a 2-QMDS code $((5, 1, 3))_{9^1 3^4}$ from Theorem 1, and it has the same length and minimum distance as a quantum MDS code $((5, 3, 3))_{9^1 3^4}$. Therefore, it has the same error-correction capability as the quantum MDS code, and its encoding efficiency is close to that of the quantum MDS code while its dimension decreases by two. Thus, resources can be saved.

Furthermore, by combining the orthogonal arrays obtained from difference schemes with the multiplication and expansive replacement methods, we have the following result.

Theorem 2. *There exist two* $(s-1)$*-QMDS codes* $((5, 1, 3))_{(2s)^1 s^4}$ *and* $((6, 1, 3))_{(2s)^1 s^3 s_1^1 s_2^1}$ *for* $s = s_1 s_2$*. In addition, an ordinary QECC* $((4+n, 1, 3))_{(2s)^1 s^3 s_1^1 s_2^1 \cdots s_n^1}$ *exists for* $s = s_1 s_2 \cdots s_n$ *and* $n \geq 3$*.*

Proof. By Lemma 1, there exists a difference scheme $D(2s, 2s, s)$ for any prime power s. Then, there exists an $A = (\mathbf{(2s)} \otimes \mathbf{1}_s, D(2s, 2s, s) \oplus \mathbf{(s)})$ as a saturated $OA(2s^2, 2s+1, (2s)^1 s^{2s}, 2)$. We have $MD(A) = 2s - 1$ from Lemma 6. For any prime power s, we can obtain an $OA(2s^2, 5, (2s)^1 s^4, 2)$ with $MD = 3$ by deleting $2s - 4$ s-level columns from A. Now we prove that there exists an $OA(2s^2, 5, (2s)^1 s^4, 2)$ with $MD = 3$ for $s = u_1^{l_1} u_2^{l_2} \cdots u_v^{l_v}$, where $u_1, u_2, \ldots, u_v$ are all primes; $v \geq 2$; $2 \leq u_1 < u_2 < \cdots < u_v \leq s$; and $l_1, l_2, \ldots, l_v \geq 1$. We consider the following three cases:

(1) When $u_1 > 2$, there exists an $OA(2(u_1^{l_1})^2, 5, (2u_1^{l_1})^1 (u_1^{l_1})^4, 2)$ with $MD = 3$. There exists an $OA((u_j^{l_j})^2, 5, u_j^{l_j}, 2)$ with $MD \geq 3$ for $2 \leq j \leq v$ from Lemmas 7 and 3. By Lemma 8, there exists an $OA(2s^2, 5, (2s)^1 s^4, 2)$ with $MD = 3$.

(2) When $u_1 = 2$, and $u_2 = 3$, we need to prove that there exists an $OA(2s^2, 5, (2s)^1 s^4, 2)$ with $MD = 3$. We consider the following three cases:

(a) When $l_1 = l_2 = 1$, there exists an $OA(72, 5, 12^1 6^4, 2)$ with $MD = 3$ in [37] and an $OA((u_j^{l_j})^2, 5, u_j^{l_j}, 2)$ with $MD \geq 3$ for $3 \leq j \leq v$ from Lemmas 7 and 3. (b) When $l_1 = 1$, and $l_2 \geq 2$, there exists an $OA(8, 5, 4^1 2^4, 2)$ with $MD = 3$ and an $OA((u_j^{l_j})^2, 5, u_j^{l_j}, 2)$ with $MD \geq 3$ for $2 \leq j \leq v$ from Lemmas 7 and 3. (c) When $l_1 \geq 2$, and $l_2 = 1$, there exists an $OA(2u_2^2, 5, (2u_2)^1 u_2^4, 2) = OA(18, 5, 6^1 3^4, 2)$ with $MD = 3$ and an $OA((u_j^{l_j})^2, 5, u_j^{l_j}, 2)$ with $MD \geq 3$ for $j = 1$ and $3 \leq j \leq v$ from Lemmas 7 and 3. Then, there exists an $OA(2s^2, 5, (2s)^1 s^4, 2)$ with $MD = 3$ from Lemma 8.

(3) When $u_1 = 2$, and $u_2 > 3$, there exists an $OA(2(u_1^{l_1})^2, 5, (2u_1^{l_1})^1 (u_1^{l_1})^4, 2)$ with $MD = 3$ for $l_1 \geq 1$ and an $OA((u_j^{l_j})^2, 5, u_j^{l_j}, 2)$ with $MD \geq 3$ for $2 \leq j \leq v$ from Lemmas 7 and 3. By Lemma 8, there exists an $OA(2s^2, 5, (2s)^1 s^4, 2)$ with $MD = 3$.

Thus, an $OA(2s^2, 4+n, (2s)^1 s^3 s_1^1 s_2^1 \cdots s_n^1, 2)$ with $MD = 3$ for $s = s_1 s_2 \cdots s_n$ can be obtained after an s-level column of $OA(2s^2, 5, (2s)^1 s^4, 2)$ is replaced by an $OA(s, n, s_1^1 s_2^1 \cdots s_n^1, n)$ from Lemma 4. Therefore, a QECC $((4+n, 1, 3))_{(2s)^1 s^3 s_1^1 s_2^1 \cdots s_n^1}$ exists for $s = s_1 s_2 \cdots s_n$ from Lemma 5. When $n = 1, 2$, it is an $(s-1)$-QMDS code from Definition 8. □

Theorem 2 is similar to Theorem 1. The only change is that the maximum dimension of the subsystem changes from s^2 to $2s$. For example, we can obtain a 1-QMDS code $((5, 1, 3))_{4^1 2^4}$ from Theorem 2; it is a near quantum MDS code whose encoding efficiency is closest to that of a quantum MDS code.

Table 2 in Appendix B lists plenty of new m-QMDS codes and ordinary QECCs including many infinite classes of such codes constructed by Theorems 1 and 2. When $n = 1, 2$, all the QECCs are m-QMDS codes. When $n \geq 3$, all the QECCs are ordinary.

Theorems 3 and 4 are based on the existence of an orthogonal array.

Theorem 3. *If there exists an $OA(s^d, 2d, s, d)$, then there exists an $(s_1 - 1)$-QMDS code $((2d + 1, 1, d + 1))_{s^{2d-1}(\frac{s}{s_1})^1 s_1^1}$ for $s_1|s$ and $s \geq s_1^2$.*

Proof. By Lemma 3, the minimal distance of $OA(s^d, 2d, s, d)$ is $d+1$. We can obtain an $A = OA(s^d, 2d+1, s^{2d-1}(\frac{s}{s_1})^1 s_1^1, d)$ with $MD(A) = d + 1$ after an s-level column of $OA(s^d, 2d, s, d)$ is replaced by an $OA(s, 2, (\frac{s}{s_1})^1 s_1^1, d)$ from Lemma 4. By Lemma 5 and Definition 8, there exists an $(s_1 - 1)$-QMDS code $((2d + 1, 1, d + 1))_{s^{2d-1}(\frac{s}{s_1})^1 s_1^1}$. □

Aside from the two m-QMDS codes $((5, 1, 3))_{(s^2)^1 s^4}$ and $((5, 1, 3))_{(2s)^1 s^4}$ in Theorems 1 and 2, respectively, the m-QMDS code $((2d + 1, 1, d + 1))_{s^{2d-1}(\frac{s}{s_1})^1 s_1^1}$ with flexible length and minimum distance can be obtained from Theorem 3. In all the literature, quantum MDS codes $((2d, 1, d + 1))_{s_1 \cdots s_{2d}}$ and $((2d + 1, 2, d + 1))_{s^{2d-1}(\frac{s}{2})^1 2^1}$ have not been found so far. The 1-QMDS code $((2d + 1, 1, d + 1))_{s^{2d-1}(\frac{s}{2})^1 2^1}$ from Theorem 3 has its length increased by only 1 compared with $((2d, 1, d + 1))_{s^{2d-1}(\frac{s}{2})^1}$, $((2d, 1, d + 1))_{s^{2d-1}2^1}$, and $((2d, 1, d + 1))_{s^{2d-2}(\frac{s}{2})^1 2^1}$, and its dimension decreases by only 1 compared with $((2d + 1, 2, d + 1))_{s^{2d-1}(\frac{s}{2})^1 2^1}$. However, its error-correction capability remains unchanged, and its encoding efficiency is closest to that of the quantum MDS codes.

The following corollary follows immediately from Theorem 3.

Corollary 1. *If $s_1|s$, and $s \geq s_1^2$, then*

(i) an $(s_1 - 1)$-QMDS code $((3, 1, 2))_{s^1(\frac{s}{s_1})^1 s_1^1}$ exists for $s \geq 4$.

(ii) an $(s_1 - 1)$-QMDS code $((5, 1, 3))_{s^3(\frac{s}{s_1})^1 s_1^1}$ exists for $s \geq 4$ and $s \neq 6$.

(iii) an $(s_1 - 1)$-QMDS code $((7, 1, 4))_{s^5(\frac{s}{s_1})^1 s_1^1}$ exists with $s = u_1 u_2 \cdots u_v$ for prime powers $u_1, u_2, \ldots, u_v$ and $\min\{u_1, u_2, \ldots, u_v\} \geq 4$.

(iv) an $(s_1 - 1)$-QMDS code $((2d + 1, 1, d + 1))_{s^{2d-1}(\frac{s}{s_1})^1 (s_1)^1}$ exists with $d \geq 4$ and $s = u_1 u_2 \cdots u_v$ for prime powers $u_1, u_2, \ldots, u_v$ and $\min\{u_1, u_2, \ldots, u_v\} \geq 2d - 1$.

Proof. (i) By Lemma 7, there exists an $OA(s, 2, s, 1)$ for $s \geq 4$.

(ii) From Lemma 7, an $OA(s^2, 4, s, 2)$ exists for a prime power s. When $s = 10$, we have an $OA(100, 4, 10, 2)$ in [37]. From [29], we have the following conclusions: (1) An $OA(s^2, n_1, s, 2)$ exists if and only if $n_1 - 2$ pairwise orthogonal Latin squares of order s exist. (2) There exist more than two pairwise orthogonal Latin squares of order $s \geq 12$, which is not a prime power. Then, there exists an $OA(s^2, 4, s, 2)$ for $s \geq 4$ and $s \neq 6$.

(iii) By Lemma 7, there exists an $OA(u_i^3, 6, u_i, 3)$ with a prime power $u_i \geq 4$ for $1 \leq i \leq v$. By Lemma 8, there exists an $OA(s^3, 6, s, 3)$ with $s = u_1 u_2 \cdots u_v$ for prime powers $u_1, u_2, \ldots, u_v$ and $\min\{u_1, u_2, \ldots, u_v\} \geq 4$.

(iv) By Lemma 7, there exists an $OA(u_i^d, 2d, u_i, d)$ with a prime power $u_i \geq 2d - 1$ for $1 \leq i \leq v$. By Lemma 8, there exists an $OA(s^d, 2d, s, d)$ with $s = u_1 u_2 \cdots u_v$ for prime powers $u_1, u_2, \ldots, u_v$ and $\min\{u_1, u_2, \ldots, u_v\} \geq 2d - 1$.

It follows from Theorem 3 that the corollary holds. □

We have listed a large number of 1-QMDS codes and 2-QMDS codes in Tables 3 and 4 of Appendix B, respectively.

The following theorem implies that infinite classes of m-QMDS codes over mixed alphabets with more dimensions exist.

Theorem 4. *Suppose that an* $OA(s^{d+l}, 2d+2l+1, s, d+l)$ *exists for integer* $d \geq 1$ *and* $l \geq 0$. *Let* $2 \leq s_1 \leq s_2 \leq \ldots \leq s_{n_1}$, $2 \leq q_1 \leq q_2 \leq \ldots \leq q_{n_2}$, *where* $(s_1 s_2 \cdots s_{n_1})|s$, $n_1 \geq 1$, *and* $s = q_1 q_2 \cdots q_{n_2}$, $n_2 \geq 2$. *Then, the following statements hold.*

(i) If $l \geq 1$, *there exist two QECCs* $((2d+l+n_1, s^l, d+1))_{s^{2d+l} s_1^1 s_2^1 \cdots s_{n_1}^1}$ *and* $((2d+l-1+n_1+n_2, s^l, d+1))_{s^{2d+l-1} s_1^1 s_2^1 \cdots s_{n_1}^1 q_1^1 q_2^1 \cdots q_{n_2}^1}$. *Particularly, there exist two* $(s_1-1)s^l$*-QMDS codes,* $((2d+l+1, s^l, d+1))_{s^{2d+l} s_1^1}$ *and* $((2d+l+n_2, s^l, d+1))_{s^{2d+l-1} s_1^1 q_2^1 \cdots q_{n_2}^1}$ *for* $s_1 \leq q_1 q_2$, *and two* $(s_1 s_2 - 1)s^l$*-QMDS codes,* $((2d+l+2, s^l, d+1))_{s^{2d+l} s_1^1 s_2^1}$ *and* $((2d+l+1+n_2, s^l, d+1))_{s^{2d+l-1} s_1^1 s_2^1 q_1^1 q_2^1 \cdots q_{n_2}^1}$ *for* $s_1, s_2 \leq q_1$.

(ii) If $l = 0$, *there exist two QECCs* $((2d+n_1, 1, d+1))_{s^{2d} s_1^1 s_2^1 \cdots s_{n_1}^1}$ *and* $((2d-1+n_1+n_2, 1, d+1))_{s^{2d-1} s_1^1 s_2^1 \cdots s_{n_1}^1 q_1^1 q_2^1 \cdots q_{n_2}^1}$. *Particularly, there exists an* (s_1-1)*-QMDS code* $((2d+1, 1, d+1))_{s^{2d} s_1^1}$, *an* $(s_1 s_2 - 1)$*-QMDS code* $((2d+2, 1, d+1))_{s^{2d} s_1^1 s_2^1}$, *and a* $(\frac{q_1 q_2 s_1}{w} - 1)$*-QMDS code* $((2d+2, 1, d+1))_{s^{2d-1} s_1^1 q_1^1 q_2^1}$ *for* $w = \max\{s_1, q_2\}$.

Proof. Let $A = OA(s^{d+l}, 2d+2l+1, s, d+l)$. By Lemma 3, $MD(A) = d+l+2$. By row permutations,

$$A = \begin{pmatrix} \mathbb{Z}_s^l \otimes \mathbf{1}_{s^d} & \overline{A} \end{pmatrix} = \begin{pmatrix} (00\cdots0) \otimes \mathbf{1}_{s^d} & A_1 \\ (00\cdots1) \otimes \mathbf{1}_{s^d} & A_2 \\ \vdots & \vdots \\ (s-1, s-1, \cdots, s-1) \otimes \mathbf{1}_{s^d} & A_{s^l} \end{pmatrix}.$$

By Lemma 9, $A_1, A_2, \ldots, A_{s^l}$ are all $OA(s^d, 2d+l+1, s, d)$, and $MD(\overline{A}) = d+2$. We can obtain a

$$B = OA(s^{d+l}, 2d+l+1, s^{2d+l} s_1^1 s_2^1 \cdots s_{n_1}^1, d+l)$$

with an orthogonal partition $\{B_1, B_2, \ldots, B_{s^l}\}$ of strength d after an s-level column of $\overline{A} = OA(s^{d+l}, 2d+l+1, s, d+l)$ is replaced by an $OA(s, n_1, s_1^1 s_2^1 \cdots s_{n_1}^1, d+l)$ from Lemma 4. Further, $MD(B) = d+1$. When $l \geq 0$, there exists a QECC $((2d+l+n_1, s^l, d+1))_{s^{2d+l} s_1^1 s_2^1 \cdots s_{n_1}^1}$ from Lemma 5. Similarly, We can obtain a

$$C = OA(s^{d+l}, 2d+l-1+n_1+n_2, s^{2d+l-1} s_1^1 s_2^1 \cdots s_{n_1}^1 q_1^1 q_2^1 \cdots q_{n_2}^1, d+l)$$

with $MD(C) = d+1$ after an s-level column of B is replaced by an $OA(s, n_2, q_1^1 q_2^1 \cdots q_{n_2}^1, d+l)$ from Lemma 4, and C has an orthogonal partition $\{C_1, C_2, \ldots, C_{s^l}\}$ of strength d. We can obtain the desired codes by Lemma 5 and Definition 8. □

Example 1. *Take* $s = 12$, $d = 1$, $l = 1$ *in Theorem 4 (i). We can obtain all codes in Table 5 of Appendix B and their orthogonal bases. For example, the code* $((4, 12, 2))_{12^3 2^1}$ *has an orthogonal basis* $\{|\phi_1\rangle, \ldots, |\phi_{12}\rangle\}$, *where* $|\phi_1\rangle = |0,0,0,0\rangle + |1,6,3,1\rangle + |2,8,6,0\rangle + |3,2,1,1\rangle + |4,7,9,0\rangle + |5,1,11,1\rangle + |6,9,2,0\rangle + |7,11,8,1\rangle + |8,4,5,0\rangle + |9,10,4,1\rangle + |10,5,7,1\rangle + |11,3,10,0\rangle$,
$|\phi_2\rangle = |0,2,6,1\rangle + |1,1,1,0\rangle + |2,7,4,1\rangle + |3,9,7,0\rangle + |4,3,2,1\rangle + |5,8,10,0\rangle + |6,4,11,0\rangle + |7,10,3,0\rangle + |8,6,9,1\rangle + |9,5,0,0\rangle + |10,11,5,1\rangle + |11,0,8,1\rangle$,
$|\phi_3\rangle = |0,9,11,0\rangle + |1,3,7,1\rangle + |2,2,2,0\rangle + |3,8,5,1\rangle + |4,10,8,0\rangle + |5,4,3,1\rangle + |6,1,9,1\rangle + |7,5,6,0\rangle +$

$|8,11,4,0\rangle + |9,7,10,1\rangle + |10,0,1,0\rangle + |11,6,0,1\rangle$,
$|\phi_4\rangle = |0,5,4,1\rangle+|1,10,6,0\rangle+|2,4,8,1\rangle+|3,3,3,0\rangle+|4,9,0,1\rangle+|5,11,9,0\rangle+|6,7,1,1\rangle+|7,2,10,1\rangle+$
$|8,0,7,0\rangle + |9,6,5,0\rangle + |10,8,11,1\rangle + |11,1,2,0\rangle$,
$|\phi_5\rangle = |0,6,10,0\rangle+|1,0,5,1\rangle+|2,11,7,0\rangle+|3,5,9,1\rangle+|4,4,4,0\rangle+|5,10,1,1\rangle+|6,2,3,0\rangle+|7,8,2,1\rangle+$
$|8,3,11,1\rangle + |9,1,8,0\rangle + |10,7,0,0\rangle + |11,9,6,1\rangle$,
$|\phi_6\rangle = |0,11,2,1\rangle+|1,7,11,0\rangle+|2,1,0,1\rangle+|3,6,8,0\rangle+|4,0,10,1\rangle+|5,5,5,0\rangle+|6,10,7,1\rangle+|7,3,4,0\rangle+$
$|8,9,3,1\rangle + |9,4,6,1\rangle + |10,2,9,0\rangle + |11,8,1,0\rangle$,
$|\phi_7\rangle = |0,3,8,1\rangle+|1,5,2,0\rangle+|2,10,11,1\rangle+|3,4,10,0\rangle+|4,11,1,0\rangle+|5,9,4,1\rangle+|6,6,6,1\rangle+|7,0,9,0\rangle+$
$|8,2,0,1\rangle + |9,8,7,0\rangle + |10,1,3,1\rangle + |11,7,5,0\rangle$,
$|\phi_8\rangle = |0,10,5,1\rangle+|1,4,9,1\rangle+|2,0,3,0\rangle+|3,11,6,1\rangle+|4,5,11,0\rangle+|5,6,2,0\rangle+|6,8,0,0\rangle+|7,7,7,1\rangle+$
$|8,1,10,0\rangle + |9,3,1,1\rangle + |10,9,8,0\rangle + |11,2,4,1\rangle$,
$|\phi_9\rangle = |0,7,3,0\rangle+|1,11,0,1\rangle+|2,5,10,1\rangle+|3,1,4,0\rangle+|4,6,7,1\rangle+|5,0,6,0\rangle+|6,3,5,1\rangle+|7,9,1,0\rangle+$
$|8,8,8,1\rangle + |9,2,11,0\rangle + |10,4,2,1\rangle + |11,10,9,0\rangle$,
$|\phi_{10}\rangle = |0,1,7,0\rangle+|1,8,4,0\rangle+|2,6,1,1\rangle+|3,0,11,1\rangle+|4,2,5,0\rangle+|5,7,8,1\rangle+|6,11,10,0\rangle+|7,4,0,1\rangle+$
$|8,10,2,0\rangle + |9,9,9,1\rangle + |10,3,6,0\rangle + |11,5,3,1\rangle$,
$|\phi_{11}\rangle = |0,8,9,1\rangle+|1,2,8,0\rangle+|2,9,5,0\rangle+|3,7,2,1\rangle+|4,1,6,1\rangle+|5,3,0,0\rangle+|6,0,4,1\rangle+|7,6,11,0\rangle+$
$|8,5,1,1\rangle + |9,11,3,0\rangle + |10,10,10,1\rangle + |11,4,7,0\rangle$,
and $|\phi_{12}\rangle = |0,4,1,0\rangle + |1,9,10,1\rangle + |2,3,9,0\rangle + |3,10,0,0\rangle + |4,8,3,1\rangle + |5,2,7,1\rangle + |6,5,8,0\rangle +$
$|7,1,5,1\rangle + |8,7,6,0\rangle + |9,0,2,1\rangle + |10,6,4,0\rangle + |11,11,11,1\rangle$.

Table 5 in Appendix B lists plenty of new quantum error-correcting codes:

$$((2d+l+n_1, s^l, d+1))_{s^{2d+l}s_1^1s_2^1\cdots s_{n_1}^1}$$

and

$$((2d+l-1+n_1+n_2, s^l, d+1))_{s^{2d+l-1}s_1^1s_2^1\cdots s_{n_1}^1q_1^1q_2^1\cdots q_{n_2}^1}$$

are obtained by Theorem 4 (i) with $s = 12$, $d = 1$, and $l = 1$. Here, "$*$" indicates that the QECC is an m-QMDS code.

Example 2. *Take $s = 12$ and $d = 2$ in Theorem 4 (ii). We can obtain all codes in Table 6 of Appendix B and their orthogonal bases. For example, the 1-QMDS code $((5,1,3))_{12^42^1}$ has a basis state* $|\phi\rangle = |0,0,0,0,0\rangle + |0,1,6,3,1\rangle + |0,2,8,6,0\rangle + |0,3,2,1,1\rangle + |0,4,7,9,0\rangle + |0,5,1,11,1\rangle +$
$|0,6,9,2,0\rangle + |0,7,11,8,1\rangle + |0,8,4,5,0\rangle + |0,9,10,4,1\rangle + |0,10,5,7,1\rangle + |0,11,3,10,0\rangle +$
$|1,0,2,6,1\rangle + |1,1,1,1,0\rangle + |1,2,7,4,1\rangle + |1,3,9,7,0\rangle + |1,4,3,2,1\rangle + |1,5,8,10,0\rangle +$
$|1,6,4,11,0\rangle + |1,7,10,3,0\rangle + |1,8,6,9,1\rangle + |1,9,5,0,0\rangle + |1,10,11,5,1\rangle + |1,11,0,8,1\rangle +$
$|2,0,9,11,0\rangle + |2,1,3,7,1\rangle + |2,2,2,2,0\rangle + |2,3,8,5,1\rangle + |2,4,10,8,0\rangle + |2,5,4,3,1\rangle +$
$|2,6,1,9,1\rangle + |2,7,5,6,0\rangle + |2,8,11,4,0\rangle + |2,9,7,10,1\rangle + |2,10,0,1,0\rangle + |2,11,6,0,1\rangle +$
$|3,0,5,4,1\rangle + |3,1,10,6,0\rangle + |3,2,4,8,1\rangle + |3,3,3,3,0\rangle + |3,4,9,0,1\rangle + |3,5,11,9,0\rangle +$
$|3,6,7,1,1\rangle + |3,7,2,10,1\rangle + |3,8,0,7,0\rangle + |3,9,6,5,0\rangle + |3,10,8,11,1\rangle + |3,11,1,2,0\rangle +$
$|4,0,6,10,0\rangle + |4,1,0,5,1\rangle + |4,2,11,7,0\rangle + |4,3,5,9,1\rangle + |4,4,4,4,0\rangle + |4,5,10,1,1\rangle +$
$|4,6,2,3,0\rangle + |4,7,8,2,1\rangle + |4,8,3,11,1\rangle + |4,9,1,8,0\rangle + |4,10,7,0,0\rangle + |4,11,9,6,1\rangle +$
$|5,0,11,2,1\rangle + |5,1,7,11,0\rangle + |5,2,1,0,1\rangle + |5,3,6,8,0\rangle + |5,4,0,10,1\rangle + |5,5,5,5,0\rangle +$
$|5,6,10,7,1\rangle + |5,7,3,4,0\rangle + |5,8,9,3,1\rangle + |5,9,4,6,1\rangle + |5,10,2,9,0\rangle + |5,11,8,1,0\rangle +$
$|6,0,3,8,1\rangle + |6,1,5,2,0\rangle + |6,2,10,11,1\rangle + |6,3,4,10,0\rangle + |6,4,11,1,0\rangle + |6,5,9,4,1\rangle +$
$|6,6,6,6,1\rangle + |6,7,0,9,0\rangle + |6,8,2,0,1\rangle + |6,9,8,7,0\rangle + |6,10,1,3,1\rangle + |6,11,7,5,0\rangle +$

$|7,0,10,5,1\rangle + |7,1,4,9,1\rangle + |7,2,0,3,0\rangle + |7,3,11,6,1\rangle + |7,4,5,11,0\rangle + |7,5,6,2,0\rangle + |7,6,8,0,0\rangle + |7,7,7,7,1\rangle + |7,8,1,10,0\rangle + |7,9,3,1,1\rangle + |7,10,9,8,0\rangle + |7,11,2,4,1\rangle + |8,0,7,3,0\rangle + |8,1,11,0,1\rangle + |8,2,5,10,1\rangle + |8,3,1,4,0\rangle + |8,4,6,7,1\rangle + |8,5,0,6,0\rangle + |8,6,3,5,1\rangle + |8,7,9,1,0\rangle + |8,8,8,8,1\rangle + |8,9,2,11,0\rangle + |8,10,4,2,1\rangle + |8,11,10,9,0\rangle + |9,0,1,7,0\rangle + |9,1,8,4,0\rangle + |9,2,6,1,1\rangle + |9,3,0,11,1\rangle + |9,4,2,5,0\rangle + |9,5,7,8,1\rangle + |9,6,11,10,0\rangle + |9,7,4,0,1\rangle + |9,8,10,2,0\rangle + |9,9,9,9,1\rangle + |9,10,3,6,0\rangle + |9,11,5,3,1\rangle + |10,0,8,9,1\rangle + |10,1,2,8,0\rangle + |10,2,9,5,0\rangle + |10,3,7,2,1\rangle + |10,4,1,6,1\rangle + |10,5,3,0,0\rangle + |10,6,0,4,1\rangle + |10,7,6,11,0\rangle + |10,8,5,1,1\rangle + |10,9,11,3,0\rangle + |10,10,10,10,1\rangle + |10,11,4,7,0\rangle + |11,0,4,1,0\rangle + |11,1,9,10,1\rangle + |11,2,3,9,0\rangle + |11,3,10,0,0\rangle + |11,4,8,3,1\rangle + |11,5,2,7,1\rangle + |11,6,5,8,0\rangle + |11,7,1,5,1\rangle + |11,8,7,6,0\rangle + |11,9,0,2,1\rangle + |11,10,6,4,0\rangle + |11,11,11,11,1\rangle$.

In Theorem 4 (ii), let $s = 12$, and $d = 2$. New QECCs $((4+n_1,1,3))_{12^4 s_1^1 s_2^1 \cdots s_{n_1}^1}$ and $((3+n_1+n_2,1,3))_{12^3 s_1^1 s_2^1 \cdots s_{n_1}^1 q_1^1 q_2^1 \cdots q_{n_2}^1}$ are obtained. There exists an m-QMDS code $((4+n_1,1,d+1))_{12^4 s_1^1 s_2^1 \cdots s_{n_1}^1}$ for $n_1 = 1,2$, where $m = s_1 s_2 \cdots s_{n_1} - 1$, and an m-QMDS code $((6,1,d+1))_{12^3 s_1^1 q_1^1 q_2^1}$ with $m = (\frac{q_1 q_2 s_1}{w} - 1)$ for $w = \max\{s_1, q_2\}$. Table 6 in Appendix B lists these codes. Here, "$*$" indicates that the QECC is an m-QMDS code.

The following two corollaries follow immediately from Theorem 4.

Corollary 2. *Suppose that $d \geq 1$ and $2 \leq s_1 \leq s_2 \leq \ldots \leq s_{n_1}$, $2 \leq q_1 \leq q_2 \leq \ldots \leq q_{n_2}$ are all integers and that $s = u_1 u_2 \cdots u_v$ for prime powers $u_1, u_2, \ldots, u_v$ and $\min\{u_1, u_2, \ldots, u_v\} \geq 2d + 2l$. If $(s_1 s_2 \cdots s_{n_1}) | s$ for $n_1 \geq 1$, and $s = q_1 q_2 \cdots q_{n_2}$ for $n_2 \geq 2$, then*

(i) when $l \geq 1$, there exist two QECCs $((2d+l+n_1, s^l, d+1))_{s^{2d+l} s_1^1 s_2^1 \cdots s_{n_1}^1}$ and $((2d+l-1+n_1+n_2, s^l, d+1))_{s^{2d+l-1} s_1^1 s_2^1 \cdots s_{n_1}^1 q_1^1 q_2^1 \cdots q_{n_2}^1}$. Particularly, there exist two $(s_1-1)s^l$-QMDS codes $((2d+l+1, s^l, d+1))_{s^{2d+l} s_1^1}$ and $((2d+l+n_2, s^l, d+1))_{s^{2d+l-1} s_1^1 q_1^1 q_2^1 \cdots q_{n_2}^1}$ for $s_1 \leq q_1 q_2$, and there exist two $(s_1 s_2 - 1)s^l$-QMDS codes $((2d+l+2, s^l, d+1))_{s^{2d+l} s_1^1 s_2^1}$ and $((2d+l+1+n_2, s^l, d+1))_{s^{2d+l-1} s_1^1 s_2^1 q_1^1 q_2^1 \cdots q_{n_2}^1}$ for $s_1, s_2 \leq q_1$.

(ii) when $l = 0$, there exist two QECCs $((2d+n_1, 1, d+1))_{s^{2d} s_1^1 s_2^1 \cdots s_{n_1}^1}$ and $((2d-1+n_1+n_2, 1, d+1))_{s^{2d-1} s_1^1 s_2^1 \cdots s_{n_1}^1 q_1^1 q_2^1 \cdots q_{n_2}^1}$. Particularly, there exists an (s_1-1)-QMDS code $((2d+1, 1, d+1))_{s^{2d} s_1^1}$, an $(s_1 s_2 - 1)$-QMDS code $((2d+2, 1, d+1))_{s^{2d} s_1^1 s_2^1}$, and a $(\frac{q_1 q_2 s_1}{w} - 1)$-QMDS code $((2d+2, 1, d+1))_{s^{2d-1} s_1^1 q_1^1 q_2^1}$ for $w = \max\{s_1, q_2\}$.

Proof. By Lemma 7, $OA(u_1^{d+l}, 2d+2l+1, u_1, d+l), \ldots, OA(u_{v-1}^{d+l}, 2d+2l+1, u_{v-1}, d+l)$ and $OA(u_v^{d+l}, 2d+2l+1, u_v, d+l)$ exist. By Lemma 8, an $OA(s^{d+l}, 2d+2l+1, s, d+l)$ exists. It follows from Theorem 4 that the corollary holds. □

Example 3. *Take $s = 16$, $d = 4$, $l = 1$ in Corollary 2 (i). We can obtain five 16-QMDS codes,* $((10,16,5))_{16^9 2^1}$, $((11,16,5))_{16^8 8^1 2^2}$, $((11,16,5))_{16^8 4^2 2^1}$, $((12,16,5))_{16^8 4^1 2^3}$, $((13,16,5))_{16^8 2^5}$*; eight 48-QMDS codes,* $((10,16,5))_{16^9 4^1}$, $((11,16,5))_{16^8 8^1 4^1 2^1}$, $((11,16,5))_{16^8 4^3}$, $((12,16,5))_{16^8 4^2 2^2}$, $((13,16,5))_{16^8 4^1 2^4}$, $((11,16,5))_{16^9 2^2}$, $((12,16,5))_{16^8 8^1 2^2}$, $((14,16,5))_{16^8 2^6}$*; five 112-QMDS codes,* $((10,16,5))_{16^9 8^1}$, $((11,16,5))_{16^8 8^2 2^1}$, $((11,16,5))_{16^8 8^1 4^2}$, $((11,16,5))_{16^9 4^1 2^1}$, $((12,16,5))_{16^8 4^3 2^1}$*; four 240-QMDS codes,* $((11,16,5))_{16^9 8^1 2^1}$, $((12,16,5))_{16^8 8^1 4^2 2^1}$, $((11,16,5))_{16^9 4^2}$, $((12,16,5))_{16^8 4^4}$*; and seventeen ordinary QECCs,* $((12,16,5))_{16^8 8^1 4^1 2^2}$, $((12,16,5))_{16^9 2^3}$, $((12,16,5))_{16^8 8^2 2^2}$, $((12,16,5))_{16^8 8^1 4^2 2^1}$, $((12,16,5))_{16^9 4^1 2^2}$, $((13,16,5))_{16^8 8^1 2^4}$, $((13,16,5))_{16^8 4^2 2^3}$, $((13,16,5))_{16^8 8^1 4^1 2^3}$, $((13,16,5))_{16^8 4^3 2^2}$, $((13,16,5))_{16^9 2^4}$, $((14,16,5))_{16^8 4^1 2^5}$, $((14,16,5))_{16^8 8^1 2^5}$, $((14,16,5))_{16^8 4^2 2^4}$, $((14,16,5))_{16^8 4^2 2^4}$, $((15,16,5))_{16^8 2^7}$, $((15,16,5))_{16^8 4^1 2^6}$, $((16,16,5))_{16^8 2^8}$.

Example 4. *Take $s = 16$, $d = 4$, $l = 0$ in Corollary 2 (ii). We have a 1-QMDS code,* $((9,1,5))_{16^8 2^1}$*; three 3-QMDS codes,* $((10,1,5))_{16^7 8^1 2^2}$, $((9,1,5))_{16^8 4^1}$, $((10,1,5))_{16^8 2^2}$*; four 7-QMDS codes,* $((10,1,5))_{16^7 4^2 2^1}$, $((10,1,5))_{16^7 8^1 4^1 2^1}$, $((9,1,5))_{16^8 8^1}$, $((10,1,5))_{16^8 4^1 2^1}$*; five 15-QMDS codes,* $((10,1,5))_{16^7 4^3}$, $((10,1,5))_{16^7 8^2 2^1}$, $((10,1,5))_{16^7 8^1 4^2}$, $((10,1,5))_{16^8 8^1 2^1}$, $((10,1,5))_{16^8 4^2}$.

We also have twenty-four ordinary QECCs, $((11,1,5))_{16^7 4^1 2^3}$, $((11,1,5))_{16^7 8^1 2^3}$, $((11,1,5))_{16^8 2^3}$, $((11,1,5))_{16^7 4^2 2^2}$, $((11,1,5))_{16^7 8^1 4^1 2^2}$, $((11,1,5))_{16^8 4^1 2^2}$, $((11,1,5))_{16^7 8^2 2^2}$, $((11,1,5))_{16^7 8^1 4^2 2^1}$, $((11,1,5))_{16^7 4^3 2^1}$, $((11,1,5))_{16^7 4^4}$, $((12,1,5))_{16^7 2^5}$, $((12,1,5))_{16^7 4^1 2^4}$, $((12,1,5))_{16^7 8^1 2^4}$, $((12,1,5))_{16^8 2^4}$, $((12,1,5))_{16^7 8^1 4^1 2^3}$, $((12,1,5))_{16^7 4^2 2^3}$, $((12,1,5))_{16^7 4^3 2^2}$, $((13,1,5))_{16^7 8^1 2^5}$, $((13,1,5))_{16^7 4^1 2^5}$, $((13,1,5))_{16^7 2^6}$, $((13,1,5))_{16^7 4^2 2^4}$, $((14,1,5))_{16^7 4^1 2^6}$, $((14,1,5))_{16^7 2^7}$, $((15,1,5))_{16^7 2^8}$.

Example 5. *Take $s = 56$, $d = 2$, and $l = 1$ in Corollary 2 (i). We can obtain the following forty-five m-QMDS codes and fifty-six ordinary QECCs.*

(1) Seven 56-QMDS codes, $((6,56,3))_{56^5 2^1}$, $((7,56,3))_{56^4 28^1 2^2}$, $((7,56,3))_{56^4 14^1 4^1 2^1}$, $((7,56,3))_{56^4 8^1 7^1 2^1}$, $((8,56,3))_{56^4 14^1 2^3}$, $((8,56,3))_{56^4 7^1 4^1 2^2}$, $((9,56,3))_{56^4 7^1 2^4}$*; twelve 168-QMDS codes,* $((6,56,3))_{56^5 4^1}$, $((7,56,3))_{56^4 28^1 4^1 2^1}$, $((7,56,3))_{56^4 14^1 4^2}$, $((7,56,3))_{56^4 8^1 7^1 4^1}$, $((8,56,3))_{56^4 14^1 4^1 2^2}$, $((8,56,3))_{56^4 7^1 4^2 2^1}$, $((9,56,3))_{56^4 7^1 4^1 2^3}$, $((7,56,3))_{56^5 2^2}$, $((8,56,3))_{56^4 28^1 2^3}$, $((8,56,3))_{56^4 8^1 7^1 2^2}$, $((9,56,3))_{56^4 14^1 2^4}$, $((10,56,3))_{56^4 7^1 2^5}$*; five 336-QMDS codes,* $((6,56,3))_{56^5 7^1}$, $((7,56,3))_{56^4 28^1 7^1 2^1}$, $((7,56,3))_{56^4 14^1 7^1 4^1}$, $((7,56,3))_{56^4 8^1 7^2}$, $((8,56,3))_{56^4 7^2 4^1 2^1}$*; seven 392-QMDS codes,* $((6,56,3))_{56^5 8^1}$, $((7,56,3))_{56^4 28^1 8^1 2^1}$, $((7,56,3))_{56^4 14^1 8^1 4^1}$, $((7,56,3))_{56^4 8^2 7^1}$, $((8,56,3))_{56^4 8^1 7^1 4^1 2^1}$, $((7,56,3))_{56^5 4^1 2^1}$, $((8,56,3))_{56^4 14^1 4^2 2^1}$*; six 728-QMDS codes,* $((6,56,3))_{56^5 14^1}$, $((7,56,3))_{56^4 28^1 14^1 2^1}$, $((7,56,3))_{56^4 14^2 4^1}$, $((7,56,3))_{56^4 14^1 8^1 7^1}$, $((7,56,3))_{56^5 7^1 2^1}$, $((8,56,3))_{56^4 8^1 7^2 2^1}$*; seven 1512-QMDS codes,* $((6,56,3))_{56^5 28^1}$, $((7,56,3))_{56^4 28^2 2^1}$, $((7,56,3))_{56^4 28^1 14^1 4^1}$, $((7,56,3))_{56^4 28^1 8^1 7^1}$, $((7,56,3))_{56^5 14^1 2^1}$, $((7,56,3))_{56^5 7^1 4^1}$, $((8,56,3))_{56^4 8^1 7^2 4^1}$*; three 3080-QMDS codes* $((7,56,3))_{56^5 28^1 2^1}$, $((7,56,3))_{56^5 14^1 4^1}$, $((7,56,3))_{56^5 8^1 7^1}$.

(2) There are fifty-six ordinary QECCs, $((8,56,3))_{56^5 2^3}$, $((8,56,3))_{56^4 8^2 7^2}$, $((8,56,3))_{56^5 7^1 2^2}$, $((8,56,3))_{56^4 14^2 2^2}$, $((8,56,3))_{56^4 14^2 4^2}$, $((8,56,3))_{56^5 14^1 2^2}$, $((8,56,3))_{56^4 28^2 2^2}$, $((8,56,3))_{56^5 7^1 4^1 2^1}$, $((8,56,3))_{56^4 14^1 8^1 2^2}$, $((8,56,3))_{56^4 28^1 4^1 2^2}$, $((8,56,3))_{56^4 28^1 7^1 2^2}$, $((8,56,3))_{56^4 14^2 4^1 2^1}$, $((8,56,3))_{56^4 14^1 7^1 4^2}$, $((8,56,3))_{56^4 28^1 14^1 2^2}$, $((8,56,3))_{56^4 14^1 7^1 4^1 2^1}$, $((8,56,3))_{56^4 14^1 8^1 7^1 2^1}$, $((8,56,3))_{56^4 14^1 8^1 7^1 4^1}$, $((8,56,3))_{56^4 28^1 7^1 4^1 2^1}$, $((8,56,3))_{56^4 28^1 8^1 7^1 2^1}$, $((8,56,3))_{56^4 28^1 14^1 4^1 2^1}$, $((9,56,3))_{56^4 7^2 2^3}$, $((9,56,3))_{56^5 7^1 2^3}$, $((9,56,3))_{56^4 28^1 2^4}$, $((9,56,3))_{56^4 14^2 2^3}$, $((9,56,3))_{56^4 8^1 7^1 2^3}$, $((9,56,3))_{56^4 7^1 4^2 2^2}$, $((9,56,3))_{56^4 7^2 4^1 2^2}$, $((9,56,3))_{56^4 7^2 4^2 2^1}$, $((9,56,3))_{56^4 8^1 7^2 2^2}$, $((9,56,3))_{56^4 14^1 4^1 2^3}$, $((9,56,3))_{56^4 14^1 7^1 2^3}$, $((9,56,3))_{56^4 28^1 7^1 2^3}$, $((9,56,3))_{56^4 14^2 4^1 2^2}$, $((9,56,3))_{56^4 14^2 4^1 2^2}$, $((9,56,3))_{56^4 28^1 14^1 2^3}$, $((9,56,3))_{56^4 8^1 7^2 4^1 2^1}$, $((9,56,3))_{56^4 8^1 7^2 4^1 2^1}$, $((9,56,3))_{56^4 14^1 7^1 4^1 2^2}$, $((9,56,3))_{56^4 14^1 8^1 7^1 2^2}$, $((9,56,3))_{56^4 28^1 7^1 4^1 2^2}$, $((9,56,3))_{56^4 14^1 7^1 4^2 2^1}$, $((10,56,3))_{56^4 7^2 2^4}$, $((10,56,3))_{56^4 14^1 2^5}$, $((10,56,3))_{56^4 14^2 2^4}$, $((10,56,3))_{56^4 7^1 4^1 2^4}$, $((10,56,3))_{56^4 7^2 4^1 2^3}$, $((10,56,3))_{56^4 8^1 7^2 2^3}$, $((10,56,3))_{56^4 7^2 4^2 2^2}$, $((10,56,3))_{56^4 14^1 7^1 2^4}$, $((10,56,3))_{56^4 28^1 7^1 2^4}$, $((10,56,3))_{56^4 14^1 7^1 4^1 2^3}$, $((11,56,3))_{56^4 7^1 2^6}$, $((11,56,3))_{56^4 7^2 2^5}$, $((11,56,3))_{56^4 7^2 4^1 2^4}$, $((11,56,3))_{56^4 14^1 7^1 2^5}$, $((12,56,3))_{56^4 7^2 2^6}$.

In Table 7 of Appendix B, new QECCs $((2d+l+n_1, s^l, d+1))_{s^{2d+l} s_1^1 s_2^1 \cdots s_{n_1}^1}$ are obtained by Corollary 2. There exists an m-QMDS code $((2d+l+n_1, s^l, d+1))_{s^{2d+l} s_1^1 s_2^1 \cdots s_{n_1}^1}$ for $n_1 = 1, 2$, where $m = (s_1 \cdots s_{n_1} - 1)s^l$. Here, "$*$" indicates that the QECC is an m-QMDS code.

Corollary 3. *Let $s \geq 4$, and $s \neq 6, 10$. There exists an $(s_1 - 1)$-QMDS code* $((5,1,3))_{s^4 s_1^1}$ *for $s_1 | s$ and an $(s_1 s_2 - 1)$-QMDS code* $((6,1,3))_{s^4 s_1^1 s_2^1}$ *for $(s_1 s_2) | s$. Further, an ordinary QECC* $((4+n,1,3))_{s^4 s_1^1 s_2^1 \cdots s_n^1}$ *exists for $n \geq 3$.*

Proof. It is similar to Corollary 1 (ii). Because there exists an $OA(s^2, 5, s, 2)$ for $s \geq 4$ and $s \neq 6, 10$, it follows from Theorem 4 that the corollary holds. □

Example 6. *Take $s = 9$ and $s_1 = 3$. Corollary 3 produces a 2-QMDS code $((5, 1, 3))_{9^4 3^1}$ with a basis state* $|\phi\rangle = |00000\rangle + |01231\rangle + |02162\rangle + |03670\rangle + |04811\rangle + |05742\rangle + |06350\rangle + |07581\rangle + |08422\rangle + |10272\rangle + |11110\rangle + |12041\rangle + |13852\rangle + |14780\rangle + |15621\rangle + |16502\rangle + |17430\rangle + |18361\rangle + |20151\rangle + |21082\rangle + |22220\rangle + |23701\rangle + |24632\rangle + |25860\rangle + |26471\rangle + |27312\rangle + |28540\rangle + |30681\rangle + |31822\rangle + |32750\rangle + |33331\rangle + |34562\rangle + |35400\rangle + |36011\rangle + |37242\rangle + |38170\rangle + |40830\rangle + |41761\rangle + |42602\rangle + |43510\rangle + |44441\rangle + |45372\rangle + |46280\rangle + |47121\rangle + |48052\rangle + |50712\rangle + |51640\rangle + |52871\rangle + |53482\rangle + |54320\rangle + |55551\rangle + |56132\rangle + |57060\rangle + |58201\rangle + |60342\rangle + |61570\rangle + |62411\rangle + |63022\rangle + |64250\rangle + |65181\rangle + |66662\rangle + |67800\rangle + |68731\rangle + |70521\rangle + |71452\rangle + |72380\rangle + |73261\rangle + |74102\rangle + |75030\rangle + |76841\rangle + |77772\rangle + |78610\rangle + |80460\rangle + |81301\rangle + |82532\rangle + |83140\rangle + |84071\rangle + |85212\rangle + |86720\rangle + |87651\rangle + |88882\rangle.$

Table 8 in Appendix B lists a lot of new 2-QMDS codes $((2d + 1, 1, d + 1))_{s^{2d}3^1}$ from Corollaries 2 and 3.

By using the expansive replacement method, we can obtain a lot of m-QMDS codes from known quantum MDS codes.

Theorem 5. *Let $s_1 > s_2$. Suppose that a quantum MDS code $((2d + n, s_2^n, d + 1))_{s_1^{2d} s_2^n}$ is constructed by Lemma 5, where $s_1 = q_1 q_2 \cdots q_{n_1}$, and $q_1 \leq q_2 \leq \cdots \leq q_{n_1}$ for $n_1 \geq 2$. Then, when $n_1 = 2$, there exists an $s_2^n(\frac{q_1 q_2}{w} - 1)$-QMDS code $((2d + 1 + n, s_2^n, d + 1))_{s_1^{2d-1} s_2^n q_1^1 q_2^1}$, where $w = \max\{s_2, q_2\}$ and when $n_1 \geq 3$, an ordinary QECC $((2d - 1 + n + n_1, s_2^n, d + 1))_{s_1^{2d-1} s_2^n q_1^1 q_2^1 \cdots q_{n_1}^1}$.*

Proof. By Lemma 5, there exists an $OA(r, 2d + n, s_1^{2d} s_2^n, t)$ with $MD = h$ and an orthogonal partition $\{A_1, A_2, \cdots, A_{s_2^n}\}$ of strength t' and $d + 1 = \min\{t' + 1, h\}$. By Lemma 4, we can obtain an $OA(r, 2d - 1 + n + n_1, s_1^{2d-1} s_2^n q_1^1 q_2^1 \cdots q_{n_1}^1, t)$ with $MD = h$ and an orthogonal partition $\{B_1, B_2, \ldots, B_{s_2^n}\}$ of strength t' after an s_1-level column of $OA(r, 2d + n, s_1^{2d} s_2^n, t)$ is replaced by an $OA(s_1, n_1, q_1^1 q_2^1 \cdots q_{n_1}^1, t)$.

By Lemma 5 and Definition 8, when $n_1 = 2$, there exists an $s_2^n(\frac{s_1}{w} - 1)$-QMDS code $((2d + 1 + n, s_2^n, d + 1))_{s_1^{2d-1} s_2^n q_1^1 q_2^1}$, where $w = \max\{s_2, q_2\}$, and when $n_1 \geq 3$, there exists an ordinary QECC $((2d - 1 + n + n_1, s_2^n, d + 1))_{s_1^{2d-1} s_2^n q_1^1 q_2^1 \cdots q_{n_1}^1}$. □

Example 7. *Let $4 = q_1 \times q_2 = 2 \times 2$. With the quantum MDS code $((7, 8, 3))_{4^4 2^3}$ in [28], Theorem 5 produces an 8-QMDS code $((8, 8, 3))_{4^3 2^5}$ with eight basis states:* $|\phi_1\rangle = |00000000\rangle + |01101011\rangle + |02210101\rangle + |03311110\rangle + |10110110\rangle + |11011101\rangle + |12300011\rangle + |13201000\rangle + |20211011\rangle + |21310000\rangle + |22001110\rangle + |23100101\rangle + |30301101\rangle + |31200110\rangle + |32111000\rangle + |33010011\rangle$, $|\phi_2\rangle = |00000111\rangle + |01101100\rangle + |02210010\rangle + |03311001\rangle + |10110001\rangle + |11011010\rangle + |12300100\rangle + |13201111\rangle + |20211100\rangle + |21310111\rangle + |22001001\rangle + |23100010\rangle + |30301010\rangle + |31200001\rangle + |32111111\rangle + |33010100\rangle$, $|\phi_3\rangle = |00111010\rangle + |01010001\rangle + |02301111\rangle + |03200100\rangle + |10001100\rangle + |11100111\rangle + |12211001\rangle + |13310010\rangle + |20300001\rangle + |21201010\rangle + |22110100\rangle + |23011111\rangle + |30210111\rangle + |31311100\rangle + |32000010\rangle + |33101001\rangle$, $|\phi_4\rangle = |00111101\rangle + |01010110\rangle + |02301000\rangle + |03200011\rangle + |10001011\rangle + |11100000\rangle + |12211110\rangle + |13310101\rangle + |20300110\rangle + |21201101\rangle + |22110011\rangle + |23011000\rangle + |30210000\rangle + |31311011\rangle + |32000101\rangle + |33101110\rangle$, $|\phi_5\rangle = |00201001\rangle + |01300010\rangle + |02011100\rangle + |03110111\rangle + |10311111\rangle + |11210100\rangle + |12101010\rangle + |13000001\rangle + |20010010\rangle + |21111001\rangle + |22200111\rangle + |23301100\rangle + |30100100\rangle + |31001111\rangle + |32310001\rangle + |33211010\rangle$, $|\phi_6\rangle = |00201110\rangle + |01300101\rangle + |02011011\rangle + |03110000\rangle + |10311000\rangle + |11210011\rangle + |12101101\rangle + |13000110\rangle + |20010101\rangle + |21111110\rangle +$

$|22200000\rangle + |23301011\rangle + |30100011\rangle + |31001000\rangle + |32310110\rangle + |33211101\rangle$, $|\phi_7\rangle = |00310011\rangle + |01211000\rangle + |02100110\rangle + |03001101\rangle + |10200101\rangle + |11301110\rangle + |12010000\rangle + |13111011\rangle + |20101000\rangle + |21000011\rangle + |22311101\rangle + |23210110\rangle + |30011110\rangle + |31110101\rangle + |32201011\rangle + |33300000\rangle$ *and* $|\phi_8\rangle = |00310100\rangle + |01211111\rangle + |02100001\rangle + |03001010\rangle + |10200010\rangle + |11301001\rangle + |12010111\rangle + |13111100\rangle + |20101111\rangle + |21000100\rangle + |22311010\rangle + |23210001\rangle + |30011001\rangle + |31110010\rangle + |32201100\rangle + |33300111\rangle$.

For a given length, minimum distance, and mixed alphabets, the larger the dimension, the higher the encoding efficiency. It is expected that we often face a more complicated situation when the encoded states belong to heterogeneous systems which have different local dimensions. However, it is more difficult to construct such an ordinary QECC, particularly an m-QMDS code over mixed alphabets with higher dimension. By using Theorems 1, 2, and 3, we have constructed m-QMDS codes over mixed alphabets with a dimension of 1. By using Theorems 4 and 5, we have further constructed a large number of m-QMDS codes over mixed alphabets with dimension greater than 1. They are also flexible in length, minimum distance, mixed alphabets, and dimension. In Theorem 4 (ii), we construct numerous infinite classes of m-QMDS codes that include the existing infinite classes of near quantum MDS codes in [28] as special cases. However, their properties and performance are closest to those of quantum MDS codes. For example, the 8-QMDS code $((8, 8, 3))_{4^3 2^5}$ in Example 7 has much higher encoding efficiency than an ordinary QECC $((8, 1, 3))_{4^3 2^5}$.

Lemma 5 and Definition 8 imply the following theorem.

Theorem 6. *Assume that $s_1 \leq s_2 \leq \cdots \leq s_n$ and that an $OA(r, n, s_1 s_2 \cdots s_n, t)$ exists with $MD = h$ and an orthogonal partition $\{A_1, \ldots, A_K\}$ of strength t'. Let $d + 1 = \min\{t' + 1, h\}$. Then, there exists an m-QMDS code $((n, K, d + 1))_{s_1 s_2 \cdots s_n}$ if $K = s_1 s_2 \cdots s_{n-2d} - m$ with $1 \leq m \leq s_3 \cdots s_{n-2d}(s_1 s_2 - 1)$ for $n \geq 2d + 3$, $1 \leq m \leq s_1 s_2 - 1$ for $n = 2d + 2$, or $1 \leq m \leq s_1 - 1$ for $n = 2d + 1$.*

Proof. By Lemma 5, there exists an $((n, K, d + 1))_{s_1 s_2 \cdots s_n}$ QECC. There exists an m-QMDS code $((n, K, d + 1))_{s_1 s_2 \cdots s_n}$ from Definition 8. □

Example 8. *By Theorem 6, we can construct a 9-QMDS code $((5, 9, 2))_{3^4 2^1}$ with the following nine logical codewords: $|\varphi_1\rangle = |00000\rangle + |00110\rangle + |11020\rangle + |11221\rangle + |22101\rangle + |22211\rangle$, $|\varphi_2\rangle = |00220\rangle + |01010\rangle + |11100\rangle + |12201\rangle + |20111\rangle + |22021\rangle$, $|\varphi_3\rangle = |01120\rangle + |01200\rangle + |12110\rangle + |12011\rangle + |20001\rangle + |20221\rangle$, $|\varphi_4\rangle = |02020\rangle + |02100\rangle + |10010\rangle + |10211\rangle + |21121\rangle + |21201\rangle$, $|\varphi_5\rangle = |02210\rangle + |10120\rangle + |20100\rangle + |02221\rangle + |11001\rangle + |21011\rangle$, $|\varphi_6\rangle = |10200\rangle + |20020\rangle + |21000\rangle + |01211\rangle + |02111\rangle + |12121\rangle$, $|\varphi_7\rangle = |11210\rangle + |20210\rangle + |22120\rangle + |01021\rangle + |02001\rangle + |10101\rangle$, $|\varphi_8\rangle = |12000\rangle + |21220\rangle + |22010\rangle + |00121\rangle + |00201\rangle + |11111\rangle$, $|\varphi_9\rangle = |12220\rangle + |21110\rangle + |22200\rangle + |00011\rangle + |01101\rangle + |10021\rangle$.*

We summarize the main families of m-QMDS codes constructed by all theorems in Table 1. The table lists the code length, encoding dimension, minimum distance, alphabet sizes, and the corresponding value of m. It is worth noting that the alphabet sizes are not necessarily required to be prime powers, providing greater flexibility in the choice of local dimensions.

Table 1. Summary of the constructed m-QMDS codes $((n, K, d+1))_{s_1 s_2 \cdots s_n}$ by the various theorems.

n	K	$d+1$	Alphabet sizes	m	Theorem
5	1	3	$(s^2)^1 s^4$	$s-1$	Theorem 1
6	1	3	$(s^2)^1 s^3 s_1^1 s_2^1$	$s-1$	Theorem 1
5	1	3	$(2s)^1 s^4$	$s-1$	Theorem 2
6	1	3	$(2s)^1 s^3 s_1^1 s_2^1$	$s-1$	Theorem 2
$2d+1$	1	$d+1$	$s^{2d-1}(\frac{s}{s_1})^1 s_1^1$	s_1-1	Theorem 3
$2d+l+1$	s^l	$d+1$	$s^{2d+l} s_1^1$	$(s_1-1)s^l$	Theorem 4(i)
$2d+l+n_2$	s^l	$d+1$	$s^{2d+l-1} s_1^1 q_1^1 \cdots q_{n_2}^1$	$(s_1-1)s^l$	Theorem 4(i)
$2d+l+2$	s^l	$d+1$	$s^{2d+l} s_1^1 s_2^1$	$(s_1 s_2-1)s^l$	Theorem 4(i)
$2d+l+1+n_2$	s^l	$d+1$	$s^{2d+l-1} s_1^1 s_2^1 q_1^1 \cdots q_{n_2}^1$	$(s_1 s_2-1)s^l$	Theorem 4(i)
$2d+1$	1	$d+1$	$s^{2d} s_1^1$	s_1-1	Theorem 4(ii)
$2d+2$	1	$d+1$	$s^{2d} s_1^1 s_2^1$	$s_1 s_2-1$	Theorem 4(ii)
$2d+2$	1	$d+1$	$s^{2d-1} s_1^1 q_1^1 q_2^1$	$\frac{q_1 q_2 s_1}{w=\max\{s_1,q_2\}}-1$	Theorem 4(ii)
$2d+1+n$	s_2^n	$d+1$	$s_1^{2d-1} s_2^n q_1^1 q_2^1$	$s_2^n(\frac{q_1 q_2}{w=\max\{s_2,q_2\}}-1)$	Theorem 5
n	K	$d+1$	$s_1 s_2 \cdots s_n$	m	Theorem 6

4. Comparisons with existing QECCs

Compared with the existing QECCs over mixed alphabets [17, 26–28], the new m-QMDS codes we have constructed not only possess higher flexibility in terms of code length, dimension of the encoding state, minimum distance, and alphabet sizes, but also have better properties.

The codes constructed in this paper have higher encoding efficiency than the codes in [27]. For a code $((n, K, d))_{s_1 \cdots s_n}$, if n, d, and $s_1, \ldots, s_n$ are fixed, then a larger K indicates higher encoding efficiency. We have $K = 3$ for the code $((5, 3, 2))_{3^4 2^1}$ in [27], whereas $K = 9$ for the code $((5, 9, 2))_{3^4 2^1}$ in Example 8.

The codes constructed in this paper have simpler mathematical expression than the codes in [26]. For example, for the 9-QMDS code $((5, 9, 2))_{3^4 2^1}$ in Example 8, each logical codeword has six computational basis states. In contrast, for the 9-QMDS code $((5, 9, 2))_{3^4 2^1}$ in [26], each logical codeword has 162 computational basis states. Without taking the normalization into account, the coefficients of each codeword in Example 8 are all equal to 1, whereas those of each codeword in [26] contain cube roots of unity. Here, we only give a logical codeword of the code in [26], $|\psi_1\rangle$; the other eight codewords are given in Appendix A.

$|\psi_1\rangle = \frac{1}{27\sqrt{-3\omega}}(|00000\rangle + |00001\rangle + |00010\rangle + \omega|00011\rangle + |00020\rangle + \omega^2|00021\rangle + |00100\rangle + |00101\rangle + \omega|00110\rangle + \omega^2|00111\rangle + \omega^2|00120\rangle + \omega|00121\rangle + |00200\rangle + |00201\rangle + \omega^2|00210\rangle + |00211\rangle + \omega|00220\rangle + |00221\rangle + |01000\rangle + |01001\rangle + |01010\rangle + \omega|01011\rangle + |01020\rangle + \omega^2|01021\rangle + \omega|01100\rangle + \omega|01101\rangle + \omega^2|01110\rangle + |01111\rangle + |01120\rangle + \omega^2|01121\rangle + \omega^2|01200\rangle + \omega^2|01201\rangle + \omega|01210\rangle + \omega^2|01211\rangle + |01220\rangle + \omega^2|01221\rangle + |02000\rangle + |02001\rangle + |02010\rangle + \omega|02011\rangle + |02020\rangle + \omega^2|02021\rangle + \omega^2|02100\rangle + \omega^2|02101\rangle + |02110\rangle + \omega|02111\rangle + \omega|02120\rangle + |02121\rangle + \omega|02200\rangle + \omega|02201\rangle + |02210\rangle + \omega|02211\rangle + \omega^2|02220\rangle + \omega|02221\rangle + |10000\rangle + \omega|10001\rangle + |10010\rangle + \omega^2|10011\rangle + |10020\rangle + |10021\rangle + |10100\rangle + \omega|10101\rangle +$

$\omega|10110\rangle+|10111\rangle+\omega^2|10120\rangle+\omega^2|10121\rangle+|10200\rangle+\omega|10201\rangle+\omega^2|10210\rangle+\omega|10211\rangle+\omega|10220\rangle+$
$\omega|10221\rangle+\omega|11000\rangle+\omega^2|11001\rangle+\omega|11010\rangle+|11011\rangle+\omega|11020\rangle+\omega|11021\rangle+\omega^2|11100\rangle+|11101\rangle+$
$|11110\rangle+\omega^2|11111\rangle+\omega|11120\rangle+\omega|11121\rangle+|11200\rangle+\omega|11201\rangle+\omega^2|11210\rangle+\omega|11211\rangle+\omega|11220\rangle+$
$\omega|11221\rangle+\omega^2|12000\rangle+|12001\rangle+\omega^2|12010\rangle+\omega|12011\rangle+\omega^2|12020\rangle+\omega^2|12021\rangle+\omega|12100\rangle+$
$\omega^2|12101\rangle+\omega^2|12110\rangle+\omega|12111\rangle+|12120\rangle+|12121\rangle+|12200\rangle+\omega|12201\rangle+\omega^2|12210\rangle+\omega|12211\rangle+$
$\omega|12220\rangle+\omega|12221\rangle+|20000\rangle+\omega^2|20001\rangle+|20010\rangle+|20011\rangle+|20020\rangle+\omega|20021\rangle+|20100\rangle+$
$\omega^2|20101\rangle+\omega|20110\rangle+\omega|20111\rangle+\omega^2|20120\rangle+|20121\rangle+|20200\rangle+\omega^2|20201\rangle+\omega^2|20210\rangle+\omega^2|20211\rangle+$
$\omega|20220\rangle+\omega^2|20221\rangle+\omega^2|21000\rangle+\omega|21001\rangle+\omega^2|21010\rangle+\omega^2|21011\rangle+\omega^2|21020\rangle+|21021\rangle+|21100\rangle+$
$\omega^2|21101\rangle+\omega|21110\rangle+\omega|21111\rangle+\omega^2|21120\rangle+|21121\rangle+\omega|21200\rangle+|21201\rangle+|21210\rangle+|21211\rangle+$
$\omega^2|21220\rangle+|21221\rangle+\omega|22000\rangle+|22001\rangle+\omega|22010\rangle+\omega|22011\rangle+\omega|22020\rangle+\omega^2|22021\rangle+|22100\rangle+$
$\omega^2|22101\rangle+\omega|22110\rangle+\omega|22111\rangle+\omega^2|22120\rangle+|22121\rangle+\omega^2|22200\rangle+\omega|22201\rangle+\omega|22210\rangle+\omega|22211\rangle+$
$|22220\rangle+\omega|22221\rangle)$, where $\omega = e^{i\frac{2\pi}{3}}$.

Generally, it is difficult to compare the proposed m-QMDS codes with additive codes or QECCs over a single alphabet. However, in quantum information processing, quantum information may be encoded in quantum resources with different dimensions, and hybrid quantum systems with different local dimensions can be used to store, transmit, and process quantum information. Therefore, we need QECCs over mixed alphabets which are more flexible than that over a single alphabet. For example, compared with the existing codes $((4,4,2))_2$, $((4,2,2))_2$, and $((4,1,2))_2$ in [38], the m-QMDS code $((4,K,2))_{4^2 2^2}$ for $(m,K) = (0,4),(1,3),(2,2),(3,1)$ can be constructed by the proposed methods. Moreover, the resulting codewords have simpler mathematical representations, involving fewer computational basis states with all coefficients equal to 1, up to normalization. Such representations may facilitate the implementation of the codewords and the design of the corresponding quantum encoding circuits.

5. Conclusions

Quantum error-correcting code theory deals with the problem of encoding quantum states into qudits such that a small number of errors can be detected, measured, and efficiently corrected. A QECC over mixed alphabets is a natural choice in practice because different subsystems require different dimensions. Therefore, Wang et al. [26] introduced such a code. However, they only built codes over mixed alphabets of distance 2 and two codes of distance 3. Generally, it is difficult to construct such codes. Thus, the existence and construction of QECCs $((n,K,d+1))s_1 s_2 \cdots s_n$ for $d \geq 2$ are still an open question since it was proposed in [26]. Usually, it is devoted to constructing quantum MDS codes because of their optimality. It is more challenging , however, to do this than to construct ordinary QECCs. Therefore, in the paper, we introduce the definition of an m-QMDS code. Its error correction capability and encoding efficiency are better than those of an ordinary QECC and close to those of a quantum MDS code. We can not only construct ordinary QECCs but also obtain numerous m-QMDS codes over mixed alphabets by using the construction methods of orthogonal arrays such as deleting columns, orthogonal partitions, difference schemes, and expansive replacement method. The codes listed in Tables 2–8 of Appendix B are a part of the codes we can obtain.

Our results constitute a significant advance in the field. We can construct a large number of m-QMDS codes over mixed alphabets with minimum distances of at least 3, including the existing codes

in [28] as special cases. Furthermore, we can construct many infinite classes of m-QMDS codes. The proposed method provides not only existence results but also constructive results, which provide a solid answer to the open problem.

These quantum codes have a lot of advantages. It is more convenient to use the m-QMDS codes constructed in this paper than those in [26] to design arithmetic and logic units based on quantum technologies. This is because each basis state of the codes in this paper is a uniform state which has far fewer terms, and the coefficients of each term are all equal to 1 without taking the normalization into account. However, the mathematical expression for the code in [26] is more complex. In practice, of course, one also wants codes that are easy to encode and decode. The codes constructed in this paper are practically relevant or needed.

It is known that m-MDS codes have properties similar to those of MDS codes. When $m = 1$, they find applications in secret sharing schemes [39–41]. Because m-QMDS codes share similar properties with quantum MDS codes, it is expected that these codes could also be applied to the construction of quantum secret sharing schemes.

These codes and methods also lay a foundation for constructing quantum MDS codes. In the future, we will study the construction of quantum MDS $((n, K, d + 1))s_1 s_2 \cdots s_n$ from asymmetric orthogonal arrays.

Author contributions

Mengqian Chen: Conceptualization, methodology, validation, investigation, resources, writing-original draft, writing-review & editing; Shanqi Pang: Conceptualization, software, validation, resources, writing-review & editing, supervision, project administration, funding acquisition; Rong Yan: Methodology, software, validation, resources, data curation, writing-review & editing, supervision; Yan Zhu: Software, validation, data curation, writing-review & editing, visualization, supervision. All authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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Appendix A: The other eight logical codewords of QECC $((5,9,2))_{3^4 2^1}$

$|\psi_2\rangle = \frac{1}{27\sqrt{-3\omega}}(|00000\rangle + |00001\rangle + \omega^2|00010\rangle + |00011\rangle + \omega|00020\rangle + |00021\rangle + |00100\rangle + |00101\rangle + |00110\rangle + \omega|00111\rangle + |00120\rangle + \omega^2|00121\rangle + |00200\rangle + |00201\rangle + \omega|00210\rangle + \omega^2|00211\rangle + \omega^2|00220\rangle + \omega|00221\rangle + \omega|01000\rangle + \omega|01001\rangle + |01010\rangle + \omega|01011\rangle + \omega^2|01020\rangle + \omega|01021\rangle + \omega^2|01100\rangle + \omega^2|01101\rangle + \omega^2|01110\rangle + |01111\rangle + \omega^2|01120\rangle + \omega|01121\rangle + |01200\rangle + |01201\rangle + \omega|01210\rangle + \omega^2|01211\rangle + \omega^2|01220\rangle + \omega|01221\rangle + \omega^2|02000\rangle + \omega^2|02001\rangle + \omega|02010\rangle + \omega^2|02011\rangle + |02020\rangle + \omega^2|02021\rangle + \omega|02100\rangle + \omega|02101\rangle + \omega|02110\rangle + \omega^2|02111\rangle + \omega|02120\rangle + |02121\rangle + |02200\rangle + |02201\rangle + \omega|02210\rangle + \omega^2|02211\rangle + \omega^2|02220\rangle + \omega|02221\rangle + |10000\rangle + \omega|10001\rangle + \omega^2|10010\rangle + \omega|10011\rangle + \omega|10020\rangle + \omega|10021\rangle + |10100\rangle + \omega|10101\rangle + |10110\rangle + \omega^2|10111\rangle + |10120\rangle + |10121\rangle + |10200\rangle + \omega|10201\rangle + \omega|10210\rangle + |10211\rangle + \omega^2|10220\rangle + \omega^2|10221\rangle + \omega^2|11000\rangle + |11001\rangle + \omega|11010\rangle + |11011\rangle + |11020\rangle + |11021\rangle + |11100\rangle + \omega|11101\rangle + |11110\rangle + \omega^2|11111\rangle + |11120\rangle + |11121\rangle + \omega|11200\rangle + \omega^2|11201\rangle + \omega^2|11210\rangle + \omega|11211\rangle + |11220\rangle + |11221\rangle + \omega|12000\rangle + \omega^2|12001\rangle + |12010\rangle + \omega^2|12011\rangle + \omega^2|12020\rangle + \omega^2|12021\rangle + |12100\rangle + \omega|12101\rangle + |12110\rangle + \omega^2|12111\rangle + |12120\rangle + |12121\rangle + \omega^2|12200\rangle + |12201\rangle + |12210\rangle + \omega^2|12211\rangle + \omega|12220\rangle + \omega|12221\rangle + |20000\rangle + \omega^2|20001\rangle + \omega^2|20010\rangle + \omega^2|20011\rangle + \omega|20020\rangle + \omega^2|20021\rangle + |20100\rangle + \omega^2|20101\rangle + |20110\rangle + |20111\rangle + |20120\rangle + \omega|20121\rangle + |20200\rangle + \omega^2|20201\rangle + \omega|20210\rangle + \omega|20211\rangle + \omega^2|20220\rangle + |20221\rangle + |21000\rangle + \omega^2|21001\rangle + \omega^2|21010\rangle + \omega^2|21011\rangle + \omega|21020\rangle + \omega^2|21021\rangle + \omega|21100\rangle + |21101\rangle + \omega|21110\rangle + \omega|21111\rangle + \omega|21120\rangle + \omega^2|21121\rangle + \omega^2|21200\rangle + \omega|21201\rangle + |21210\rangle + |21211\rangle + \omega|21220\rangle + \omega^2|21221\rangle + |22000\rangle + \omega^2|22001\rangle + \omega^2|22010\rangle + \omega^2|22011\rangle + \omega|22020\rangle + \omega^2|22021\rangle + \omega^2|22100\rangle + \omega|22101\rangle + \omega^2|22110\rangle + \omega^2|22111\rangle + \omega^2|22120\rangle + |22121\rangle + \omega|22200\rangle + |22201\rangle + \omega^2|22210\rangle + \omega^2|22211\rangle + |22220\rangle + \omega|22221\rangle)$,

$|\psi_3\rangle = \frac{1}{27\sqrt{-3\omega}}(|00000\rangle + |00001\rangle + \omega|00010\rangle + \omega^2|00011\rangle + \omega^2|00020\rangle + \omega|00021\rangle + \omega|00100\rangle + \omega|00101\rangle + |00110\rangle + \omega|00111\rangle + \omega^2|00120\rangle + \omega|00121\rangle + \omega^2|00200\rangle + \omega^2|00201\rangle + \omega^2|00210\rangle + |00211\rangle + \omega^2|00220\rangle + \omega|00221\rangle + \omega^2|01000\rangle + \omega^2|01001\rangle + |01010\rangle + \omega|01011\rangle + \omega|01020\rangle + |01021\rangle + \omega|01100\rangle + \omega|01101\rangle + |01110\rangle + \omega|01111\rangle + \omega^2|01120\rangle + \omega|01121\rangle + |01200\rangle + |01201\rangle + |01210\rangle + \omega|01211\rangle +$

$|01220\rangle+\omega^2|01221\rangle+\omega|02000\rangle+\omega|02001\rangle+\omega^2|02010\rangle+|02011\rangle+|02020\rangle+\omega^2|02021\rangle+\omega|02100\rangle+$
$\omega|02101\rangle+|02110\rangle+\omega|02111\rangle+\omega^2|02120\rangle+\omega|02121\rangle+\omega|02200\rangle+\omega|02201\rangle+\omega|02210\rangle+\omega^2|02211\rangle+$
$\omega|02220\rangle+|02221\rangle+|10000\rangle+\omega|10001\rangle+\omega|10010\rangle+|10011\rangle+\omega^2|10020\rangle+\omega^2|10021\rangle+\omega|10100\rangle+$
$\omega^2|10101\rangle+|10110\rangle+\omega^2|10111\rangle+\omega^2|10120\rangle+\omega^2|10121\rangle+\omega^2|10200\rangle+|10201\rangle+\omega^2|10210\rangle+\omega|10211\rangle+$
$\omega^2|10220\rangle+\omega^2|10221\rangle+|11000\rangle+\omega|11001\rangle+\omega|11010\rangle+|11011\rangle+\omega^2|11020\rangle+\omega^2|11021\rangle+\omega^2|11100\rangle+$
$|11101\rangle+\omega|11110\rangle+|11111\rangle+|11120\rangle+|11121\rangle+\omega|11200\rangle+\omega^2|11201\rangle+\omega|11210\rangle+|11211\rangle+$
$\omega|11220\rangle+\omega|11221\rangle+|12000\rangle+\omega|12001\rangle+\omega|12010\rangle+|12011\rangle+\omega^2|12020\rangle+\omega^2|12021\rangle+|12100\rangle+$
$\omega|12101\rangle+\omega^2|12110\rangle+\omega|12111\rangle+\omega|12120\rangle+\omega|12121\rangle+|12200\rangle+\omega|12201\rangle+|12210\rangle+\omega^2|12211\rangle+$
$|12220\rangle+|12221\rangle+|20000\rangle+\omega^2|20001\rangle+\omega|20010\rangle+\omega|20011\rangle+\omega^2|20020\rangle+|20021\rangle+\omega|20100\rangle+$
$|20101\rangle+|20110\rangle+|20111\rangle+\omega^2|20120\rangle+|20121\rangle+\omega^2|20200\rangle+\omega|20201\rangle+\omega^2|20210\rangle+\omega^2|20211\rangle+$
$\omega^2|20220\rangle+|20221\rangle+\omega|21000\rangle+|21001\rangle+\omega^2|21010\rangle+\omega^2|21011\rangle+|21020\rangle+\omega|21021\rangle+|21100\rangle+$
$\omega^2|21101\rangle+\omega^2|21110\rangle+\omega^2|21111\rangle+\omega|21120\rangle+\omega^2|21121\rangle+\omega^2|21200\rangle+\omega|21201\rangle+\omega^2|21210\rangle+$
$\omega^2|21211\rangle+\omega^2|21220\rangle+|21221\rangle+\omega^2|22000\rangle+\omega|22001\rangle+|22010\rangle+|22011\rangle+\omega|22020\rangle+\omega^2|22021\rangle+$
$\omega^2|22100\rangle+\omega|22101\rangle+\omega|22110\rangle+\omega|22111\rangle+|22120\rangle+\omega|22121\rangle+\omega^2|22200\rangle+\omega|22201\rangle+\omega^2|22210\rangle+$
$\omega^2|22211\rangle+\omega^2|22220\rangle+|22221\rangle),$

$|\psi_4\rangle=\frac{1}{27\sqrt{-3\omega}}(|00000\rangle+|00001\rangle+\omega|00010\rangle+\omega^2|00011\rangle+\omega^2|00020\rangle+\omega|00021\rangle+|00100\rangle+|00101\rangle+$
$\omega^2|00110\rangle+|00111\rangle+\omega|00120\rangle+|00121\rangle+|00200\rangle+|00201\rangle+|00210\rangle+\omega|00211\rangle+|00220\rangle+\omega^2|00221\rangle+$
$\omega|01000\rangle+\omega|01001\rangle+\omega^2|01010\rangle+|01011\rangle+|01020\rangle+\omega^2|01021\rangle+\omega^2|01100\rangle+\omega^2|01101\rangle+\omega|01110\rangle+$
$\omega^2|01111\rangle+|01120\rangle+\omega^2|01121\rangle+|01200\rangle+|01201\rangle+|01210\rangle+\omega|01211\rangle+|01220\rangle+\omega^2|01221\rangle+$
$\omega^2|02000\rangle+\omega^2|02001\rangle+|02010\rangle+\omega|02011\rangle+\omega|02020\rangle+|02021\rangle+\omega|02100\rangle+\omega|02101\rangle+|02110\rangle+$
$\omega|02111\rangle+\omega^2|02120\rangle+\omega|02121\rangle+|02200\rangle+|02201\rangle+|02210\rangle+\omega|02211\rangle+|02220\rangle+\omega^2|02221\rangle+$
$\omega|10000\rangle+\omega^2|10001\rangle+\omega^2|10010\rangle+\omega|10011\rangle+|10020\rangle+|10021\rangle+\omega|10100\rangle+\omega^2|10101\rangle+|10110\rangle+$
$\omega^2|10111\rangle+\omega^2|10120\rangle+\omega^2|10121\rangle+\omega|10200\rangle+\omega^2|10201\rangle+\omega|10210\rangle+|10211\rangle+\omega|10220\rangle+\omega|10221\rangle+$
$|11000\rangle+\omega|11001\rangle+\omega|11010\rangle+|11011\rangle+\omega^2|11020\rangle+\omega^2|11021\rangle+\omega|11100\rangle+\omega^2|11101\rangle+|11110\rangle+$
$\omega^2|11111\rangle+\omega^2|11120\rangle+\omega^2|11121\rangle+\omega^2|11200\rangle+|11201\rangle+\omega^2|11210\rangle+\omega|11211\rangle+\omega^2|11220\rangle+$
$\omega^2|11221\rangle+\omega^2|12000\rangle+|12001\rangle+|12010\rangle+\omega^2|12011\rangle+\omega|12020\rangle+\omega|12021\rangle+\omega|12100\rangle+\omega^2|12101\rangle+$
$|12110\rangle+\omega^2|12111\rangle+\omega^2|12120\rangle+\omega^2|12121\rangle+|12200\rangle+\omega|12201\rangle+|12210\rangle+\omega^2|12211\rangle+|12220\rangle+$
$|12221\rangle+\omega^2|20000\rangle+\omega|20001\rangle+|20010\rangle+|20011\rangle+\omega|20020\rangle+\omega^2|20021\rangle+\omega^2|20100\rangle+\omega|20101\rangle+$
$\omega|20110\rangle+\omega|20111\rangle+|20120\rangle+\omega|20121\rangle+\omega^2|20200\rangle+\omega|20201\rangle+\omega^2|20210\rangle+\omega^2|20211\rangle+\omega^2|20220\rangle+$
$|20221\rangle+\omega^2|21000\rangle+\omega|21001\rangle+|21010\rangle+|21011\rangle+\omega|21020\rangle+\omega^2|21021\rangle+|21100\rangle+\omega^2|21101\rangle+$
$\omega^2|21110\rangle+\omega^2|21111\rangle+\omega|21120\rangle+\omega^2|21121\rangle+\omega|21200\rangle+|21201\rangle+\omega|21210\rangle+\omega|21211\rangle+\omega|21220\rangle+$
$\omega^2|21221\rangle+\omega^2|22000\rangle+\omega|22001\rangle+|22010\rangle+|22011\rangle+\omega|22020\rangle+\omega^2|22021\rangle+\omega|22100\rangle+|22101\rangle+$
$|22110\rangle+|22111\rangle+\omega^2|22120\rangle+|22121\rangle+|22200\rangle+\omega^2|22201\rangle+|22210\rangle+|22211\rangle+|22220\rangle+\omega|22221\rangle),$

$|\psi_5\rangle=\frac{1}{27\sqrt{-3\omega}}(|00000\rangle+\omega^2|00001\rangle+\omega^2|00010\rangle+\omega^2|00011\rangle+\omega|00020\rangle+\omega^2|00021\rangle+\omega^2|00100\rangle+$
$\omega|00101\rangle+\omega^2|00110\rangle+\omega^2|00111\rangle+\omega^2|00120\rangle+|00121\rangle+\omega|00200\rangle+|00201\rangle+\omega^2|00210\rangle+\omega^2|00211\rangle+$
$|00220\rangle+\omega|00221\rangle+|01000\rangle+\omega^2|01001\rangle+\omega^2|01010\rangle+\omega^2|01011\rangle+\omega|01020\rangle+\omega^2|01021\rangle+|01100\rangle+$
$\omega^2|01101\rangle+|01110\rangle+|01111\rangle+|01120\rangle+\omega|01121\rangle+|01200\rangle+\omega^2|01201\rangle+\omega|01210\rangle+\omega|01211\rangle+$
$\omega^2|01220\rangle+|01221\rangle+|02000\rangle+\omega^2|02001\rangle+\omega^2|02010\rangle+\omega^2|02011\rangle+\omega|02020\rangle+\omega^2|02021\rangle+\omega|02100\rangle+$
$|02101\rangle+\omega|02110\rangle+\omega|02111\rangle+\omega|02120\rangle+\omega^2|02121\rangle+\omega^2|02200\rangle+\omega|02201\rangle+|02210\rangle+|02211\rangle+$
$\omega|02220\rangle+\omega^2|02221\rangle+\omega|10000\rangle+\omega|10001\rangle+|10010\rangle+\omega|10011\rangle+\omega^2|10020\rangle+\omega|10021\rangle+|10100\rangle+$
$|10101\rangle+|10110\rangle+\omega|10111\rangle+|10120\rangle+\omega^2|10121\rangle+\omega^2|10200\rangle+\omega^2|10201\rangle+|10210\rangle+\omega|10211\rangle+$
$\omega|10220\rangle+|10221\rangle+\omega^2|11000\rangle+\omega^2|11001\rangle+\omega|11010\rangle+\omega^2|11011\rangle+|11020\rangle+\omega^2|11021\rangle+\omega^2|11100\rangle+$
$\omega^2|11101\rangle+\omega^2|11110\rangle+|11111\rangle+\omega^2|11120\rangle+\omega|11121\rangle+\omega^2|11200\rangle+\omega^2|11201\rangle+|11210\rangle+\omega|11211\rangle+$

$\omega|11220\rangle + |11221\rangle + |12000\rangle + |12001\rangle + \omega^2|12010\rangle + |12011\rangle + \omega|12020\rangle + |12021\rangle + \omega|12100\rangle + \omega|12101\rangle + \omega|12110\rangle + \omega^2|12111\rangle + \omega|12120\rangle + |12121\rangle + \omega^2|12200\rangle + \omega^2|12201\rangle + |12210\rangle + \omega|12211\rangle + \omega|12220\rangle + |12221\rangle + \omega^2|20000\rangle + |20001\rangle + \omega|20010\rangle + |20011\rangle + |20020\rangle + |20021\rangle + \omega|20100\rangle + \omega^2|20101\rangle + \omega|20110\rangle + |20111\rangle + \omega|20120\rangle + \omega|20121\rangle + |20200\rangle + \omega|20201\rangle + \omega|20210\rangle + |20211\rangle + \omega^2|20220\rangle + \omega^2|20221\rangle + \omega|21000\rangle + \omega^2|21001\rangle + |21010\rangle + \omega^2|21011\rangle + \omega^2|21020\rangle + \omega^2|21021\rangle + \omega|21100\rangle + \omega^2|21101\rangle + \omega|21110\rangle + |21111\rangle + \omega|21120\rangle + \omega|21121\rangle + \omega|21200\rangle + \omega^2|21201\rangle + \omega^2|21210\rangle + \omega|21211\rangle + |21220\rangle + |21221\rangle + |22000\rangle + \omega|22001\rangle + \omega^2|22010\rangle + \omega|22011\rangle + \omega|22020\rangle + \omega|22021\rangle + \omega|22100\rangle + \omega^2|22101\rangle + \omega|22110\rangle + |22111\rangle + \omega|22120\rangle + \omega|22121\rangle + \omega^2|22200\rangle + |22201\rangle + |22210\rangle + \omega^2|22211\rangle + \omega|22220\rangle + \omega|22221\rangle),$

$|\psi_6\rangle = \frac{1}{27\sqrt{-3\omega}}(|00000\rangle + |00001\rangle + |00010\rangle + \omega|00011\rangle + |00020\rangle + \omega^2|00021\rangle + \omega^2|00100\rangle + \omega^2|00101\rangle + |00110\rangle + \omega|00111\rangle + \omega|00120\rangle + |00121\rangle + \omega|00200\rangle + \omega|00201\rangle + |00210\rangle + \omega|00211\rangle + \omega^2|00220\rangle + \omega|00221\rangle + \omega^2|01000\rangle + \omega^2|01001\rangle + \omega^2|01010\rangle + |01011\rangle + \omega^2|01020\rangle + \omega|01021\rangle + \omega^2|01100\rangle + \omega^2|01101\rangle + |01110\rangle + \omega|01111\rangle + \omega|01120\rangle + |01121\rangle + \omega^2|01200\rangle + \omega^2|01201\rangle + \omega|01210\rangle + \omega^2|01211\rangle + |01220\rangle + \omega^2|01221\rangle + \omega|02000\rangle + \omega|02001\rangle + \omega|02010\rangle + \omega^2|02011\rangle + \omega|02020\rangle + |02021\rangle + \omega^2|02100\rangle + \omega^2|02101\rangle + |02110\rangle + \omega|02111\rangle + \omega|02120\rangle + |02121\rangle + |02200\rangle + |02201\rangle + \omega^2|02210\rangle + |02211\rangle + \omega|02220\rangle + |02221\rangle + \omega|10000\rangle + \omega^2|10001\rangle + \omega|10010\rangle + |10011\rangle + \omega|10020\rangle + \omega|10021\rangle + |10100\rangle + \omega|10101\rangle + \omega|10110\rangle + |10111\rangle + \omega^2|10120\rangle + \omega^2|10121\rangle + \omega^2|10200\rangle + |10201\rangle + \omega|10210\rangle + |10211\rangle + |10220\rangle + |10221\rangle + \omega|11000\rangle + \omega^2|11001\rangle + \omega|11010\rangle + |11011\rangle + \omega|11020\rangle + \omega|11021\rangle + \omega|11100\rangle + \omega^2|11101\rangle + \omega^2|11110\rangle + \omega|11111\rangle + |11120\rangle + |11121\rangle + \omega^|11200\rangle + \omega^2|11201\rangle + |11210\rangle + \omega^2|11211\rangle + \omega^2|11220\rangle + \omega^2|11221\rangle + \omega|12000\rangle + \omega^2|12001\rangle + \omega|12010\rangle + |12011\rangle + \omega|12020\rangle + \omega|12021\rangle + \omega^2|12100\rangle + |12101\rangle + |12110\rangle + \omega^2|12111\rangle + \omega|12120\rangle + \omega|12121\rangle + |12200\rangle + \omega|12201\rangle + \omega^2|12210\rangle + \omega|12211\rangle + \omega|12220\rangle + \omega|12221\rangle + \omega^2|20000\rangle + \omega|20001\rangle + \omega^2|20010\rangle + \omega^2|20011\rangle + \omega^2|20020\rangle + |20021\rangle + \omega|20100\rangle + |20101\rangle + \omega^2|20110\rangle + \omega^2|20111\rangle + |20120\rangle + \omega|20121\rangle + |20200\rangle + \omega^2|20201\rangle + \omega^2|20210\rangle + \omega^2|20211\rangle + \omega|20220\rangle + \omega^2|20221\rangle + |21000\rangle + \omega^2|21001\rangle + |21010\rangle + |21011\rangle + |21020\rangle + \omega|21021\rangle + |21100\rangle + \omega^2|21101\rangle + \omega|21110\rangle + \omega|21111\rangle + \omega^2|21120\rangle + |21121\rangle + |21200\rangle + \omega^2|21201\rangle + \omega^2|21210\rangle + \omega^2|21211\rangle + \omega|21220\rangle + \omega^2|21221\rangle + \omega|22000\rangle + |22001\rangle + \omega|22010\rangle + \omega|22011\rangle + \omega|22020\rangle + \omega^2|22021\rangle + \omega^2|22100\rangle + \omega|22101\rangle + |22110\rangle + |22111\rangle + \omega|22120\rangle + \omega^2|22121\rangle + |22200\rangle + \omega^2|22201\rangle + \omega^2|22210\rangle + \omega^2|22211\rangle + \omega|22220\rangle + \omega^2|22221\rangle),$

$|\psi_7\rangle = \frac{1}{27\sqrt{-3\omega}}(|00000\rangle + |00001\rangle + \omega|00010\rangle + \omega^2|00011\rangle + \omega^2|00020\rangle + \omega|00021\rangle + \omega^2|00100\rangle + \omega^2|00101\rangle + \omega|00110\rangle + \omega^2|00111\rangle + |00120\rangle + \omega^2|00121\rangle + \omega|00200\rangle + \omega|00201\rangle + \omega|00210\rangle + \omega^2|00211\rangle + \omega|00220\rangle + |00221\rangle + |01000\rangle + |01001\rangle + \omega|01010\rangle + \omega^2|01011\rangle + \omega^2|01020\rangle + \omega|01021\rangle + |01100\rangle + |01101\rangle + \omega^2|01110\rangle + |01111\rangle + \omega|01120\rangle + |01121\rangle + |01200\rangle + |01201\rangle + |01210\rangle + \omega|01211\rangle + |01220\rangle + \omega^2|01221\rangle + |02000\rangle + |02001\rangle + \omega|02010\rangle + \omega^2|02011\rangle + \omega^2|02020\rangle + \omega|02021\rangle + \omega|02100\rangle + \omega|02101\rangle + |02110\rangle + \omega|02111\rangle + \omega^2|02120\rangle + \omega|02121\rangle + \omega^2|02200\rangle + \omega^2|02201\rangle + \omega^2|02210\rangle + |02211\rangle + \omega^2|02220\rangle + \omega|02221\rangle + \omega^2|10000\rangle + |10001\rangle + |10010\rangle + \omega^2|10011\rangle + \omega|10020\rangle + \omega|10021\rangle + \omega|10100\rangle + \omega^2|10101\rangle + |10110\rangle + \omega^2|10111\rangle + \omega^2|10120\rangle + \omega^2|10121\rangle + |10200\rangle + \omega|10201\rangle + |10210\rangle + \omega^2|10211\rangle + |10220\rangle + |10221\rangle + |11000\rangle + \omega|11001\rangle + \omega|11010\rangle + |11011\rangle + \omega^2|11020\rangle + \omega^2|11021\rangle + |11100\rangle + \omega|11101\rangle + \omega^2|11110\rangle + \omega|11111\rangle + \omega|11120\rangle + \omega|11121\rangle + |11200\rangle + \omega|11201\rangle + |11210\rangle + \omega^2|11211\rangle + |11220\rangle + |11221\rangle + \omega|12000\rangle + \omega^2|12001\rangle + \omega^2|12010\rangle + \omega|12011\rangle + |12020\rangle + |12021\rangle + \omega^2|12100\rangle + |12101\rangle + \omega|12110\rangle + |12111\rangle + |12120\rangle + |12121\rangle + |12200\rangle + \omega|12201\rangle + |12210\rangle + \omega^2|12211\rangle + |12220\rangle + |12221\rangle + \omega|20000\rangle + |20001\rangle + \omega^2|20010\rangle + \omega^2|20011\rangle + |20020\rangle + \omega|20021\rangle + |20100\rangle + \omega^2|20101\rangle + \omega^2|20110\rangle + \omega^2|20111\rangle + \omega|20120\rangle + \omega^2|20121\rangle + \omega^2|20200\rangle + \omega|20201\rangle + \omega^2|20210\rangle + \omega^2|20211\rangle + \omega^2|20220\rangle + |20221\rangle + |21000\rangle + \omega^2|21001\rangle + \omega|21010\rangle + \omega|21011\rangle + \omega^2|21020\rangle + |21021\rangle +$

$|21100\rangle+\omega^2|21101\rangle+\omega^2|21110\rangle+\omega^2|21111\rangle+\omega|21120\rangle+\omega^2|21121\rangle+|21200\rangle+\omega^2|21201\rangle+|21210\rangle+$
$|21211\rangle+|21220\rangle+\omega|21221\rangle+\omega^2|22000\rangle+\omega|22001\rangle+|22010\rangle+|22011\rangle+\omega|22020\rangle+\omega^2|22021\rangle+$
$|22100\rangle+\omega^2|22101\rangle+\omega^2|22110\rangle+\omega^2|22111\rangle+\omega|22120\rangle+\omega^2|22121\rangle+\omega|22200\rangle+|22201\rangle+\omega|22210\rangle+$
$\omega|22211\rangle+\omega|22220\rangle+\omega^2|22221\rangle)$,

$|\psi_8\rangle=\frac{1}{27\sqrt{-3\omega}}(|00000\rangle+\omega^2|00001\rangle+|00010\rangle+|00011\rangle+|00020\rangle+\omega|00021\rangle+\omega|00100\rangle+|00101\rangle+$
$\omega^2|00110\rangle+\omega^2|00111\rangle+|00120\rangle+\omega|00121\rangle+\omega^2|00200\rangle+\omega|00201\rangle+\omega|00210\rangle+\omega|00211\rangle+|00220\rangle+$
$\omega|00221\rangle+\omega|01000\rangle+|01001\rangle+\omega|01010\rangle+\omega|01011\rangle+\omega|01020\rangle+\omega^2|01021\rangle+|01100\rangle+\omega^2|01101\rangle+$
$\omega|01110\rangle+\omega|01111\rangle+\omega^2|01120\rangle+|01121\rangle+\omega^2|01200\rangle+\omega|01201\rangle+\omega|01210\rangle+\omega|01211\rangle+|01220\rangle+$
$\omega|01221\rangle+\omega^2|02000\rangle+\omega|02001\rangle+\omega^2|02010\rangle+\omega^2|02011\rangle+\omega^2|02020\rangle+|02021\rangle+\omega^2|02100\rangle+\omega|02101\rangle+$
$|02110\rangle+|02111\rangle+\omega|02120\rangle+\omega^2|02121\rangle+\omega^2|02200\rangle+\omega|02201\rangle+\omega|02210\rangle+\omega|02211\rangle+|02220\rangle+$
$\omega|02221\rangle+\omega^2|10000\rangle+\omega^2|10001\rangle+\omega^2|10010\rangle+|10011\rangle+\omega^2|10020\rangle+\omega|10021\rangle+|10100\rangle+|10101\rangle+$
$\omega|10110\rangle+\omega^2|10111\rangle+\omega^2|10120\rangle+\omega|10121\rangle+\omega|10200\rangle+\omega|10201\rangle+|10210\rangle+\omega|10211\rangle+\omega^2|10220\rangle+$
$\omega|10221\rangle+\omega|11000\rangle+\omega|11001\rangle+\omega|11010\rangle+\omega^2|11011\rangle+\omega|11020\rangle+|11021\rangle+|11100\rangle+|11101\rangle+$
$\omega|11110\rangle+\omega^2|11111\rangle+\omega^2|11120\rangle+\omega|11121\rangle+\omega^2|11200\rangle+\omega^2|11201\rangle+\omega|11210\rangle+\omega^2|11211\rangle+|11220\rangle+$
$\omega^2|11221\rangle+|12000\rangle+|12001\rangle+|12010\rangle+\omega|12011\rangle+|12020\rangle+\omega^2|12021\rangle+|12100\rangle+|12101\rangle+\omega|12110\rangle+$
$\omega^2|12111\rangle+\omega^2|12120\rangle+\omega|12121\rangle+|12200\rangle+|12201\rangle+\omega^2|12210\rangle+|12211\rangle+\omega|12220\rangle+|12221\rangle+$
$\omega|20000\rangle+\omega^2|20001\rangle+\omega|20010\rangle+|20011\rangle+\omega|20020\rangle+\omega|20021\rangle+\omega^2|20100\rangle+|20101\rangle+|20110\rangle+$
$\omega^2|20111\rangle+\omega|20120\rangle+\omega|20121\rangle+|20200\rangle+\omega|20201\rangle+\omega^2|20210\rangle+\omega|20211\rangle+\omega|20220\rangle+\omega|20221\rangle+$
$\omega|21000\rangle+\omega^2|21001\rangle+\omega|21010\rangle+|21011\rangle+\omega|21020\rangle+\omega|21021\rangle+|21100\rangle+\omega|21101\rangle+\omega|21110\rangle+$
$|21111\rangle+\omega^2|21120\rangle+\omega^2|21121\rangle+\omega^2|21200\rangle+|21201\rangle+\omega|21210\rangle+|21211\rangle+|21220\rangle+|21221\rangle+$
$\omega|22000\rangle+\omega^2|22001\rangle+\omega|22010\rangle+|22011\rangle+\omega|22020\rangle+\omega|22021\rangle+\omega|22100\rangle+\omega^2|22101\rangle+\omega^2|22110\rangle+$
$\omega|22111\rangle+|22120\rangle+|22121\rangle+\omega|22200\rangle+\omega^2|22201\rangle+|22210\rangle+\omega^2|22211\rangle+\omega^2|22220\rangle+\omega^2|22221\rangle)$,

$|\psi_9\rangle=\frac{1}{27\sqrt{-3\omega}}(|00000\rangle+|00001\rangle+\omega^2|00010\rangle+|00011\rangle+\omega|00020\rangle+|00021\rangle+\omega|00100\rangle+\omega|00101\rangle+$
$\omega|00110\rangle+\omega^2|00111\rangle+\omega|00120\rangle+|00121\rangle+\omega^2|00200\rangle+\omega^2|00201\rangle+|00210\rangle+\omega|00211\rangle+\omega|00220\rangle+$
$|00221\rangle+\omega^2|01000\rangle+\omega^2|01001\rangle+\omega|01010\rangle+\omega^2|01011\rangle+|01020\rangle+\omega^2|01021\rangle+\omega|01100\rangle+\omega|01101\rangle+$
$\omega|01110\rangle+\omega^2|01111\rangle+\omega|01120\rangle+|01121\rangle+|01200\rangle+|01201\rangle+\omega|01210\rangle+\omega^2|01211\rangle+\omega^2|01220\rangle+$
$\omega|01221\rangle+\omega|02000\rangle+\omega|02001\rangle+|02010\rangle+\omega|02011\rangle+\omega^2|02020\rangle+\omega|02021\rangle+\omega|02100\rangle+\omega|02101\rangle+$
$\omega|02110\rangle+\omega^2|02111\rangle+\omega|02120\rangle+|02121\rangle+\omega|02200\rangle+\omega|02201\rangle+\omega^2|02210\rangle+|02211\rangle+|02220\rangle+$
$\omega^2|02221\rangle+\omega^2|10000\rangle+|10001\rangle+\omega|10010\rangle+|10011\rangle+|10020\rangle+|10021\rangle+|10100\rangle+\omega|10101\rangle+|10110\rangle+$
$\omega^2|10111\rangle+|10120\rangle+|10121\rangle+\omega|10200\rangle+\omega^2|10201\rangle+\omega^2|10210\rangle+\omega|10211\rangle+|10220\rangle+|10221\rangle+$
$\omega^2|11000\rangle+|11001\rangle+\omega|11010\rangle+|11011\rangle+|11020\rangle+|11021\rangle+\omega|11100\rangle+\omega^2|11101\rangle+\omega|11110\rangle+$
$|11111\rangle+\omega|11120\rangle+\omega|11121\rangle+|11200\rangle+\omega|11201\rangle+\omega|11210\rangle+|11211\rangle+\omega^2|11220\rangle+\omega^2|11221\rangle+$
$\omega^2|12000\rangle+|12001\rangle+\omega|12010\rangle+|12011\rangle+|12020\rangle+|12021\rangle+\omega^2|12100\rangle+|12101\rangle+\omega^2|12110\rangle+$
$\omega|12111\rangle+\omega^2|12120\rangle+\omega^2|12121\rangle+\omega^2|12200\rangle+|12201\rangle+|12210\rangle+\omega^2|12211\rangle+\omega|12220\rangle+\omega|12221\rangle+$
$\omega|20000\rangle+|20001\rangle+|20010\rangle+|20011\rangle+\omega^2|20020\rangle+|20021\rangle+\omega^2|20100\rangle+\omega|20101\rangle+\omega^2|20110\rangle+$
$\omega^2|20111\rangle+\omega^2|20120\rangle+|20121\rangle+|20200\rangle+\omega^2|20201\rangle+\omega|20210\rangle+\omega|20211\rangle+\omega^2|20220\rangle+|20221\rangle+$
$\omega^2|21000\rangle+\omega|21001\rangle+\omega|21010\rangle+\omega|21011\rangle+|21020\rangle+\omega|21021\rangle+\omega|21100\rangle+|21101\rangle+\omega|21110\rangle+$
$\omega|21111\rangle+\omega|21120\rangle+\omega^2|21121\rangle+|21200\rangle+\omega^2|21201\rangle+\omega|21210\rangle+\omega|21211\rangle+\omega^2|21220\rangle+|21221\rangle+$
$|22000\rangle+\omega^2|22001\rangle+\omega^2|22010\rangle+\omega^2|22011\rangle+\omega|22020\rangle+\omega^2|22021\rangle+|22100\rangle+\omega^2|22101\rangle+|22110\rangle+$
$|22111\rangle+|22120\rangle+\omega|22121\rangle+|22200\rangle+\omega^2|22201\rangle+\omega|22210\rangle+\omega|22211\rangle+\omega^2|22220\rangle+|22221\rangle)$.

Appendix B: Tables

Table 2. New m-QMDS codes and ordinary QECCs. When $n = 1, 2$, all the QECCs are m-QMDS. When $n \geq 3$, all the QECCs are ordinary. Here, "$*$" indicates that the QECC is an m-QMDS code.

s	$s_1 s_2 \cdots s_n$	$((4+n,1,3))_{(s^2)^1 s^3 s_1^1 \cdots s_n^1}$ by Th 1		$((4+n,1,3))_{(2s)^1 s^3 s_1^1 \cdots s_n^1}$ by Th 2		m
2	2	$((5,1,3))_{4^1 2^4}$	$*$	$((5,1,3))_{4^1 2^4}$	$*$	1
3	3	$((5,1,3))_{9^1 3^4}$	$*$	$((5,1,3))_{6^1 3^4}$	$*$	2
4	4	$((5,1,3))_{16^1 4^4}$	$*$	$((5,1,3))_{8^1 4^4}$	$*$	3
4	2×2	$((6,1,3))_{16^1 4^3 2^2}$	$*$	$((6,1,3))_{8^1 4^3 2^2}$	$*$	3
5	5	$((5,1,3))_{25^1 5^4}$	$*$	$((5,1,3))_{10^1 5^4}$	$*$	4
6	6	$((5,1,3))_{36^1 6^4}$	$*$	$((5,1,3))_{12^1 6^4}$	$*$	5
6	2×3	$((6,1,3))_{36^1 6^3 3^1 2^1}$	$*$	$((6,1,3))_{12^1 6^3 3^1 2^1}$	$*$	5
7	7	$((5,1,3))_{49^1 7^4}$	$*$	$((5,1,3))_{14^1 7^4}$	$*$	6
8	8	$((5,1,3))_{64^1 8^4}$	$*$	$((5,1,3))_{16^1 8^4}$	$*$	7
8	2×4	$((6,1,3))_{64^1 8^3 4^1 2^1}$	$*$	$((6,1,3))_{16^1 8^3 4^1 2^1}$	$*$	7
8	$2 \times 2 \times 2$	$((7,1,3))_{64^1 8^3 2^3}$		$((7,1,3))_{16^1 8^3 2^3}$		
9	9	$((5,1,3))_{81^1 9^4}$	$*$	$((5,1,3))_{18^1 9^4}$	$*$	8
9	3×3	$((6,1,3))_{81^1 9^3 3^2}$	$*$	$((6,1,3))_{18^1 9^3 3^2}$	$*$	8
10	10	$((5,1,3))_{100^1 10^4}$	$*$	$((5,1,3))_{20^1 10^4}$	$*$	9
10	2×5	$((6,1,3))_{100^1 10^3 5^1 2^1}$	$*$	$((6,1,3))_{20^1 10^3 5^1 2^1}$	$*$	9
11	11	$((5,1,3))_{121^1 11^4}$	$*$	$((5,1,3))_{22^1 11^4}$	$*$	10
12	12	$((5,1,3))_{144^1 12^4}$	$*$	$((5,1,3))_{24^1 12^4}$	$*$	11
12	2×6	$((6,1,3))_{144^1 12^3 6^1 2^1}$	$*$	$((6,1,3))_{24^1 12^3 6^1 2^1}$	$*$	11
12	3×4	$((6,1,3))_{144^1 12^3 4^1 3^1}$	$*$	$((6,1,3))_{24^1 12^3 4^1 3^1}$	$*$	11
$s \geq 13$	s	$((5,1,3))_{(s^2)^1 s^4}$	$*$	$((5,1,3))_{(2s)^1 s^4}$	$*$	$s-1$
$s \geq 13$	$s_1 s_2$	$((6,1,3))_{(s^2)^1 s^3 s_1 s_2}$	$*$	$((6,1,3))_{(2s)^1 s^3 s_1 s_2}$	$*$	$s-1$
$s \geq 13$	$s_1 s_2 \cdots s_n$	$((4+n,1,3))_{(s^2)^1 s^3 s_1^1 s_2^1 \cdots s_n^1}$ for $n \geq 3$		$((4+n,1,3))_{(2s)^1 s^3 s_1^1 s_2^1 \cdots s_n^1}$ for $n \geq 3$		

Table 3. New 1-QMDS codes $((2d+1,1,d+1))_{s^{2d-1}(\frac{s}{2})^1 2^1}$ from Corollary 1.

d	s	$((2d+1,1,d+1))_{s^{2d-1}(\frac{s}{2})^1 2^1}$
1	4	$((3,1,2))_{4^1 2^2}$
1	6	$((3,1,2))_{6^1 3^1 2^1}$
1	8	$((3,1,2))_{8^1 4^1 2^1}$
1	10	$((3,1,2))_{10^1 5^1 2^1}$
1	12	$((3,1,2))_{12^1 6^1 2^1}$
1	14	$((3,1,2))_{14^1 7^1 2^1}$
1	$s = 2k (k \geq 8)$	$((3,1,2))_{s^1(\frac{s}{2})^1 2^1}$
2	4	$((5,1,3))_{4^3 2^2}$
2	8	$((5,1,3))_{8^3 4^1 2^1}$
2	10	$((5,1,3))_{10^3 5^1 2^1}$

Continued on next page

Table 3 continued

d	s	$((2d+1,1,d+1))_{s^{2d-1}(\frac{s}{2})^1 2^1}$
2	12	$((5,1,3))_{12^3 6^1 2^1}$
2	16	$((5,1,3))_{16^3 8^1 2^1}$
2	18	$((5,1,3))_{18^3 9^1 2^1}$
2	$s=2k(k\geq 10)$	$((5,1,3))_{s^3(\frac{s}{2})^1 2^1}$
3	8	$((7,1,4))_{8^5 4^1 2^1}$
3	16	$((7,1,4))_{16^5 8^1 2^1}$
3	20	$((7,1,4))_{20^5 10^1 2^1}$
3	28	$((7,1,4))_{28^5 14^1 2^1}$
3	32	$((7,1,4))_{32^5 16^1 2^1}$
3	36	$((7,1,4))_{36^5 18^1 2^1}$
3	40	$((7,1,4))_{40^5 20^1 2^1}$
3	44	$((7,1,4))_{44^5 22^1 2^1}$
3	52	$((7,1,4))_{52^5 26^1 2^1}$
3	56	$((7,1,4))_{56^5 28^1 2^1}$
3	$s=4\times 2^p\times u_1\times u_2\times\cdots\times u_v$ ($p\geq 0$, u_i is a prime power and $u_i\geq 5$)	$((7,1,4))_{s^5(\frac{s}{2})^1 2^1}$
4	8	$((9,1,5))_{8^7 4^1 2^1}$
4	16	$((9,1,5))_{16^7 8^1 2^1}$
4	32	$((9,1,5))_{32^7 16^1 2^1}$
4	56	$((9,1,5))_{56^7 28^1 2^1}$
4	64	$((9,1,5))_{64^7 32^1 2^1}$
4	72	$((9,1,5))_{72^7 36^1 2^1}$
4	88	$((9,1,5))_{88^7 44^1 2^1}$
4	$s=8\times 2^p\times u_1\times u_2\times\cdots\times u_v$ ($p\geq 0$, u_i is a prime power and $u_i\geq 7$)	$((9,1,5))_{s^7(\frac{s}{2})^1 2^1}$
5	16	$((11,1,6))_{16^9 8^1 2^1}$
5	32	$((11,1,6))_{32^9 16^1 2^1}$
5	64	$((11,1,6))_{64^9 32^1 2^1}$
5	128	$((11,1,6))_{128^9 64^1 2^1}$
5	144	$((11,1,6))_{144^9 72^1 2^1}$
5	$s=16\times 2^p\times u_1\times u_2\times\cdots\times u_v$ ($p\geq 0$, u_i is a prime power and $u_i\geq 9$)	$((11,1,6))_{s^7(\frac{s}{2})^1 2^1}$
...	...	...

Table 4. New 2-QMDS codes $((2d+1,1,d+1))_{s^{2d-1}(\frac{s}{3})^1 3^1}$ from Corollary 1.

d	s	$((2d+1,1,d+1))_{s^{2d-1}(\frac{s}{3})^1 3^1}$
1	9	$((3,1,2))_{9^1 3^2}$
1	12	$((3,1,2))_{12^1 4^1 3^1}$
1	15	$((3,1,2))_{15^1 5^1 3^1}$

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Table 4 continued

d	s	$((2d+1,1,d+1))_{s^{2d-1}(\frac{s}{3})^1 3^1}$
1	18	$((3,1,2))_{18^1 6^1 3^1}$
1	21	$((3,1,2))_{21^1 7^1 3^1}$
1	$s = 3k(k \geq 8)$	$((3,1,2))_{s^1(\frac{s}{3})^1 3^1}$
2	9	$((5,1,3))_{9^3 3^2}$
2	12	$((5,1,3))_{12^3 4^1 3^1}$
2	15	$((5,1,3))_{15^3 5^1 3^1}$
2	18	$((5,1,3))_{18^3 6^1 3^1}$
2	21	$((5,1,3))_{21^3 7^1 3^1}$
2	$s = 3k(k \geq 8)$	$((5,1,3))_{s^3(\frac{s}{3})^1 3^1}$
3	9	$((7,1,4))_{8^5 4^1 2^1}$
3	27	$((7,1,4))_{27^5 9^1 3^1}$
3	36	$((7,1,4))_{36^5 12^1 3^1}$
3	45	$((7,1,4))_{45^5 15^1 3^1}$
3	63	$((7,1,4))_{63^5 21^1 3^1}$
3	$s = 9 \times 3^p \times u_1 \times u_2 \times \cdots \times u_v$ ($p \geq 0$, u_i is a prime power and $u_i \geq 4$)	$((7,1,4))_{s^5(\frac{s}{3})^1 3^1}$
4	9	$((9,1,5))_{9^7 3^2}$
4	27	$((9,1,5))_{27^7 9^1 3^1}$
4	63	$((9,1,5))_{63^7 21^1 3^1}$
4	72	$((9,1,5))_{72^7 24^1 3^1}$
4	81	$((9,1,5))_{81^7 27^1 3^1}$
4	99	$((9,1,5))_{99^7 33^1 3^1}$
4	117	$((9,1,5))_{117^7 39^1 3^1}$
4	$s = 9 \times 3^p \times u_1 \times u_2 \times \cdots \times u_v$ ($p \geq 0$, u_i is a prime power and $u_i \geq 7$)	$((9,1,5))_{s^7(\frac{s}{3})^1 3^1}$
5	9	$((11,1,6))_{9^9 3^2}$
5	27	$((11,1,6))_{27^9 9^1 3^1}$
5	81	$((11,1,6))_{81^9 27^1 3^1}$
5	99	$((11,1,6))_{99^9 33^1 3^1}$
5	117	$((11,1,6))_{117^9 39^1 3^1}$
5	144	$((11,1,6))_{144^9 48^1 3^1}$
5	153	$((11,1,6))_{153^9 51^1 3^1}$
5	$s = 9 \times 3^p \times u_1 \times u_2 \times \cdots \times u_v$ ($p \geq 0$, u_i is a prime power and $u_i \geq 9$)	$((11,1,6))_{s^7(\frac{s}{3})^1 3^1}$
...	...	...

Table 5. New QECCs. Here, "$*$" indicates that the QECC is an m-QMDS code.

m	$s_1 \cdots s_{n_1}$	$((3+n_1, 12, 2))_{12^3 s_1^1 \cdots s_{n_1}^1}$		$q_1 \cdots q_{n_2}$	$((2+n_1+n_2, 12, 2))_{12^2 s_1^1 \cdots s_{n_1}^1 q_1^1 \cdots q_{n_2}^1}$	
				6×2	$((5, 12, 2))_{12^2 6^1 2^2}$	$*$
12	2	$((4, 12, 2))_{12^3 2^1}$	$*$	4×3	$((5, 12, 2))_{12^2 4^1 3^1 2^1}$	$*$
				$3 \times 2 \times 2$	$((6, 12, 2))_{12^2 3^1 2^3}$	$*$
				6×2	$((5, 12, 2))_{12^2 6^1 3^1 2^1}$	$*$
24	3	$((4, 12, 2))_{12^3 3^1}$	$*$	4×3	$((5, 12, 2))_{12^2 4^1 3^2}$	$*$
				$3 \times 2 \times 2$	$((6, 12, 2))_{12^2 3^2 2^2}$	$*$
				6×2	$((5, 12, 2))_{12^2 6^1 4^1 2^1}$	$*$
36	4	$((4, 12, 2))_{12^3 4^1}$	$*$	4×3	$((5, 12, 2))_{12^2 4^2 3^1}$	$*$
				$3 \times 2 \times 2$	$((6, 12, 2))_{12^2 4^1 3^1 2^2}$	$*$
				6×2	$((5, 12, 2))_{12^2 6^2 2^1}$	$*$
60	6	$((4, 12, 2))_{12^3 6^1}$	$*$	4×3	$((5, 12, 2))_{12^2 6^1 4^1 3^1}$	$*$
				$3 \times 2 \times 2$	$((6, 12, 2))_{12^2 6^1 3^1 2^2}$	
				6×2	$((6, 12, 2))_{12^2 6^2 2^2}$	
132	6×2	$((5, 12, 2))_{12^3 6^1 2^1}$	$*$	4×3	$((6, 12, 2))_{12^2 6^1 4^1 3^1 2^1}$	
				$3 \times 2 \times 2$	$((7, 12, 2))_{12^2 6^1 3^1 2^3}$	
				6×2	$((6, 12, 2))_{12^2 6^1 4^1 3^1 2^1}$	
132	4×3	$((5, 12, 2))_{12^3 4^1 3^1}$	$*$	4×3	$((6, 12, 2))_{12^2 4^2 3^2}$	
				$3 \times 2 \times 2$	$((7, 12, 2))_{12^2 4^1 3^2 2^2}$	
				6×2	$((6, 12, 2))_{12^2 6^1 3^1 2^2}$	
60	3×2	$((5, 12, 2))_{12^3 3^1 2^1}$	$*$	4×3	$((6, 12, 2))_{12^2 4^1 3^2 2^1}$	$*$
				$3 \times 2 \times 2$	$((7, 12, 2))_{12^2 3^2 2^3}$	
				6×2	$((6, 12, 2))_{12^2 6^1 2^3}$	$*$
36	2×2	$((5, 12, 2))_{12^3 2^2}$	$*$	4×3	$((6, 12, 2))_{12^2 4^1 3^1 2^2}$	$*$
				$3 \times 2 \times 2$	$((7, 12, 2))_{12^2 3^1 2^4}$	$*$
				6×2	$((7, 12, 2))_{12^2 6^1 3^1 2^3}$	
132	$3 \times 2 \times 2$	$((6, 12, 2))_{12^3 3^1 2^2}$		4×3	$((7, 12, 2))_{12^2 4^1 3^2 2^2}$	
				$3 \times 2 \times 2$	$((8, 12, 2))_{12^2 3^2 2^4}$	

Table 6. QECCs we obtain by Theorem 4 (ii) for $s = 12$ and $d = 2$. Here, "$*$" indicates that the QECC is an m-QMDS code.

$s_1 \cdots s_{n_1}$	m	$((4+n_1, 1, 3))_{12^4 s_1^1 \cdots s_{n_1}^1}$		$q_1 \cdots q_{n_2}$	m	$((3+n_1+n_2, 1, 3))_{12^3 s_1^1 \cdots s_{n_1}^1 q_1^1 \cdots q_{n_2}^1}$	
				6×2	3	$((6, 1, 3))_{12^3 6^1 2^2}$	$*$
2	1	$((5, 1, 3))_{12^4 2^1}$	$*$	4×3	5	$((6, 1, 3))_{12^3 4^1 3^1 2^1}$	$*$
				$3 \times 2 \times 2$		$((7, 1, 3))_{12^3 3^1 2^3}$	
				6×2	5	$((6, 1, 3))_{12^3 6^1 3^1 2^1}$	$*$
3	2	$((5, 1, 3))_{12^4 3^1}$	$*$	4×3	8	$((6, 1, 3))_{12^3 4^1 3^2}$	$*$
				$3 \times 2 \times 2$		$((7, 1, 3))_{12^3 3^2 2^2}$	

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Table 6 continued

$s_1 \cdots s_{n_1}$	m	$((4+n_1,1,3))_{12^4 s_1^1 \cdots s_{n_1}^1}$		$q_1 \cdots q_{n_2}$	m	$((3+n_1+n_2,1,3))_{12^3 s_1^1 \cdots s_{n_1}^1 q_1^1 \cdots q_{n_2}^1}$	
				6×2	7	$((6,1,3))_{12^3 6^1 4^1 2^1}$	*
4	3	$((5,1,3))_{12^4 4^1}$	*	4×3	11	$((6,1,3))_{12^3 4^2 3^1}$	*
				$3 \times 2 \times 2$		$((7,1,3))_{12^3 4^1 3^1 2^2}$	
				6×2	11	$((6,1,3))_{12^3 6^2 2^1}$	*
6	5	$((5,1,3))_{12^4 6^1}$	*	4×3	11	$((6,1,3))_{12^3 6^1 4^1 3^1}$	*
				$3 \times 2 \times 2$		$((7,1,3))_{12^3 6^1 3^1 2^2}$	
				6×2		$((7,1,3))_{12^3 6^2 2^2}$	
6×2	11	$((6,1,3))_{12^4 6^1 2^1}$	*	4×3		$((7,1,3))_{12^3 6^1 4^1 3^1 2^1}$	
				$3 \times 2 \times 2$		$((8,1,3))_{12^3 6^1 3^1 2^3}$	
				6×2		$((7,1,3))_{12^3 6^1 4^1 3^1 2^1}$	
4×3	11	$((6,1,3))_{12^4 4^1 3^1}$	*	4×3		$((7,1,3))_{12^3 4^2 3^2}$	
				$3 \times 2 \times 2$		$((8,1,3))_{12^3 4^1 3^2 2^2}$	
				6×2		$((7,1,3))_{12^3 6^1 3^1 2^2}$	
3×2	5	$((6,1,3))_{12^4 3^1 2^1}$	*	4×3		$((7,1,3))_{12^3 4^1 3^2 2^1}$	
				$3 \times 2 \times 2$		$((8,1,3))_{12^3 3^2 2^3}$	
				6×2		$((7,1,3))_{12^3 6^1 2^3}$	
2×2	3	$((6,1,3))_{12^4 2^2}$	*	4×3		$((7,1,3))_{12^3 4^1 3^1 2^2}$	
				$3 \times 2 \times 2$		$((8,1,3))_{12^3 3^1 2^4}$	
				6×2		$((8,1,3))_{12^3 6^1 3^1 2^3}$	
$3 \times 2 \times 2$	11	$((7,1,3))_{12^4 3^1 2^2}$	*	4×3		$((8,1,3))_{12^3 4^1 3^2 2^2}$	
				$3 \times 2 \times 2$		$((9,1,3))_{12^3 3^2 2^4}$	

Table 7. New QECCs over mixed alphabets. Here, "$*$" indicates that the QECC is an m-QMDS code.

l	d	s	$s_1 s_2 \cdots s_{n_1}$	$((2d+l+n_1, s^l, d+1))_{s^{2d+l} s_1^1 \cdots s_{n_1}^1}$		m
0	1	4	2×2	$((4,1,2))_{4^2 2^2}$	*	3
0	1	6	3	$((3,1,2))_{6^2 3^1}$	*	2
0	1	6	2×3	$((4,1,2))_{6^2 2^1 3^1}$	*	5
0	1	8	4	$((3,1,2))_{8^2 4^1}$	*	3
0	1	8	4×2	$((4,1,2))_{8^2 4^1 2^1}$	*	7
0	1	8	2×2	$((4,1,2))_{8^2 2^2}$	*	3
0	1	8	$2 \times 2 \times 2$	$((5,1,2))_{8^2 2^3}$		
0	1	$s \geq 9$	s_1	$((3,1,2))_{s^2 s_1^1}$	*	$s_1 - 1$
0	1	$s \geq 9$	$s_1 s_2$	$((4,1,2))_{s^2 s_1^1 s_2^1}$	*	$s_1 s_2 - 1$
0	1	$s \geq 9$	$s_1 s_2 \cdots s_{n_1}$ for $n_1 \geq 3$	$((2+n_1,1,2))_{s^2 s_1^1 s_2^1 \cdots s_{n_1}^1}$		
0	2	4	2×2	$((6,1,3))_{4^4 2^2}$	*	3
0	2	8	4	$((5,1,3))_{8^4 4^1}$	*	3

Continued on next page

Table 7 continued

l	d	s	$s_1 s_2 \cdots s_{n_1}$	$((2d+l+n_1, s^l, d+1))_{s^{2d+l} s_1^1 \cdots s_{n_1}^1}$		m
0	2	8	4×2	$((6,1,3))_{8^4 4^1 2^1}$	*	7
0	2	8	2×2	$((6,1,3))_{8^4 2^2}$	*	3
0	2	8	$2 \times 2 \times 2$	$((7,1,3))_{8^4 2^3}$		
0	2	$s \geq 9$	s_1	$((5,1,3))_{s^4 s_1^1}$	*	$s_1 - 1$
0	2	$s \geq 9$	$s_1 s_2$	$((6,1,3))_{s^4 s_1^1 s_2^1}$	*	$s_1 s_2 - 1$
0	2	$s \geq 9$	$s_1 s_2 \cdots s_{n_1}$ for $n_1 \geq 3$	$((4+n_1,1,3))_{s^4 s_1^1 s_2^1 \cdots s_{n_1}^1}$		
0	3	8	4	$((7,1,4))_{8^6 4^1}$	*	3
0	3	8	4×2	$((8,1,4))_{8^6 4^1 2^1}$	*	7
0	3	8	2×2	$((8,1,4))_{8^6 2^2}$	*	3
0	3	8	$2 \times 2 \times 2$	$((9,1,4))_{8^6 2^3}$		
0	3	9	3	$((7,1,4))_{9^6 3^1}$	*	2
0	3	9	3×3	$((8,1,4))_{9^6 3^2}$	*	8
0	3	$s \geq 16$	s_1	$((7,1,4))_{s^6 s_1^1}$	*	$s_1 - 1$
0	3	$s \geq 16$	$s_1 s_2$	$((8,1,4))_{s^6 s_1^1 s_2^1}$	*	$s_1 s_2 - 1$
0	3	$s \geq 16$	$s_1 s_2 \cdots s_{n_1}$ for $n_1 \geq 3$	$((6+n_1,1,4))_{s^6 s_1^1 s_2^1 \cdots s_{n_1}^1}$		
0	4	8	4	$((9,1,5))_{8^8 4^1}$	*	3
0	4	8	4×2	$((10,1,5))_{8^8 4^1 2^1}$	*	7
0	4	8	2×2	$((10,1,5))_{8^8 2^2}$	*	3
0	4	8	$2 \times 2 \times 2$	$((11,1,5))_{8^8 2^3}$		
0	4	9	3	$((9,1,5))_{9^8 3^1}$	*	2
0	4	9	3×3	$((10,1,5))_{9^8 3^2}$	*	8
0	4	$s \geq 16$	s_1	$((9,1,5))_{s^8 s_1^1}$	*	$s_1 - 1$
0	4	$s \geq 16$	$s_1 s_2$	$((10,1,5))_{s^8 s_1^1 s_2^1}$	*	$s_1 s_2 - 1$
0	4	$s \geq 16$	$s_1 s_2 \cdots s_{n_1}$ for $n_1 \geq 3$	$((8+n_1,1,5))_{s^8 s_1^1 s_2^1 \cdots s_{n_1}^1}$		
0	$d \geq 5$	$s \geq 16$	s_1	$((2d+1,1,d+1))_{s^{2d} s_1^1}$	*	$s_1 - 1$
0	$d \geq 5$	$s \geq 16$	$s_1 s_2$	$((2d+2,1,d+1))_{s^{2d} s_1^1 s_2^1}$	*	$s_1 s_2 - 1$
0	$d \geq 5$	$s \geq 16$	$s_1 s_2 \cdots s_{n_1}$ for $n_1 \geq 3$	$((2d+n_1,1,d+1))_{s^{2d} s_1^1 s_2^1 \cdots s_{n_1}^1}$		
1	1	4	2	$((4,4,2))_{4^3 2^1}$	*	4
1	1	4	2×2	$((5,4,2))_{4^3 2^2}$	*	12
1	1	8	2	$((4,8,2))_{8^3 2^1}$	*	8
1	1	8	4	$((4,8,2))_{8^3 4^1}$	*	24
1	1	8	4×2	$((5,8,2))_{8^3 4^1 2^1}$	*	56
1	1	8	2×2	$((5,8,2))_{8^3 2^2}$	*	24
1	1	8	$2 \times 2 \times 2$	$((6,8,2))_{8^3 2^3}$		
1	1	$s \geq 9$	s_1	$((4,s,2))_{s^3 s_1^1}$	*	$(s_1 - 1)s$

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Table 7 continued

l	d	s	$s_1 s_2 \cdots s_{n_1}$	$((2d+l+n_1, s^l, d+1))_{s^{2d+l} s_1^1 \cdots s_{n_1}^1}$		m
1	1	$s \geq 9$	$s_1 s_2$	$((5, s, 2))_{s^3 s_1^1 s_2^1}$	*	$(s_1 s_2 - 1)s$
1	1	$s \geq 9$	$s_1 s_2 \cdots s_{n_1}$ for $n_1 \geq 3$	$((3+n_1, s, 2))_{s^3 s_1^1 s_2^1 \cdots s_{n_1}^1}$		
1	2	4	2	$((6,4,3))_{4^4 2^1}$	*	4
1	2	4	2×2	$((7,4,3))_{4^4 2^2}$	*	12
1	2	8	2	$((6,8,3))_{8^4 2^1}$	*	4
1	2	8	4	$((6,8,3))_{8^4 4^1}$	*	12
1	2	8	4×2	$((7,8,3))_{8^4 4^1 2^1}$	*	56
1	2	8	2×2	$((7,8,3))_{8^4 2^2}$	*	24
1	2	8	$2 \times 2 \times 2$	$((8,8,3))_{8^4 2^3}$		
1	2	$s \geq 9$	s_1	$((6, s, 3))_{s^5 s_1^1}$	*	$(s_1 - 1)s$
1	2	$s \geq 9$	$s_1 s_2$	$((7, s, 3))_{s^5 s_1^1 s_2^1}$	*	$(s_1 s_2 - 1)s$
1	2	$s \geq 9$	$s_1 s_2 \cdots s_{n_1}$ for $n_1 \geq 3$	$((5+n_1, s, 3))_{s^5 s_1^1 s_2^1 \cdots s_{n_1}^1}$		
1	3	8	2	$((8,8,4))_{8^7 2^1}$	*	8
1	3	8	4	$((8,8,4))_{8^7 4^1}$	*	24
1	3	8	4×2	$((9,8,4))_{8^7 4^1 2^1}$	*	56
1	3	8	2×2	$((9,8,4))_{8^7 2^2}$	*	24
1	3	8	$2 \times 2 \times 2$	$((10,8,4))_{8^7 2^3}$		
1	3	9	3	$((8,9,4))_{9^7 3^1}$	*	18
1	3	9	3×3	$((9,9,4))_{9^7 3^2}$	*	72
1	3	$s \geq 16$	s_1	$((8, s, 4))_{s^7 s_1^1}$	*	$(s_1 - 1)s$
1	3	$s \geq 16$	$s_1 s_2$	$((9, s, 4))_{s^7 s_1^1 s_2^1}$	*	$(s_1 s_2 - 1)s$
1	3	$s \geq 16$	$s_1 s_2 \cdots s_{n_1}$ for $n_1 \geq 3$	$((7+n_1, s, 4))_{s^7 s_1^1 s_2^1 \cdots s_{n_1}^1}$		
1	4	16	2	$((10,16,5))_{16^9 2^1}$	*	16
1	4	16	4	$((10,16,5))_{16^9 4^1}$	*	48
1	4	16	8	$((10,16,5))_{16^9 8^1}$	*	112
1	4	16	8×2	$((11,16,5))_{16^9 8^1 2^1}$	*	240
1	4	16	4×4	$((11,16,5))_{16^9 4^2}$	*	240
1	4	16	4×2	$((11,16,5))_{16^9 4^1 2^1}$	*	112
1	4	16	2×2	$((11,16,5))_{16^9 2^2}$	*	48
1	4	16	$4 \times 2 \times 2$	$((12,16,5))_{16^9 4^1 2^2}$		
1	4	16	$2 \times 2 \times 2$	$((12,16,5))_{16^9 2^3}$		
1	4	16	$2 \times 2 \times 2 \times 2$	$((13,16,5))_{16^9 2^4}$		
1	4	$s \geq 27$	s_1	$((10, s, 5))_{s^9 s_1^1}$	*	$(s_1 - 1)s$
1	4	$s \geq 27$	$s_1 s_2$	$((11, s, 5))_{s^9 s_1^1 s_2^1}$	*	$(s_1 s_2 - 1)s$
1	4	$s \geq 27$	$s_1 s_2 \cdots s_{n_1}$ for $n_1 \geq 3$	$((9+n_1, s, 5))_{s^9 s_1^1 s_2^1 \cdots s_{n_1}^1}$		

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Table 7 continued

l	d	s	$s_1 s_2 \cdots s_{n_1}$	$((2d+l+n_1, s^l, d+1))_{s^{2d+l} s_1^1 \cdots s_{n_1}^1}$		m
1	$d \geq 5$	$s \geq 16$	s_1	$((2d+2, s, d+1))_{s^{2d+1} s_1^1}$	*	$(s_1 - 1)s$
1	$d \geq 5$	$s \geq 16$	$s_1 s_2$	$((2d+3, s, d+1))_{s^{2d+1} s_1^1 s_2^1}$	*	$(s_1 s_2 - 1)s$
1	$d \geq 5$	$s \geq 16$	$s_1 s_2 \cdots s_{n_1}$ for $n_1 \geq 3$	$((2d+1+n_1, s, d+1))_{s^{2d+1} s_1^1 s_2^1 \cdots s_{n_1}^1}$		
$l \geq 2$	$d \geq 1$	$s \geq 8$	s_1	$((2d+l+1, s^l, d+1))_{s^{2d+l} s_1^1}$	*	$(s_1 - 1)s^l$
$l \geq 2$	$d \geq 1$	$s \geq 8$	$s_1 s_2$	$((2d+l+2, s^l, d+1))_{s^{2d+l} s_1^1 s_2^1}$	*	$(s_1 s_2 - 1)s^l$
$l \geq 2$	$d \geq 1$	$s \geq 8$	$s_1 s_2 \cdots s_{n_1}$ for $n_1 \geq 3$	$((2d+l+n_1, s^l, d+1))_{s^{2d+l} s_1^1 s_2^1 \cdots s_{n_1}^1}$		

Table 8. New 2-QMDS codes.

d	s	$((2d+1, 1, d+1))_{s^{2d}3^1}$
1	6	$((3,1,2))_{6^2 3^1}$
1	9	$((3,1,2))_{9^2 3^1}$
1	12	$((3,1,2))_{12^2 3^1}$
1	15	$((3,1,2))_{15^2 3^1}$
1	18	$((3,1,2))_{18^2 3^1}$
1	21	$((3,1,2))_{21^2 3^1}$
1	$s = 3k (k \geq 8)$	$((3,1,2))_{s^2 3^1}$
2	9	$((5,1,3))_{9^4 3^1}$
2	12	$((5,1,3))_{12^4 3^1}$
2	15	$((5,1,3))_{15^4 3^1}$
2	18	$((5,1,3))_{18^4 3^1}$
2	21	$((5,1,3))_{21^4 3^1}$
2	$s = 3k (k \geq 8)$	$((5,1,3))_{s^4 3^1}$
3	9	$((7,1,4))_{9^6 3^1}$
3	27	$((7,1,4))_{27^6 3^1}$
3	45	$((7,1,4))_{45^6 3^1}$
3	63	$((7,1,4))_{63^6 3^1}$
3	72	$((7,1,4))_{72^6 3^1}$
3	81	$((7,1,4))_{81^6 3^1}$
3	$s = 9 \times 3^p \times u_1 \times u_2 \times \cdots \times u_v$ ($p \geq 0$, u_i is a prime power and $u_i \geq 5$)	$((7,1,4))_{s^6 3^1}$
4	9	$((9,1,5))_{9^8 3^1}$
4	27	$((9,1,5))_{27^8 3^1}$
4	63	$((9,1,5))_{63^8 3^1}$
4	72	$((9,1,5))_{72^8 3^1}$
4	81	$((9,1,5))_{81^8 3^1}$

Continued on next page

Table 8 continued

d	s	$((2d+1,1,d+1))_{s^{2d}3^1}$
4	$s = 9 \times 3^p \times u_1 \times u_2 \times \cdots \times u_v$ ($p \geq 0$, u_i is a prime power and $u_i \geq 8$)	$((9,1,5))_{s^8 3^1}$
5	27	$((11,1,6))_{27^{10}3^1}$
5	81	$((11,1,6))_{81^{10}3^1}$
5	243	$((11,1,6))_{243^{10}3^1}$
5	297	$((11,1,6))_{297^{10}3^1}$
5	$s = 27 \times 3^p \times u_1 \times u_2 \times \cdots \times u_v$ ($p \geq 0$, u_i is a prime power and $u_i \geq 11$)	$((11,1,6))_{s^{10}3^1}$
...	...	...

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