



*Theory article***Stability of a Poiseuille-type velocity-stress subsystem for an electrohydrodynamic polymeric fluid****Xuelin Guan***

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Abstract: We study the long-time behaviour of a Poiseuille-type flow of a weakly conducting incompressible polymeric fluid governed by the Pokrovskii-Vinogradov model. Under the Poiseuille ansatz, the electrohydrodynamic system separates into velocity-stress and charge-potential subsystems. In the constant-relaxation regime with zero initial transverse normal stress, the velocity-shear subsystem reduces to a damped wave equation. If the pressure gradient has a finite limit and the residual forcing vanishes, the unique energy solution converges in $H_0^1(0, 1) \times L_2(0, 1)$ to the corresponding stationary state. For the homogeneous deviation problem, a modified Lyapunov functional yields exponential stability, while modal analysis determines the supremal decay rate. The same representation provides a fading-memory estimate and a standard L_∞ input-to-state stability bound. Under additional regularity, the shear and streamwise normal stresses converge in $L_2(0, 1)$, and the transverse normal stress remains zero. Staggered-grid computations support the predicted decay rates. The results concern streamwise-independent perturbations and constant coefficients.

Keywords: Pokrovskii-Vinogradov model; Poiseuille-type flow; damped wave equation; exponential stability; input-to-state stability

Mathematics Subject Classification: 35B35, 35B40, 35Q35, 76A10, 35L20

Abbreviations:

EHD: electrohydrodynamic
ISS: input-to-state stability
PDE: partial differential equation

1. Introduction

Polymeric melts and solutions exhibit relaxation, normal-stress effects, and flow-induced anisotropy that cannot be captured by Newtonian constitutive laws. A mesoscopic description of these phenomena is provided by the Pokrovskii-Vinogradov model [1–3], in which the rheological state is represented by a symmetric anisotropy tensor associated with the configuration of polymer macromolecules. For weakly conducting fluids, this model can be coupled to equations for the space-charge density and electric potential. Stationary electrohydrodynamic states were studied in [4], whereas the nonstationary charge-potential problem for a Poiseuille-type flow was investigated numerically in [5].

Stationary channel flows of the Pokrovskii-Vinogradov model have been examined from both constructive and stability perspectives. Explicit Poiseuille- and Couette-type solutions were obtained in [6], and the numerical selection of stationary Poiseuille branches through a nonstationary relaxation process was considered in [7]. Spectral and Lyapunov analyses revealed instability of stationary flows and of the rest state [8–10]. Corresponding spectral results for the electrohydrodynamic model were derived in [11], while parameter-dependent stability conditions for a magnetohydrodynamic extension were obtained in [12]. These studies demonstrate that stability depends essentially on the constitutive regime, electromagnetic coupling, and admissible perturbation class. The present analysis is restricted to streamwise-independent Poiseuille-type motions, which form an invariant subclass of the full two-dimensional problem. The present results concern only this subclass and do not address streamwise-dependent perturbations.

The constitutive framework underlying the Oldroyd-B model originates from the classical formulation of Oldroyd [13]. Transient Oldroyd-B-type flows have also been studied in pressure-driven and other elementary configurations [14, 15]. Exact solutions for wall-driven Stokes start-up flows were obtained in [16], while numerical and analytical studies of pressure-driven start-up and pulsating channel flows were presented in [17]. These works provide explicit or numerical descriptions of evolving velocity and shear-stress fields. A related long-time problem was considered in [18], where a nonzero initial transverse normal stress generates a non-autonomous velocity equation with exponentially varying coefficients. Its modal solution is expressed in terms of Bessel functions, and the velocity approaches the rest state in the homogeneous setting.

Here we consider a different limiting regime. The orientation- and size-dependent constitutive corrections are suppressed by setting $\beta = k = 0$, with the physical meaning of these parameters described in [3]. The resulting relaxation coefficients are constant: The closed velocity-shear subsystem is linear, whereas the streamwise normal stress is recovered from a triangular equation with a quadratic source term.

In addition, the initial transverse normal stress is assumed to vanish, as in the conventional initially unstressed configuration used in Oldroyd-B start-up problems [16]. This condition defines an invariant class and leads to a velocity equation with constant coefficients. The flow is driven by a longitudinal pressure gradient $A(t)$ that approaches a generally nonzero limit $\hat{A}$.

Under the Poiseuille ansatz, the longitudinal velocity-stress subsystem decouples exactly from the charge-potential subsystem: The electric field has no longitudinal component, and the convective charge term vanishes. The electric variables therefore affect only the reconstruction of the transverse pressure and do not enter the reduced longitudinal evolution. This geometric decoupling permits a rigorous analysis of the mechanical subsystem without neglecting terms from its governing equations.

The purpose of this paper is to identify the stationary state selected by the limiting pressure gradient and to establish convergence of the complete reduced velocity-stress state. We first construct the explicit nonzero stationary Poiseuille profile, including its velocity, shear stress, and streamwise normal stress. For a time-dependent pressure gradient satisfying $A(t) \rightarrow \hat{A}$, the stationary profile is the equilibrium of the limiting autonomous system rather than an exact equilibrium of the non-autonomous problem. We prove that decay of the residual forcing generated by the time-dependent pressure gradient implies convergence of the velocity to the stationary profile in $H_0^1(0, 1)$, convergence of its time derivative to zero in $L_2(0, 1)$, and uniform convergence with respect to the transverse coordinate.

The resulting fading-memory convolution estimate implies a standard L_∞ input-to-state stability (ISS) bound with respect to this residual forcing; see [19] for the ISS framework for infinite-dimensional systems.

For the homogeneous deviation problem, we establish global exponential stability for all positive values of the reduced parameters and determine the supremal modal decay rate.

Under additional regularity assumptions, the shear and streamwise normal stresses converge in $L_2(0, 1)$ to their stationary values, while the transverse normal stress remains identically zero. The proof uses two complementary approaches. A modified Lyapunov functional establishes exponential stability directly in the energy space without separating the modal regimes, while the Fourier representation identifies the supremal decay rate and provides the Duhamel estimates needed for the nonhomogeneous problem.

In particular, the slow decay rate approaches the Newtonian Stokes rate in the small-Weissenberg limit. Numerical calculations support the analytical rates and illustrate the start-up dynamics and parameter dependence.

The paper is organised as follows. Section 2 introduces the full electrohydrodynamic model and derives the Poiseuille reduction. Section 3 constructs the stationary state and states the main convergence and stability results. Section 4 develops the modal and energy estimates. Numerical results are presented in Section 5, and the scope and possible extensions of the analysis are discussed in Section 6.

2. The EHD model and its Poiseuille reduction

2.1. The model and its notation

We consider a weakly conducting incompressible polymeric fluid in a plane parallel-plate capacitor. In dimensionless variables, the two-dimensional EHD model reads [4, 5, 18]

$$\operatorname{div} \vec{u} = u_x + v_y = 0, \quad (2.1)$$

$$\frac{d\vec{u}}{dt} + \nabla P = \operatorname{div} \Pi - \sigma_{Eq} \vec{Q}, \quad (2.2)$$

$$\frac{da_{11}}{dt} - 2A_1 u_x - 2a_{12} u_y + \mathcal{L}_{11} = 0, \quad (2.3)$$

$$\frac{da_{12}}{dt} - A_1 v_x - A_2 u_y + \tilde{K}_I a_{12} = 0, \quad (2.4)$$

$$\frac{da_{22}}{dt} - 2A_2 v_y - 2a_{12} v_x + \mathcal{L}_{22} = 0, \quad (2.5)$$

$$\frac{dq}{dt} - b \operatorname{div}(q\vec{Q}) = 0, \quad (2.6)$$

$$\Delta_{x,y}\Phi = -q. \quad (2.7)$$

Here t is time, x is the coordinate along the channel, and y is the transverse coordinate. The velocity is $\vec{u} = (u, v)$, P is the pressure, $q > 0$ is the volume charge density, Φ is the electric potential, and $\vec{Q} = \nabla\Phi$ is the electric field. The quantities a_{ij} are the components of the symmetric anisotropy tensor, while $\alpha_{ij} = a_{ij}/\operatorname{Re}$ are the scaled stress components, with $\alpha_{12} = \alpha_{21}$. Thus

$$\Pi = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12} & \alpha_{22} \end{pmatrix},$$

where α_{12} is the shear stress and α_{11}, α_{22} are the normal stresses. We use $d/dt = \partial_t + (\vec{u}, \nabla)$ and $\Delta_{x,y} = \partial_{xx} + \partial_{yy}$.

Let $I = a_{11} + a_{22}$. The relaxation coefficients are

$$K_I = W^{-1} + \frac{\bar{k}}{3}I, \quad \bar{K}_I = K_I + \beta I.$$

The remaining constitutive quantities are $A_1 = a_{11} + W^{-1}$, $A_2 = a_{22} + W^{-1}$, $\mathcal{L}_{11} = K_I a_{11} + \beta(a_{11}^2 + a_{12}^2)$, and $\mathcal{L}_{22} = K_I a_{22} + \beta(a_{12}^2 + a_{22}^2)$.

The dimensionless parameters and modal quantities are collected in Table 1. Here $\hat{l}$, ρ , u_H , η_0 , τ_0 , U_e , K , ε_0 , and ε denote, respectively, the channel width, fluid density, characteristic velocity, initial shear viscosity, relaxation time, dimensional anode potential, ion mobility, vacuum permittivity, and dielectric permittivity.

Table 1. Nomenclature of the principal dimensionless parameters and modal quantities.

Symbol	Meaning
Re	Reynolds number, $\operatorname{Re} = \rho u_H \hat{l} / \eta_0$
W	Weissenberg number, $W = \tau_0 u_H / \hat{l}$
κ	Reduced viscosity parameter, $\kappa^2 = W^{-1} / \operatorname{Re}$
β	Macromolecular-coil orientation parameter, $0 \leq \beta < 1$
$\bar{k}$	Coil-size correction parameter, $\bar{k} = k - \beta$
σ_E	Electric-to-inertial force ratio, $\sigma_E = \varepsilon_0 \varepsilon U_e^2 / [\rho(u_H \hat{l})^2]$
b	Dimensionless ion mobility, $b = K U_e / (u_H \hat{l})$
a_{inj}	Charge-injection coefficient, $a_{\text{inj}} > 0$
γ	Damping half-rate, $\gamma = (2W)^{-1}$
ω_n	n th modal frequency, $\omega_n = \kappa n \pi$
n^*	Critical modal index, $n^* = \gamma / (\kappa \pi)$

The variables u , v , a_{ij} , t , x , y , P , q , and Φ are nondimensionalised using the characteristic scales u_H , $W/3$, $\hat{l}/u_H$, $\hat{l}$, ρu_H^2 , $\varepsilon_0 \varepsilon U_e / \hat{l}^2$, and U_e , respectively.

The capacitor plates are located at $y = 0$ and $y = 1$. The boundary conditions are

$$\left. \begin{aligned} \vec{u} &= 0, & y &= 0, 1, \\ \Phi &= 0, & y &= 0, \\ \Phi &= 1, \quad q = a_{\text{inj}}\Phi_y, & y &= 1. \end{aligned} \right\}.$$

The regime $a_{\text{inj}} \ll 1$ corresponds to weak charge injection.

2.2. Exact decoupling under the Poiseuille ansatz

We specialise to Poiseuille-type flow: The velocity, stresses, charge density, and electric potential are independent of x , and the transverse velocity vanishes. Let $\mathcal{P}(t, y)$ denote the transverse pressure component. Let $A(t) = -P_x$ denote the prescribed longitudinal pressure gradient. The ansatz is

$$u = u(t, y), \quad v \equiv 0, \quad a_{ij} = a_{ij}(t, y), \quad P = \mathcal{P}(t, y) - A(t)x, \quad \Phi = \Phi(t, y), \quad q = q(t, y). \quad (2.8)$$

Under the Poiseuille ansatz (2.8), the mechanical and electric parts of the system decouple exactly. Indeed, since $\Phi = \Phi(t, y)$,

$$\vec{Q} = \nabla\Phi = (0, \Phi_y), \quad -\sigma_E q \vec{Q} = (0, -\sigma_E q \Phi_y),$$

so the Coulomb force has no longitudinal component. Hence the x -component of the momentum Eq (2.2), which governs $u(t, y)$, contains no electric term, whereas the y -component only determines the transverse pressure through

$$\mathcal{P}_y = (\alpha_{22})_y - \sigma_E q \Phi_y.$$

Conversely, because $q = q(t, y)$ and $v \equiv 0$,

$$(\vec{u}, \nabla)q = uq_x + vq_y = 0,$$

and therefore $dq/dt = q_t$. Using (2.7),

$$\text{div}(q\vec{Q}) = (q\Phi_y)_y = \Phi_y q_y - q^2,$$

so the charge equation contains only q and Φ . Consequently, the reduced EHD system splits into a velocity-stress subsystem for (u, a_{ij}) and a charge-potential subsystem for (q, Φ) . If x -dependence is restored, the longitudinal Coulomb term $-\sigma_E q \Phi_x$ reappears, and the two subsystems are coupled again.

2.3. The reduced subsystem and the constant-relaxation limit

For the scaled stresses define

$$\mathcal{D} = \alpha_{11}\alpha_{22} - \alpha_{12}^2.$$

Under (2.8), system (2.1)–(2.7) becomes

$$u_t - (\alpha_{12})_y = A(t), \quad (2.9)$$

$$(\alpha_{12})_t - (\alpha_{22} + \kappa^2)u_y + \widetilde{K}_I \alpha_{12} = 0, \quad (2.10)$$

$$(\alpha_{11})_t - 2\alpha_{12} u_y + \widetilde{K}_I \alpha_{11} - \beta \text{Re } \mathcal{D} = 0, \quad (2.11)$$

$$(\alpha_{22})_t + \widetilde{K}_I \alpha_{22} - \beta \text{Re } \mathcal{D} = 0, \quad (2.12)$$

$$q_t - b \Phi_y q_y + b q^2 = 0, \quad (2.13)$$

$$\Phi_{yy} = -q. \quad (2.14)$$

The system is supplemented by

$$u|_{y=0,1} = 0; \quad \Phi|_{y=0} = 0, \quad \Phi|_{y=1} = 1; \quad q = a_{\text{inj}} \Phi_y \quad \text{at } y = 1, \quad t > 0,$$

and by the initial data

$$u|_{t=0} = u_0, \quad \alpha_{ij}|_{t=0} = (\alpha_{ij})_0, \quad 0 < y < 1.$$

We study the velocity-stress subsystem (2.9)–(2.12). To distinguish this reduced subsystem from the full EHD model, from this point on we denote its velocity and scaled stress components by

$$U = u, \quad A_{ij} = \alpha_{ij},$$

respectively. The stress components A_{ij} are distinct from the constitutive quantities A_1 and A_2 introduced in Section 2.1.

Remark 2.1 (On the coefficient in (2.10)). The factor $(\alpha_{22} + \kappa^2)$ multiplying u_y follows from the reduction of (2.3)–(2.5) under (2.8). The effective shear modulus is the sum of the dimensionless viscosity contribution κ^2 and the transverse normal stress α_{22} [4,5,18]. In the limiting regime considered below, this factor reduces to the constant κ^2 .

Equations (2.10)–(2.12) contain no spatial derivatives of the stress components. Consequently, the stress evolution equations require no independent boundary conditions: At each fixed y , they form a system of evolution equations in t , supplied with the initial data $(\alpha_{ij})_0(y)$. The only spatial boundary conditions in the velocity-stress subsystem are the homogeneous Dirichlet conditions on the velocity. For strong solutions, the initial data must additionally satisfy the corresponding compatibility conditions at the channel walls.

We now specify the constitutive regime and the invariant initial-data class used in the stability analysis. In the mesoscopic Pokrovskii-Vinogradov model, the parameter β accounts for the contribution associated with the orientation of the macromolecular coils, whereas k accounts for the contribution associated with their size [3]. We consider the limiting regime in which both corrections are suppressed:

$$\beta = 0, \quad \bar{k} = 0, \quad (\alpha_{22})_0 \equiv 0. \quad (2.15)$$

Since $\bar{k} = k - \beta$, the first two conditions in (2.15) are equivalent to

$$k = \beta = 0.$$

These assumptions define a constant-relaxation benchmark regime of the mesoscopic model. Indeed, when $k = \beta = 0$, the state-dependent and quadratic orientation terms disappear and

$$K_I = W^{-1}, \quad \widetilde{K}_I = W^{-1}.$$

Consequently,

$$\mathcal{L}_{11} = W^{-1}a_{11}, \quad \mathcal{L}_{22} = W^{-1}a_{22},$$

while

$$A_1 = a_{11} + W^{-1}, \quad A_2 = a_{22} + W^{-1}.$$

Thus the relaxation coefficients are constant, and the closed equations for the velocity and shear stress form a linear subsystem. The streamwise normal stress is then recovered from a triangular equation whose source $2A_{12}U_y$ is quadratic; it does not feed back into the velocity-shear subsystem. Under the Poiseuille ansatz, these equations yield the Oldroyd-B-type limiting structure obtained after the contributions associated with coil orientation and coil size have been removed.

The same constant-relaxation system also arises as the leading-order problem in the regular small- β expansion considered in [18].

Using the definitions of the scaled stresses and $\varkappa$, we have

$$\frac{A_2}{\text{Re}} = \alpha_{22} + \varkappa^2,$$

which explains the coefficient appearing in (2.10).

The condition $(\alpha_{22})_0 \equiv 0$ concerns the initial transverse normal-stress component. It includes, as a particular case, the conventional initially unstressed start-up configuration frequently used in Oldroyd-B shear-flow problems [16]. This start-up configuration provides the physical motivation for the zero initial transverse stress; the invariance of this condition follows directly from the present equations.

Indeed, after imposing $k = \beta = 0$, Eq (2.12) becomes

$$(A_{22})_t + W^{-1}A_{22} = 0. \quad (2.16)$$

Solving (2.16) gives

$$A_{22}(y, t) = e^{-W^{-1}t}(\alpha_{22})_0(y). \quad (2.17)$$

Equation (2.17) shows that

$$(\alpha_{22})_0 \equiv 0 \quad \implies \quad A_{22}(y, t) \equiv 0 \quad (t \geq 0, 0 \leq y \leq 1).$$

Hence the zero transverse normal-stress condition defines an invariant class of the reduced dynamics. Note that no corresponding zero condition is imposed on the initial shear stress $(\alpha_{12})_0$ or on the initial streamwise normal stress $(\alpha_{11})_0$.

3. Stationary state and main stability result

3.1. The stationary Poiseuille profile

Under (2.15), the velocity-stress subsystem becomes

$$\begin{cases} U_t - (A_{12})_y = A(t), \\ (A_{12})_t - (A_{22} + \varkappa^2)U_y + W^{-1}A_{12} = 0, \\ (A_{11})_t - 2A_{12}U_y + W^{-1}A_{11} = 0, \\ (A_{22})_t + W^{-1}A_{22} = 0, \quad t > 0, 0 < y < 1, \end{cases} \quad (3.1)$$

with

$$U|_{y=0,1} = 0, \quad U|_{t=0} = u_0(y), \quad A_{ij}|_{t=0} = (\alpha_{ij})_0(y).$$

Since $(\alpha_{22})_0 \equiv 0$, the last equation of (3.1) gives $A_{22} \equiv 0$. Eliminating A_{12} from the first two equations then yields

$$U_{tt} - \kappa^2 U_{yy} + W^{-1} U_t = A'(t) + W^{-1} A(t) \quad (3.2)$$

on $(0, 1) \times (0, \infty)$, with $U|_{y=0,1} = 0$.

Assume now that $A(t) \rightarrow \hat{A}$ as $t \rightarrow \infty$. The stationary velocity of the limiting autonomous system satisfies $-\kappa^2 \hat{U}_{yy} = W^{-1} \hat{A}$ with $\hat{U}(0) = \hat{U}(1) = 0$ and is therefore

$$\hat{U}(y) = \frac{W^{-1} \hat{A}}{2\kappa^2} y(1-y) = \frac{\hat{A} \operatorname{Re}}{2} y(1-y). \quad (3.3)$$

The remaining equations in (3.1) yield the corresponding stationary stresses

$$\hat{A}_{12}(y) = \frac{\hat{A}}{2}(1-2y), \quad \hat{A}_{11}(y) = \frac{\hat{A}^2}{2\kappa^2}(1-2y)^2, \quad \hat{A}_{22}(y) = 0. \quad (3.4)$$

The profiles (3.3) and (3.4) constitute a stationary solution of the reduced velocity-stress system for the constant pressure gradient $A(t) \equiv \hat{A}$. Their construction alone does not establish stability. The purpose of the following analysis is to determine whether solutions arising from general admissible initial data approach this stationary profile.

When $A(t) \equiv \hat{A}$, the profile is an equilibrium of the autonomous reduced system, and the homogeneous deviation equation studied below describes its stability with respect to Poiseuille-type perturbations. In this case, we prove global exponential stability of the velocity component in

$$H_0^1(0, 1) \times L_2(0, 1).$$

For a general pressure gradient satisfying $A(t) \rightarrow \hat{A}$, the profile is the equilibrium of the limiting autonomous system rather than an exact stationary solution of the time-dependent system. The corresponding result is therefore formulated as asymptotic convergence to the limiting stationary state under a vanishing residual forcing.

3.2. The deviation equation and the compatibility of the data

Define the residual forcing

$$\Theta(t) = A'(t) + W^{-1}(A(t) - \hat{A})$$

and the velocity deviation $Z(y, t) = U(y, t) - \hat{U}(y)$.

In pressure-driven polymer processing, $A(t)$ represents the start-up or adjustment of the driving pressure in a channel, as occurs in coating and extrusion operations. The residual $\Theta(t)$ is the part of that pressure history that is not absorbed by the limiting stationary balance; its decay therefore determines whether the observed transient is forcing-controlled or relaxation-controlled. Subtracting the stationary equation from (3.2) gives

$$Z_{tt} - \kappa^2 Z_{yy} + W^{-1} Z_t = \Theta(t), \quad Z(0, t) = Z(1, t) = 0. \quad (3.5)$$

The initial conditions for (3.5) are derived from the velocity-stress subsystem evaluated at $t = 0$. The first equation of (3.1) gives $U_t|_{t=0} = A(0) + d(\alpha_{12})_0/dy$, and hence

$$Z(y, 0) = Z_0(y) = u_0(y) - \hat{U}(y), \quad Z_t(y, 0) = Z_1(y) = A(0) + \frac{d(\alpha_{12})_0}{dy}(y). \quad (3.6)$$

Thus $Z_0 \in H_0^1(0, 1)$ follows from $u_0 \in H_0^1(0, 1)$, while $Z_1 \in L_2(0, 1)$ follows from $(\alpha_{12})_0 \in H^1(0, 1)$. The stronger assumption $Z_1 \in H_0^1(0, 1)$ used below requires $(\alpha_{12})_0 \in H^2(0, 1)$ together with the corresponding wall compatibility. The remaining datum $(\alpha_{11})_0$ is taken in $L_2(0, 1)$, and $(\alpha_{22})_0 \equiv 0$ by (2.15).

Throughout the analysis, we use the norm $\|v\|_{H_0^1} = \|v_y\|_{L_2}$ on $H_0^1(0, 1)$, which is equivalent to the standard Sobolev norm by Poincaré's inequality.

3.3. The main result

With the notation of Table 1, define

$$\sigma_* = \begin{cases} \gamma - \sqrt{\gamma^2 - \kappa^2 \pi^2}, & \gamma > \kappa \pi, \\ \gamma, & \gamma \leq \kappa \pi. \end{cases} \quad (3.7)$$

Theorem 3.1. Assume (2.15), $\kappa > 0$, $W > 0$, and let

$$A \in C^1([0, \infty)), \quad \Theta \in L_\infty(0, \infty).$$

Assume that A converges pointwise and that the residual forcing decays in the essential-supremum sense; namely,

$$\lim_{t \rightarrow \infty} A(t) = \hat{A}, \quad \lim_{T \rightarrow \infty} \|\Theta\|_{L_\infty(T, \infty)} = 0.$$

Let

$$u_0 \in H_0^1(0, 1), \quad (\alpha_{12})_0 \in H^1(0, 1).$$

Then the closed linear velocity-shear subsystem admits a unique energy solution, and

$$\|U(\cdot, t) - \hat{U}\|_{H_0^1} + \|U_t(\cdot, t)\|_{L_2} \longrightarrow 0, \quad t \rightarrow \infty, \quad (3.8)$$

and

$$\max_{0 \leq y \leq 1} |U(y, t) - \hat{U}(y)| \longrightarrow 0, \quad t \rightarrow \infty. \quad (3.9)$$

For every $\sigma \in (0, \sigma_*)$, with σ_* defined by (3.7), there exists $M_\sigma > 0$ such that

$$\|U(\cdot, t) - \hat{U}\|_{H_0^1} + \|U_t(\cdot, t)\|_{L_2} \leq M_\sigma e^{-\sigma t} (\|Z_0\|_{H_0^1} + \|Z_1\|_{L_2}) + M_\sigma \int_0^t e^{-\sigma(t-s)} |\Theta(s)| \, ds, \quad t \geq 0. \quad (3.10)$$

Moreover, (3.10) implies the standard exponential L_∞ -ISS estimate

$$\|U(\cdot, t) - \hat{U}\|_{H_0^1} + \|U_t(\cdot, t)\|_{L_2} \leq M_\sigma e^{-\sigma t} (\|Z_0\|_{H_0^1} + \|Z_1\|_{L_2}) + \frac{M_\sigma}{\sigma} \|\Theta\|_{L_\infty(0, t)}, \quad t \geq 0. \quad (3.11)$$

Indeed, the first term on the right defines a class- $\mathcal{KL}$ function of the initial-state norm, and the last term defines a class- $\mathcal{K}_\infty$ gain of the input norm.

If $\Theta \equiv 0$, the velocity deviation is exponentially stable for every $\sigma < \sigma_*$, and σ_* is the supremal exponential decay rate.

If, in addition,

$$\begin{aligned} u_0 &\in H^2(0, 1), & (\alpha_{12})_0 &\in H^2(0, 1), \\ (\alpha_{11})_0 &\in L_2(0, 1), & Z_1 &\in H_0^1(0, 1), \end{aligned}$$

and

$$\Theta \in W^{1,1}(0, \infty) \cap W^{1,\infty}(0, \infty),$$

then

$$\|A_{12}(\cdot, t) - \hat{A}_{12}\|_{L_2} + \|A_{11}(\cdot, t) - \hat{A}_{11}\|_{L_2} \longrightarrow 0, \quad t \rightarrow \infty, \quad (3.12)$$

while

$$A_{22}(y, t) \equiv 0.$$

Thus the first part of the theorem is an energy-space statement for the velocity-shear state. The L_2 convergence of the full stress state, including A_{11} , is asserted only under the additional regularity above, which makes the quadratic source $A_{12}U_y$ an L_2 function.

4. Proof of the main stability theorem

4.1. Modal representation

Expand

$$Z(y, t) = \sum_{n=1}^{\infty} z_n(t) \sin(n\pi y),$$

where

$$z_{0n} = 2 \int_0^1 Z_0(y) \sin(n\pi y) \, dy, \quad z_{1n} = 2 \int_0^1 Z_1(y) \sin(n\pi y) \, dy.$$

Since Θ is independent of y , its sine coefficients are

$$c_n = 2 \int_0^1 \sin(n\pi y) \, dy = \frac{2(1 - (-1)^n)}{n\pi}.$$

The modal coefficients then satisfy

$$z_n'' + W^{-1}z_n' + \omega_n^2 z_n = c_n \Theta(t), \quad z_n(0) = z_{0n}, \quad z_n'(0) = z_{1n}. \quad (4.1)$$

Let C_n and S_n be the two fundamental solutions of the homogeneous modal equation

$$w'' + W^{-1}w' + \omega_n^2 w = 0,$$

normalised by

$$C_n(0) = 1, \quad C_n'(0) = 0, \quad S_n(0) = 0, \quad S_n'(0) = 1,$$

respectively. Their explicit overdamped, critical, and underdamped formulas, together with the characteristic roots $r_{n,\pm}$, are collected in Appendix A. With this notation, Duhamel's formula gives

$$z_n(t) = C_n(t)z_{0n} + S_n(t)z_{1n} + c_n \int_0^t S_n(t-s)\Theta(s) \, ds. \quad (4.2)$$

The complete series representation is stated in Proposition A.1; only the unified bounds below are needed in the main proof.

Lemma 4.1. *For every $\sigma \in (0, \sigma_*)$, there exists $C_\sigma > 0$ such that*

$$|C_n(t)| + |S'_n(t)| + n|S_n(t)| + \frac{|C'_n(t)|}{n} \leq C_\sigma e^{-\sigma t}, \quad n \geq 1, \quad t \geq 0. \quad (4.3)$$

Moreover,

$$\sigma_* = \min_{n \geq 1} (-\max \operatorname{Re} r_{n,\pm}). \quad (4.4)$$

Proof. For $\omega_n < \gamma$, the overdamped formula in Appendix A gives

$$-\max r_{n,\pm} = \gamma - \sqrt{\gamma^2 - \omega_n^2},$$

and

$$\frac{d}{d\omega} (\gamma - \sqrt{\gamma^2 - \omega^2}) = \frac{\omega}{\sqrt{\gamma^2 - \omega^2}} > 0.$$

Hence the smallest decay exponent in this regime is attained at $n = 1$.

For $\omega_n > \gamma$,

$$\operatorname{Re} r_{n,\pm} = -\gamma,$$

and

$$\frac{\sqrt{\omega_n^2 - \gamma^2}}{n} \rightarrow \kappa\pi, \quad \frac{n}{\sqrt{\omega_n^2 - \gamma^2}} \rightarrow \frac{1}{\kappa\pi}.$$

Thus the corresponding coefficients are uniformly bounded in n .

For $\omega_n = \gamma$,

$$te^{-\gamma t} \leq \frac{1}{e(\gamma - \sigma)} e^{-\sigma t}, \quad 0 < \sigma < \gamma.$$

Combining these estimates over the three modal regimes gives (4.3) and (4.4). $\square$

For $\gamma \gg \kappa\pi$,

$$\sigma_* = \frac{\kappa^2 \pi^2}{2\gamma} + \frac{\kappa^4 \pi^4}{8\gamma^3} + O(\gamma^{-5}) = \frac{\pi^2}{\operatorname{Re}} \left[1 + \frac{\pi^2 W}{\operatorname{Re}} + O\left(\frac{W^2}{\operatorname{Re}^2}\right) \right]. \quad (4.5)$$

Expansion (4.5) yields

$$\sigma_* \longrightarrow \frac{\pi^2}{\operatorname{Re}} \quad \text{as} \quad \frac{W}{\operatorname{Re}} \longrightarrow 0.$$

4.2. A complementary Lyapunov estimate

The modal representation developed in Section 4.1 is needed to identify the supremal decay rate and to obtain explicit convolution estimates. Exponential stability in the energy space, however, can also be established directly, without separating the modal regimes. The following Lyapunov argument provides such an independent estimate for all positive values of W and Re . Besides complementing the spectral analysis, this formulation is more suitable for extensions to problems with variable coefficients, for which the Fourier modes generally do not decouple. For $\Theta \equiv 0$, define

$$\mathcal{L}(t) = \frac{1}{2} \int_0^1 (Z_t^2 + \kappa^2 Z_y^2) \, dy. \quad (4.6)$$

Differentiating (4.6) gives

$$\frac{d\mathcal{L}}{dt} = -W^{-1} \int_0^1 Z_t^2 \, dy \leq 0. \quad (4.7)$$

The dissipation identity (4.7) shows that $\mathcal{L}$ is nonincreasing. Choose ζ so that

$$0 < \zeta < \min \left\{ W^{-1}, \frac{1}{2}, \frac{\pi^2 \kappa^2}{2} \right\}.$$

For such a ζ , set

$$\mathcal{E}_\zeta(t) = \mathcal{L}(t) + \zeta \int_0^1 ZZ_t \, dy + \frac{\zeta W^{-1}}{2} \int_0^1 Z^2 \, dy. \quad (4.8)$$

Young's and Poincaré's inequalities give, for some $C_\zeta > 0$,

$$\frac{1}{2} \mathcal{L}(t) \leq \mathcal{E}_\zeta(t) \leq C_\zeta \mathcal{L}(t). \quad (4.9)$$

Differentiating (4.8),

$$\frac{d\mathcal{E}_\zeta}{dt} = -(W^{-1} - \zeta) \int_0^1 Z_t^2 \, dy - \zeta \kappa^2 \int_0^1 Z_y^2 \, dy. \quad (4.10)$$

Consequently, (4.10) and the equivalence (4.9) imply

$$\mathcal{E}_\zeta(t) \leq \mathcal{E}_\zeta(0) e^{-c_\zeta t} \quad (4.11)$$

for some $c_\zeta > 0$. Thus (4.9)–(4.11) provide an independent proof of exponential stability for the homogeneous velocity equation. In the proof of Theorem 3.1, the modal estimate will refine this qualitative conclusion by identifying the supremal decay rate and treating the residual forcing.

4.3. Regularity estimate

The higher regularity assumed below is used only in Section 4.4 to obtain uniform control of Z_y and thereby control the quadratic product $R_{12}Z_y$ when recovering the streamwise normal stress A_{11} ; it is not needed for the velocity stability and convergence estimates. These hypotheses are the standard regularity and wall-compatibility conditions for a strong solution of the one-dimensional damped-wave problem, whereas the velocity stability result itself remains valid for the energy-space data stated in the first part of Theorem 3.1.

Lemma 4.2. *Let*

$$Z_0 \in H^2(0, 1) \cap H_0^1(0, 1), \quad Z_1 \in H_0^1(0, 1), \quad \Theta \in W^{1,1}(0, \infty) \cap W^{1,\infty}(0, \infty).$$

Then

$$\sup_{t \geq 0} \|Z(\cdot, t)\|_{H^2} < \infty$$

and

$$\sup_{t \geq 0} \|Z_y(\cdot, t)\|_{L_\infty} < \infty, \quad \|Z_y(\cdot, t)\|_{L_\infty} \longrightarrow 0.$$

The proof of Lemma 4.2 is given in Appendix B.

4.4. Proof of Theorem 3.1

Proof. For $\Theta \equiv 0$, the Lyapunov estimate (4.9)–(4.11) already establishes global exponential stability in $H_0^1(0, 1) \times L_2(0, 1)$ for every $\kappa > 0$ and $W > 0$. We now use the modal representation to identify the supremal decay rate and treat the nonhomogeneous problem. For terminological precision, we first retain the sharper fading-memory convolution estimate and then derive its standard ISS consequence.

Parseval's identity gives

$$\begin{aligned} \|Z(\cdot, t)\|_{H_0^1}^2 &= \frac{\pi^2}{2} \sum_{n=1}^{\infty} n^2 |z_n(t)|^2, \\ \|Z_t(\cdot, t)\|_{L_2}^2 &= \frac{1}{2} \sum_{n=1}^{\infty} |z'_n(t)|^2. \end{aligned} \quad (4.12)$$

From (4.2) and Lemma 4.1,

$$n|z_n(t)| + |z'_n(t)| \leq C_\sigma e^{-\sigma t} (n|z_{0n}| + |z_{1n}|) + C_\sigma |c_n| \int_0^t e^{-\sigma(t-s)} |\Theta(s)| \, ds. \quad (4.13)$$

Since

$$\sum_{n=1}^{\infty} n^2 |z_{0n}|^2 < \infty, \quad \sum_{n=1}^{\infty} |z_{1n}|^2 < \infty, \quad \sum_{n=1}^{\infty} |c_n|^2 < \infty,$$

Eqs (4.12) and (4.13) yield

$$\|Z(\cdot, t)\|_{H_0^1} + \|Z_t(\cdot, t)\|_{L_2} \leq M_\sigma e^{-\sigma t} (\|Z_0\|_{H_0^1} + \|Z_1\|_{L_2}) + M_\sigma \int_0^t e^{-\sigma(t-s)} |\Theta(s)| \, ds. \quad (4.14)$$

This is (3.10).

To justify the ISS terminology precisely, observe that

$$\int_0^t e^{-\sigma(t-s)} |\Theta(s)| \, ds \leq \|\Theta\|_{L_\infty(0,t)} \int_0^t e^{-\sigma(t-s)} \, ds \leq \frac{1}{\sigma} \|\Theta\|_{L_\infty(0,t)}.$$

Substitution into (4.14) gives (3.11). Hence (3.10) is more precisely a convolution-type, or fading-memory, input estimate, and it implies the standard L_∞ -ISS bound rather than merely resembling one.

Let $\varepsilon > 0$. By the tail condition on Θ , choose $T > 0$ such that

$$\|\Theta\|_{L_\infty(T,\infty)} < \varepsilon.$$

Then, for $t \geq T$,

$$\begin{aligned} \int_0^t e^{-\sigma(t-s)} |\Theta(s)| \, ds &\leq \frac{\|\Theta\|_{L_\infty(0,T)}}{\sigma} e^{-\sigma(t-T)} + \frac{\|\Theta\|_{L_\infty(T,\infty)}}{\sigma} \\ &\leq \frac{\|\Theta\|_{L_\infty(0,T)}}{\sigma} e^{-\sigma(t-T)} + \frac{\varepsilon}{\sigma}. \end{aligned}$$

Therefore

$$\|Z(\cdot, t)\|_{H_0^1} + \|Z_t(\cdot, t)\|_{L_2} \longrightarrow 0.$$

Since

$$H_0^1(0, 1) \hookrightarrow C[0, 1],$$

Eq (3.9) follows. For $\Theta \equiv 0$, (4.14) gives exponential decay, while (4.4) identifies the supremal exponential decay rate σ_* .

Define

$$R_{12} = A_{12} - \hat{A}_{12}, \quad R_{11} = A_{11} - \hat{A}_{11}.$$

Since $A_{22} \equiv 0$, subtraction of the stationary equations from (3.1) gives

$$(R_{12})_t + W^{-1}R_{12} = \kappa^2 Z_y.$$

Hence

$$R_{12}(t) = e^{-W^{-1}t} R_{12}(0) + \kappa^2 \int_0^t e^{-W^{-1}(t-s)} Z_y(s) \, ds.$$

Using (3.8),

$$\|R_{12}(\cdot, t)\|_{L_2} \longrightarrow 0. \quad (4.15)$$

For R_{11} ,

$$(R_{11})_t + W^{-1}R_{11} = G_{11},$$

where

$$G_{11} = 2\hat{A}_{12}Z_y + 2\hat{U}_y R_{12} + 2R_{12}Z_y.$$

Lemma 4.2 and (4.15) give

$$\|G_{11}(\cdot, t)\|_{L_2} \leq 2\|\hat{A}_{12}\|_{L_\infty} \|Z_y(\cdot, t)\|_{L_2} + 2\|\hat{U}_y\|_{L_\infty} \|R_{12}(\cdot, t)\|_{L_2} + 2\|R_{12}(\cdot, t)\|_{L_2} \|Z_y(\cdot, t)\|_{L_\infty} \longrightarrow 0.$$

Consequently,

$$R_{11}(t) = e^{-W^{-1}t} R_{11}(0) + \int_0^t e^{-W^{-1}(t-s)} G_{11}(s) \, ds,$$

and

$$\|R_{11}(\cdot, t)\|_{L_2} \longrightarrow 0. \quad (4.16)$$

Equations (4.15) and (4.16) and $A_{22} \equiv 0$ prove (3.12). $\square$

5. Quantitative numerical assessment

This section gives two focused tests of Theorem 3.1. Both use an independent physical-space discretisation of the velocity-stress system (3.1); neither the eliminated Eq (3.2) nor the modal formulas in Section 4.1 enter the time-stepping algorithm. The first test measures the dependence of the decay exponent on W and Re , while the second examines the competition between the intrinsic relaxation rate and a time-dependent pressure gradient.

5.1. Numerical method and parameter-dependent decay

Let N_y be the number of uniform cells and $h = N_y^{-1}$. The velocity is stored at the nodes $y_j = jh$, whereas the stresses are stored at the cell centres $y_{j+1/2} = (j + 1/2)h$. With $U_0 = U_{N_y} = 0$, the method of lines is

$$\begin{cases} \dot{U}_j = \frac{A_{12,j+1/2} - A_{12,j-1/2}}{h} + A(t), & j = 1, \dots, N_y - 1, \\ \dot{A}_{12,j+1/2} = (A_{22,j+1/2} + \kappa^2) \frac{U_{j+1} - U_j}{h} - W^{-1} A_{12,j+1/2}, \\ \dot{A}_{11,j+1/2} = 2A_{12,j+1/2} \frac{U_{j+1} - U_j}{h} - W^{-1} A_{11,j+1/2}, \\ \dot{A}_{22,j+1/2} = -W^{-1} A_{22,j+1/2}. \end{cases} \quad (5.1)$$

No stress boundary values are introduced. The centred staggered differences retain the cancellation between the velocity and shear-stress equations. The system is advanced by DOP853 [20] with $\text{rtol} = 10^{-10}$ and $\text{atol} = 10^{-12}$. A preliminary benchmark with $\text{Re} = 10$, $W = 10/(16\pi^2)$, and modes $n = 1, 2, 3$ gives an observed spatial order of 2.00 for both U and A_{12} on the grids $N_y = 100, 200, 400, 800$. The following tests use $N_y = 400$. All numerical trajectories below are computed from (5.1). We write U_h and $A_{ij,h}$ for the resulting discrete fields. For a grid function V_h , the discrete norms used below are

$$\|V_h\|_{L_{2,h}}^2 = h \sum_j |V_j|^2, \quad \|U_h\|_{H_{0,h}^1}^2 = h \sum_{j=0}^{N_y-1} \left| \frac{U_{j+1} - U_j}{h} \right|^2.$$

For the parameter study, take $A(t) \equiv 0$ and

$$U(y, 0) = \sin(\pi y) + \frac{1}{2} \sin(2\pi y) + \frac{1}{4} \sin(3\pi y), \quad A_{12}(y, 0) = A_{11}(y, 0) = A_{22}(y, 0) = 0.$$

The first modal coefficient is recovered from the physical-space solution by

$$z_1^h(t) = 2h \sum_{j=1}^{N_y-1} U_j(t) \sin(\pi y_j).$$

The predicted slowest modal decay exponent is

$$\sigma_* = \begin{cases} \gamma - \sqrt{\gamma^2 - \omega_1^2}, & \omega_1 < \gamma, \\ \gamma, & \omega_1 \geq \gamma. \end{cases} \quad (5.2)$$

For an overdamped first mode, the exponent is fitted from $\log |z_1^h(t)|$ after the fast component has become negligible. For an underdamped first mode, the fit uses the discrete oscillation envelope

$$\mathcal{A}_{1,h}(t) = \left[|z_1^h(t)|^2 + \frac{|(z_1^h)'(t) + \gamma z_1^h(t)|^2}{\omega_{1,h}^2 - \gamma^2} \right]^{1/2}, \quad \omega_{1,h} = \frac{2\kappa}{h} \sin \frac{\pi h}{2}.$$

We perform the two scans

$$W = 0.1, \quad \text{Re} = 5, 10, 20, 40,$$

and

$$\text{Re} = 10, \quad W = 0.05, 0.10, 0.20, W_c, 0.40, 0.80, \quad W_c = \frac{10}{4\pi^2}.$$

Figure 1 shows the normalised discrete H_0^1 velocity error. The measured slopes follow the parameter dependence predicted by (5.2); σ_{num} denotes the exponent obtained from the corresponding numerical fit.

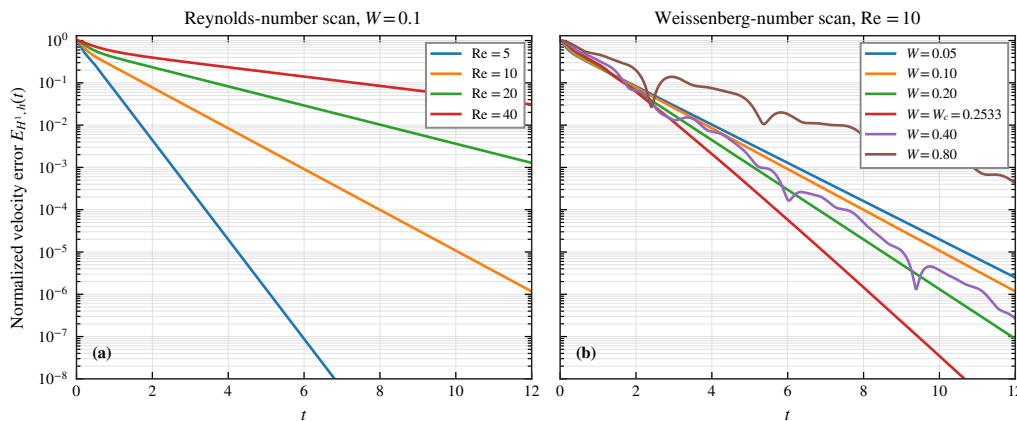


Figure 1. Normalised H_0^1 velocity errors obtained from the independent staggered-grid solver: (a) Reynolds-number scan; (b) Weissenberg-number scan.

Outside the critical case, the largest relative discrepancy in Table 2 is 1.1×10^{-5} . At $W = W_c$, the first mode contains the expected algebraic factor $te^{-\gamma t}$ and is therefore not fitted by a pure exponential.

Table 2. Predicted and measured first-mode decay exponents for representative values in the two parameter scans.

Scan	Value	n^*	Regime	σ_*	σ_{num}
$W = 0.1$	$\text{Re} = 5$	1.1254	overdamped	2.706359	2.706334
	$\text{Re} = 10$	1.5915	overdamped	1.110219	1.110212
	$\text{Re} = 40$	3.1831	overdamped	0.253149	0.253147
$\text{Re} = 10$	$W = 0.05$	2.2508	overdamped	1.041161	1.041155
	$W = 0.20$	1.1254	overdamped	1.353179	1.353165
	$W = W_c$	1.0000	critical	1.973921	—
	$W = 0.40$	0.7958	underdamped	1.250000	1.250000
	$W = 0.80$	0.5627	underdamped	0.625000	0.625000

5.2. Forced start-up and convergence to the stationary state

We prescribe the time-dependent pressure gradient

$$A(t) = \hat{A}(1 - e^{-\lambda t}), \quad \Theta(t) = (\lambda - W^{-1})\hat{A}e^{-\lambda t}. \quad (5.3)$$

Here $\lambda > 0$ is the pressure-gradient relaxation rate, and $\Theta \neq 0$ whenever $\lambda \neq W^{-1}$. We take

$$\text{Re} = 10, \quad W = 0.1, \quad \kappa^2 = 1, \quad \hat{A} = 1,$$

and start from rest:

$$U(y, 0) = A_{12}(y, 0) = A_{11}(y, 0) = A_{22}(y, 0) = 0.$$

The stationary state is

$$\hat{U}(y) = 5y(1 - y), \quad \hat{A}_{12}(y) = \frac{1}{2}(1 - 2y), \quad \hat{A}_{11}(y) = \frac{1}{2}(1 - 2y)^2, \quad \hat{A}_{22} = 0.$$

To measure convergence of the complete velocity-stress state, define

$$E_U(t) = \left(\|U_h - \hat{U}\|_{H_{0,h}^1}^2 + \|\dot{U}_h\|_{L_{2,h}}^2 \right)^{1/2},$$

$$E_{12}(t) = \|A_{12,h} - \hat{A}_{12}\|_{L_{2,h}},$$

$$E_{11}(t) = \|A_{11,h} - \hat{A}_{11}\|_{L_{2,h}}.$$

For these parameters,

$$\sigma_* = 5 - \sqrt{25 - \pi^2} = 1.11021908.$$

We compare a forcing-controlled case, $\lambda = 0.5 < \sigma_*$, with a system-controlled case, $\lambda = 2 > \sigma_*$. Since

$$\sigma_{\text{obs}} = \min\{\lambda, \sigma_*\}, \quad (5.4)$$

the two cases distinguish decay imposed by the pressure-gradient relaxation from decay imposed by the velocity-stress system itself. The residual forcing is given explicitly by (5.3), so its decay rate is λ . Figure 2 includes the normalised residual $|\Theta(t)|/|\Theta(0)| = e^{-\lambda t}$ together with the normalised velocity and stress errors.

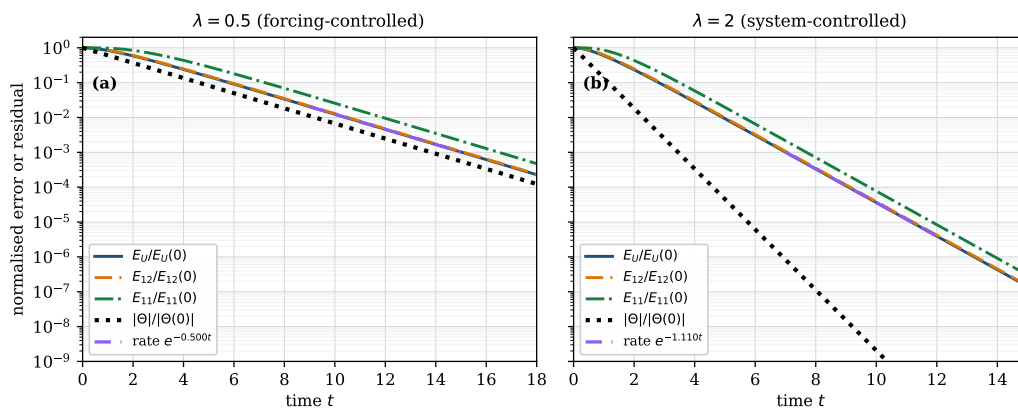


Figure 2. Normalised velocity and stress errors for the nontrivially forced start-up: (a) $\lambda = 0.5$, for which the forcing controls the convergence; (b) $\lambda = 2$, for which the intrinsic exponent σ_* controls the convergence. The thick dotted curve is $|\Theta(t)|/|\Theta(0)| = e^{-\lambda t}$, and the dashed reference segments have the rates predicted by (5.4).

The measured rates in Table 3 agree with (5.4). In addition, $A_{22,h}$ remains zero to machine precision. Thus this experiment supports both the forcing-dependent velocity decay and convergence of the stress components, rather than only the characteristic roots of the eliminated velocity equation.

Table 3. Predicted and measured start-up convergence rates.

λ	predicted	E_U	E_{12}	E_{11}
0.5	0.500000	0.499903	0.499900	0.499254
2	1.110219	1.110134	1.110152	1.110132

6. Discussion and open problems

6.1. Scope and limitations

The analysis is restricted to the x -independent Poiseuille class and the constant-coefficient regime (2.15). It therefore does not address streamwise-dependent perturbations, the full quasilinear Pokrovskii-Vinogradov system, or two-dimensional regimes in which the electric and mechanical subsystems are actively coupled. The exponential estimate concerns the velocity state, whereas the stresses are shown to converge asymptotically. The stress convergence requires the additional regularity stated in Theorem 3.1; it is not part of the basic energy-space conclusion.

Relaxing the constant-coefficient assumption leads to the problem described below.

6.2. The variable-coefficient problem

If $(\alpha_{22})_0 \neq 0$, define the time-dependent principal coefficient

$$m(y, t) = \kappa^2 + e^{-W^{-1}t}(\alpha_{22})_0(y).$$

The velocity equation then has the form

$$U_{tt} - \partial_y(m(y, t)U_y) + W^{-1}U_t = A'(t) + W^{-1}A(t). \quad (6.1)$$

For the deviation $Z = U - \hat{U}$ from the limiting stationary profile, the corresponding equation is

$$Z_{tt} - \partial_y(m(y, t)Z_y) + W^{-1}Z_t = \Theta(t) + \partial_y((m(y, t) - \kappa^2)\hat{U}_y). \quad (6.2)$$

In (6.2), the second term on the right-hand side decays exponentially but retains the spatial dependence of $(\alpha_{22})_0$. For general $(\alpha_{22})_0$, the operator is non-autonomous and does not diagonalise in the sine basis, so the Fourier modes are coupled.

Equation (6.1) is therefore a genuinely non-autonomous velocity equation. If $(\alpha_{22})_0$ is spatially homogeneous, the sine basis still diagonalises the operator, and the modal stiffness depends on time through the exponentially relaxing normal stress. The resulting Bessel-type equations are those studied in [18].

For a general bounded $(\alpha_{22})_0$ satisfying $\kappa^2 + (\alpha_{22})_0(y) \geq \nu > 0$, replacing κ^2 by $m(y, t)$ in the Lyapunov functional produces the additional term

$$\frac{1}{2} \int_0^1 m_t Z_y^2 dy.$$

Since $m_t = -W^{-1}e^{-W^{-1}t}(\alpha_{22})_0$ is integrable in time, a natural route is to combine a weighted Lyapunov estimate with Gronwall's inequality. Establishing uniform stability and convergence under minimal regularity assumptions remains an open problem.

If the x -independence assumption is removed, a possible route towards the fully coupled EHD problem is to regard the mechanical variables and the charge-potential variables as two interconnected subsystems. The coupling inputs are generated by the longitudinal Coulomb force $f_E = -\sigma_E q \Phi_x$ and by the convective charge transport $g_q = -(\vec{u}, \nabla)q$. The convolution estimate (3.10) and its standard L_∞ -ISS consequence (3.11) provide prototype bounds for the reduced mechanical component.

A small-gain programme would require compatible input-to-state estimates for both subsystems in appropriate function spaces, including the injection boundary condition and the quasilinear stress terms. If the corresponding gain operators Γ_m and Γ_e satisfy

$$(\Gamma_m \circ \Gamma_e)(s) < s \quad \text{for every } s > 0,$$

then a small-gain theorem could yield stability of the coupled interconnection. Small-gain-based input-to-state analysis has also been used for other interconnected dynamical systems; see, for example, [21]. This is only a methodological roadmap: Proving the required PDE estimates for the full EHD model remains open.

7. Conclusions

We have studied the Poiseuille-type velocity-stress subsystem of the EHD polymeric-fluid model in the limiting regime $\beta = 0$, $\bar{k} = 0$, and $(\alpha_{22})_0 \equiv 0$. Under the Poiseuille ansatz, the EHD system decouples exactly. If the applied pressure gradient approaches a finite limit and the residual forcing tends to zero, the velocity converges in $H_0^1(0, 1)$, and hence uniformly, to the explicit stationary Poiseuille profile. In the homogeneous case, the velocity deviation is exponentially stable for every positive parameter pair. Under additional regularity, the stress components converge in $L_2(0, 1)$ to their stationary values.

The modified Lyapunov functional gives a direct energy-space stability estimate, while the modal analysis identifies the supremal exponential decay rate σ_* . For the residual forcing, the modal argument yields a fading-memory convolution estimate that implies the standard L_∞ -ISS bound.

The numerical experiments support the predicted dependence of the decay on W and Re and the predicted competition between the pressure-gradient relaxation rate and the intrinsic exponent. In the small-Weissenberg regime, $\sigma_* \sim \pi^2/\text{Re}$, so the terminal decay approaches the Newtonian diffusive time scale, whereas a slowly relaxing pressure gradient can determine the observed convergence rate. The variable-coefficient problem described in Section 6, the full quasilinear constitutive system, and the coupled EHD problem remain natural directions for future work.

Author contributions

Xuelin Guan: Conceptualization, methodology, formal analysis, writing—original draft, and writing—review and editing. The author has read and approved the final manuscript.

Data availability

No external data were used. The code that generates the numerical results and figures of Section 5 is available from the author on request.

Use of Generative-AI tools declaration

The author declares that artificial intelligence (AI) tools were used to assist with the formatting of the manuscript. The author takes full responsibility for the content of this article.

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Conflict of interest

The author declares no conflict of interest.

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Appendix

A. Modal classification and the explicit solution

The characteristic roots of the homogeneous part of (4.1) are

$$r_{n,\pm} = -\gamma \pm \sqrt{\gamma^2 - \omega_n^2}.$$

They are real and distinct for $n < n^*$, repeated for $n = n^*$, and complex conjugate for $n > n^*$. Set

$$\Delta_n = \sqrt{\gamma^2 - \omega_n^2} \quad (n < n^*), \quad \bar{\Delta}_n = \sqrt{\omega_n^2 - \gamma^2} \quad (n > n^*).$$

The normalised fundamental solutions introduced in Section 4.1 are

$$\begin{aligned} n < n^* : \quad C_n(t) &= e^{-\gamma t} \left(\cosh(\Delta_n t) + \frac{\gamma}{\Delta_n} \sinh(\Delta_n t) \right), \quad S_n(t) = \frac{e^{-\gamma t} \sinh(\Delta_n t)}{\Delta_n}; \\ n = n^* : \quad C_n(t) &= e^{-\gamma t} (1 + \gamma t), \quad S_n(t) = t e^{-\gamma t}; \\ n > n^* : \quad C_n(t) &= e^{-\gamma t} \left(\cos(\bar{\Delta}_n t) + \frac{\gamma}{\bar{\Delta}_n} \sin(\bar{\Delta}_n t) \right), \quad S_n(t) = \frac{e^{-\gamma t} \sin(\bar{\Delta}_n t)}{\bar{\Delta}_n}. \end{aligned} \quad (\text{A.1})$$

The middle line of (A.1) is present only when n^* is an integer.

Proposition A.1 (Explicit solution of the deviation problem). *Let $\varkappa > 0$, $W > 0$, $Z_0 \in H_0^1(0, 1)$, $Z_1 \in L_2(0, 1)$, and $\Theta \in L_\infty(0, \infty)$. Then (3.5) with the initial data (3.6) admits a unique energy solution*

$$Z \in C([0, \infty); H_0^1(0, 1)) \cap C^1([0, \infty); L_2(0, 1)).$$

For every $T > 0$, the following series converges in

$$C([0, T]; H_0^1(0, 1)) \cap C^1([0, T]; L_2(0, 1))$$

and represents that solution:

$$Z(y, t) = \sum_{n=1}^{\infty} \left[C_n(t) z_{0n} + S_n(t) z_{1n} + c_n \int_0^t S_n(t-s) \Theta(s) \, ds \right] \sin(n\pi y), \quad (\text{A.2})$$

where z_{0n} , z_{1n} , and c_n are the coefficients introduced in Section 4.1 and C_n, S_n are given by (A.1).

Representation (A.2) displays the three modal regimes without repeating them in the main proof. Low-order modes decay at the rates $\gamma - \Delta_n$, high-order modes share the envelope $e^{-\gamma t}$, and the critical mode carries the factor $t e^{-\gamma t}$. Consequently, the estimates of Lemma 4.1 are stated for exponents strictly below σ_* . In every regime, $\operatorname{Re} r_{n,\pm} < 0$, so no mode grows. This is the constant-coefficient counterpart of the Bessel-function classification obtained in [18] for the non-autonomous problem.

B. Proof of Lemma 4.2

Proof. Set

$$V = Z_t.$$

Differentiating (3.5) gives

$$V_{tt} - \varkappa^2 V_{yy} + W^{-1} V_t = \Theta'(t), \quad V(0, t) = V(1, t) = 0, \quad (\text{A.3})$$

with

$$V(\cdot, 0) = Z_1, \quad V_t(\cdot, 0) = \varkappa^2 (Z_0)_{yy} - W^{-1} Z_1 + \Theta(0).$$

Hence

$$V(\cdot, 0) \in H_0^1(0, 1), \quad V_t(\cdot, 0) \in L_2(0, 1).$$

Applying (4.3) to (A.3),

$$\|V(\cdot, t)\|_{H_0^1} + \|V_t(\cdot, t)\|_{L_2} \leq C_\sigma e^{-\sigma t} \left(\|Z_1\|_{H_0^1} + \|V_t(\cdot, 0)\|_{L_2} \right) + C_\sigma \int_0^t e^{-\sigma(t-s)} |\Theta'(s)| \, ds.$$

Since $\Theta' \in L_1(0, \infty)$,

$$\int_0^t e^{-\sigma(t-s)} |\Theta'(s)| \, ds \longrightarrow 0.$$

Therefore

$$\sup_{t \geq 0} \left(\|Z_t(\cdot, t)\|_{H_0^1} + \|Z_{tt}(\cdot, t)\|_{L_2} \right) < \infty,$$

and

$$\|Z_t(\cdot, t)\|_{H_0^1} + \|Z_{tt}(\cdot, t)\|_{L_2} \longrightarrow 0.$$

From (3.5),

$$\kappa^2 \|Z_{yy}(\cdot, t)\|_{L_2} \leq \|Z_{tt}(\cdot, t)\|_{L_2} + W^{-1} \|Z_t(\cdot, t)\|_{L_2} + |\Theta(t)|.$$

Thus

$$\sup_{t \geq 0} \|Z(\cdot, t)\|_{H^2} < \infty.$$

The one-dimensional Gagliardo-Nirenberg inequality yields, with a constant $C > 0$ independent of t ,

$$\|Z_y(\cdot, t)\|_{L_\infty} \leq C \|Z_y(\cdot, t)\|_{L_2}^{1/2} \|Z_y(\cdot, t)\|_{H^1}^{1/2}. \quad (\text{A.4})$$

The second factor in (A.4) is uniformly bounded, whereas

$$\|Z_y(\cdot, t)\|_{L_2} \longrightarrow 0.$$

Hence

$$\sup_{t \geq 0} \|Z_y(\cdot, t)\|_{L_\infty} < \infty, \quad \|Z_y(\cdot, t)\|_{L_\infty} \longrightarrow 0.$$

□



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