



*Research article***Boundary geometry and directional modulational instability in a generalized (2+1)-dimensional Davey-Stewartson system****Chun Tong***

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Abstract: We study a generalized (2+1)-dimensional Davey–Stewartson system with arbitrary-power and double-power nonlinearities. For a general propagation direction, the amplitude coordinate $\rho = |u|^p$ reduces the traveling-wave problem to a planar system that is singular at $\rho = 0$ for every real $p > 1$. A time rescaling yields a polynomial system with an invariant boundary. We clarify the physical domain and uniqueness properties of this regularization, rederive the known arbitrary-power first integral on $\rho > 0$, and classify seven complete configurations of positive interior roots, including the no-positive-root case and two double-root boundaries. A representative $p = 2$ solitary wave is verified analytically and by direct pseudospectral simulation of the original partial differential equations, with relative L^2 error 3.31×10^{-7} at $t = 1$. Linearization about a continuous wave gives a directional modulational-instability criterion that separates the effects of local nonlinearities and mean-field coupling. A two-dimensional simulation with oblique perturbation vector (2, 1) gives growth rate 3.60555, agreeing with the theoretical value $\sqrt{13}$ at $p = 2$; a separate $p = 3/2$ test also agrees with its prediction. Equal-magnitude oblique perturbations can grow or oscillate solely because of their orientation. Boundary-reaching orbit segments are identified, but extending them by zero generally does not produce classical compactly supported solutions. A weak-solution analysis and nonlinear stability theory remain open.

Keywords: Davey-Stewartson system; arbitrary-power nonlinearity; singular dynamical system; boundary regularization; modulational instability; first integral

Mathematics Subject Classification: 35C07, 35B35, 34C23, 37K10

1. Introduction

The Davey-Stewartson (DS) equations are fundamental models for weakly nonlinear wave packets in finite-depth water and resonant long-short wave interactions [1, 2]. Dispersive surface-wave models and symbolic constructions in finite water depth provide further physical context [3, 4]. Their standard

cubic forms support solitons, dromions, lumps, and rogue waves [5–7]. Recent studies of DS-type systems include exact soliton constructions and stochastic sensitivity analysis [8, 9]. Other work develops rogue curves and their large-time dynamics, as well as spatially bounded rogue waves in the DS I equation [10–12].

Arbitrary-power nonlinearities are used phenomenologically when a cubic constitutive law is insufficient. In nonlinear optics, they represent non-Kerr or effective power-law responses; in plasma envelope models, they can encode nonstandard equations of state or distribution-dependent responses; and in condensed-matter mean-field models, they provide an interpolation between polynomial interaction laws. In this paper, u denotes the complex wave envelope and v denotes the real, slowly varying mean field or potential. The exponent p controls both the local response $|u|^p u$ and the source $|u|^p$ of the mean field. Fractional p should therefore be interpreted as an effective dimensionless constitutive exponent, not as a universal material constant. Quantitative parameter ranges require calibration for a specific physical system; our analysis treats the dimensionless range $p > 1$ and records this modeling limitation explicitly.

We consider

$$\begin{cases} iu_t + u_{xx} + u_{yy} + \gamma|u|^p u + \alpha uv + \delta|u|^{2p} u = 0, \\ v_{xx} + v_{yy} = \beta(|u|^p)_{xx}. \end{cases} \quad (1.1)$$

Here x and y are the spatial coordinates, t is time, $u(x, y, t) \in \mathbb{C}$ is the wave envelope, and $v(x, y, t) \in \mathbb{R}$ is the mean field. The real exponent $p > 1$ specifies the power-law response. The real coefficients γ and δ weight the local nonlinearities $|u|^p u$ and $|u|^{2p} u$, respectively. α is the coupling coefficient between the envelope and the mean field, and β weights the mean-field source. Subscripts denote partial derivatives. All variables and coefficients are dimensionless. Expansion and auxiliary-equation approaches produce explicit wave profiles for generalized DS equations with power-law nonlinearities [13, 14]. Unlike a prescribed-profile construction, a dynamical-system analysis organizes the accessible orbits before inverting them. Wang et al. [15] already applied that approach to the same DS model; our analysis builds on, rather than originates, their reduction.

Table 1 states the relation to the closest literature. In particular, [15] already gives the polynomial regularization, its equilibrium Jacobian, the same first integral (their Eqs (4.6) and (4.7)), and bifurcation phase portraits for selected parameter regimes. We do not claim priority for those results. The present work clarifies the domain and boundary limits of that reduction, enumerates the positive interior-root configurations, extends the traveling direction and real exponent range, and derives and tests the directional MI criterion at the PDE level. We do not claim to recover every exact solution generated by other methods.

This study has four aims. First, it derives the traveling reduction for an arbitrary direction rather than only $x+y$. Second, it distinguishes the singular system on $\rho > 0$ from its previously reported polynomial regularization on $\rho \geq 0$ and states what can and cannot be concluded at the boundary. Third, it rederives the known arbitrary-power first integral and establishes the completeness of seven physical positive-root configurations. Fourth, it derives the directional modulational-instability criterion and tests the original PDE for both integer and noninteger p .

Table 1. Comparison with representative analytical studies. The final column identifies the issue addressed here rather than ranking the methods.

References	Scope and technique	Relation to the present analysis
[3,4]	Dispersive DS systems; reductive perturbation and symbolic solutions	Physical DS context without the arbitrary- p boundary coordinate used here
[13,14]	Power-law DS models; expansion and auxiliary equations	Explicit families from prescribed trial functions, rather than an orbit classification
[15]	Same DS model; regularized system, equilibrium Jacobian, first integral, selected bifurcation portraits, and exact profiles	Those ingredients are prior work; here we analyze boundary admissibility, enumerate all positive-root configurations, extend direction and real $p > 1$, and test directional MI in the full PDE

2. The model and traveling-wave reduction

2.1. Traveling-wave ansatz

Let $\ell, m \in \mathbb{R}$, with $(\ell, m) \neq (0, 0)$, specify the direction of variation of the traveling profile, and set

$$u(x, y, t) = \rho^{1/p}(\chi) e^{i\vartheta}, \quad v(x, y, t) = \psi(\chi), \quad \chi = \ell x + m y - ct, \quad \vartheta = kx + \lambda y - \omega t. \quad (2.1)$$

Here χ is the traveling coordinate and ϑ is the carrier phase. The real constants k and λ are the carrier wavenumbers in the x and y directions, respectively. ω is the carrier angular frequency; c is the translation parameter in χ ; the profile velocity normal to a level set of χ is $c/\sqrt{D}$. The real functions $\rho(\chi) = |u|^p \geq 0$ and $\psi(\chi)$ are the intensity-related profile and the mean-field profile, respectively. We write $D := \ell^2 + m^2 > 0$ for the squared norm of the direction vector and $\eta := \ell^2/D \in [0, 1]$ for its squared x -component fraction. Primes below denote differentiation with respect to χ . The choice $\rho = |u|^p$ is useful because both $|u|^p$ and $|u|^{2p}$ become the polynomial variables ρ and ρ^2 , while the mean-field source is also linear in ρ . This is an algebraic reason for the fractional-power amplitude; it is not a consequence of a scaling symmetry.

Writing $R(\chi) = \rho^{1/p}(\chi)$ for the nonnegative envelope amplitude gives

$$R' = \frac{1}{p} \rho^{1/p-1} \rho', \quad R'' = \frac{1}{p} \rho^{1/p-1} \rho'' + \frac{1}{p} \left(\frac{1}{p} - 1 \right) \rho^{1/p-2} (\rho')^2. \quad (2.2)$$

The imaginary part of the first PDE is $[-c + 2(k\ell + \lambda m)]R' = 0$; hence, a nonconstant profile requires

$$c = 2(k\ell + \lambda m). \quad (2.3)$$

The second PDE becomes $D\psi'' = \beta\ell^2\rho''$. After two integrations,

$$\psi = \beta\eta\rho + d_0 + d_1\chi. \quad (2.4)$$

Here $d_0, d_1 \in \mathbb{R}$ are integration constants. For bounded or localized waves on the whole real line, $d_1 = 0$, and the constant d_0 can be absorbed into ω ; we therefore set $d_0 = d_1 = 0$ below. The real part is

$$(\omega - k^2 - \lambda^2)R + DR'' + (\gamma\rho + \alpha\psi + \delta\rho^2)R = 0. \quad (2.5)$$

Substituting (2.2) and (2.4) into (2.5), dividing by $D\rho^{1/p-1}/p$, and moving the nonlinear terms to the right gives

$$\rho'' = \frac{a(\rho')^2}{\rho} + A\rho^3 + B\rho^2 + C\rho, \quad (2.6)$$

where

$$a = 1 - \frac{1}{p}, \quad A = -\frac{p\delta}{D}, \quad B = -\frac{p}{D}(\gamma + \alpha\beta\eta), \quad C = \frac{p}{D}(k^2 + \lambda^2 - \omega). \quad (2.7)$$

The diagonal reduction is the special case $\ell = m = 1$, for which $D = 2$, $\eta = 1/2$, and $c = 2(k + \lambda)$. Thus, the diagonal choice is not needed for the subsequent analysis; the propagation direction enters only through A , B , and C .

Remark 1. The term $a(\rho')^2/\rho$ records the loss of invertibility of $R \mapsto \rho = R^p$ at $R = 0$. For $p = 1$ it vanishes. For every $p > 1$, including noninteger p , Eq (2.6) is a smooth ODE only on $\rho > 0$. The singularity is therefore a property of the amplitude coordinate and must not be confused with a proof of singular behavior in the original PDE.

2.2. The reduced planar system

In the phase plane, y denotes the auxiliary variable ρ' , rather than the spatial coordinate. Setting $y = \rho'$ recasts (2.6) as the first-order pair

$$\frac{d\rho}{d\chi} = y, \quad \frac{dy}{d\chi} = \frac{ay^2 + \rho^2(A\rho^2 + B\rho + C)}{\rho}. \quad (2.8)$$

The vector field in (2.8) is undefined at $\rho = 0$, so it is not mathematically correct to call that boundary an ordinary invariant line of the original χ -system. For each initial condition with $\rho_0 > 0$, however, the vector field is smooth and the Picard-Lindelöf theorem gives a unique local solution up to the first time it approaches the boundary or becomes unbounded.

Define ζ by $d\chi = \rho d\zeta$. This change is legitimate only while $\rho > 0$ and gives the polynomial system

$$\frac{d\rho}{d\zeta} = \rho y, \quad \frac{dy}{d\zeta} = ay^2 + \rho^2(A\rho^2 + B\rho + C). \quad (2.9)$$

On $\rho > 0$ the two systems trace the same oriented curves and differ only in their parametrization.

Proposition 1 (Boundary invariance and limits of the regularization). *System (2.9) has a unique local solution through every point of $\mathbb{R}^2$. Its line $\rho = 0$ is invariant, and a solution with $\rho_0 > 0$ cannot cross that line at finite ζ . These statements do not provide a unique continuation of (2.8) through $\rho = 0$ in the original variable χ .*

Proof. The right-hand side of (2.9) is polynomial and therefore locally Lipschitz. Moreover,

$$\rho(\zeta) = \rho_0 \exp\left(\int_{\zeta_0}^{\zeta} y(s) ds\right), \quad (2.10)$$

as long as the solution is finite. Hence, the sign of ρ is preserved; if $\rho_0 = 0$, uniqueness gives $\rho \equiv 0$. On the boundary, $y' = ay^2$, so points $(0, y_0)$ with $y_0 \neq 0$ move along the boundary and only $(0, 0)$ is an equilibrium. Finally, $d\chi/d\zeta = \rho$ vanishes on the boundary. The time map is not invertible there, and the boundary flow of (2.9) has no corresponding vector field in (2.8). A continuation across $\rho = 0$ would therefore require an additional weak or free-boundary formulation. $\square$

3. Verified symmetry subgroup and scope of the reduction

We only use symmetries that can be verified directly; no claim of a complete Lie-point-symmetry classification is made. Equation (1.1) is invariant under translations in x , y , and t and under the global phase $u \mapsto e^{i\theta}u$. It is also invariant under the two-parameter Galilean transformation

$$\begin{aligned}\widetilde{u}(x, y, t) &= e^{i[s_1x + s_2y - (s_1^2 + s_2^2)t]} u(x - 2s_1t, y - 2s_2t, t), \\ \widetilde{v}(x, y, t) &= v(x - 2s_1t, y - 2s_2t, t),\end{aligned}\tag{3.1}$$

for $s_1, s_2 \in \mathbb{R}$. Substitution shows that the terms generated by differentiating the phase cancel the translated first derivatives, while $|\widetilde{u}| = |u|$ preserves both nonlinear powers and the mean-field source.

There is no single fixed-coefficient scaling symmetry when both γ and δ are nonzero. Indeed, if spatial and temporal scaling make u_t and Δu homogeneous of weight two, then matching $|u|^p u$ requires one amplitude weight, while matching $|u|^{2p} u$ requires half that weight. The two requirements are incompatible for $p > 0$. A scaling can be recovered in special one-power submodels or by transforming the coefficients, but not for the general equation with fixed nonzero γ and δ .

Consequently, the fractional power in (2.1) is justified by the polynomialization $\rho = |u|^p$, not by a claimed scaling generator. The velocity condition (2.3) and the general direction (ℓ, m) follow from direct substitution. We make no assertion that (2.1) exhausts all possible symmetry reductions.

4. First integrals for arbitrary power p

Wang et al. [15] reported the same first-integral formula in their Eq (4.7) under the assumption that p is a positive integer. The following direct integrating-factor proof emphasizes that the identity holds for every real $p > 1$ on $\rho > 0$ and identifies its normalized uniqueness class. This is a clarification and extension of the domain, not a claim that the integral itself is new.

Theorem 1. *Let $p > 1$, so $a = 1 - 1/p \in (0, 1)$. On every solution interval of (2.8) contained in $\rho > 0$,*

$$H_p(\rho, y) = y^2 \rho^{-2a} - \frac{2A}{4-2a} \rho^{4-2a} - \frac{2B}{3-2a} \rho^{3-2a} - \frac{2C}{2-2a} \rho^{2-2a} = h.\tag{4.1}$$

In the normalized class $y^2 \rho^{-2a} + F(\rho)$, this first integral is unique up to an additive constant. More generally, any differentiable one-to-one function of H_p represents the same level-set foliation.

Proof. On any subinterval where $y \neq 0$, set $z(\rho) = y^2$. Then, $dz/d\rho = 2y dy/d\rho$, and (2.8) gives

$$\frac{dz}{d\rho} - \frac{2a}{\rho} z = 2\rho(A\rho^2 + B\rho + C).$$

Because $\rho > 0$, the integrating factor ρ^{-2a} is real, positive, and smooth. Thus,

$$\frac{d}{d\rho}(z\rho^{-2a}) = 2A\rho^{3-2a} + 2B\rho^{2-2a} + 2C\rho^{1-2a},$$

and integration gives (4.1). The denominators are nonzero for $a \in (0, 1)$. At a turning point $y = 0$, direct differentiation of (4.1) along (2.8) gives $dH_p/d\chi = 0$, so the same constant extends continuously across the turning point. Finally, the first-order linear equation for z has a unique solution for prescribed (ρ_0, z_0) on $\rho > 0$. Fixing the coefficient of $y^2\rho^{-2a}$ to one therefore fixes F up to the integration constant. $\square$

Remark 2. Using $a = 1 - 1/p$, equation (4.1) can also be written

$$H_p = y^2\rho^{-2+2/p} - \frac{Ap}{p+1}\rho^{2+2/p} - \frac{2Bp}{p+2}\rho^{1+2/p} - Cp\rho^{2/p}. \quad (4.2)$$

For $p = 2$ it reduces to $H_2 = y^2/\rho - (2A/3)\rho^3 - B\rho^2 - 2C\rho$. More generally, Eq (4.1) is algebraically identical to Eq (4.7) of [15] after collecting powers of ρ .

5. Bifurcation of the singular planar system

Let $f(\rho) = A\rho^2 + B\rho + C$. The Jacobian of (2.9) is

$$M(\rho_*, y_*) = \begin{pmatrix} y_* & \rho_* \\ 4A\rho_*^3 + 3B\rho_*^2 + 2C\rho_* & 2ay_* \end{pmatrix}. \quad (5.1)$$

In the physical interior $\rho > 0$, every equilibrium has the form $E_r = (r, 0)$, where $f(r) = 0$. At a simple positive root,

$$M(r, 0) = \begin{pmatrix} 0 & r \\ r^2 f'(r) & 0 \end{pmatrix}, \quad \lambda^2 = r^3 f'(r), \quad \text{tr } M(r, 0) = 0. \quad (5.2)$$

The equilibrium Jacobian also appears in [15]. Here we apply it only to physical interior roots. Thus, $f'(r) > 0$ gives a saddle and $f'(r) < 0$ gives a linear center; nodes and foci are excluded because the trace is exactly zero, not merely because a first integral exists. Near a nondegenerate center, the level sets of H_p are closed curves, which confirms the nonlinear center classification. If $f'(r) = 0$, the equilibrium is degenerate and linearization does not classify it. The origin $E_0 = (0, 0)$ is also degenerate, but Proposition 1 shows that it must be analyzed as a boundary equilibrium of the regularized problem rather than as an interior equilibrium of the singular χ -system.

The number and type of positive roots now give a complete classification. A quadratic has zero, one, or two positive roots. If it has two simple roots $0 < r_1 < r_2$, then

$$f'(r_1) = A(r_1 - r_2), \quad f'(r_2) = A(r_2 - r_1), \quad (5.3)$$

so their types are fixed by the sign of A . If $AC < 0$, the roots have opposite signs and there is exactly one positive root, whose type is again fixed by A . If $AC > 0$, two positive roots require $-B/A > 0$ and $\Delta := B^2 - 4AC > 0$. Equality $\Delta = 0$ gives a positive double root when $-B/(2A) > 0$. These alternatives produce the seven configurations in Table 2; these exhaust the possibilities for $A \neq 0$ and $C \neq 0$.

The codimension-one surface $C = 0$ places one root at the boundary; the second root $-B/A$ is positive exactly when $AB < 0$. The surface $A = 0$ reduces f to a linear polynomial. These are limiting parameter surfaces, not additional interior root configurations. Figure 1 shows the classification in the (B, C) -plane for normalized values $A = \pm 1$.

Table 2. Complete positive-root classification for $A \neq 0$ and $C \neq 0$. The boundary equilibrium $E_0 = (0, 0)$ of the regularized system is present in every row.

Case	Conditions	Positive equilibria and type
1	All remaining $A \neq 0, C \neq 0$ regimes	No positive interior equilibrium
2	$A > 0, C < 0$	One positive saddle
3	$A < 0, C > 0$	One positive center
4	$A > 0, B < 0, C > 0, \Delta > 0$	r_1 center and r_2 saddle
5	$A < 0, B > 0, C < 0, \Delta > 0$	r_1 saddle and r_2 center
6	$A > 0, B < 0, C > 0, \Delta = 0$	One positive degenerate double root
7	$A < 0, B > 0, C < 0, \Delta = 0$	One positive degenerate double root

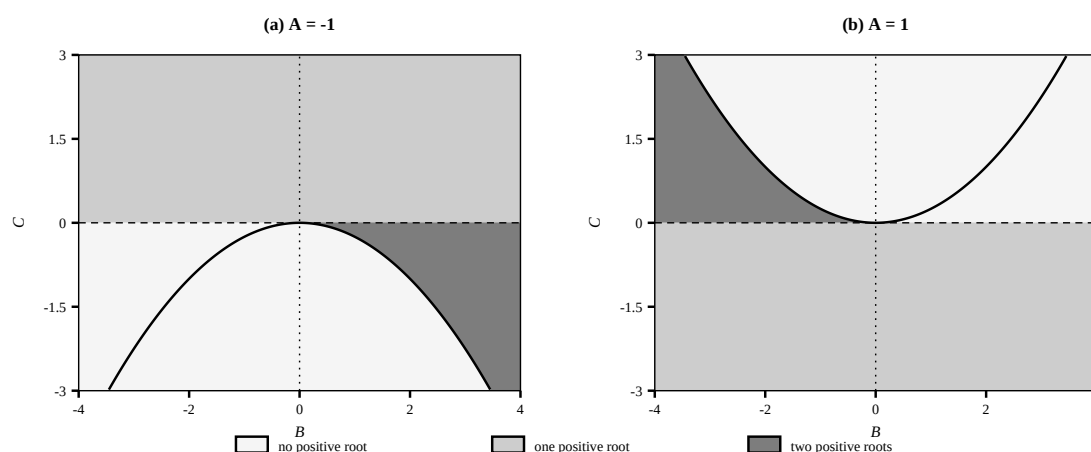


Figure 1. Number of positive roots of $f(\rho) = A\rho^2 + B\rho + C$ for (a) $A = -1$ and (b) $A = 1$. The solid parabola is $\Delta = 0$, the dashed line is $C = 0$, and the dotted line is $B = 0$. For $A = -1$, a single positive root is a center and two positive roots are ordered saddle-center. For $A = 1$, a single positive root is a saddle and two positive roots are ordered center-saddle. The grayscale shading is keyed below the panels.

6. Traveling-wave quadrature and a verified solitary wave

Before using it in the quadrature, define

$$Q_p(\rho; h) := h + \frac{Ap}{p+1}\rho^{2+2/p} + \frac{2Bp}{p+2}\rho^{1+2/p} + C\rho^{2/p}. \quad (6.1)$$

Equation (4.2) is equivalent to

$$(\rho')^2 = \rho^{2a} Q_p(\rho; h), \quad \chi - \chi_0 = \pm \int_{\rho_0}^{\rho} \frac{s^{-a} ds}{\sqrt{Q_p(s; h)}}. \quad (6.2)$$

The admissible intervals are precisely those on which $Q_p \geq 0$. Simple positive zeros are turning points; two such turning points surrounding a center generate a periodic quadrature, while a repeated zero can generate a homoclinic or heteroclinic limit. This root statement is more general than a single closed-form ansatz and remains valid for noninteger p . For $p = 2$,

$$(\rho')^2 = \rho \left[h + \frac{2A}{3} \rho^3 + B\rho^2 + 2C\rho \right], \quad (6.3)$$

so generic bounded profiles are obtained by an elliptic integral. Closed-form inversion requires parameter-specific root conditions and is not needed for the classification.

6.1. Explicit solitary wave and analytical substitution

A simple closed form is available for $p = 2$, $B = 0$, $A < 0$, $C > 0$, and $h = 0$:

$$\rho(\chi) = \left(\frac{3C}{|A|} \right)^{1/2} \operatorname{sech}(\sqrt{2C}\chi). \quad (6.4)$$

For an explicit solution of the original PDE, choose

$$\ell = 1, \quad m = k = \lambda = 0, \quad p = 2, \quad \alpha = \beta = \delta = 1, \quad \gamma = -1, \quad \omega = -\frac{1}{2}. \quad (6.5)$$

Then, $A = -2$, $B = 0$, $C = 1$, $\chi = x$, and $v = \rho$. The solution is

$$u(x, y, t) = R(x)e^{it/2}, \quad R(x) = \left(\frac{3}{2} \right)^{1/4} \operatorname{sech}^{1/2}(\sqrt{2}x), \quad v(x, y, t) = R^2(x). \quad (6.6)$$

The cubic terms cancel because $\gamma|u|^2u + uv = 0$. Moreover,

$$\frac{R''}{R} = \frac{1}{2} - \frac{3}{2} \operatorname{sech}^2(\sqrt{2}x) = \frac{1}{2} - R^4, \quad (6.7)$$

and therefore substitution into the first PDE gives

$$e^{-it/2} \left(iu_t + u_{xx} - |u|^2u + uv + |u|^4u \right) = R'' - \frac{1}{2}R + R^5 = 0. \quad (6.8)$$

The second PDE reduces to $v_{xx} = (|u|^2)_{xx}$ and is also satisfied identically. This supplies an explicit analytical back-substitution rather than only a symbolic residual statement.

6.2. Behavior at the singular boundary

The first integral also determines the edge regularity.

Proposition 2. *Let $p > 1$ and let a positive traveling profile on a branch of $H_p = h > 0$ tend to $\rho = 0$ at an edge. The edge occurs at finite $\chi = \chi_e$, and extending $u = \rho^{1/p}e^{i\theta}$ by zero across that edge does not give a classical solution of (1.1).*

Proof. Equation (4.2) gives $y^2 = h\rho^{2-2/p}[1 + o(1)]$ as $\rho \rightarrow 0^+$. On a branch approaching the edge, $y = \rho'$ has a fixed sign sufficiently close to zero, so integration yields

$$\rho' = \pm \sqrt{h}\rho^{1-1/p}[1 + o(1)], \quad \rho^{1/p} = \frac{\sqrt{h}}{p}|\chi - \chi_e|[1 + o(1)]. \quad (6.9)$$

Consequently $R = \rho^{1/p}$ has a nonzero one-sided χ -derivative of magnitude $\sqrt{h}/p$. Its zero extension has a derivative jump, so u is not even continuously differentiable across the edge and cannot be a classical solution of the second-order PDE. $\square$

If $h = 0$ and $C > 0$, then $(\rho')^2 \sim Cp\rho^2$, so the boundary is approached exponentially at infinite $|\chi|$ rather than at a compact edge. Accordingly, we refer to finite-edge curves only as boundary-reaching orbit segments. Whether a particular gluing defines a weak solution depends on an explicitly chosen weak formulation and transmission condition, which are outside the present analysis.

7. Modulational instability of the plane-wave background

To distinguish the continuous-wave (CW) amplitude from the reduction coefficient A , write the amplitude as $\mathcal{A} > 0$. A CW state is

$$u_0(t) = \mathcal{A}e^{i\omega t}, \quad v_0 = v_0^{\text{const}}, \quad \omega = \gamma\mathcal{A}^p + \alpha v_0^{\text{const}} + \delta\mathcal{A}^{2p}. \quad (7.1)$$

Because the elliptic mean-field equation determines v only up to a constant, we set $v_0^{\text{const}} = 0$ and absorb any other choice into ω .

Let

$$u(x, y, t) = [\mathcal{A} + a_1(x, y, t) + ia_2(x, y, t)]e^{i\omega t}, \quad v(x, y, t) = v_0 + b(x, y, t), \quad (7.2)$$

where a_1 , a_2 , and b are real. Using $|u|^p = \mathcal{A}^p + p\mathcal{A}^{p-1}a_1 + O(a_1^2 + a_2^2)$ and retaining linear terms gives

$$\begin{cases} a_{2,t} = \Delta_{\perp}a_1 + p\mathcal{A}^p(\gamma + 2\delta\mathcal{A}^p)a_1 + \alpha\mathcal{A}b, \\ a_{1,t} = -\Delta_{\perp}a_2, \\ \Delta_{\perp}b = p\beta\mathcal{A}^{p-1}a_{1,xx}, \end{cases} \quad (7.3)$$

where $\Delta_{\perp} = \partial_{xx} + \partial_{yy}$. To avoid confusion with the carrier wavenumber k in (2.1), use the perturbation vector $\mathbf{q} = (q_x, q_y)$ and $q^2 = q_x^2 + q_y^2$. For nonzero modes ($q > 0$) proportional to $\exp[i(q_x x + q_y y - \Omega t)]$ where Ω is the perturbation frequency, the elliptic equation gives

$$b = p\beta\mathcal{A}^{p-1}\frac{q_x^2}{q^2}a_1. \quad (7.4)$$

Eliminating a_2 and b yields the anisotropic dispersion relation

$$\Omega^2 = q^4 - p\mathcal{A}^p(\gamma + 2\delta\mathcal{A}^p)q^2 - \alpha\beta p\mathcal{A}^p q_x^2. \quad (7.5)$$

The sign in (7.4) is positive because both Laplacians contribute a minus sign in Fourier space. Substitution into the amplitude equation nevertheless gives the negative mean-field term in (7.5).

Let $\cos \theta = q_x/q$. Then,

$$\Omega^2 = q^2 \left[q^2 - p\mathcal{A}^p F(\theta, \mathcal{A}) \right], \quad F(\theta, \mathcal{A}) = \gamma + 2\delta\mathcal{A}^p + \alpha\beta \cos^2 \theta. \quad (7.6)$$

For a fixed direction, the CW is unstable exactly when $F > 0$, and the unstable band and maximum gain are

$$0 < q^2 < p\mathcal{A}^p F, \quad q_{\max}^2 = \frac{p\mathcal{A}^p F}{2}, \quad g_{\max} = \frac{p\mathcal{A}^p F}{2}. \quad (7.7)$$

Proposition 3. *The CW background is modulationally unstable in direction θ if and only if $\gamma + 2\delta\mathcal{A}^p + \alpha\beta \cos^2 \theta > 0$. The one-dimensional x -directed criterion is recovered at $\theta = 0$, while a purely transverse perturbation with $q_x = 0$ does not couple to the mean field at linear order.*

For clarity, the preceding criterion means that at least one nonzero wave-number magnitude in that direction has positive gain; an individual mode must also lie in the band (7.7). The angular sector itself has a simple closed classification.

Corollary 1. *Set $G = \gamma + 2\delta\mathcal{A}^p$ and $K = \alpha\beta$. At least one direction is unstable if and only if $G + \max\{K, 0\} > 0$, and every direction is unstable if and only if $G + \min\{K, 0\} > 0$. If $K \neq 0$ and $-G/K \in [0, 1]$, neutral angular boundaries satisfy $\cos^2 \theta = -G/K$.*

Proof. The function $F(\theta, \mathcal{A}) = G + K \cos^2 \theta$ is affine in $\cos^2 \theta \in [0, 1]$, so its maximum and minimum are $G + \max\{K, 0\}$ and $G + \min\{K, 0\}$. The stated strict inequalities and zero set follow. $\square$

The quantity $\gamma + 2\delta\mathcal{A}^p$ is the effective local focusing coefficient. Positive $\alpha\beta$ enhances focusing most strongly for x -directed perturbations; negative $\alpha\beta$ can suppress those directions while leaving transverse perturbations unchanged. The exponent p affects both the prefactor $p\mathcal{A}^p$ and the double-power correction $2\delta\mathcal{A}^p$. Consequently, increasing p does not have a universal monotone effect: it depends on whether the dimensionless amplitude is below or above one and on the signs of δ and $\alpha\beta$. The classical cubic DS limit is $p = 2$ and $\delta = 0$, for which (7.6) reduces to the familiar directional focusing/defocusing criterion.

8. Direct numerical validation of the original PDE

We used a Fourier pseudospectral Strang-splitting method on the original PDE rather than on the reduced ODE or the linearized equations. The linear half-step multiplies Fourier mode (q_x, q_y) by $\exp[-i(q_x^2 + q_y^2)\Delta t/2]$. At the nonlinear step, the elliptic equation is solved spectrally using $\widehat{v} = \beta q_x^2(q_x^2 + q_y^2)^{-1}|\widehat{u}|^p$ for nonzero modes, and the phase

$$\exp \left\{ i \left[\gamma |u|^p + \alpha v + \delta |u|^{2p} \right] \Delta t \right\} \quad (8.1)$$

is applied pointwise. The second linear half-step completes one time step.

For the y -independent solitary wave (6.6), the domain was $[-30, 30]$ with 2048 Fourier points, $\Delta t = 5 \times 10^{-4}$, and final time $t = 1$. The larger domain suppresses the periodic-boundary truncation error. The decaying choice of the otherwise arbitrary mean-field constant gives $v = |u|^2$. The relative error

$$\frac{\|u_{\text{num}}(\cdot, 1) - u_{\text{exact}}(\cdot, 1)\|_2}{\|u_{\text{exact}}(\cdot, 1)\|_2} = 3.31 \times 10^{-7}, \quad (8.2)$$

and the maximum amplitude error is 1.52×10^{-7} .

For the genuinely two-dimensional MI test, we used the periodic domain $[0, 2\pi]^2$, a 64×64 Fourier grid, $\Delta t = 2 \times 10^{-4}$, $t \leq 1$, and

$$p = 2, \quad \mathcal{A} = 1, \quad \gamma = \delta = \alpha = \beta = 1, \quad u(x, y, 0) = 1 + 10^{-7} \cos(2x + y). \quad (8.3)$$

The zero Fourier mode of v was set to zero. With $(q_x, q_y) = (2, 1)$, Eq (7.5) gives $\Omega^2 = -13$ and theoretical growth $g = \sqrt{13} = 3.6055513$. The real Fourier sideband follows its linear prediction $a_1(0) \cosh(\sqrt{13}t)$ with relative curve error 5.87×10^{-7} . Inferring the growth from its endpoint ratio gives $g_{\text{num}} = 3.6055506$.

To check a noninteger exponent in the original PDE, we repeated the same two-dimensional experiment with $p = 3/2$, leaving all other parameters and the initial perturbation unchanged. Now $\Omega^2 = -7/2$, so $g = \sqrt{7/2} = 1.8708287$. The measured endpoint growth is 1.8708280, and the relative sideband-curve error is 4.89×10^{-7} . Table 3 reports three time steps at fixed spatial grids for both exponents. The errors decrease by factors between 3.45 and 4.35 under time-step halving, consistent with the second-order splitting scheme over this short interval but not a formal order proof. At the baseline time step, halving spatial resolution changes the $p = 2$ errors by less than 4×10^{-9} and the $p = 3/2$ MI error by 1.37×10^{-8} . Figure 2 shows the baseline $p = 2$ runs. Numerical samples are identified by open markers to keep the data distinguishable where the curves overlap; the supplied script reproduces all reported values. The fractional- p test checks linear MI only, not a solitary-wave family or nonlinear stability.

To isolate the directional effect rather than merely verify growth in one oblique mode, we also set $p = 2$, $\mathcal{A} = 1$, $\gamma = -2$, $\delta = 0$, $\alpha = 6$, and $\beta = 1$ on the same 64×64 periodic grid. Here $G = -2$, $K = 6$, and Corollary 1 gives the neutral boundary $\cos^2 \theta = 1/3$. The modes $(q_x, q_y) = (2, 1)$ and $(1, 2)$ have the same $q^2 = 5$ but lie on opposite sides of this boundary: Equation (7.5) predicts $\Omega^2 = -3$ and 33, respectively. Direct evolution from $u(x, y, 0) = 1 + 10^{-7} \cos(q_x x + q_y y)$ at $\Delta t = 2 \times 10^{-4}$ up to $t = 1$ gives normalized real sideband factors 2.9145754 and 0.8584160, versus the linear predictions $\cosh(\sqrt{3}) = 2.9145774$ and $\cos(\sqrt{33}) = 0.8584160$. Thus, equal-magnitude oblique modes can grow or oscillate solely because of their orientation. These small-perturbation, short-time checks do not establish nonlinear stability.

Table 3. Time-step refinement for direct PDE checks at $t = 1$. The solitary-wave error uses $p = 2$ on $[-30, 30]$ with 2048 points. MI entries are relative Fourier-sideband curve errors on $[0, 2\pi]^2$ with a 64×64 grid; both MI tests use the same coefficients and mode, differing only in p .

Soliton Δt	Soliton error	MI Δt	MI error ($p = 2$)	MI error ($p = 3/2$)
10^{-3}	1.32×10^{-6}	4×10^{-4}	2.32×10^{-6}	1.99×10^{-6}
5×10^{-4}	3.31×10^{-7}	2×10^{-4}	5.87×10^{-7}	4.89×10^{-7}
2.5×10^{-4}	8.27×10^{-8}	10^{-4}	1.35×10^{-7}	1.42×10^{-7}

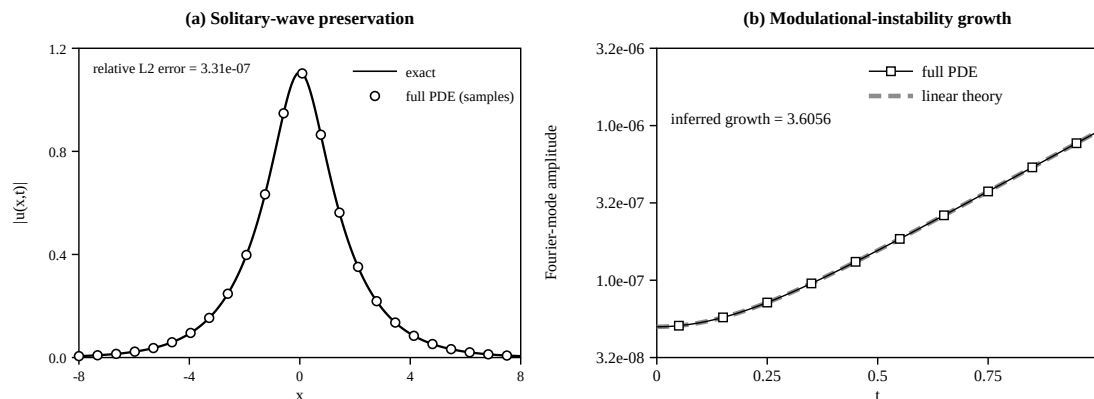


Figure 2. Direct simulations of the original PDE. (a) The exact $p = 2$ solitary amplitude (6.6) (black solid line) and the numerical amplitude at $t = 1$ (open circles) for the parameters (6.5). (b) Growth of the oblique $(q_x, q_y) = (2, 1)$ Fourier sideband for $p = 2$, $\mathcal{A} = 1$, and $\gamma = \delta = \alpha = \beta = 1$: the full-PDE result is shown by a black solid line with open squares, and the linear prediction $a_1(0) \cosh(\sqrt{13}t)$ by a thick gray dashed line. Only selected numerical samples are marked. The agreement causes the curves to overlap at plotting resolution; the distinct markers and explicit legends identify both data sets in grayscale.

9. Conclusions

The amplitude variable $\rho = |u|^p$ converts the arbitrary- and double-power nonlinearities into a polynomial but makes the traveling-wave ODE singular at $\rho = 0$ for $p > 1$. On $\rho > 0$, the time rescaling $d\chi = \rho d\zeta$ gives an equivalent polynomial orbit problem. Its boundary is invariant by uniqueness, although the degenerate time map does not define a continuation of the original singular ODE across that boundary.

The previously reported first integral extends directly to every real $p > 1$ on $\rho > 0$. Classifying the positive roots of $A\rho^2 + B\rho + C$ gives seven complete interior equilibrium configurations, with the surfaces $A = 0$ and $C = 0$ treated as degree-reduction and boundary-root limits. A representative $p = 2$ solitary wave satisfies the original PDE exactly and is preserved by direct numerical evolution. The CW calculation identifies the effective directional coefficient $\gamma + 2\delta\mathcal{A}^p + \alpha\beta \cos^2 \theta$ and its angular instability sectors. Direct two-dimensional tests find the predicted MI growth at $p = 2$ and $p = 3/2$, and contrasting growth versus oscillation for equal-magnitude oblique modes on opposite sides of a neutral angular boundary.

The present results have four limitations. First, the reduction describes one-dimensional traveling profiles embedded in two spatial dimensions and does not cover general two-dimensional coherent structures. Second, the numerical tests establish short-time preservation of one solitary wave and early linear MI growth; they are not a nonlinear orbital-stability proof. Third, finite-edge orbit segments generally fail to give classical compactly supported PDE solutions when glued to zero, and no weak-solution theory is supplied. Fourth, the dimensionless arbitrary-power model has not been calibrated to a particular optical, plasma, or condensed-matter experiment. Future work should

formulate admissible weak transmission conditions, perform long-time two-dimensional simulations with oblique perturbations, determine nonlinear stability of the solitary family, and connect p and the coefficients to measured constitutive laws.

Use of Generative-AI tools declaration

The author used OpenAI's Codex to assist with formatting, language editing, and reference organization during preparation of the final manuscript. The author is responsible for the content of the article.

Conflict of interest

The author declares no competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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