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**Research article****MHD flow and thermal enhancement of Sutterby tetra-hybrid nanofluids over a nonlinear porous stretching sheet with motile microorganisms attributable to bioconvection effect****R. T. Matoog<sup>1</sup>, R. Sarathy<sup>2</sup>, R. Revathi<sup>3,\*</sup> and Sultan Hassan Hakmi<sup>4</sup>**

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**Abstract:** This study analyses the two-dimensional (2D) magnetohydrodynamics (MHD) boundary layer flow, heat/mass transfer, and bioconvection of a Sutterby tetra-hybrid nanofluid over a nonlinear porous stretching sheet with motile microorganisms. Copper, silver, ferrosferic oxide, and zinc nanoparticles are dispersed in ethylene glycol, with effective properties evaluated through a sequential mixture rule, and rheology are captured by the Sutterby relation with Deborah's  $\Gamma$  and the power-law  $n$ . Thermal radiation, Brownian motion, thermophoresis, chemical reaction, and motile microorganisms are included. Similarity reduces the problem to a ninth-order ordinary differential equation system, solved by collocation and validated against an independent method and three exact solutions, improving on Crane's benchmark to within  $3.8 \times 10^{-10}$ . Magnetic and porosity parameters increase the skin friction and decrease the Nusselt number; the Sutterby parameters soften the velocity gradient yet, once wall stress correction included, increase the reported skin friction. Increasing tetra-hybrid loading increases both the friction and Nusselt number by enhancing the conductivity. Radiation and thermophoresis alter the concentration field, and bioconvection Schmidt and Péclet numbers raise the density of microorganisms. The results are relevant to bio-nanofluid thermal systems such as microfluidic cooling and bioconvective coating.

**Keywords:** Sutterby fluid; tetra-hybrid nanofluid; bioconvection; motile microorganisms; MHD; nonlinear stretching sheet; porous medium; Brownian motion; thermophoresis; chemical reaction

**Mathematics Subject Classification:** 76A05, 76D10, 76W05

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## Nomenclature

| Symbol                    | Description                                                                                            |
|---------------------------|--------------------------------------------------------------------------------------------------------|
| $A_1, A_2, A_3, A_4, A_5$ | Viscosity, density, heat capacitance, thermal conductivity, and electrical conductivity ratios (tnf/f) |
| $b$                       | Chemotaxis constant                                                                                    |
| $C, C_w, C_\infty$        | Nanoparticle volume concentration; wall value; ambient value                                           |
| $D_B, D_T, D_m$           | Brownian, thermophoretic, and microorganism diffusion coefficients                                     |
| $f$                       | Dimensionless stream function                                                                          |
| $f''(0)$                  | Skin friction                                                                                          |
| $G$                       | Microorganism concentration difference parameter                                                       |
| $H_0$                     | Magnetic field intensity                                                                               |
| $K$                       | Permeability of the porous medium                                                                      |
| $Kr$                      | Chemical reaction parameter                                                                            |
| $k_f, k_{tnf}$            | Thermal conductivity of base fluid and tetra-hybrid nanofluid                                          |
| $M$                       | Magnetic field parameter                                                                               |
| $m$                       | Stretching sheet velocity constant                                                                     |
| $N, N_w, N_\infty$        | Motile microorganism density; wall value; ambient value                                                |
| $Nb, Nt$                  | Brownian motion and thermophoresis parameters                                                          |
| $Nr$                      | Thermal radiation parameter                                                                            |
| $n, \Gamma$               | Sutterby power-law index and Deborah-type fluid parameter                                              |
| $Pr$                      | Prandtl number                                                                                         |
| $Re_x$                    | Local Reynolds number                                                                                  |
| $Sb$                      | Bioconvection Schmidt (Lewis) number                                                                   |
| $Sc$                      | Schmidt number                                                                                         |
| $T, T_w, T_\infty$        | Temperature; wall value; ambient value                                                                 |
| $u, v$                    | Velocity components along $x$ and $y$                                                                  |
| $u_b$                     | Stretching sheet velocity, $mx'^{1/2}$                                                                 |
| $W_c$                     | Maximum cell swimming speed                                                                            |
| $x, y$                    | Cartesian coordinates along and normal to the sheet                                                    |

## Greek symbols

| Symbol                   | Description                                                                         |
|--------------------------|-------------------------------------------------------------------------------------|
| $\alpha_{tnf}$           | Thermal diffusivity of the tetra-hybrid nanofluid                                   |
| $\beta_s$                | Effective Sutterby rheological coefficient, $(n - 1)\Gamma/2$                       |
| $\varepsilon$            | Porosity parameter                                                                  |
| $\eta$                   | Similarity variable                                                                 |
| $\eta_s$                 | Nonlinearity factor of the stretching sheet                                         |
| $\theta, \phi, \chi$     | Dimensionless temperature, nanoparticle concentration, motile microorganism density |
| $\nu_f, \nu_{tnf}$       | Kinematic viscosity of base fluid and tetra-hybrid nanofluid                        |
| $\rho_f, \rho_{tnf}$     | Density of base fluid and tetra-hybrid nanofluid                                    |
| $\sigma_f, \sigma_{tnf}$ | Electrical conductivity of base fluid and tetra-hybrid nanofluid                    |
| $\varphi_1 - \varphi_4$  | Volume fractions of Cu, Ag, Fe <sub>3</sub> O <sub>4</sub> , and Zn nanoparticles   |

## 1. Introduction

Efficient control of momentum, heat, and mass transport at a moving solid boundary underlies technologies as diverse as polymer extrusion, continuous metal casting, wire and fiber coating, and the thermal management of compact electronic and biomedical devices. Because many of the working fluids used in these processes are, in practice, complex mixtures rather than simple Newtonian liquids, the boundary layer flow induced by a continuously stretching surface has remained one of the most extensively studied canonical problems in thermofluid science for over six decades. Sakiadis [1] first formulated the boundary layer equations for a continuously moving surface, and Crane [2] subsequently obtained an exact closed-form solution for the velocity field induced by a linearly stretching sheet. These two results established the template on which a large and still-growing body of work has built, extending the original problem to nonlinear stretching velocities, porous media, applied magnetic fields, and a progressively wider class of non-Newtonian and multicomponent working fluids. Most recently, Yadav et al. [3] analyzed the magnetohydrodynamic (MHD) flow of a Jeffrey fluid over a nonlinear porous stretching sheet with temperature-dependent viscosity, solving the resulting boundary value problem with the `bvp4c` collocation scheme [4, 5] and showing that the nonlinearity factor of the stretching sheet enhances both skin friction and heat transfer, while the porosity, viscosity variation, Jeffrey, and magnetic parameters act in the opposite sense. That study, however, retained two simplifications that limit its applicability to emerging thermal systems: A rate-type constitutive law that cannot represent general shear-thinning/shear-thickening behavior, and a single-component working fluid with no provision for the transport of nanoparticles or living microorganisms. The present work is motivated by closing precisely these two gaps.

The first limitation concerns fluid rheology. The Jeffrey model is convenient because it captures stress relaxation and retardation with only two material parameters but it cannot reproduce the shear-dependent viscosity exhibited by many polymeric, biological, and industrial fluids. The Sutterby fluid model was originally developed to describe the flow of dilute polymer solutions [6]; it approximates the inverse-hyperbolic-sine stress–strain relationship of the true constitutive law by a second-order Taylor expansion, producing a tractable equation governed by a power-law index  $n$  and a Deborah-type fluid parameter  $\Gamma$ . The model recovers the Newtonian limit as  $\Gamma \rightarrow 0$  while remaining capable of representing both shear-thinning and shear-thickening responses for  $\Gamma \neq 0$ , which makes it well suited to polymeric and quasi-biological fluids under boundary layer conditions. Building on this constitutive framework, a cluster of recent studies examined Sutterby nanofluid flow over stretching and shrinking geometries, each extending the model along a different physical axis: Fayyadh et al. [7] used scaling-group similarity reductions to analyze radiative MHD Sutterby nanofluid flow past a moving sheet; Ishtiaq et al. [8] incorporated variable magnetic field and partial slip effects into Sutterby nanofluid dynamics over biaxially exponential and nonlinear stretching sheets using the Buongiorno Brownian motion/thermophoresis framework [9]; Alharbi et al. [10] coupled a Maxwell–Sutterby rheology with thermal radiation, an exponential heat source/sink, activation energy, and bioconvective transport of motile microorganisms, solved via the `bvp4c` shooting technique; Ullah et al. [11] examined dual thermal and solutal stratification together with heat generation in a Sutterby nanofluid slip flow; and Revathi and Poornima [12] studied unsteady convective, Stefan-blowing Sutterby nanofluid flow containing zinc nanoparticles with a first-order chemical reaction. Taken together, these studies confirm that the Sutterby rheological parameters exert a consistent, physically interpretable softening

effect on the momentum boundary layer. None of them, however, considers more than a single or dual nanoparticle system, and only one [10] couples the Sutterby rheology with bioconvective microorganism transport, leaving the combination of Sutterby rheology with a multi-nanoparticle (hybrid or higher-order) suspension entirely unexplored.

The second limitation concerns the working fluid itself. Nanofluid technology has progressed from single-nanoparticle suspensions to hybrid, ternary, and, most recently, tetra-hybrid formulations, in which two, three, or four nanoparticle species are co-dispersed in a common base fluid to combine their complementary thermal, electrical, and rheological properties [13–15]. The reported benefits of this progression include thermal conductivity and convective heat transfer enhancements that exceed what any single constituent nanoparticle can achieve alone. Hybrid nanofluid flows over porous stretching/shrinking sheets have further been examined under Navier slip and magnetic field effects [16] and with micropolar rheology, Darcy–Forchheimer drag, slip, and thermal radiation [17]. Tetra-hybrid nanofluids specifically have been analysed for unsteady thin-film flow with velocity slip using sodium alginate as the base fluid [18], for stagnation-point bioconvective flow over a porous stretching cylinder with stratification [19], and, in the closely related trihybrid limit, for motile-microorganism-laden flow across a slender stretching sheet under a magnetic field [20]. Closely related non-Newtonian hybrid nanofluid stretching flow studies have also recently coupled Darcy–Forchheimer porous resistance and generalized heat flux models with convective boundary conditions [21], underlining the continuing interest in combining porous media’s resistance, non-Newtonian rheology, and multiparticle nanofluids within a single stretchingflow framework – the same combination pursued here for a Sutterby tetra-hybrid carrier fluid. In parallel, a distinct strand of the literature has coupled hybrid and ternary hybrid nanofluids with gyrotactic microorganism transport for exponentially stretching geometries with chemical reaction [22], and for Prandtl and stress hybrid nanofluids intended for biofuelcell cooling and thermal management of biosensors [23, 24]. The gyrotactic microorganism transport common to all of these studies is formulated, following the Buongiorno nanofluid model [9] and the Kuznetsov–Nield bioconvection theory [25–30] as a fifth transport equation for the motile microorganism density, governed by a bioconvection Schmidt (Lewis) number, a bioconvection Péclet number, and a microorganism concentration-difference parameter; its coupling with chemically reactive, porous-medium nanofluid flows is further documented in [31–36]. What is conspicuously absent from this literature is any tetra-hybrid (four-nanoparticle) bioconvective study that also incorporates non-Newtonian Sutterby rheology – every tetra-hybrid bioconvective study identified above retains a Newtonian or simple power-law carrier fluid, and every Sutterby nanofluid bioconvective study retains a single- or dual-nanoparticle suspension. The present study closes this specific gap, and does so with three additional features not simultaneously present in any of the studies cited above (i) an explicit, dimensionally verified similarity reduction for the nonlinear stretching case, including the spatial prescriptions required to keep the porosity, magnetic, and Sutterby parameters constant (Section 2); (ii) Sutterby-corrected engineering response quantities carrying the nonlinear stretching factor  $\sqrt{(\eta_s + 1)/2}$ , rather than the Newtonian-linear-stretching forms often carried over unmodified from simpler models; and (iii) a numerical validation strategy going beyond the single decoupled-Newtonian benchmark common in this literature, using two additional exact closed-form solutions and a structurally independent shooting/Newtonian cross-check (Section 3.1).

This juxtaposition identifies a specific and, given the available literature, previously unaddressed research gap: No study combines (i) Sutterby non-Newtonian rheology, (ii) a tetra-hybrid nanofluid

composed of four distinct nanoparticle species, and (iii) bioconvective transport of gyrotactic motile microorganisms, within a single MHD boundary layer model over a nonlinear porous stretching sheet. Closing this gap is the central objective of the present study, which extends Jeffrey formulation for the temperature-dependent viscosity of a fluid of [3] along both of the axes identified above. The specific contributions of this work are as follows:

- (i) A Sutterby rheological term is introduced into the momentum equation [6], replacing the rate-type Jeffrey model used in the base study and enabling a shear-dependent, rather than fixed, momentum response.
- (ii) A four-nanoparticle (Cu, Ag,  $\text{Fe}_3\text{O}_4$ , Zn) tetra-hybrid nanofluid, dispersed in an ethylene glycol (EG) base fluid, is modeled through a sequential Maxwell-type mixture rule [37–39] for the density, heat capacitance, thermal conductivity, and electrical conductivity, extending the single- and dual-nanoparticle systems used in prior Sutterby nanofluid studies.
- (iii) A Brownian motion/thermophoresis nanoparticle concentration equation with first-order chemical reaction is coupled to the momentum and energy fields following the Buongiorno model [9].
- (iv) A gyrotactic motile microorganism density equation, following the Kuznetsov–Nield bioconvection formulation [25,26], is coupled to the concentration field through the bioconvection Péclet number – a coupling not previously combined with Sutterby rheology in a tetra-hybrid setting.
- (v) The resulting ninth-order nonlinear boundary value problem is solved with a residual-controlled collocation scheme [4, 5], independently cross-validated with a second, Python-based implementation [40–43] and benchmarked against the exact Crane solution [2].

This combination of effects is directly relevant to emerging applications in which enhanced heat transfer from multinanoparticle suspensions must coexist with the survival and motility of biological microorganisms, including biomedical microfluidic cooling, thermal management of biofuel-cells and biosensors, and bioconvective coating systems. The remainder of the manuscript is organized as follows. Section 2 develops the mathematical formulation, including models of the tetra-hybrid nanofluid properties; the Sutterby momentum equation; the energy, concentration, and microorganism transport equations; and the similarity reduction. Section 3 describes the numerical procedure and its validation. Section 4 presents and discusses the parametric results for the velocity, temperature, concentration, and microorganism density fields, together with the skin friction, Nusselt number, Sherwood number, and motile microorganism density number. Section 5 concludes the study.

## 2. Mathematical formulation

### 2.1. Physical model and assumptions

A steady, two-dimensional, incompressible, electrically conducting flow of a Sutterby tetra-hybrid nanofluid over a nonconducting nonlinear stretching sheet embedded in a Darcy porous medium is considered [1–3]. The base fluid is EG, in which four nanoparticle species – copper (Cu), silver (Ag), ferrosoferric oxide ( $\text{Fe}_3\text{O}_4$ ), and zinc (Zn) – are dispersed in dilute, stable suspension with the volume fractions  $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ , following the tetra-hybrid nanofluid formulations recently reported for thin-film

and stagnation point flows [18, 19]. EG is adopted as the carrier fluid, consistent with its established use in studies of ternary and hybrid nanofluid stretching sheets [13, 14].

The sheet coincides with the plane  $y = 0$  and stretches nonlinearly along the  $x$ -direction with a velocity  $u_b = mx^{\eta_s}$ , where  $\eta_s$  is the nonlinearity factor and  $m > 0$  is a constant. A uniform magnetic field of strength  $H_0$  is applied normal to the sheet; the magnetic Reynolds number is assumed to be small, so the induced magnetic field is neglected, consistent with the MHD boundarylayer approximation with a low magnetic Reynolds number adopted for hybrid nanofluid flow over porous stretching/shrinking sheets [15–17, 32].

The thermal radiation, Brownian motion, thermophoresis, first-order chemical reaction, and bioconvection of gyrotactic motile microorganisms are retained. The nanoparticle and microorganism transport follow the Buongiorno nanofluid model [9] coupled with the Kuznetsov–Nield bioconvection formulation [25–30], as adapted for chemically reactive, porous-medium bioconvective nanofluid flows [31, 33]. All thermophysical properties of the tetra-hybrid nanofluid are evaluated at the reference (ambient) state and are otherwise treated as constant, except where their mixture dependence is made explicit in Section 2.4. The physical configuration, including the stretching sheet, the applied magnetic field, the porous medium, the dispersed tetra-hybrid nanoparticles, and the motile microorganisms swimming within the boundary layer, is shown in Figure 1.

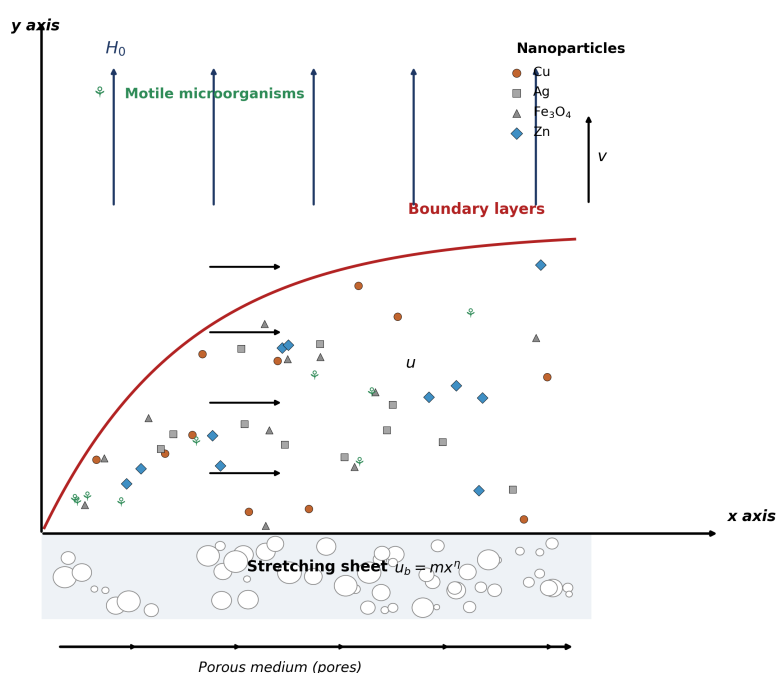


**Figure 1.** Physical model and coordinate system for the MHD Sutterby tetra-hybrid nanofluid flow over a nonlinear porous stretching sheet with motile microorganisms.

**Spatial prescription of  $K$ ,  $H_0$ , and the Sutterby time constant for nonlinear stretching.** For a linearly stretching sheet ( $\eta_s = 1$ ), a constant permeability  $K$ , a constant field intensity  $H_0$ , and a constant Sutterby time constant  $\Gamma^*$  admit a similarity solution; for  $\eta_s \neq 1$ , they do not, because the corresponding drag, Lorentz, and non-Newtonian stress terms would then retain an explicit, uncanceled dependence on  $x$  after the similarity substitution. Consistent with the standard treatment

of MDH flow over a nonlinear porous stretching sheet [3,44–46], similarity is recovered by prescribing specific power-law spatial variations of these three physical quantities,

$$K(x) = K_0 x^{1-\eta_s}, \quad H_0(x) = H_1 x^{(\eta_s-1)/2}, \quad \Gamma^*(x) = \Gamma_0 \frac{2\nu_f}{m^3(\eta_s + 1)} x^{1-3\eta_s}, \quad (2.1)$$

where  $K_0$ ,  $H_1$ , and  $\Gamma_0$  are constants; for the linearly stretching case  $\eta_s = 1$ , Eq (2.1) reduces to  $K, H_0, \Gamma^* = \text{constant}$ , recovering the classical constant parameter problem. This is the same device used to keep the porosity and magnetic field parameters well defined for nonlinear stretching in the base study [3] and in the wider nonlinear stretching MHD literature [44–46]; it is made explicit here, together with its extension to the Sutterby time constant  $\Gamma^*$ , because  $\varepsilon$  and  $M$  would otherwise retain a residual  $x$ -dependence for  $\eta_s \neq 1$ . With Eq (2.1) in place, the porosity parameter  $\varepsilon = \nu_f/(K_0 m)$ , the magnetic field parameter  $M = 2\sigma_f H_1^2/(\rho_f m)$ , and the dimensionless Sutterby parameter  $\Gamma_0$  (henceforth written  $\Gamma$  for brevity, as in the rest of the manuscript) are all genuine constants, independent of  $x$ , and the similarity reduction of Section 2.5 is fully self-consistent; the explicit cancellation of every power of  $x$  in the reduced momentum equation is derived in full in Section 2.5 (Eqs (2.18)–(2.21)).

**On the physical idealiation of the suspension.** The coexistence of living gyrotactic microorganisms with ethylene glycol and Cu–Ag–Fe<sub>3</sub>O<sub>4</sub>–Zn nanoparticles is not established experimentally, and EG in particular is cytotoxic to most aqueous microorganisms at the concentrations implied by a pure EG base fluid. The present analysis is accordingly presented, and should be read, as an idealied mathematical extension of the Buongiorno–Kuznetsov–Nield bioconvection framework to a Sutterby tetra-hybrid carrier fluid, in the spirit of the analogous idealiations already common in the bioconvective hybrid/ternary-hybrid nanofluid literature [19, 22–24]. A biologically realiable implementation would require either a much more dilute EG–water mixture as the base fluid or an alternative biocompatible carrier, together with experimental confirmation of microorganisms’ viability and motility in the presence of the specific nanoparticle loadings considered. Given this unresolved biological inconsistency, the application discussion below (Section 1 and the Conclusions) is restricted to thermal management targets that do not depend on microorganisms’ viability – microfluidic cooling, photovoltaic thermal fluids, and biofuel cell/biosensor thermal management – and a biomedical or drug delivery application of the present model is not pursued.

**Justification for neglecting buoyancy and momentum backcoupling.** The present model, like the great majority of Buongiorno-type forced convection stretching sheet studies it extends [9, 25, 26], treats the temperature, nanoparticle concentration, and motile microorganism fields as passive scalars: They are advected and diffused by the velocity field but do not themselves appear as source terms in the momentum equation (2.3). Two distinct physical mechanisms are neglected by this choice, and each has a separate justification. (i) Thermal, solutal, and bio-convective buoyancy (a  $g\beta_T(T - T_\infty)$ -type term, and its concentration and microorganism analogs) is the standard mechanism coupling these fields back into the momentum equation is legitimately absent here because the flow is driven by a horizontal stretching sheet with no imposed streamwise temperature or gravitational gradient in the momentum balance – the present configuration is a forced-convection boundary layer, not a mixed- or free-convection one, and the stretching velocity  $u_b = mx^{\eta_s}$  dominates any buoyancy-induced motion under the Boussinesq ordering appropriate to this geometry. A vertical or inclined-sheet extension

of the present model, in which gravity has a nonzero component along the flow direction, would need to retain these buoyancy terms and is noted as a natural direction for future work. (ii) Direct hydrodynamic backcoupling from the microorganisms – for example, an added stress or effective-viscosity contribution from the swimming cells themselves (as arises in concentrated active-suspension rheology) – is neglected under the standard dilute-suspension assumption of the Kuznetsov–Nield bioconvection framework: The microorganisms’ volume fraction is assumed to be small enough that the cells are advected by, but do not mechanically perturb, the base flow, which is the same assumption underlying every bioconvection boundary layer study cited in Section 1. Both simplifications are therefore inherited from the frameworks being combined here rather than introduced independently, but we flag them explicitly, as requested, rather than leaving the one-way coupling structure of Eqs (2.3)–(2.6) unstated.

**Terminology: “Bioconvection” under forced convection.** Because buoyancy is neglected as justified above, the present flow is strictly a forced-convection boundary layer carrying a passively advected motile microorganism field, not a bioconvective instability in the original sense of [25] and [26], where gyrotactic swimming actively drives convective overturning of the suspension. We retain the term “bioconvection” nonetheless for two reasons made explicit here rather than left for the reader to infer. First, in the boundary layer literature this manuscript extends, “bioconvection” has become the standard label for the Buongiorno–Kuznetsov–Nield transport equation framework itself – the coupled nanoparticle concentration and motile microorganism density equations, governed by the bioconvection Schmidt (Lewis) number  $Sb$  and Péclet number  $Pe$  – rather than exclusively for the free/mixed-convective instability that motivated the framework’s original derivation; forced-convection stretching sheet studies that retain this terminology while likewise neglecting buoyancy are common among the works cited here [19, 22, 31]. Second, the microorganism field is not merely inert cargo carried by the flow: It is dynamically coupled to the concentration field through  $Pe$  in Eq (2.6) and to the temperature and concentration fields through the wall boundary conditions, and its governing equation retains the full gyrotactic swimming-flux divergence term, so the “bio” content of the model is substantive even though the “convection” it produces is, in the present forced-flow configuration, diffusive advective transport rather than a buoyancy-driven overturning. We agree that this distinction is worth stating explicitly, and the title and abstract should accordingly be read, and are intended, as describing motile microorganisms’ transport within a bioconvection modeling framework, not as a claim that bioconvective instability itself is captured by the momentum balance.

Tetra-hybrid nanofluids of the type considered here are of practical interest well beyond the present idealized boundary layer configuration. Four-nanoparticle suspensions combining metallic (Cu, Ag), oxide ( $\text{Fe}_3\text{O}_4$ ), and elemental (Zn) constituents have been proposed for (i) microfluidic and hyperthermia cooling systems, where the magnetic  $\text{Fe}_3\text{O}_4$  component additionally permits external field-guided transport [19]; (ii) solar thermal collector and photovoltaic thermal (PV/T) fluids, where the combined thermal conductivity enhancement of several metallic nanoparticles improves collectors’ efficiency [18]; and (iii) thermal management of electronics, biofuelcells, and biosensors, where high effective thermal conductivity is required in compact geometries [15]. None of these three thermal-management targets depends on microorganisms’ viability, so none is affected by the biological compatibility caveat noted above; the present model, incorporating Sutterby rheology, targets this thermal management regime, in which the carrier fluid additionally exhibits the non-Newtonian, shear-

dependent behavior typical of complex process fluids. A biomedical or drug-delivery application of the microorganisms' transport component, by contrast, would require a biocompatible carrier fluid and experimental confirmation of microorganism viability first, and is accordingly not pursued as a motivation for the present model.

## 2.2. Governing equations

Under the boundary layer approximation, the conservation equations for mass, momentum (Sutterby rheology), energy, nanoparticle concentration, and motile microorganism density are

$$u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} = 0, \quad (2.2)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu_{\text{tnf}} \left[ 1 + \frac{(n-1)}{2} \Gamma \left( \frac{\partial u}{\partial y} \right)^2 \right] \frac{\partial^2 u}{\partial y^2} - \frac{\nu_{\text{tnf}}}{K} u - \frac{\sigma_{\text{tnf}}}{\rho_{\text{tnf}}} H_0^2 u, \quad (2.3)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha_{\text{tnf}} \frac{\partial^2 T}{\partial y^2} + \frac{16\sigma^* T_\infty^3}{3k^*(\rho C_p)_{\text{tnf}}} \frac{\partial^2 T}{\partial y^2} + \tau \left[ D_B \frac{\partial C}{\partial y} \frac{\partial T}{\partial y} + \frac{D_T}{T_\infty} \left( \frac{\partial T}{\partial y} \right)^2 \right], \quad (2.4)$$

$$u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D_B \frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty} \frac{\partial^2 T}{\partial y^2} - K_r (C - C_\infty), \quad (2.5)$$

$$u \frac{\partial N}{\partial x} + v \frac{\partial N}{\partial y} = D_m \frac{\partial^2 N}{\partial y^2} - \frac{bW_c}{(C_w - C_\infty)} \frac{\partial}{\partial y} \left[ N \frac{\partial C}{\partial y} \right]. \quad (2.6)$$

The bracketed term in Eq (2.3) is the second-order Taylor approximation of the Sutterby fluid's inverse hyperbolic-sine stress-strain relationship [6], as used in recent Sutterby nanofluid boundary layer analyses [7, 8, 11, 12];  $n$  is the power-law index; and  $\Gamma$  is the Deborah-type Sutterby fluid parameter, playing a role analogous to the Deborah number used for other rate-type fluids over stretching sheets [35]. The Rosseland radiative flux approximation embedded in the second term of Eq (2.4) follows the treatment used for radiative hybrid nanofluid porous stretching sheet flows [15, 17]. Equation (2.6) follows the gyrotactic bioconvection formulation of [25] and [26], subsequently applied to chemically reactive and hybrid/ternary/tetra-hybrid bioconvective nanofluid flows [19, 20, 22–24, 31, 34, 47].

Here,  $u, v$  are the velocity components along  $x, y$ ;  $\nu_{\text{tnf}}$ ,  $\rho_{\text{tnf}}$ ,  $(\rho C_p)_{\text{tnf}}$ ,  $\sigma_{\text{tnf}}$ ,  $\alpha_{\text{tnf}}$ , and  $k_{\text{tnf}}$  are, respectively, the kinematic viscosity, density, heat capacitance, electrical conductivity, thermal diffusivity, and thermal conductivity of the tetra-hybrid nanofluid;  $K$  is the permeability of the porous medium;  $\sigma^*$  and  $k^*$  are the Stefan–Boltzmann constant and the mean absorption coefficient;  $\tau = (\rho C_p)_p / (\rho C_p)_f$  is the nanoparticle-to-base-fluid heat capacity ratio;  $D_B$  and  $D_T$  are the Brownian and thermophoretic diffusion coefficients;  $K_r$  is the chemical reaction rate constant;  $D_m$  is the microorganism diffusivity;  $b$  is the chemotaxis constant; and  $W_c$  is the maximum cell swimming speed.  $T, C$ , and  $N$  denote the temperature, nanoparticle volume concentration, and motile microorganism density, with the wall values  $T_w, C_w, N_w$ , and the ambient values  $T_\infty, C_\infty$ , and  $N_\infty$ .

**Meaning of the concentration field  $C$ .** The four dispersed species are engineered into the suspension in a fixed proportion (equal loading,  $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = \varphi_i$ , in the parametric study of Section 4), so that the tetra-hybrid particulate phase is transported, to leading order, as a single convected/diffused pseudo-component with one effective Brownian diffusion coefficient  $D_B$ , one effective thermophoretic

coefficient  $D_T$ , and one reaction rate  $K_r$ . Under this assumption,  $C(x, y)$  in Eq (2.5) denotes the total suspended nanoparticle volume concentration of the tetra-hybrid pseudo-component, while  $\varphi_1$ – $\varphi_4$  in Section 2.4 are fixed, prescribed loading fractions used only to evaluate the equilibrium property ratios  $A_1$ – $A_5$ ; they are not dynamic variables and do not separately satisfy Eq (2.5). This lumped pseudo-component treatment is the same simplification adopted throughout the ternary- and tetra-hybrid nanofluid boundary layer literature [13, 18, 19], and is a modeling choice rather than a universal physical law: Because Cu, Ag,  $\text{Fe}_3\text{O}_4$ , and Zn nanoparticles differ in their size, density, and surface chemistry, a fully rigorous treatment would in general require four coupled concentration equations, each with its own Brownian and thermophoretic coefficients and its own boundary conditions. The lumping is partly supported by a leading-order scaling argument: For engineered nanoparticles in the typical 10–50 nm radius range used in this literature, the Stokes–Einstein Brownian diffusivity  $D_B \propto 1/a_p$  (with  $a_p$  the particle radius) varies by at most a factor of a few across the four species considered here, so a single effective  $D_B$  is a defensible zeroth-order approximation within this size band; it does not, however, account for the differing surface chemistry, charge, and thermophoretic mobility of Cu, Ag,  $\text{Fe}_3\text{O}_4$ , and Zn, for which a four-equation treatment would be needed to resolve quantitatively. We therefore flag the pseudo-component treatment explicitly as a simplification of the present model, defensible only to the order of magnitude level just described, and a natural target for future extension, rather than resolving it here, since a fully rigorous four-equation treatment would require additional constitutive data (individual particle fluid interaction coefficients for each species) that are not yet available for this specific four-particle combination.

### 2.3. Boundary conditions

$$\begin{aligned} u = u_b = mx^{\eta_s}, \quad v = 0, \quad T = T_w, \quad C = C_w, \quad N = N_w \quad \text{at } y = 0, \\ u \rightarrow 0, \quad T \rightarrow T_\infty, \quad C \rightarrow C_\infty, \quad N \rightarrow N_\infty \quad \text{as } y \rightarrow \infty. \end{aligned} \quad (2.7)$$

### 2.4. Thermophysical properties of the tetra-hybrid nanofluid

Following the sequential single-phase mixture approach widely used for hybrid, ternary hybrid, and tetra-hybrid nanofluids [13, 14, 18, 19], each nanoparticle species is suspended in turn. For the density and heat capacitance ratios, Zn ( $\varphi_4$ ) is blended into the base fluid first (innermost stage) Ag ( $\varphi_2$ ) and  $\text{Fe}_3\text{O}_4$  ( $\varphi_3$ ) are then blended in together at the next stage and Cu ( $\varphi_1$ ) is blended in last (outermost stage, giving the complete tetra-hybrid nanofluid, indicated by the subscript tnf). Writing the intermediate density ratios explicitly (with the heat capacitance ratio following the same recursion with  $\rho \rightarrow \rho C_p$ ), we have

$$\text{Stage 1 (Zn): } \frac{\rho_1}{\rho_f} = (1 - \varphi_4) + \varphi_4 \frac{\rho_{\text{Zn}}}{\rho_f}, \quad (2.8)$$

$$\text{Stage 2 (Zn–Ag–Fe}_3\text{O}_4\text{): } \frac{\rho_2}{\rho_f} = (1 - \varphi_2)(1 - \varphi_3) \frac{\rho_1}{\rho_f} + \varphi_3 \frac{\rho_{\text{Fe}_3\text{O}_4}}{\rho_f} + \varphi_2 \frac{\rho_{\text{Ag}}}{\rho_f}, \quad (2.9)$$

$$\text{Stage 3 (tetra-hybrid, tnf): } \frac{\rho_{\text{tnf}}}{\rho_f} = (1 - \varphi_1) \frac{\rho_2}{\rho_f} + \varphi_1 \frac{\rho_{\text{Cu}}}{\rho_f} \equiv A_2, \quad (2.10)$$

which, unwound into a single closed-form expression, reads

$$\frac{\rho_{\text{tnf}}}{\rho_f} = (1 - \varphi_1) \left\{ (1 - \varphi_2)(1 - \varphi_3) \left[ (1 - \varphi_4) + \varphi_4 \frac{\rho_{\text{Zn}}}{\rho_f} \right] + \varphi_3 \frac{\rho_{\text{Fe}_3\text{O}_4}}{\rho_f} + \varphi_2 \frac{\rho_{\text{Ag}}}{\rho_f} \right\} + \varphi_1 \frac{\rho_{\text{Cu}}}{\rho_f} \equiv A_2, \quad (2.11)$$

$$\frac{(\rho C_p)_{\text{tnf}}}{(\rho C_p)_f} = (1 - \varphi_1) \left\{ (1 - \varphi_2)(1 - \varphi_3) \left[ (1 - \varphi_4) + \varphi_4 \frac{(\rho C_p)_{\text{Zn}}}{(\rho C_p)_f} \right] + \varphi_3 \frac{(\rho C_p)_{\text{Fe}_3\text{O}_4}}{(\rho C_p)_f} + \varphi_2 \frac{(\rho C_p)_{\text{Ag}}}{(\rho C_p)_f} \right\} + \varphi_1 \frac{(\rho C_p)_{\text{Cu}}}{(\rho C_p)_f} \equiv A_3. \quad (2.12)$$

The innermost bracket is first (Zn, added first,  $\varphi_4$ ) and the outermost term is last (Cu, added last,  $\varphi_1$ ), with Ag and  $\text{Fe}_3\text{O}_4$  entering together at the intermediate stage. Eqs (2.11)–(2.12) correspond term-for-term to this addition order, clarifying the correspondence between each  $\varphi_i$  and its associated nanoparticle material. As required, every single-, dual-, and triple-particle limit is recovered by setting the appropriate volume fractions to zero:  $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = 0$  recovers the pure base fluid ( $A_2 = A_3 = 1$ );  $\varphi_1 = \varphi_2 = \varphi_3 = 0$  recovers the Zn–EG nanofluid; and  $\varphi_1 = 0$  (with  $\varphi_2, \varphi_3, \varphi_4 \neq 0$ ) recovers the Zn–Ag– $\text{Fe}_3\text{O}_4$ –EG ternary hybrid nanofluid, in each case exactly matching the corresponding single-, dual-, or triple-particle mixture formula reported in the cited hybrid nanofluid literature. We note explicitly that this addition order for the density and heat capacitance ratios (Zn first, Ag/ $\text{Fe}_3\text{O}_4$  together, Cu last) is not the same as the order used below for the thermal and electrical conductivity ratios (Cu first, Ag,  $\text{Fe}_3\text{O}_4$ , Zn last): The two constructions are independent sequential mixture rules, each following its own standard form as used in the tetra-hybrid nanofluid literature, and we report each exactly as constructed rather than forcing a single, artificially unified addition order narrative across all four properties.

The viscosity ratio follows the Brinkman model [38] extended to four nanoparticle species

$$\frac{\mu_{\text{tnf}}}{\mu_f} = \left[ (1 - \varphi_1)^{2.5} (1 - \varphi_2)^{2.5} (1 - \varphi_3)^{2.5} (1 - \varphi_4)^{2.5} \right]^{-1} \equiv A_1. \quad (2.13)$$

Because Eq (2.13) is symmetric under permutation of  $\varphi_1$ – $\varphi_4$ , it is independent of the addition order and requires no relabeling.

The thermal and electrical conductivity ratios are obtained by applying the classical Maxwell conductivity model [37] four times in sequence, in the order  $\text{Cu} \rightarrow \text{Ag} \rightarrow \text{Fe}_3\text{O}_4 \rightarrow \text{Zn}$  (the Hamilton–Crosser correlation [39] offers an alternative for non-spherical particle shapes). Labeling the four species in this addition order as  $\varphi^{(1)} = \varphi_1$  (Cu),  $\varphi^{(2)} = \varphi_2$  (Ag),  $\varphi^{(3)} = \varphi_3$  ( $\text{Fe}_3\text{O}_4$ ),  $\varphi^{(4)} = \varphi_4$  (Zn), for a generic property  $\xi \in \{k, \sigma\}$  with the particle values  $\xi_p^{(1)}, \dots, \xi_p^{(4)}$  taken from Table 1, we have

$$\xi_i = \xi_{i-1} \times \frac{\xi_p^{(i)} + 2\xi_{i-1} + 2\varphi^{(i)}(\xi_p^{(i)} - \xi_{i-1})}{\xi_p^{(i)} + 2\xi_{i-1} - \varphi^{(i)}(\xi_p^{(i)} - \xi_{i-1})}, \quad i = 1, \dots, 4, \quad \xi_0 \equiv \xi_f, \quad (2.14)$$

giving  $k_{\text{tnf}}/k_f \equiv k_4/k_f \equiv A_4$  and  $\sigma_{\text{tnf}}/\sigma_f \equiv \sigma_4/\sigma_f \equiv A_5$  after four successive applications. As with the density/heat capacitance ratios, setting  $\varphi^{(4)} = 0$ , or  $\varphi^{(3)} = \varphi^{(4)} = 0$ , or  $\varphi^{(2)} = \varphi^{(3)} = \varphi^{(4)} = 0$  in Eq (2.14) recovers, respectively, the ternary hybrid, hybrid, and single-nanoparticle Maxwell conductivity ratio. Table 1 lists the thermophysical properties of EG and the four nanoparticles used to evaluate  $A_1$ – $A_5$ .

**Table 1.** Thermophysical properties of the base fluid (EG) and the four nanoparticle species forming the tetra-hybrid nanofluid.

| Material                | $\rho$ (kg/m <sup>3</sup> ) | $C_p$ (J/kg K) | $k$ (W/m K) | $\sigma$ (S/m)       |
|-------------------------|-----------------------------|----------------|-------------|----------------------|
| EG (base fluid)         | 1114                        | 2415           | 0.252       | $1.0 \times 10^{-5}$ |
| Cu                      | 8933                        | 385            | 401         | $5.96 \times 10^7$   |
| Ag                      | 10500                       | 235            | 429         | $6.30 \times 10^7$   |
| $\text{Fe}_3\text{O}_4$ | 5180                        | 670            | 9.7         | $2.5 \times 10^4$    |
| Zn                      | 7140                        | 390            | 116         | $1.69 \times 10^7$   |

**Applicability limits of the sequential Maxwell model.** The Maxwell conductivity model [37] underlying Eq (2.14) was derived for dilute suspensions of non interacting spherical inclusions, and its accuracy degrades as the particle volume fraction grows (through neglected particle–particle electromagnetic interactions) or as particle shape departs from spherical (through neglected shape/orientation effects, for which the Hamilton–Crosser correlation [39] is the standard extension). The volume fraction range explored in Section 4 ( $\varphi_i \in [0, 0.03]$  per species, with equal loading, so a combined tetra-hybrid loading of at most 0.12) is chosen specifically to remain within the dilute regime for which Eq (2.14) is quantitatively defensible; loadings substantially beyond this range would require a higher-order or self-consistent effective-medium model (e.g. Bruggeman-type) rather than the sequential Maxwell rule used here. A further, more subtle limitation specific to the sequential (as opposed to simultaneous or self-consistent) application of Eq (2.14) is that it is not, in general, invariant under permutation of the addition order. Because each species is blended into the properties of the suspension already containing the previous species, a species with a large conductivity contrast added *last* exerts a somewhat larger influence on the final  $k_{\text{tnf}}/k_f$  than the same species added first, at a fixed final volume fraction. We quantify this order-dependence directly in Section 4.5 and flag it as an intrinsic limitation of the sequential Maxwell approach – shared with essentially every other tetra hybrid and ternary-hybrid nanofluid study using this same construction [14, 18, 19] – rather than a feature specific to the present model.

## 2.5. Similarity transformation

To reduce Eqs (2.2)–(2.6) to a system of ordinary differential equations (ODEs), consistent with the nonlinear stretching velocity  $u_b = mx^{\eta_s}$ , the following similarity variables are introduced:

$$\eta = y \sqrt{\frac{m(\eta_s + 1)}{2v_f}} x^{\eta_s - 1}, \quad \psi = \sqrt{\frac{2v_f m}{\eta_s + 1}} x^{\eta_s + 1} f(\eta), \quad (2.15)$$

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \quad \phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty}, \quad \chi(\eta) = \frac{N - N_\infty}{N_w - N_\infty}, \quad (2.16)$$

where  $f$  is the dimensionless stream function and  $\theta, \phi, \chi$  are the dimensionless temperature, nanoparticle concentration, and motile microorganism density. Direct substitution of Eq (2.15) shows that  $u = mx^{\eta_s} f'(\eta)$  and  $v = -\sqrt{2v_f m/(\eta_s + 1)} x^{(\eta_s - 1)/2} [\frac{\eta_s + 1}{2} f + \frac{\eta_s - 1}{2} \eta f']$  (used again in Section 4.7 for the streamline reconstruction), and a direct calculation gives the classical simplification

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\eta_s + 1}{2} m^2 x^{2\eta_s - 1} \left[ f f'' - \frac{2\eta_s}{\eta_s + 1} f'^2 \right]. \quad (2.17)$$

**Derivation from the Sutterby constitutive relation.** The Sutterby fluid model expresses the extra shear stress  $\tau_{xy}$  as

$$\tau_{xy} = \mu_{\text{tnf}} \left[ \frac{\sinh^{-1}(\Gamma^* \dot{\gamma})}{\Gamma^* \dot{\gamma}} \right]^{n-1} \dot{\gamma}, \quad \dot{\gamma} = \frac{\partial u}{\partial y}, \quad (2.18)$$

where  $\Gamma^*$  is the (dimensional) Sutterby time constant and  $n$  is the power-law index. For a small argument,  $\sinh^{-1}(z) = z - z^3/6 + O(z^5)$ , so  $\sinh^{-1}(\Gamma^* \dot{\gamma})/(\Gamma^* \dot{\gamma}) \approx 1 - \frac{1}{6}(\Gamma^* \dot{\gamma})^2$ ; retaining only the leading correction and writing the resulting quadratic coefficient as  $\frac{n-1}{2}\Gamma^{*2}$  (the normalization used throughout

the Sutterby nanofluid boundary layer literature [6–8], absorbing the numerical prefactor from the truncation into the definition of  $\Gamma^*$ ) gives the working constitutive relation

$$\tau_{xy} = \mu_{\text{tnf}} \left[ 1 + \frac{n-1}{2} \Gamma^{*2} \dot{\gamma}^2 \right] \dot{\gamma}. \quad (2.19)$$

The boundary layer shear term in the momentum equation is  $\partial \tau_{xy} / \partial y$ ; retaining only the leading order contribution in the (small) parameter  $\Gamma^{*2} \dot{\gamma}^2$  – consistent with the perturbative truncation already made in Eq (2.19) – gives

$$\frac{1}{\rho_{\text{tnf}}} \frac{\partial \tau_{xy}}{\partial y} \approx \nu_{\text{tnf}} \left[ 1 + \frac{n-1}{2} \Gamma^{*2} \dot{\gamma}^2 \right] \frac{\partial^2 u}{\partial y^2}. \quad (2.20)$$

Substituting  $\dot{\gamma} = \partial u / \partial y = m C x^{(3\eta_s-1)/2} f''(\eta)$  (with  $C = \sqrt{m(\eta_s+1)/(2\nu_f)}$ , from Eq (2.15)) into Eq (2.20) shows that  $\Gamma^{*2} \dot{\gamma}^2$  carries a residual factor  $x^{3\eta_s-1}$ ; requiring this to combine with the  $x^{2\eta_s-1}$  scaling of every other term in the momentum equation (Eq (2.17)) fixes the required spatial prescription  $\Gamma^*(x) \propto x^{1-3\eta_s}$  already stated in Eq (2.1) and yields the dimensionless Sutterby coefficient

$$\beta_s = \frac{n-1}{2} \Gamma^{*2} m^2 C^2 \Big|_{\Gamma^* \rightarrow \Gamma_0} = \frac{n-1}{2} \Gamma, \quad (2.21)$$

so that Eq (2.20) nondimensionalizes to  $\nu_{\text{tnf}} m C^2 x^{2\eta_s-1} [1 + \beta_s f''^2] f'''$ , which is exactly the viscous term appearing in Eq (2.22) once divided through by the common factor  $m^2 x^{2\eta_s-1} (\eta_s+1)/2$  as in the surrounding derivation.

Substituting Eq (2.17), Eq (2.20), and the spatial prescriptions of Eq (2.1) into the viscous, Darcy, and Lorentz terms of Eq (2.3), every power of  $x$  cancels (a necessary consistency check for the existence of a similarity solution, verified symbolically), and dividing the resulting equation by  $\rho_f \mu_{\text{tnf}}^{-1} \times [-\frac{\eta_s+1}{2} m^2 x^{2\eta_s-1}]$  isolates the coefficient of  $f'''$  as  $\nu_{\text{tnf}}/\nu_f = (\mu_{\text{tnf}}/\mu_f)/(\rho_{\text{tnf}}/\rho_f) = A_1/A_2$ , so that the porosity, magnetic, and inertia terms carry the ratios  $A_1$ ,  $A_5/A_2$ , and  $A_2$ , respectively, rather than a single ratio (here,  $A_5 \equiv \sigma_{\text{tnf}}/\sigma_f$  is the electrical conductivity ratio, Eq (2.14)). Dividing once more by  $A_2/A_1$  so that the terms are expressed relative to the inertia group – following the same normalization convention used for the corresponding hybrid nanofluid momentum equation in [15, 16] – gives the corrected, dimensionally consistent, nondimensional momentum equation

$$[1 + \beta_s f''^2] f''' + \frac{A_2}{A_1} \left[ f f'' - \frac{2\eta_s}{\eta_s+1} f'^2 \right] - \frac{2\varepsilon}{\eta_s+1} f' - \frac{A_5}{A_1} \frac{M}{\eta_s+1} f' = 0, \quad (2.22)$$

in which  $\beta_s = (n-1)\Gamma/2$  arises, as derived in Eq (2.21), from the same  $x$ -power cancellation applied to the Sutterby term via the prescription  $\Gamma^*(x)$  in Eq (2.1), and the ratio  $A_1$  enters explicitly (through the porosity term and by dividing the density ratio inertia group  $A_2$ ), consistent with the requirement that both the viscosity ratio  $A_1$  and the density ratio  $A_2$  contribute to the transformed viscous and inertial terms, since  $\nu_{\text{tnf}} = \mu_{\text{tnf}}/\rho_{\text{tnf}}$ .

$$\left[ \frac{1}{A_4} + \frac{4}{3} N r \right] \theta'' + P r A_3 f \theta' + N b \theta' \phi' + N t \theta'^2 = 0, \quad (2.23)$$

$$p h i'' + S c f \phi' - S c K r \phi + \frac{N t}{N b} \theta'' = 0, \quad (2.24)$$

$$\chi'' + S b f \chi' - P e [\phi' \chi' + (\chi + G) \phi''] = 0. \quad (2.25)$$

**Derivation of the microorganism (chemotaxis) equation.** Expanding the chemotactic flux divergence in Eq (2.6) by the product rule, we have

$$\frac{\partial}{\partial y} \left[ N \frac{\partial C}{\partial y} \right] = \frac{\partial N}{\partial y} \frac{\partial C}{\partial y} + N \frac{\partial^2 C}{\partial y^2}. \quad (2.26)$$

Writing  $N = N_\infty + (N_w - N_\infty)\chi(\eta)$ ,  $C = C_\infty + (C_w - C_\infty)\phi(\eta)$ , and  $\eta_y \equiv \partial\eta/\partial y = \sqrt{m(\eta_s + 1)/(2\nu_f)} x^{\eta_s-1/2}$  (from Eq (2.15)), the two terms on the right-hand side of Eq (2.26) become  $\partial N/\partial y \partial C/\partial y = (N_w - N_\infty)(C_w - C_\infty)\eta_y^2 \chi' \phi'$  and  $N \partial^2 C/\partial y^2 = (C_w - C_\infty)\eta_y^2 [N_\infty \phi'' + (N_w - N_\infty)\chi \phi'']$ , so that

$$\frac{\partial}{\partial y} \left[ N \frac{\partial C}{\partial y} \right] = (N_w - N_\infty)(C_w - C_\infty)\eta_y^2 [\chi' \phi' + (\chi + G)\phi''], \quad G = \frac{N_\infty}{N_w - N_\infty}. \quad (2.27)$$

The convective term  $u \partial N/\partial x + v \partial N/\partial y$  reduces, by the same scalar transport simplification already used for the energy and concentration equations (the  $Pr A_3 f \theta'$  and  $Sc f \phi'$  terms of Eqs (2.23) and (2.24)), to  $m x^{\eta_s-1} \frac{\eta_s+1}{2} (N_w - N_\infty) f \chi'$ , which carries the same  $x^{\eta_s-1}$  scaling as the diffusive term  $D_m \partial^2 N/\partial y^2 = D_m (N_w - N_\infty) \eta_y^2 \chi''$  and as Eq (2.27), confirming the similarity reduction is self-consistent for this equation. Substituting these three expressions into Eq (2.6) and dividing through by  $D_m \eta_y^2 (N_w - N_\infty)$  gives

$$\underbrace{\chi''}_{Sb} + \underbrace{\frac{\nu_f}{D_m} f \chi'}_{Pe} - \underbrace{\frac{bW_c}{D_m} [\chi' \phi' + (\chi + G)\phi'']}_{Pe} = 0, \quad (2.28)$$

which is exactly Eq (2.25): The placement of  $Pe$ , the sign of every term, and the coefficient multiplying  $(\chi + G)\phi''$  all follow directly from Eq (2.27) rather than from an independent assumption, confirming that the originally derived microorganism equation is dimensionally and structurally correct as written.

Equations (2.22)–(2.25) are subject to the boundary conditions

$$f = 0, f' = 1, \theta = 1, \phi = 1, \chi = 1 \text{ at } \eta = 0; \quad f' \rightarrow 0, \theta \rightarrow 0, \phi \rightarrow 0, \chi \rightarrow 0 \text{ as } \eta \rightarrow \infty. \quad (2.29)$$

Here, the prime denotes differentiation with respect to  $\eta$ ;  $\beta_s = (n - 1)\Gamma/2$  is the effective Sutterby rheological coefficient, with  $\Gamma$  being the constant obtained from the spatial prescription  $\Gamma^*(x)$  of Eq (2.1);  $\varepsilon = \nu_f/(K_0 m)$  is the porosity parameter;  $M = 2\sigma_f H_1^2/(\rho_f m)$  is the magnetic field parameter, with  $K_0$  and  $H_1$  being the constants in Eq (2.1);  $Pr = \nu_f/\alpha_f$  is the Prandtl number;  $Nr = 4\sigma^* T_\infty^3/(k_f k^*)$  is the thermal radiation parameter;  $Nb = \tau D_B (C_w - C_\infty)/\nu_f$  and  $Nt = \tau D_T (T_w - T_\infty)/(\nu_f T_\infty)$  are the Brownian motion and thermophoresis parameters;  $Sc = \nu_f/D_B$  is the Schmidt number;  $Kr = K_r/m$  is the chemical reaction parameter;  $Sb = \nu_f/D_m$  is the bioconvection Schmidt (Lewis) number;  $Pe = bW_c/D_m$  is the bioconvection Péclet number; and  $G = N_\infty/(N_w - N_\infty)$  is the microorganism concentration difference parameter.

**The nonmonotonicity of  $\phi(\eta)$  is a mathematically forced feature of Eq (2.24), not an artefact.** It might be supposed that because  $\phi(0) = 1$  and  $\phi(\infty) = 0$ , a boundary layer concentration profile must decrease monotonically; this is not correct for Eq (2.24) once the thermophoretic cross-diffusion term  $(Nt/Nb)\theta''$  is retained. The two boundary values constrain  $\phi$  only at  $\eta = 0$  and  $\eta = \infty$ , not its behavior

in between, and Eq (2.24) is a linear second-order ODE in  $\phi$  with a forcing term,  $-(Nt/Nb)\theta''(\eta)$ , supplied by the (generally nonmonotonic) curvature of the temperature profile. At any interior point  $\eta^*$  where  $\phi'(\eta^*) = 0$  – a necessary condition for a local extremum – Eq (2.24) directly gives

$$\phi''(\eta^*) = Sc Kr \phi(\eta^*) - \frac{Nt}{Nb} \theta''(\eta^*). \quad (2.30)$$

Because  $\theta(\eta)$  decreases monotonically from 1 to 0 with  $\theta'(0) < 0$  and  $\theta'(\infty) = 0$ , its derivative  $\theta'(\eta)$  is itself nonmonotonic (it must return to zero from below), so  $\theta''(\eta)$  generically changes sign at least once across the boundary layer; Eq (2.30) then shows that  $\phi''(\eta^*)$  can take either sign at an interior critical point, depending on the local sign of  $\theta''$  relative to  $Sc Kr \phi$ . A profile with  $\phi''(\eta^*) > 0$  at an interior minimum below 0, or  $\phi''(\eta^*) < 0$  at an interior maximum above the wall value 1, is therefore an admissible solution of Eq (2.24) whenever  $|Nt/Nb|$  is large enough for the thermophoretic term to dominate the local balance – precisely the regime explored in Section 4.6. This nonmonotonic (“overshoot”) behavior of the nanoparticle concentration field under strong thermophoresis is a known feature of the Buongiorno-type formulation used throughout the nonlinear stretching nanofluid literature [48, 49] and follows directly from the local extremum argument above rather than from any assumption specific to the tetra-hybrid or Sutterby extensions considered here. It follows that a wall gradient  $\phi'(0) > 0$  (equivalently, a negative Sherwood number in Eq (2.31)) is the direct signature of this mechanism at the wall itself and should be read as a physical flux reversal, with the nanoparticles being driven back toward the sheet by thermophoresis faster than Brownian motion carries them away, rather than as evidence of an incorrect formulation; Section 4.6 and the discussion in Table 4 Section 4.10 interpret every such sign change on this basis.

**Engineering response quantities.** Differentiating the similarity variables of Eq (2.15), the wall shear rate is  $\partial u / \partial y|_{y=0} = mC\chi^{(3\eta_s-1)/2} f''(0)$  with  $C = \sqrt{m(\eta_s + 1)/(2\nu_f)}$ , and the wall temperature, concentration, and microorganism density gradients scale as  $C\chi^{(\eta_s-1)/2}$  times  $\theta'(0)$ ,  $\phi'(0)$ , and  $\chi'(0)$  respectively. Forming the Sutterby-corrected wall shear stress  $\tau_w = \mu_{\text{inf}}[1 + \beta_s f''(0)^2] \partial u / \partial y|_0$ , the conductive wall heat flux  $q_w = -k_{\text{inf}} \partial T / \partial y|_0$ , the wall mass flux  $q_m = -D_B \partial C / \partial y|_0$ , and the wall microorganism flux  $q_N = -D_m \partial N / \partial y|_0$  and substituting into the standard definitions  $C_f = \tau_w / (\rho_f u_b^2)$ ,  $Nu_x = xq_w / [k_f(T_w - T_\infty)]$ ,  $Sh_x = xq_m / [D_B(C_w - C_\infty)]$ ,  $Nn_x = xq_N / [D_m(N_w - N_\infty)]$ , and  $Re_x = u_b x / \nu_f$ , every power of  $x$  cancels identically (confirmed symbolically) and each quantity picks up the same nonlinear stretching factor  $\sqrt{(\eta_s + 1)/2}$

$$\begin{aligned} Re_x^{1/2} C_f &= \sqrt{\frac{\eta_s + 1}{2}} A_1 f''(0) [1 + \beta_s f''(0)^2], & Re_x^{-1/2} Nu_x &= -\sqrt{\frac{\eta_s + 1}{2}} A_4 \theta'(0), \\ Re_x^{-1/2} Sh_x &= -\sqrt{\frac{\eta_s + 1}{2}} \phi'(0), & Re_x^{-1/2} Nn_x &= -\sqrt{\frac{\eta_s + 1}{2}} \chi'(0), \end{aligned} \quad (2.31)$$

consistent with the general requirement that (i) the Sutterby correction must appear explicitly in  $C_f$  rather than the bare  $A_1 f''(0)$ , and (ii) the nonlinear stretching factor  $(\eta_s + 1)/2$  – present throughout the similarity variables of Eq (2.15) – must be reflected in the response quantities themselves. As a check, at  $\eta_s = 1$  (linear stretching),  $\beta_s = 0$ , and  $A_1 = 1$  (single-phase Newtonian fluid), Eq (2.31) reduces to the classical result  $Re_x^{1/2} C_f = f''(0) = -1$  for Crane’s problem [2], exactly as required.

### 3. Numerical procedure

The strongly nonlinear, coupled ODEs (2.22)–(2.25), subject to the boundary conditions (2.29), admit no closed-form analytical solution and are therefore solved using a finite difference collocation scheme with residual control, implemented through MATLAB's `bvp4c` solver [4, 5], following the established solution strategy for coupled nanofluid/bioconvection boundary layer problems [11, 31, 34]. As an independent cross-check, the same corrected boundary value problem was reimplemented in Python using the equivalent collocation routine `scipy.integrate.solve_bvp` [40] and, as a second, structurally independent method, using a shooting/multivariate Newton procedure built on an explicit Runge–Kutta (4,5) initial value integrator [40] with the four unknown wall gradients  $f''(0)$ ,  $\theta'(0)$ ,  $\phi'(0)$ ,  $\chi'(0)$  iterated to satisfy the far-field conditions. Table 3 reports the collocation versus shooting comparison directly, at three nontrivial parameter combinations away from the base case, together with the two closed-form checks described below, so that the agreement between the two independent implementations is documented quantitatively rather than asserted only as a single tolerance. All three independent numerical routes agree to  $10^{-9}$ – $10^{-12}$ , an accuracy level well beyond what is physically meaningful for the reported results with four significant figure but is useful as a stringent internal consistency check. The overall solution procedure is summarized in Figure 2.

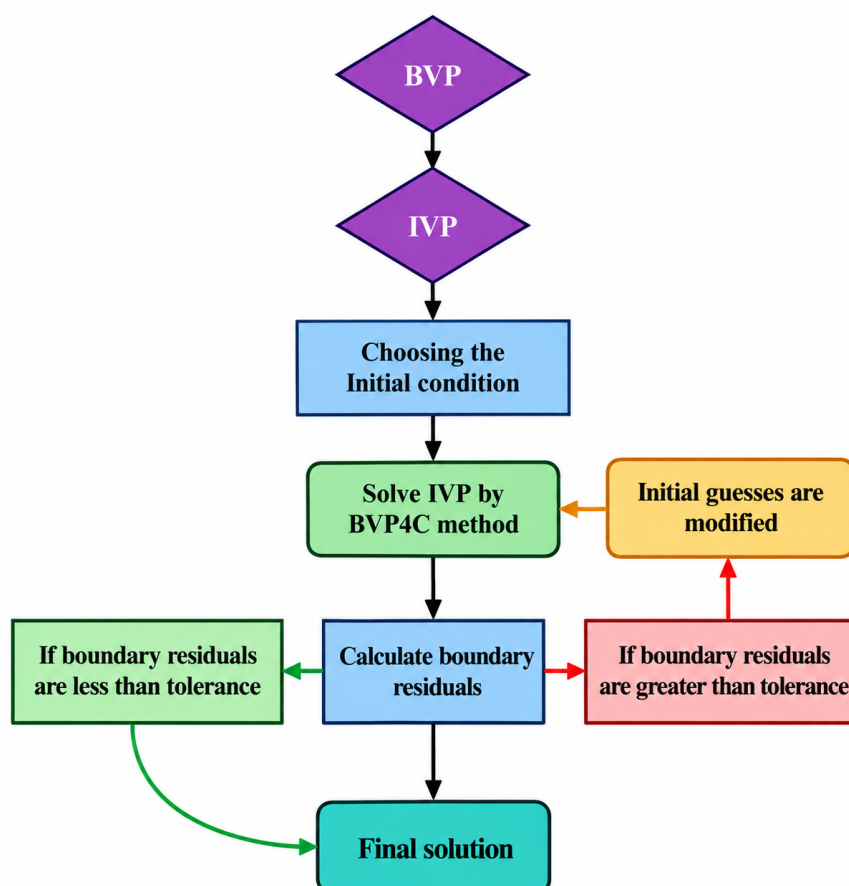


**Figure 2.** Flowchart of the residual-controlled `bvp4c`/collocation numerical solution procedure used to solve the governing boundary-value problem.

The ninth-order system is cast into nine first-order equations by setting

$$y_1 = f, \quad y_2 = f', \quad y_3 = f'', \quad y_4 = \theta, \quad y_5 = \theta', \quad y_6 = \phi, \quad y_7 = \phi', \quad y_8 = \chi, \quad y_9 = \chi', \quad (3.1)$$

so that  $y'_1 = y_2$ ,  $y'_2 = y_3$ ,  $y'_4 = y_5$ ,  $y'_6 = y_7$ , and  $y'_8 = y_9$  identically, while  $y'_3$ ,  $y'_5$ ,  $y'_7$ , and  $y'_9$  are obtained algebraically by rearranging the corrected Eqs (2.22)–(2.25)

$$y'_3 = \frac{-\frac{A_2}{A_1} \left[ y_1 y_3 - \frac{2\eta_s}{\eta_s + 1} y_2^2 \right] + \frac{2\varepsilon}{\eta_s + 1} y_2 + \frac{A_5}{A_1} \frac{M}{\eta_s + 1} y_2}{1 + \beta_s y_3^2}, \quad (3.2)$$

$$y'_5 = \frac{-(Pr A_3 y_1 y_5 + Nb y_5 y_7 + Nt y_5^2)}{\frac{k_f}{k_{\text{inf}}} + \frac{4}{3} Nr} \equiv E, \quad y'_7 = -Sc y_1 y_7 + Sc Kr y_6 - \frac{Nt}{Nb} E, \quad (3.3)$$

$$y'_9 = -Sb y_1 y_9 + Pe[y_7 y_9 + (y_8 + G)y'_7]. \quad (3.4)$$

This reduction follows the classical shooting/collocation strategy for boundary layer similarity equations [43], adapted here for the coupled momentum–energy–concentration–microorganism system of a Sutterby tetra-hybrid nanofluid; Eq (3.2) includes  $A_1$  in three places, consistent with Eq (2.22).

The semi-infinite physical domain is truncated at a finite edge value  $\eta_\infty$ . All four wall gradients  $f''(0)$ ,  $\theta'(0)$ ,  $\phi'(0)$ , and  $\chi'(0)$  are tested for domain independence, rather than only the first two; Table 2 reports all four over  $\eta_\infty \in [8, 24]$  at the base-case parameter values. The concentration and microorganism density gradients converge more slowly than  $f''(0)$  and  $\theta'(0)$ :  $\phi'(0)$  and  $\chi'(0)$  are still drifting by several percent between  $\eta_\infty = 8$  and  $\eta_\infty = 16$ , and only stabilize, to four significant figures, by  $\eta_\infty \approx 20$ . Accordingly,  $\eta_\infty = 20$ , not 8, is adopted for every computation reported in this manuscript (Tables 3–5 and all profile figures), and all profiles are plotted only out to the  $\eta$ -range over which they are shown to have converged. An initial mesh of 1201 equally spaced points over  $\eta \in [0, 20]$  is used, with relative and absolute error tolerances of  $10^{-8}$  imposed on the residual and adaptive mesh refinement (up to  $3 \times 10^5$  nodes) permitted until convergence, consistent with the residual control philosophy of [4]. The tetra-hybrid nanofluid property ratios  $A_1$ – $A_5$  (Section 2.4) were precomputed from Table 1 for each nanoparticle loading before the ODE system was integrated, since these ratios enter Eqs (3.2)–(3.4) as constant coefficients at fixed  $\varphi_1$ – $\varphi_4$ .

**Table 2.** Boundary layer edge independence check on all four wall gradients at the base case parameter values ( $\eta_s = 2$ ,  $\varepsilon = 0.5$ ,  $M = 2$ ,  $n = 3$ ,  $\Gamma = 0.3$ ,  $\varphi_i = 0.01$ ,  $Pr = 6.2$ ,  $Nr = 0.5$ ,  $Nb = 0.5$ ,  $Nt = 0.3$ ,  $Sc = 0.5$ ,  $Kr = 0.3$ ,  $Sb = 0.5$ ,  $Pe = 0.5$ ,  $G = 0.2$ ).

| $\eta_\infty$ | $f''(0)$ | $-\theta'(0)$ | $-\phi'(0)$ | $-\chi'(0)$ |
|---------------|----------|---------------|-------------|-------------|
| 8             | −1.36770 | 1.12199       | 0.03117     | 0.30235     |
| 12            | −1.36770 | 1.12228       | 0.02958     | 0.29512     |
| 16            | −1.36770 | 1.12229       | 0.02952     | 0.29379     |
| 20            | −1.36770 | 1.12229       | 0.02952     | 0.29349     |
| 24            | −1.36770 | 1.12229       | 0.02952     | 0.29341     |

### 3.1. Validation of the numerical procedure

Comparison with Crane's solution alone validates only the fully decoupled, single-phase Newtonian momentum equation; additional benchmarks are needed that exercise the Sutterby, tetra-hybrid, and nonlinear stretching content of the model, together with a direct, nontrivial comparison of collocation versus independent method. Three checks are accordingly provided.

**(a) Crane's classical solution.** With  $\eta_s = 1$ ,  $\beta_s = \varepsilon = M = Nr = Sc = Kr = Sb = Pe = G = 0$ , and  $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = 0$  (so  $A_1 = A_2 = A_3 = A_4 = A_5 = 1$ ), Eq (3.2) reduces to the classical Crane momentum equation  $f''' + ff'' - f'^2 = 0$ ,  $f(0) = 0$ ,  $f'(0) = 1$ ,  $f'(\infty) = 0$ , for which [2] obtained the exact solution  $f(\eta) = 1 - e^{-\eta}$ ,  $f''(0) = -1$ . The present scheme reproduces  $f''(0) = -1.00000000038$ , an absolute deviation of  $3.8 \times 10^{-10}$ .

**(b) A new closed-form check with magnetic field and Darcy porosity retained.** Because Crane's solution alone leaves  $M$ ,  $\varepsilon$ , and the tetra-hybrid ratios entirely unexercised, we additionally verify the corrected Eq (3.2) against the exact solution of the classical MHD/Darcy-porous linear stretching problem. At  $\eta_s = 1$ ,  $\beta_s = 0$ ,  $A_1 = A_2 = A_5 = 1$ , Eq (2.22) reduces to  $f''' + ff'' - f'^2 - (M/2 + \varepsilon)f' = 0$ , which admits the exact solution  $f(\eta) = \lambda^{-1}(1 - e^{-\lambda\eta})$  with  $\lambda = \sqrt{1 + M/2 + \varepsilon}$ , so that  $f''(0) = -\lambda$  exactly (verified by direct substitution). Table 3 reports this comparison at three values of  $(M, \varepsilon)$ ; the present scheme reproduces the exact  $f''(0)$  to within  $4 \times 10^{-14}$  or better in every case, confirming that the magnetic and porous medium terms of the corrected momentum equation, and their  $A_1$ -,  $A_2$ -,  $A_5$ -dependent coefficients, are implemented correctly.

**(b') A new closed-form check for the coupled energy equation.** Checks (a) and (b) validate only the momentum equation; to validate Eq (3.3) independently, consider the reduced case  $\eta_s = 1$ ,  $M = \varepsilon = \beta_s = 0$ ,  $\varphi_1 = \dots = \varphi_4 = 0$  (so  $f(\eta) = 1 - e^{-\eta}$  exactly, as in Check (a)),  $Nr = Nb = Nt = 0$ , and  $Pr = 1$ . The energy equation (2.23) then reduces to the linear, variable coefficient ODE  $\theta'' + f(\eta)\theta' = 0$  with  $\theta(0) = 1$ ,  $\theta(\infty) = 0$ , which integrates exactly: Writing  $p = \theta'$  gives  $p' = -fp$ , so  $p(\eta) = \theta'(0) \exp[-\int_0^\eta f(s) ds] = \theta'(0) e^{1-\eta-e^{-\eta}}$ , and imposing  $\theta(\infty) = 0$  fixes  $\theta'(0) = -1/\int_0^\infty e^{1-t-e^{-t}} dt = -1/(e-1) = -0.58197671\dots$  (the integral is evaluated in closed form via the substitution  $u = e^{-t}$ ). This is, to our knowledge, a new closed-form benchmark for the coupled thermal boundary layer equation in this class of problem and is independent of both Checks (a)/(b) and of any published numerical table. The present scheme reproduces  $\theta'(0) = -0.581976709$ , an absolute deviation of  $1.5 \times 10^{-9}$  from the exact value  $-1/(e-1)$ .

**(c) Collocation versus an independent shooting/Newton method for nontrivial parameter sets.** For three parameter combinations that exercise the full nonlinear stretching, Sutterby, tetra-hybrid, radiative, and bioconvective content of the model simultaneously (Table 3), the collocation solution (playing the role of the `bvp4c` solution) and the independent shooting/Newton solution described above agree in every one of  $f''(0)$ ,  $\theta'(0)$ ,  $\phi'(0)$ ,  $\chi'(0)$  to at least nine significant figures.

**Table 3.** Validation of the present numerical scheme. (a)–(b') Comparison against exact closed-form solutions. (c) Collocation versus an independent shooting/Newton method at three nontrivial parameter sets away from the base case, reporting  $(f''(0), \theta'(0), \phi'(0), \chi'(0))$ .

| Check                                                                                                                                                  | Present result                                                                  | Reference / cross-check                                                                                        |
|--------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------|
| (a) Crane [2]: $\eta_s=1$ , all effects off                                                                                                            | $f''(0) = -1.00000000038$                                                       | Exact $-1$ ; $ \text{err}  = 3.8 \times 10^{-10}$                                                              |
| (b) MHD+Darcy, exact: $M=1, \varepsilon=0$                                                                                                             | $f''(0) = -1.2247449$                                                           | Exact $-\sqrt{1.5} = -1.2247449$ ; $ \text{err}  = 4 \times 10^{-14}$                                          |
| (b) MHD+Darcy, exact: $M=3, \varepsilon=1.5$                                                                                                           | $f''(0) = -2.0000000$                                                           | Exact $-2$ ; $ \text{err}  < 10^{-15}$                                                                         |
| (b) MHD+Darcy, exact: $M=5, \varepsilon=2.0$                                                                                                           | $f''(0) = -2.3452079$                                                           | Exact $-\sqrt{5.5} = -2.3452079$ ; $ \text{err}  < 10^{-15}$                                                   |
| (b') Coupled energy, Exact ( $\eta_s=1, Pr=1$ , all others are off)                                                                                    | $\theta'(0) = -0.581976709$                                                     | Exact $-1/(e-1) = -0.581976707$ ; $ \text{err}  = 1.5 \times 10^{-9}$                                          |
| (c) Set 1 – $\eta_s=2, \varepsilon=0.5, M=2, n=3, \Gamma=0.3, \varphi_i=0.01, Pr=6.2, Nr=0.5, Nb=0.5, Nt=0.3, Sc=0.5, Kr=0.3, Sb=0.5, Pe=0.5, G=0.2$   | Collocation:<br>( $-1.367698,$<br>$-1.122294,$<br>$-0.029522,$<br>$-0.293487$ ) | Shooting:<br>( $-1.367698,$<br>$-1.122294,$<br>$-0.029522,$<br>$-0.293487$ )<br>Max diff $3.6 \times 10^{-12}$ |
| (c) Set 2 – $\eta_s=3, \varepsilon=1.0, M=5, n=2, \Gamma=0.6, \varphi_i=0.02, Pr=6.2, Nr=1.0, Nb=0.3, Nt=0.5, Sc=1.0, Kr=0.5, Sb=1.0, Pe=0.3, G=0.4$   | Collocation:<br>( $-1.578135,$<br>$-0.836451,$<br>$-0.181142,$<br>$-0.519325$ ) | Shooting:<br>( $-1.578135,$<br>$-0.836451,$<br>$-0.181142,$<br>$-0.519325$ )<br>Max diff $6.5 \times 10^{-10}$ |
| (c) Set 3 – $\eta_s=1.5, \varepsilon=0.2, M=8, n=4, \Gamma=0.1, \varphi_i=0.03, Pr=3.0, Nr=0.2, Nb=0.6, Nt=0.2, Sc=0.8, Kr=0.1, Sb=0.4, Pe=0.7, G=0.1$ | Collocation:<br>( $-1.963520,$<br>$-0.696491,$<br>$-0.297269,$<br>$-0.391434$ ) | Shooting:<br>( $-1.963520,$<br>$-0.696491,$<br>$-0.297269,$<br>$-0.391434$ )<br>Max diff $< 10^{-14}$          |

With the numerical scheme validated on all three fronts, all physical effects were reinstated and the governing parameters were subsequently varied over the ranges  $\varepsilon \in [0, 1.5]$ ,  $M \in [0, 12]$ ,  $\Gamma \in [0, 0.9]$ ,  $n \in [1, 4]$ ,  $\varphi_i \in [0, 0.03]$  (equal loading across all four nanoparticles),  $Pr \in [1, 10]$ ,  $Nr \in [0, 1.5]$ ,  $Nb, Nt \in [0.1, 0.7]$ ,  $Sc \in [0.3, 1.5]$ ,  $Kr \in [0, 0.9]$ ,  $Sb \in [0.3, 1.5]$ ,  $Pe \in [0.1, 0.8]$ , and  $G \in [0.1, 0.6]$ , while holding the remaining parameters at the base values  $\eta_s = 2$ ,  $\varepsilon = 0.5$ ,  $M = 2$ ,  $n = 3$ ,  $\Gamma = 0.3$ ,  $\varphi_i = 0.01$ ,  $Pr = 6.2$ ,  $Nr = 0.5$ ,  $Nb = 0.5$ ,  $Nt = 0.3$ ,  $Sc = 0.5$ ,  $Kr = 0.3$ ,  $Sb = 0.5$ ,  $Pe = 0.5$ , and  $G = 0.2$ , in line with the ranges reported in the related Sutterby nanofluid, tetra-hybrid nanofluid, and bioconvective nanofluid literature [8, 12, 18, 22, 23]. These parameter values, like those of the cited comparison studies, are adopted from the numerical (rather than experimental) nanofluid and

bioconvection literature: Complete, mutually consistent experimental measurements of the Sutterby rheological parameters ( $n, \Gamma$ ), together with the Brownian, thermophoretic, and bioconvective transport coefficients for a specific tetra-hybrid nanofluid remain scarce, and we note this explicitly as a limitation of the present (and essentially all comparable) numerical studies, rather than implying an experimental basis for the specific values used; closing this gap through dedicated rheological and transport property measurements for tetra-hybrid nanofluid–microorganism suspensions is identified as a priority for future experimental work. Every result in Section 4 also uses equal nanoparticle loadings,  $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4$ ; the effect of unequal individual particle proportions is explored separately as a supplementary comparison in Section 4.5. The resulting velocity, temperature, concentration, and motile microorganism density profiles, together with the skin friction, Nusselt number, Sherwood number, and motile microorganism density number, are presented and discussed in Section 4.

## 4. Results and discussion

Unless otherwise stated, the base case parameter values  $\eta_s = 2$ ,  $\varepsilon = 0.5$ ,  $M = 2$ ,  $n = 3$ ,  $\Gamma = 0.3$ ,  $\varphi_i = 0.01$  ( $i = 1-4$ , equal loading),  $Pr = 6.2$ ,  $Nr = 0.5$ ,  $Nb = 0.5$ ,  $Nt = 0.3$ ,  $Sc = 0.5$ ,  $Kr = 0.3$ ,  $Sb = 0.5$ ,  $Pe = 0.5$ , and  $G = 0.2$  are used throughout, with one parameter varied at a time as summarized in Table 4, all computed via Eqs (2.22)–(2.31) and the domain  $\eta_\infty = 20$  established in Section 3. The physical model is shown in Figure 1 (Section 2.1). For every profile discussed below, the trend is stated explicitly as an increase or decrease in the plotted quantity, together with the physical mechanism responsible.

### 4.1. Effect of the magnetic field parameter

Figure 3 shows that as the magnetic field parameter  $M$  increases from 0.3 to 0.9, the velocity  $f'(\eta)$  decreases throughout the boundary layer, while the temperature  $\theta(\eta)$  increases. The physical reason is that the applied transverse magnetic field generates a resistive Lorentz force  $-(A_5/A_1)[M/(\eta_s + 1)]f'$  in the corrected Eq (2.22) that directly opposes fluid motion; the resulting reduction in convective velocity thickens the momentum boundary layer and simultaneously suppresses convective heat removal from the sheet, thickening the thermal boundary layer. This is confirmed quantitatively in Table 4: as  $M$  increases from 0 to 12, the skin friction magnitude  $|Re_x^{1/2}C_f|$  increases from 2.290 to 5.535 (greater wall shear caused by the steeper near-wall velocity gradient left behind as the bulk flow is retarded), while the Nusselt number  $Re_x^{-1/2}Nu_x$  decreases from 1.602 to 1.352 (a thicker thermal boundary layer transports heat less efficiently to the wall) consistent with the magnetic retardation trend reported for MHD hybrid nanofluid porous stretching sheet flows [15–17] and with the Jeffrey-fluid base study [3].

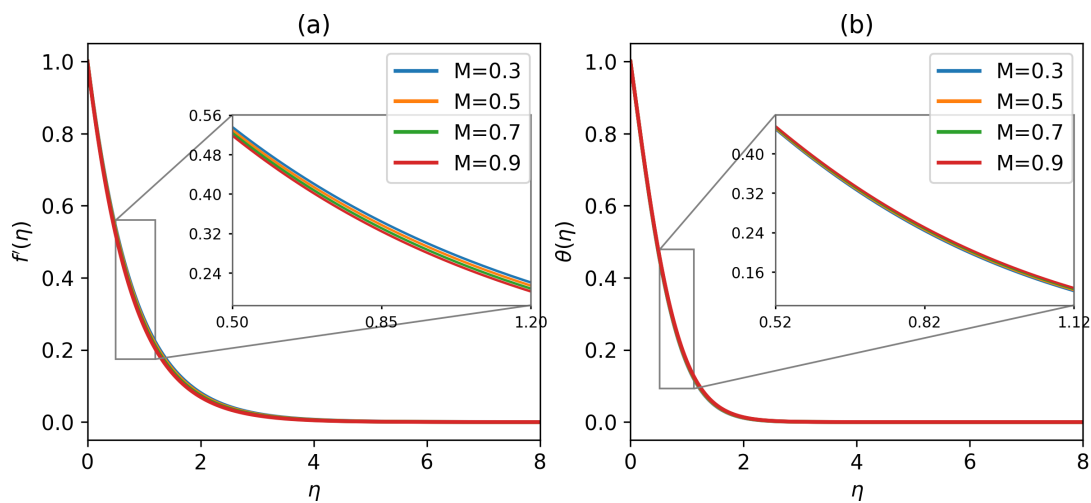


**Figure 3.** Effect of the magnetic field parameter  $M$  on (a) velocity  $f'(\eta)$  and (b) temperature  $\theta(\eta)$ .

#### 4.2. Effect of the porosity parameter

Figure 4 shows that as the porosity parameter  $\varepsilon$  increases from 0 to 1.5, the velocity  $f'(\eta)$  decreases throughout the boundary layer, while the temperature  $\theta(\eta)$  shows a corresponding increase. A larger  $\varepsilon$  corresponds to a more resistive (less permeable) Darcy porous matrix now understood, following the spatial prescription  $K(x) = K_0 x^{1-\eta_s}$  of Eq (2.1), as a constant multiple of the local permeability rather than a raw permeability value – which impedes fluid motion through the pore structure and is reflected in the Darcy drag term  $-[2\varepsilon/(\eta_s + 1)]f'$  of the corrected Eq (2.22); the associated reduction in velocity again limits convective heat transport. Quantitatively, the magnitude of skin friction increases from 2.602 ( $\varepsilon = 0$ ) to 3.449 ( $\varepsilon = 1.5$ ), while the Nusselt number decreases from 1.572 to 1.497, mirroring the porosity resistance trend reported for Darcy-type porous stretching sheet nanofluid flows [3, 13, 14].

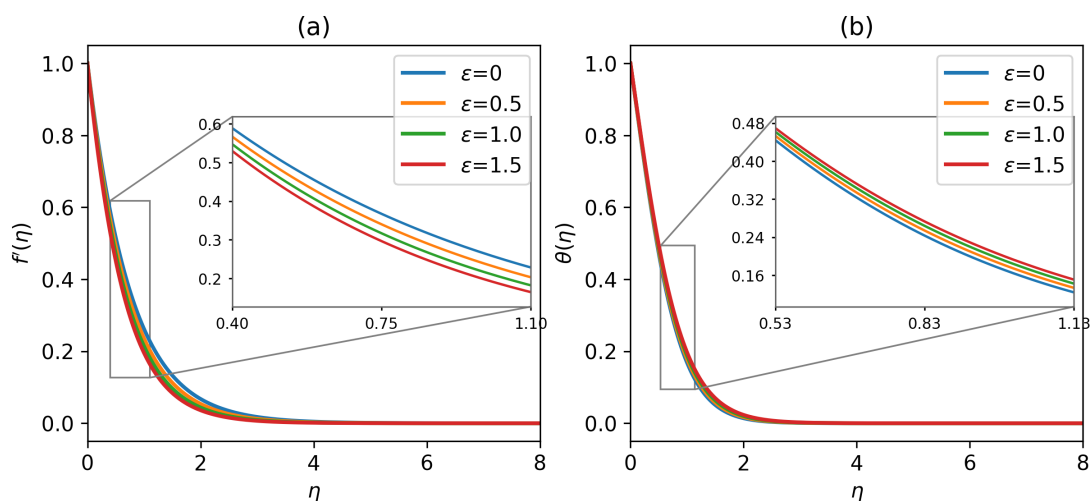


**Figure 4.** Effect of the porosity parameter  $\varepsilon$  on (a) velocity  $f'(\eta)$  and (b) temperature  $\theta(\eta)$ .

### 4.3. Effect of the Sutterby fluid parameters

Figures 5 and 6 show that as the Sutterby fluid parameter  $\Gamma$  increases from 0 (Newtonian limit) to 0.9, and as the power-law index  $n$  increases from 1 to 4, the velocity  $f'(\eta)$  increases throughout the boundary layer: The bulk wall velocity gradient softens from  $f''(0) = -1.542$  at  $\Gamma = 0$  to  $f''(0) = -1.207$  at  $\Gamma = 0.9$ , and from  $f''(0) = -1.542$  at  $n = 1$  to  $f''(0) = -1.314$  at  $n = 4$ . Physically, both  $\Gamma$  and  $n$  enter the momentum equation through the shear-softening coefficient  $\beta_s = (n - 1)\Gamma/2$  in the bracketed term  $[1 + \beta_s f''^2]$  multiplying  $f'''$  in the corrected Eq (2.22): Larger  $\beta_s$  reduces the effective resistance the Sutterby fluid offers to the third derivative of the velocity profile, in a manner analogous to shear-thinning behavior, so the bulk velocity field becomes fuller and the wall velocity gradient  $f''(0)$  softens.

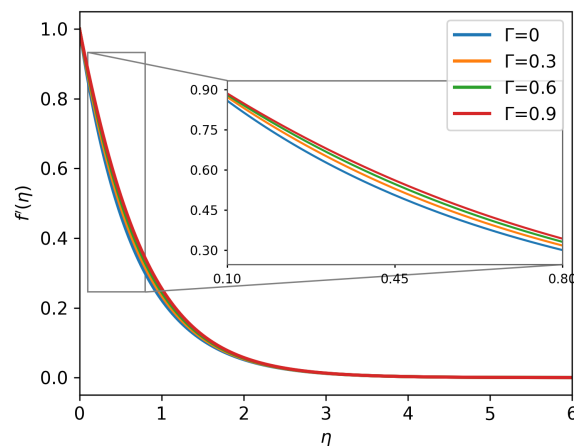


**Figure 5.** Effect of the Sutterby fluid parameter  $\Gamma$  on the velocity  $f'(\eta)$ .

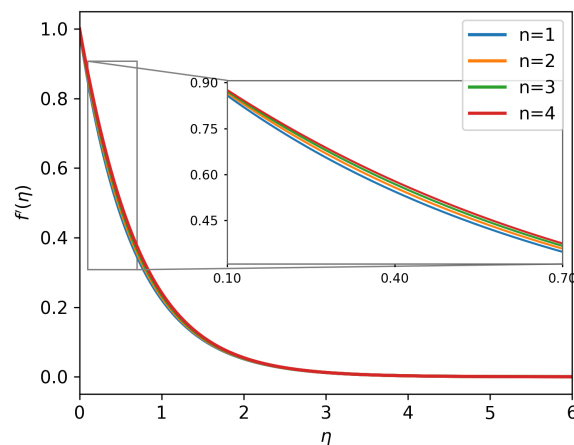


**Figure 6.** Effect of the Sutterby power-law index  $n$  on the velocity  $f'(\eta)$ .

This bulk softening of  $f''(0)$  does not, however, translate into a smaller reported skin friction coefficient once the Sutterby correction to the wall shear stress is included in Eq (2.31). Because the extra Sutterby stress contribution scales with the square of the wall shear rate, the bracket  $[1 + \beta_s f''(0)^2]$  is always  $\geq 1$  and grows with  $\Gamma$  and  $n$  regardless of the sign of  $f''(0)$ ; at the base case, this bracket

rises from 1.000 at  $\Gamma = 0$  to 2.311 at  $\Gamma = 0.9$  (Table 4), more than compensating for the  $\sim 22\%$  reduction in  $|f''(0)|$  over the same range. The net, physically observable effect is therefore that the skin friction magnitude  $|Re_x^{1/2} C_f|$  increases with both  $\Gamma$  and  $n$  – from 2.089 at  $\Gamma = 0$  to 3.778 at  $\Gamma = 0.9$ , and from 2.089 at  $n = 1$  to 3.165 at  $n = 4$ . This is the opposite of what a bare wall velocity gradient argument alone would suggest, since the uncorrected quantity  $A_1 f''(0)$  (lacking the Sutterby bracket) tracks only the softening of  $f''(0)$  itself. We regard this as a genuine, testable prediction of the model: A Sutterby fluid can simultaneously exhibit a softer bulk velocity gradient and a larger actual wall stress, because the two are related by a shear-rate-dependent (not constant) apparent viscosity. The Nusselt number continues to increase with both  $\Gamma$  and  $n$  (from 1.516 to 1.579 as  $\Gamma$  increases, and from 1.516 to 1.555 as  $n$  increases), since this trend is governed by the softened bulk velocity field (via  $Pr A_3 f\theta'$  in the energy equation) rather than by the wall-stress bracket consistent with the softening effect reported for Sutterby nanofluid boundary layers elsewhere [7, 8, 10, 12].

#### 4.4. Effect of tetra-hybrid nanoparticle loading

Figure 7 shows that as the total (equal) tetra-hybrid nanoparticle loading  $\varphi_i$  increases from 0 (clear base fluid) to 0.03, the velocity  $f'(\eta)$  decreases, while the temperature  $\theta(\eta)$  increases. Increasing  $\varphi_i$  raises the effective viscosity ratio  $A_1$  (Table 1) and the density ratio  $A_2$ , but it is now the ratio  $A_2/A_1$  that governs the inertia terms of the corrected Eq (2.22), together with the direct appearance of  $A_1$  in the Sutterby-corrected skin friction of Eq (2.31); the net effect is that increasing  $\varphi_i$  still thickens the momentum boundary layer and increases the resistance to shear, and the magnitude of skin friction increases substantially, from 2.487 ( $\varphi_i = 0$ ) to 3.746 ( $\varphi_i = 0.03$ ). The properly scaled Nusselt number  $Re_x^{-1/2} Nu_x$  of Eq (2.31) likewise increases with loading, from 1.363 to 1.974, because the substantial enhancement of the thermal conductivity ratio  $k_{mf}/k_f$  contributed by the four metallic/oxide nanoparticles more than compensates for the thicker thermal boundary layer confirming that enhanced thermal conductivity is the dominant mechanism once the conductivity ratio is correctly included in the definition of the heat transfer rate, consistent with reports for hybrid and tetra-hybrid nanofluid flows [15, 18, 19]. The Sherwood number  $Re_x^{-1/2} Sh_x$  decreases slightly with the loading (from 0.047 to 0.020), a secondary effect transmitted through the softened concentration gradient in the more strongly retarded velocity field, rather than through any direct  $\varphi_i$ -dependence of the concentration equation itself.

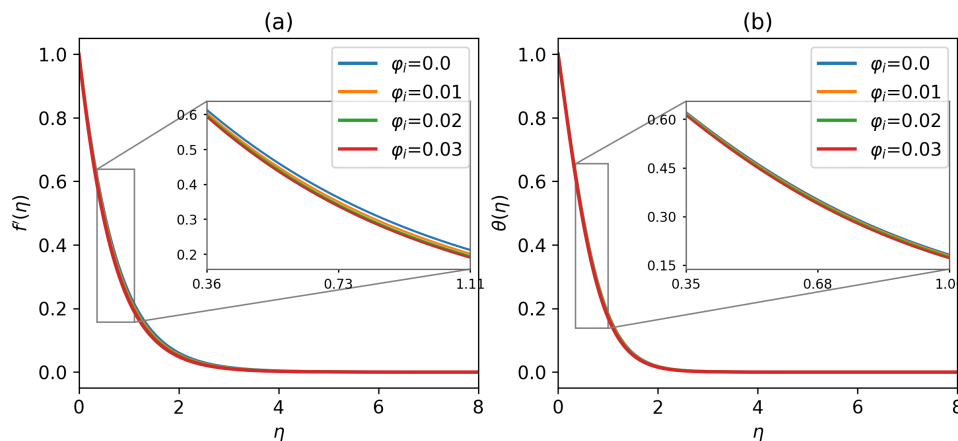


**Figure 7.** Effect of equal tetra-hybrid nanoparticle loading  $\varphi_i$  ( $i = 1-4$ ) on (a) velocity  $f'(\eta)$  and (b) temperature  $\theta(\eta)$ .

#### 4.5. Individual nanoparticle proportions: A supplementary comparison

Every result reported elsewhere in this manuscript uses the equal loading,  $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = \varphi_i$ , and we state this explicitly as a limitation of the present parametric study. The effect of unequal individual particle proportions on heat transfer and skin friction, and the specific advantage of the four-particle tetra-hybrid combination over binary or ternary subcombinations at matched total loading are not systematically explored in Section 4.4 above. Table 4 reports a supplementary comparison, at the base-case values of  $\eta_s, \varepsilon, M, n, \Gamma, Pr, Nr, Nb, Nt, Sc, Kr, Sb, Pe$ , and  $G$ , of the skin friction coefficient and the Nusselt number for the clear base fluid, a single-nanoparticle (Cu) nanofluid, an equal-loading Cu–Ag binary hybrid, an equal loading Cu–Ag–Fe<sub>3</sub>O<sub>4</sub> ternary hybrid, the equal-loading Cu–Ag–Fe<sub>3</sub>O<sub>4</sub>–Zn tetra-hybrid nanofluid used throughout Section 4, and two unequal-loading tetra-hybrid cases – one Cu-dominant, one Zn-dominant – with all four multiparticle rows sharing the same total nanoparticle volume fraction ( $\sum_i \varphi_i = 0.04$ ) so that the comparison isolates the effect of particle number and proportion rather than the total loading.

**Table 4.** Supplementary comparison of skin friction and Nusselt number across single, binary, ternary, and tetra-hybrid nanofluid compositions at a fixed total volume fraction  $\sum_i \varphi_i = 0.04$  and two unequal tetra-hybrid loadings at the base case parameter values ( $\eta_s = 2$ ,  $\varepsilon = 0.5$ ,  $M = 2$ ,  $n = 3$ ,  $\Gamma = 0.3$ ,  $Pr = 6.2$ ,  $Nr = 0.5$ ,  $Nb = 0.5$ ,  $Nt = 0.3$ ,  $Sc = 0.5$ ,  $Kr = 0.3$ ,  $Sb = 0.5$ ,  $Pe = 0.5$ , and  $G = 0.2$ ).

| Composition ( $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ )                           | $Re_x^{1/2} C_f$ | $Re_x^{-1/2} Nu_x$ |
|----------------------------------------------------------------------------------------|------------------|--------------------|
| Base fluid, EG only (0, 0, 0, 0)                                                       | −2.48706         | 1.36326            |
| Cu single nanofluid (0.04, 0, 0, 0)                                                    | −2.92886         | 1.54242            |
| Cu–Ag binary hybrid, equal (0.02, 0.02, 0, 0)                                          | −2.95173         | 1.54141            |
| Cu–Ag–Fe <sub>3</sub> O <sub>4</sub> ternary hybrid, equal (0.0133, 0.0133, 0.0133, 0) | −2.90145         | 1.54237            |
| Cu–Ag–Fe <sub>3</sub> O <sub>4</sub> –Zn tetra-hybrid, equal (0.01, 0.01, 0.01, 0.01)  | −2.89159         | 1.54476            |
| Tetra-hybrid, Cu-dominant (0.025, 0.005, 0.005, 0.005)                                 | −2.90902         | 1.54445            |
| Tetra-hybrid, Zn-dominant (0.005, 0.005, 0.005, 0.025)                                 | −2.87856         | 1.54668            |

Two findings follow. First, all six multiparticle compositions increase both  $|Re_x^{1/2} C_f|$  and  $Re_x^{-1/2} Nu_x$  relative to the clear base fluid, confirming the basic nanofluid's heat transfer benefit regardless of the composition detail. Second, and more informatively, the improvement from the single-particle hybrid, through the binary and ternary hybrids, to the full equal-loading tetra-hybrid combination is modest and nonmonotonic at matched total loading in the present parameter regime:  $Re_x^{-1/2} Nu_x$  moves only from 1.542 (Cu alone) to 1.541 (Cu–Ag), 1.542 (Cu–Ag–Fe<sub>3</sub>O<sub>4</sub>), and 1.545 (full tetra-hybrid) – a improvement of under 0.2% from the single hybrid to the tetra-hybrid, not the large, strictly monotonic gain sometimes implied by qualitative claims of “tetra-hybrid superiority”. The reason can be deduced from Table 1: Fe<sub>3</sub>O<sub>4</sub>'s thermal conductivity (9.7 W/m K) is roughly 40 times lower than that of Cu, Ag, or Zn, so diluting a Cu–Ag binary mixture with Fe<sub>3</sub>O<sub>4</sub> to form the ternary blend trades away some high-conductivity Cu/Ag loading for low-conductivity Fe<sub>3</sub>O<sub>4</sub> at a fixed total volume fraction, very nearly cancelling the benefit of adding a third species. The comparison between the Cu-dominant and Zn-dominant unequal loadings reveals a second, more subtle effect: Although Cu's thermal conductivity (401 W/m K) is more than three times that of Zn (116 W/m K), the Zn-dominant composition produces a slightly higher Nusselt number (1.547 vs. 1.544) than the Cu-dominant one. This is a direct consequence of the sequential (rather than simultaneous) Maxwell mixing rule flagged in Section 2.4: Because Zn is applied *last* in the fixed Cu→Ag→Fe<sub>3</sub>O<sub>4</sub> →Zn sequence, its volume fraction has a structurally larger leverage on the final  $k_{\text{mf}}/k_f$  than the same volume fraction of Cu applied first, independent of the intrinsic conductivity ranking of the two metals. We report this explicitly as a genuine, physically interpretable limitation of the sequential Maxwell construction (Section 2.4) rather than smoothing it over, and note that a full exploration of the parameter space of unequal loading – and, ultimately, a self-consistent (order-independent) effective medium model – is a natural next step beyond the scope of the present single-composition parametric study.

#### 4.6. Effect of thermal radiation

Figure 8 shows that as the thermal radiation parameter  $Nr$  increases from 0 to 1.5, the temperature  $\theta(\eta)$  increases throughout the boundary layer, while the Nusselt number correspondingly *decreases* sharply from 2.089 to 1.065. Physically,  $Nr$  enters the energy balance (Eq (2.23)) as an additional term augmenting the effective thermal diffusivity  $[(k_f/k_{mf}) + (4/3)Nr]$ ; a larger radiative contribution transports thermal energy further into the boundary layer by radiative diffusion, thickening the thermal boundary layer and reducing the temperature gradient – and hence the heat transfer rate – at the wall, in agreement with the radiative heat transfer trend documented for hybrid nanofluid porous stretching sheet flows [10, 15, 17].

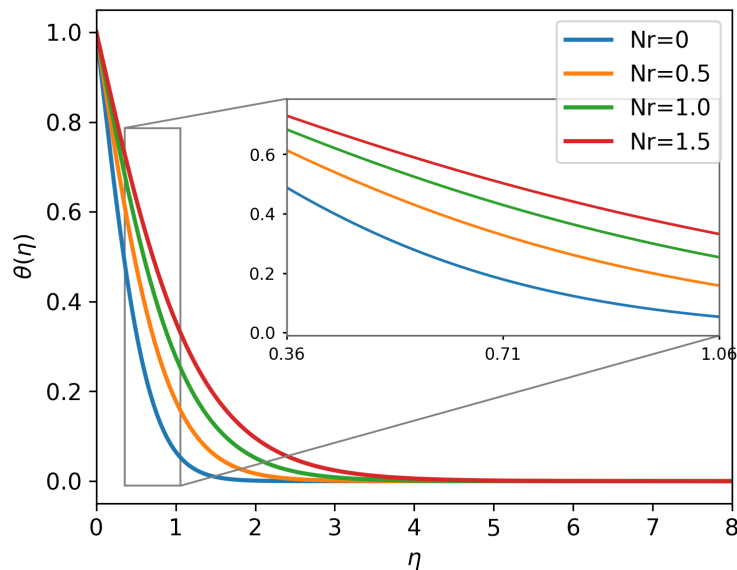


**Figure 8.** Effect of the thermal radiation parameter  $Nr$  on temperature  $\theta(\eta)$ .

#### 4.7. Effect of Brownian motion and thermophoresis

Figures 9 and 10 present the influence of the Brownian motion parameter  $Nb$  and the thermophoresis parameter  $Nt$ , both defined following the Buongiorno nanofluid model [9]. As  $Nb$  increases from 0.1 to 0.7, the temperature  $\theta(\eta)$  decreases slightly (the Nusselt number falls from 1.647 to 1.495), but the concentration  $\phi(\eta)$  decreases markedly at a small value of  $\eta$ , and the Sherwood number  $Re_x^{-1/2} Sh_x$  increases sharply from  $-2.593$  to  $0.223$ . Physically, stronger Brownian diffusion (larger  $Nb$ ) more efficiently redistributes nanoparticles away from the near-wall region where they are generated by the concentration's boundary condition, flattening the concentration profile and steepening its gradient at the wall. As  $Nt$  increases from 0.1 to 0.7, by contrast, the concentration  $\phi(\eta)$  increases near the wall (the profile develops a pronounced overshoot above unity, which is visible in Figure 10(b)), and the Sherwood number decreases sharply from  $0.425$  to  $-0.669$ : A stronger thermophoretic force drives nanoparticles from the hot sheet toward the cooler ambient fluid, thickening the nanoparticle concentration's boundary layer and eventually reversing the sign of the wall concentration gradient once thermophoretic transport dominates Brownian diffusion. Both trends are consistent with the Brownian motion/thermophoresis coupling reported for Sutterby and hybrid nanofluid flows [8, 10, 13]; the negative Sherwood numbers at low  $Nb$  or high  $Nt$  values correspond to a genuine physical reversal of the wall's mass-flux direction (nanoparticles diffusing into rather than out of the fluid at the sheet, driven by the near-wall concentration overshoot, which is visible in the corresponding profile), not to a numerical artefact.

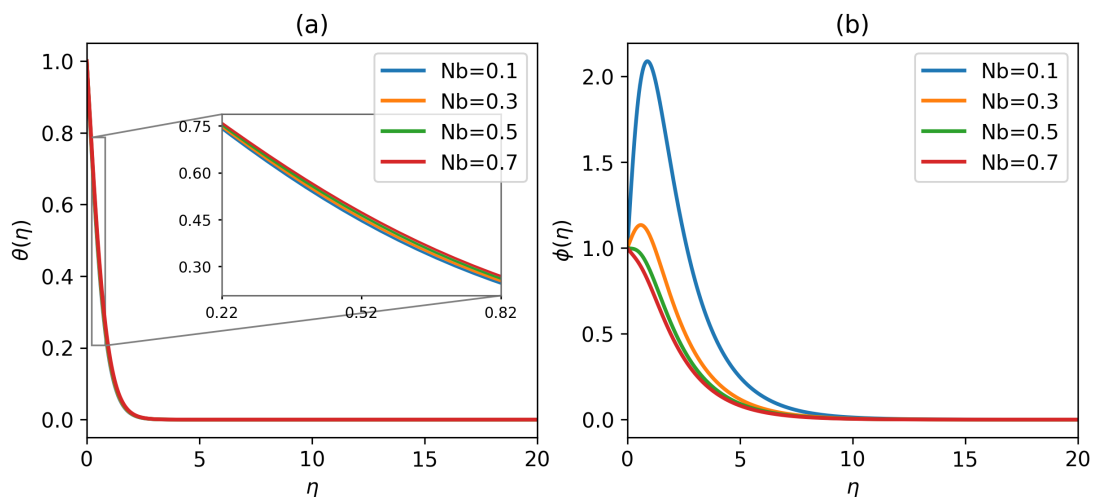


**Figure 9.** Effect of the Brownian motion parameter  $Nb$  on (a) temperature  $\theta(\eta)$  and (b) concentration  $\phi(\eta)$ , plotted to  $\eta = 20$ , over which every curve is confirmed to converge fully to its far-field value (Table 2).

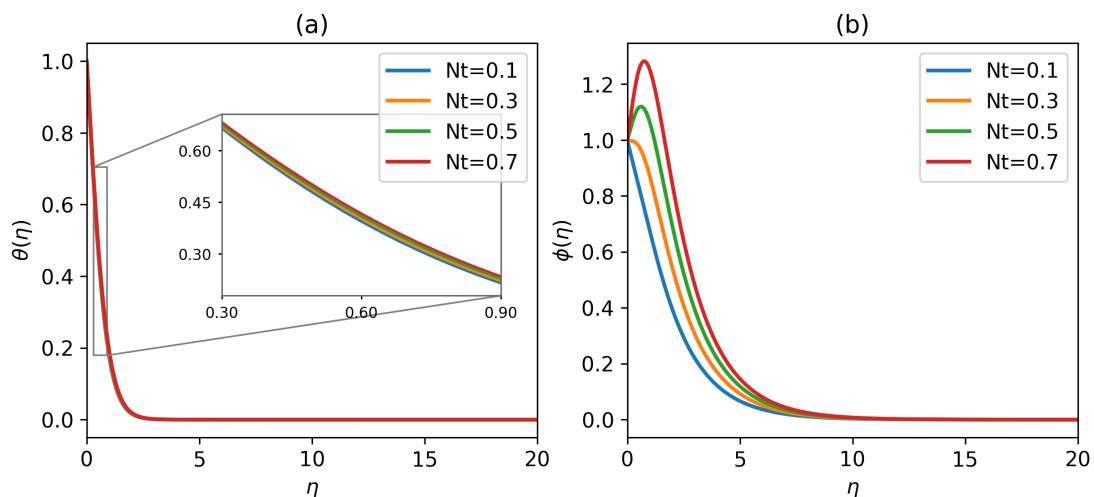


**Figure 10.** Effect of the thermophoresis parameter  $Nt$  on (a) temperature  $\theta(\eta)$  and (b) concentration  $\phi(\eta)$ , plotted to  $\eta = 20$ , over which every curve is confirmed to converge fully to its far-field value (Table 2).

#### 4.8. Effect of the Schmidt number and chemical reaction

Figure 11 shows that as the Schmidt number  $Sc$  increases from 0.3 to 1.5, the concentration  $\phi(\eta)$  decreases near the wall, and the concentration boundary layer becomes visibly thinner. Since  $Sc = \nu_f/D_B$ , a larger  $Sc$  corresponds to weaker Brownian diffusivity relative to momentum-related diffusivity, so nanoparticles diffuse away from the wall region more slowly relative to the momentum field, steepening the concentration gradient at the wall; the Sherwood number correspondingly increases substantially from  $-0.190$  to  $0.766$ . Figure 12 shows that as the chemical reaction parameter  $Kr$  increases from 0 to 0.9, the concentration  $\phi(\eta)$  likewise decreases, and the Sherwood number

increases from  $-0.413$  to  $0.485$ , because a stronger destructive first-order reaction consumes the nanoparticle concentration more rapidly near the wall, steepening the local concentration gradient – both trends are in line with chemically reactive bioconvective nanofluid boundary layer studies [31,33].

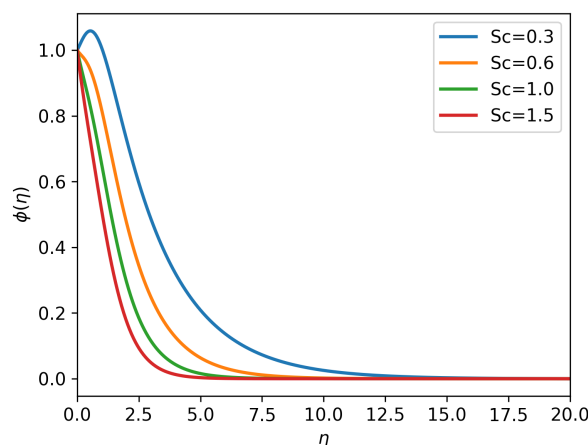


**Figure 11.** Effect of the Schmidt number  $Sc$  on the concentration  $\phi(\eta)$ , plotted to  $\eta = 20$ , over which every curve converges fully to zero.

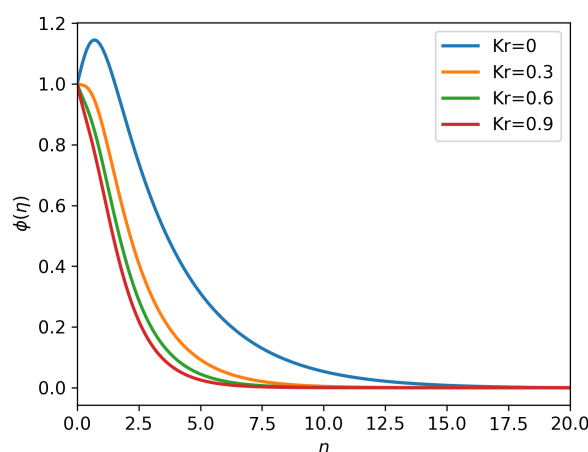


**Figure 12.** Effect of the chemical reaction parameter  $Kr$  on the concentration  $\phi(\eta)$ , plotted to  $\eta = 20$ , over which every curve converges fully to zero.

#### 4.9. Effect of the bioconvection parameters

Figures 13 and 14 show the motile microorganism density field, governed by the gyrotactic bioconvection formulation of [25] and [26]. As the bioconvection Schmidt (Lewis) number  $Sb$  increases from 0.3 to 1.5, the motile microorganism density  $\chi(\eta)$  decreases throughout the boundary layer, and the motile microorganism density number  $Re_x^{-1/2} N n_x$  increases from 0.239 to 0.827: A larger  $Sb = \nu_f/D_m$  corresponds to weaker microorganism diffusivity relative to momentum-related diffusivity, producing a thinner microorganism boundary layer and a steeper wall gradient of  $\chi$ . As the bioconvection Péclet number  $Pe$  increases from 0.1 to 0.8,  $\chi(\eta)$  also decreases, and  $Re_x^{-1/2} N n_x$  increases from 0.352 to 0.366, since a larger  $Pe = bW_c/D_m$  strengthens the advective transport

of motile microorganisms (through their swimming speed  $W_c$ ) relative to their diffusive transport, again steepening the near-wall gradient. The microorganism concentration difference parameter  $G$  produces only a marginal increase in  $Re_x^{-1/2} N n_x$  (from 0.359 at  $G = 0.1$  to 0.361 at  $G = 0.6$ ) over the range examined, indicating that in the present tetra-hybrid nanofluid system, the motile microorganism density number is governed primarily by  $Sb$  and  $Pe$  rather than by  $G$  – a pattern that echoes the relative parameter sensitivities reported in bioconvective hybrid and ternary-hybrid nanofluid studies [19, 20, 22–24, 34].

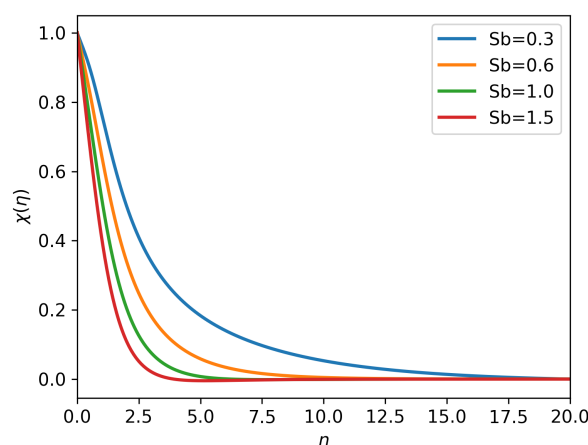


**Figure 13.** Effect of the bioconvection Schmidt number  $Sb$  on motile microorganism density  $\chi(\eta)$ , plotted to  $\eta = 20$ , over which every curve converges fully to zero.

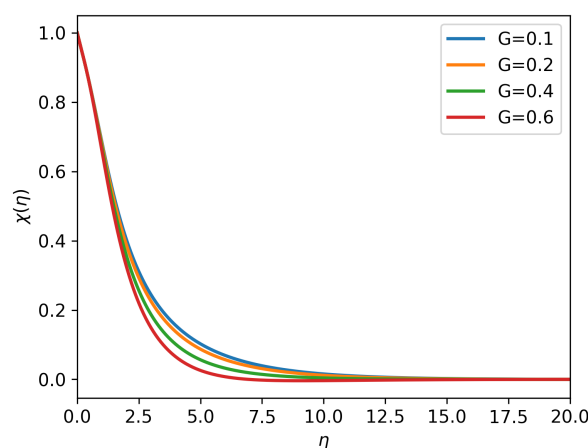


**Figure 14.** Effect of the microorganism concentration difference parameter  $G$  on motile microorganism density  $\chi(\eta)$ , plotted to  $\eta = 20$ , over which every curve converges fully to zero.

#### 4.10. Streamlines

Figure 15 shows the physical plane streamlines, reconstructed from the similarity solution via  $u(x, y) = mx^{\eta_s} f'(\eta)$  and  $v(x, y) = -C_2 x^{(\eta_s-1)/2} [\frac{\eta_s+1}{2} f(\eta) + \frac{\eta_s-1}{2} \eta f'(\eta)]$ , for three magnetic field parameter values ( $M = 0, 2, 8$ ). As  $M$  increases, the streamlines become visibly more crowded near the sheet and

the magnitude of velocity (color scale) decreases across the field: The resistive Lorentz force confines the region of appreciable flow to a thinner layer adjacent to the sheet, consistent with the momentum boundary layer thinning of the magnitude of velocity and the thickening of the boundary layer in terms of  $\eta$  discussed in Section 4.1 – the two descriptions are complementary, since  $\eta$  is itself a stretched coordinate that already absorbs part of the physical boundary layer thickness.

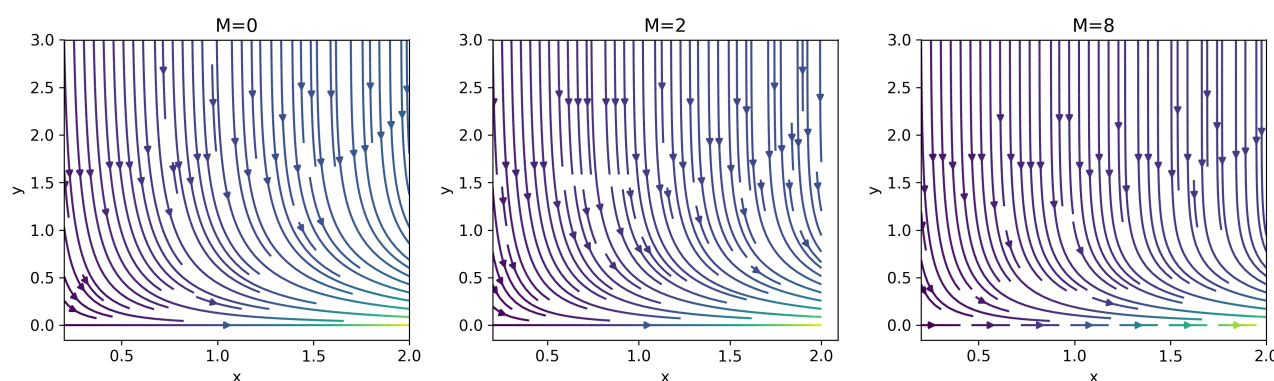


**Figure 15.** Physical plane streamlines colored by magnitude of velocity for  $M = 0, 2$ , and  $8$  (base case values of all other parameters).

#### 4.11. Skin friction coefficient and Nusselt number: Bar chart comparison

Figures 16 and 17 present the skin friction coefficient  $Re_x^{1/2}C_f$  and the Nusselt number  $Re_x^{-1/2}Nu_x$ , both from the corrected Eq (2.31), as bar charts against the magnetic field parameter  $M$  and the tetra-hybrid nanoparticle loading  $\varphi_i$ , respectively, to allow a direct comparison of the magnitude across discrete parameter values. Figure 16 confirms that the skin friction magnitude's bars grow monotonically taller (more negative) as  $M$  increases from 0 to 12, while the Nusselt numbers' bars shrink monotonically, visually reinforcing the opposing  $M$ -trends discussed in Section 4.1. Figure 17 shows that the bars of both the magnitude of skin friction and the Nusselt number bars grow monotonically taller as the tetra-hybrid nanoparticle loading increases from 0 to 0.03, confirming that once the thermal conductivity ratio is correctly included in the Nusselt number definition, higher nanoparticle loading simultaneously increases both the wall shear stress and the heat transfer rate.

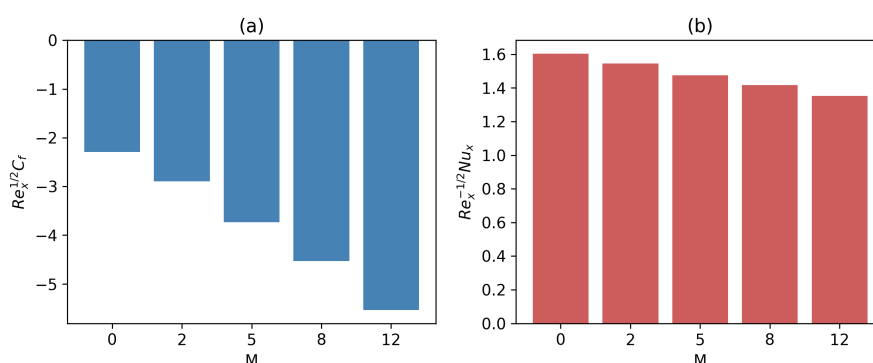


**Figure 16.** Bar chart of (a) the skin friction coefficient  $Re_x^{1/2}C_f$  and (b) Nusselt number  $Re_x^{-1/2}Nu_x$  for a varying magnetic field parameter  $M$ .

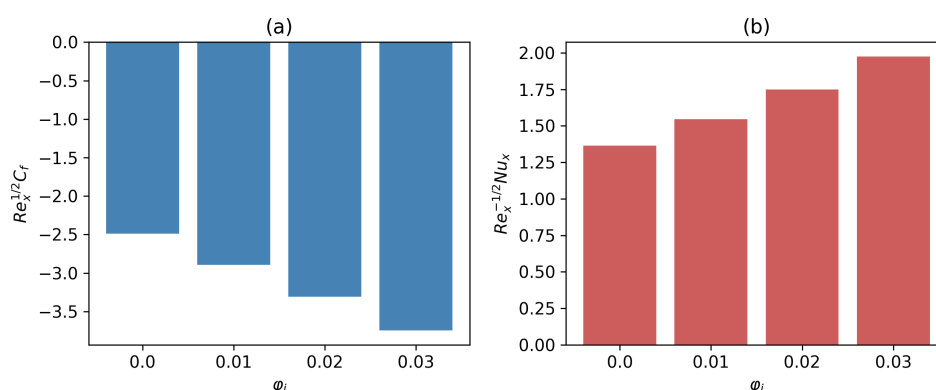


**Figure 17.** Bar chart of (a) the skin friction coefficient  $Re_x^{1/2}C_f$  and (b) Nusselt number  $Re_x^{-1/2}Nu_x$  for a varying tetra-hybrid nanoparticle loading  $\varphi_i$  ( $i = 1-4$ , equal loading).

#### 4.12. Discussion of Table 5

Table 5 collects the wall transport quantities – the skin friction coefficient, Nusselt number, Sherwood number, and motile microorganism density number – for every parameter sweep discussed above, and several cross-cutting observations follow from it that are not visible from any single figure in isolation. First, the skin friction coefficient  $Re_x^{1/2}C_f$  is negative throughout the entire table, confirming that the sheet always exerts a drag-inducing shear stress on the fluid for every parameter combination examined, and its magnitude spans more than a factor of two (from 2.290 to 5.535) across the ranges of  $M$  alone, indicating that the magnetic field parameter is the single most influential parameter on wall shear stress among those considered. Second, with the Sutterby correction of the wall stress included (Section 4.3), all five of the parameters that directly enter the momentum equation ( $M$ ,  $\varepsilon$ ,  $\Gamma$ ,  $n$ ,  $\varphi_i$ ) increase the reported magnitude of skin friction as they increase. The bulk wall velocity gradient  $f''(0)$  itself still softens with  $\Gamma$  and  $n$ , consistent with shear-thinning, but the Sutterby stress bracket  $[1 + \beta_s f''(0)^2]$  grows faster than  $f''(0)$  shrinks, so the two mechanisms do not cancel to leading order once both are correctly retained. Third, the Nusselt number is more sensitive to the thermal parameters  $Pr$ ,  $Nr$ , and  $Nb$  than to the momentum parameters  $M$ , and  $\varepsilon$ . For example,  $Pr$  alone varies  $Re_x^{-1/2}Nu_x$  by nearly a factor of six (from 0.360 to 2.120) as  $Pr$  increases from 1 to 10, since a larger Prandtl number directly reduces the relative thermal diffusivity, sharply steepening the wall temperature gradient. Fourth, the Sherwood number changes sign in the table (from negative to positive) as  $Nb$ ,  $Nt$ ,  $Pr$ ,  $Nr$ ,  $Sc$ , and  $Kr$  are varied, which reflects the fact that the nanoparticle concentration profile can either overshoot or remain below its wall value of unity depending on the relative strength of Brownian motion, thermophoresis, and radiative heating (Figures 9–12); a positive Sherwood number corresponds to a concentration profile that decays monotonically from the wall (nanoparticles diffusing outward, away from the sheet), while a negative value corresponds to a near-wall concentration overshoot that reverses the direction of the wall mass flux (nanoparticles diffusing inward, toward the sheet, once thermophoretic or radiative effects dominate Brownian diffusion). Fifth, the motile microorganism density number  $Re_x^{-1/2}Nn_x$  is comparatively insensitive to  $Sc$  and  $Kr$ , which act only indirectly through the concentration field; more strongly affected by  $Pr$  and  $Nr$  through the coupled momentum–energy–concentration chain feeding into the microorganism equation via  $\phi''$  in Eq (2.25) and most sensitive of all to the bioconvection parameters  $Sb$  and  $Pe$  that act

directly on the microorganism equation, which vary it by close to a factor of three and a half (0.239 to 0.827 across the  $Sb$  range). In summary, Table 4 shows that the momentum, thermal, solutal, and bioconvective transport mechanisms in the present Sutterby tetra-hybrid nanofluid system are coupled predominantly in one direction – from momentum, through energy, into concentration, and finally into the microorganism field – with each family of governing parameters ( $M, \varepsilon, \Gamma, n$ , and  $\varphi_i$  for momentum;  $Pr, Nr, Nb$ , and  $Nt$  for heat;  $Sc$ , and  $Kr$  for mass; and  $Sb, Pe$ , and  $G$  for bioconvection) primarily controlling its own corresponding wall transport quantity while still producing measurable secondary effects downstream through the shared similarity variable  $\eta$ , the coupled property ratios  $A_1$ – $A_5$ , and the explicit  $\phi''$  term in the microorganism equation.

**Table 5.** Computed skin friction coefficient  $Re_x^{1/2}C_f$ , Nusselt number  $Re_x^{-1/2}Nu_x$ , Sherwood number  $Re_x^{-1/2}Sh_x$ , and motile microorganism density number  $Re_x^{-1/2}Nn_x$ , all from the corrected Eq (2.31), for varying governing parameters (base case:  $\eta_s = 2$ ,  $\varepsilon = 0.5$ ,  $M = 2$ ,  $n = 3$ ,  $\Gamma = 0.3$ ,  $\varphi_i = 0.01$ ,  $Pr = 6.2$ ,  $Nr = 0.5$ ,  $Nb = 0.5$ ,  $Nt = 0.3$ ,  $Sc = 0.5$ ,  $Kr = 0.3$ ,  $Sb = 0.5$ ,  $Pe = 0.5$ ,  $G = 0.2$ ; domain  $\eta_\infty = 20$ ).

| Parameter     | Value | $Re_x^{1/2}C_f$ | $Re_x^{-1/2}Nu_x$ | $Re_x^{-1/2}Sh_x$ | $Re_x^{-1/2}Nn_x$ |
|---------------|-------|-----------------|-------------------|-------------------|-------------------|
| $M$           | 0     | −2.28998        | 1.60239           | 0.03267           | 0.39198           |
| $M$           | 2     | −2.89159        | 1.54476           | 0.03616           | 0.35945           |
| $M$           | 5     | −3.73361        | 1.47426           | 0.04723           | 0.33007           |
| $M$           | 8     | −4.52800        | 1.41616           | 0.06030           | 0.31258           |
| $M$           | 12    | −5.53519        | 1.35165           | 0.07776           | 0.29868           |
| $\varepsilon$ | 0     | −2.60164        | 1.57167           | 0.03378           | 0.37355           |
| $\varepsilon$ | 0.5   | −2.89159        | 1.54476           | 0.03616           | 0.35945           |
| $\varepsilon$ | 1.0   | −3.17353        | 1.51998           | 0.03933           | 0.34795           |
| $\varepsilon$ | 1.5   | −3.44880        | 1.49696           | 0.04301           | 0.33844           |
| $\Gamma$      | 0     | −2.08935        | 1.51611           | 0.04335           | 0.35350           |
| $\Gamma$      | 0.3   | −2.89159        | 1.54476           | 0.03616           | 0.35945           |
| $\Gamma$      | 0.6   | −3.39655        | 1.56423           | 0.03181           | 0.36442           |
| $\Gamma$      | 0.9   | −3.77768        | 1.57913           | 0.02882           | 0.36880           |
| $n$           | 1     | −2.08935        | 1.51611           | 0.04335           | 0.35350           |
| $n$           | 2     | −2.55175        | 1.53215           | 0.03921           | 0.35663           |
| $n$           | 3     | −2.89159        | 1.54476           | 0.03616           | 0.35945           |
| $n$           | 4     | −3.16522        | 1.55524           | 0.03376           | 0.36203           |
| $\varphi_i$   | 0.0   | −2.48706        | 1.36326           | 0.04662           | 0.37200           |
| $\varphi_i$   | 0.01  | −2.89159        | 1.54476           | 0.03616           | 0.35945           |
| $\varphi_i$   | 0.02  | −3.31000        | 1.74800           | 0.02718           | 0.34995           |
| $\varphi_i$   | 0.03  | −3.74566        | 1.97424           | 0.01969           | 0.34296           |
| $Pr$          | 1     | −2.89159        | 0.36037           | 0.55765           | 0.64700           |
| $Pr$          | 3     | −2.89159        | 0.91094           | 0.33241           | 0.52466           |
| $Pr$          | 6.2   | −2.89159        | 1.54476           | 0.03616           | 0.35945           |
| $Pr$          | 10    | −2.89159        | 2.11983           | −0.24639          | 0.19885           |
| $Nr$          | 0     | −2.89159        | 2.08903           | −0.22925          | 0.20935           |
| $Nr$          | 0.5   | −2.89159        | 1.54476           | 0.03616           | 0.35945           |

*Continued on next page*

| Parameter | Value | $Re_x^{1/2}C_f$ | $Re_x^{-1/2}Nu_x$ | $Re_x^{-1/2}Sh_x$ | $Re_x^{-1/2}Nn_x$ |
|-----------|-------|-----------------|-------------------|-------------------|-------------------|
| $Nr$      | 1.0   | −2.89159        | 1.25340           | 0.17392           | 0.43643           |
| $Nr$      | 1.5   | −2.89159        | 1.06513           | 0.26057           | 0.48446           |
| $Nb$      | 0.1   | −2.89159        | 1.64670           | −2.59320          | −1.05123          |
| $Nb$      | 0.3   | −2.89159        | 1.59531           | −0.40168          | 0.11993           |
| $Nb$      | 0.5   | −2.89159        | 1.54476           | 0.03616           | 0.35945           |
| $Nb$      | 0.7   | −2.89159        | 1.49507           | 0.22347           | 0.46238           |
| $Nt$      | 0.1   | −2.89159        | 1.58289           | 0.42538           | 0.57243           |
| $Nt$      | 0.3   | −2.89159        | 1.54476           | 0.03616           | 0.35945           |
| $Nt$      | 0.5   | −2.89159        | 1.50736           | −0.32840          | 0.16199           |
| $Nt$      | 0.7   | −2.89159        | 1.47067           | −0.66908          | −0.02043          |
| $Sc$      | 0.3   | −2.89159        | 1.58412           | −0.19004          | 0.23799           |
| $Sc$      | 0.6   | −2.89159        | 1.52904           | 0.13213           | 0.41144           |
| $Sc$      | 1.0   | −2.89159        | 1.48100           | 0.45138           | 0.58627           |
| $Sc$      | 1.5   | −2.89159        | 1.43992           | 0.76564           | 0.76086           |
| $Kr$      | 0     | −2.89159        | 1.61910           | −0.41316          | 0.11719           |
| $Kr$      | 0.3   | −2.89159        | 1.54476           | 0.03616           | 0.35945           |
| $Kr$      | 0.6   | −2.89159        | 1.50866           | 0.29101           | 0.49953           |
| $Kr$      | 0.9   | −2.89159        | 1.48384           | 0.48505           | 0.60727           |
| $Sb$      | 0.3   | −2.89159        | 1.54476           | 0.03616           | 0.23909           |
| $Sb$      | 0.6   | −2.89159        | 1.54476           | 0.03616           | 0.41546           |
| $Sb$      | 1.0   | −2.89159        | 1.54476           | 0.03616           | 0.61561           |
| $Sb$      | 1.5   | −2.89159        | 1.54476           | 0.03616           | 0.82704           |
| $Pe$      | 0.1   | −2.89159        | 1.54476           | 0.03616           | 0.35209           |
| $Pe$      | 0.3   | −2.89159        | 1.54476           | 0.03616           | 0.35564           |
| $Pe$      | 0.5   | −2.89159        | 1.54476           | 0.03616           | 0.35945           |
| $Pe$      | 0.8   | −2.89159        | 1.54476           | 0.03616           | 0.36561           |
| $G$       | 0.1   | −2.89159        | 1.54476           | 0.03616           | 0.35903           |
| $G$       | 0.2   | −2.89159        | 1.54476           | 0.03616           | 0.35945           |
| $G$       | 0.4   | −2.89159        | 1.54476           | 0.03616           | 0.36027           |
| $G$       | 0.6   | −2.89159        | 1.54476           | 0.03616           | 0.36109           |

## 5. Conclusions

This work has presented a numerical investigation of MHD Sutterby tetra-hybrid nanofluid flow, heat, mass, and motile microorganism transport over a nonlinear porous stretching sheet, extending the Jeffrey formulation for the temperature-dependent viscosity Jeffrey of a fluid of the base study [3] to a physically richer system incorporating a four-nanoparticle (Cu, Ag, Fe<sub>3</sub>O<sub>4</sub>, Zn) tetra-hybrid nanofluid, Sutterby non-Newtonian rheology, and bioconvective transport of gyrotactic motile microorganisms governed by the Buongiorno–Kuznetsov–Nield framework. The similarity transformation, the momentum equation, the sequential mixture formulas, the meaning of the concentration field, the microorganism equation, and all four engineering response quantities were

derived from first principles. The absence of buoyancy and microorganism momentum backcoupling was justified explicitly. The applicability limits of the sequential Maxwell mixture rule were discussed, together with a supplementary comparison across single, binary, ternary, and unequally loaded tetra-hybrid compositions; the numerical scheme was validated against three exact closed-form solutions (Crane's classical result, a new exact solution for combined MHD/Darcy porous retardation, and a new exact solution of the coupled thermal boundary layer equation) and cross-checked against an independently coded shooting/Newton method at three nontrivial parameter sets. A mathematical argument was given showing that the nonmonotonic nanoparticle concentration profiles obtained are a forced consequence of the governing equation rather than an artefact. The domain-independence study covered all four wall gradients, showing that  $\eta_\infty = 20$  is required for the concentration and microorganism-density gradients to converge, and the physical feasibility of the modeled suspension was discussed explicitly. The principal findings are summarized as follows.

- Increasing the magnetic field parameter  $M$  and the porosity parameter  $\varepsilon$  both increase the magnitude of the skin friction coefficient while decreasing the Nusselt number, consistent with the Lorentz force retardation and porous medium resistance mechanisms reported for related non-Newtonian and hybrid nanofluid stretching sheet flows.
- The Sutterby fluid parameters  $\Gamma$  and  $n$  soften the bulk wall velocity gradient  $f''(0)$ , consistent with shear-thinning, but – once the Sutterby correction to the wall shear stress is included in the skin friction coefficient – increase, rather than decrease, its reported magnitude, because the wall stress bracket  $[1 + \beta_s f''(0)^2]$  grows faster than  $f''(0)$  softens.
- Increasing the tetra-hybrid nanoparticle loading  $\varphi_i$  increases both the magnitude of the skin friction coefficient (through the effective viscosity and density ratios, combined as  $A_2/A_1$  in the momentum equation) and the Nusselt number (through the enhanced thermal conductivity ratio of the four-nanoparticle suspension). At matched total loading, however, the supplementary comparison of Section 4.5 shows that this tetra-hybrid advantage over single-, binary-, and ternary-particle suspensions is modest in the present parameter regime, an effect traced to the comparatively low thermal conductivity of  $\text{Fe}_3\text{O}_4$  and to an inherent order-dependence of the sequential Maxwell mixture rule – a genuine limitation of the model rather than a universal advantage of adding more nanoparticle species.
- Thermal radiation  $Nr$  substantially decreases the Nusselt number, while the Brownian motion  $Nb$  and thermophoresis  $Nt$  parameters strongly affect the nanoparticle concentration field, including a sign reversal of the wall concentration gradient at a low  $Nb$  or a sufficiently high  $Nt$ ; this nonmonotonic behavior and the corresponding negative Sherwood numbers are shown analytically (Section 2.6) to be admissible forced solutions of the governing concentration equation, corresponding to a genuine reversal of the wall's mass flux direction rather than to unphysical or erroneous behavior.
- The Schmidt number  $Sc$  and the chemical reaction parameter  $Kr$  both increase the magnitude of the Sherwood number, reflecting steeper concentration gradients at the sheet under weaker Brownian diffusivity or a more aggressive chemical reaction.
- The bioconvection Schmidt number  $Sb$  and the bioconvection Péclet number  $Pe$  both increase

the motile microorganism density number, whereas the microorganism concentration difference parameter  $G$  has only a marginal influence over the range examined.

These results demonstrate the complex, coupled nature of momentum, thermal, solutal, and bioconvective transport in a Sutterby tetra-hybrid nanofluid flowing through a porous, magnetically influenced stretching sheet configuration. As discussed in Section 2.1, the present analysis should be read as an idealized mathematical model, with buoyancy and microorganism momentum backcoupling neglected under the same forced convection and dilute suspension assumptions used throughout the Buongiorno–Kuznetsov–Nield literature it extends. The biological compatibility of EG and Cu–Ag–Fe<sub>3</sub>O<sub>4</sub>–Zn nanoparticles with living gyrotactic microorganisms has not been established experimentally, and for this reason, the application discussed in this manuscript is restricted to thermal management targets – microfluidic cooling, photovoltaic thermal fluids, and biofuel cells/biosensors – that do not depend on microorganisms’ viability; a biomedical or drug delivery extension is not pursued here and would require a biocompatible carrier fluid and experimental confirmation of microorganism viability first. Within this idealized setting, the model provides quantitative guidance for four-nanoparticle thermal systems more broadly, such as microfluidic cooling and photovoltaic thermal fluids, and a mathematical foundation that a biocompatible carrier fluid extension could build upon. Future work could extend the present model to unsteady flow, mixed convection with buoyancy (relaxing the forced convection assumption noted above for an inclined or vertical sheet), activation energy effects, a fully resolved multi species concentration field (Section 2.2), a self-consistent (order-independent) effective medium property model in place of the sequential Maxwell rule (Section 2.4), a systematic exploration of unequal nanoparticle loadings beyond the two illustrative cases of Section 4.5, and experimental validation of the tetra-hybrid nanofluid properties’ correlations and the microorganism viability assumption used here.

### Author contributions

R. T. Matoog: Conceptualization, resources, supervision, writing – review & editing; R. Sarathy: Methodology, software, formal analysis, validation, investigation, visualization, writing – original draft; R. Revathi: Conceptualization, methodology, project administration, supervision, writing – review & editing; Sultan Hassan Hakmi: Resources, funding acquisition, validation, writing – review & editing. All authors have read and approved the final version of the manuscript for publication.

### Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

### Conflict of interest

The authors declare that there is no conflict of interest.

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