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Research article

Complexity and symmetry: the asymmetry factor approach

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Abstract: We study the influence of the instantaneous appearance of a time-like Killing vector (TKV) in self-gravitating fluid spheres during their evolution. For doing so, we resort to the concept of the asymmetry factor, which allows the existence of any specific type of symmetry (e.g., a TKV), exclusively, for a given value of the time-like coordinate. We consider adiabatic and dissipative fluids. The analysis of different relevant physical variables in this process shows the imprints of the emergence of the TKV at some point of the evolution. Particular attention is paid to the relationship appearing between the TKV and the complexity factor describing the degree of complexity of the fluid, bringing out for the first time a link between complexity and symmetry.

Keywords: relativistic fluids; complexity; interior solutions
Mathematics Subject Classification: 83C15, 83C99, 85A99

1. Introduction

In a recent paper [1] we introduced the concept of an asymmetry factor, which is a scalar function that allows to switch on (off) a given symmetry for any specific value of the time-like coordinate. In [1] we considered the case of a conformal Killing vector, and we looked for the imprints that the instantaneous appearance of such a symmetry produces on the evolution of the fluid.

In this work we want to apply a similar scheme to the case where the involved symmetry corresponds to a TKV, focusing on the possible link between the complexity as measured by the complexity factor [2, 3], and the symmetry under consideration.

The idea of complexity in different branches of science has a long and a venerable history (see [4] and references therein). Some years ago important advances on this issue were achieved by Lopez-Ruiz and collaborators within the context of statistical physics [5–8]. This approach was later on employed by some authors to define complexity for self-gravitating systems (see [9] and references therein). However such a definition suffers from some flaws, which motivated the search for an alternative

definition of complexity for self-gravitating systems. This search led to the definition proposed in [2,3].

Thus we shall consider spherically symmetric fluid distributions which at some point of their evolution admit a TKV (the particular case of geodesic fluids has been studied in [10]). The (dis)appearance of the TKV is controlled by a tensor defined through the asymmetry factor. As we will see below, the sudden appearance of the TKV, even during an infinitesimal small interval of time, has a remarkable influence on all physical variables (in particular on the complexity factor).

We shall consider two different models. The first one is adiabatic (non-dissipative). Adiabatic models, in general, are defined up to two functions of the radial coordinate. A specific choice of these functions provides us with our first model. The evolution of such a system is analyzed in detail assuming that for some value of the time-like coordinate, the space-time admits a TKV which vanishes immediately afterward.

Next, we shall consider dissipative fluids. In this case it can be shown that the fluid cannot be shear-free, and the system is defined up to three arbitrary functions of r. Our second model is obtained from a specific choice of these three functions.

In both cases, all the physical variables are analyzed in detail. In the adiabatic case, particular attention is paid to the effects of the sudden appearance of the TKV on the behavior of the complexity factor and the anisotropic factor, whereas in the dissipative case we focus on the behavior of the temperature, the redshift and the complexity factor.

The manuscript is organized as follows: The method and the variables required for describing the evolution of a spherically symmetric fluid distribution are presented in the next section. In Section III we introduce the concept of the asymmetry factor for a time-like isometry, and express the metric variables in terms of such a factor. The non-dissipative and the dissipative cases are analyzed in IV, and V respectively. A discussion on the obtained results is presented in VI.

2. Fundamental variables and relevant equations

Let us briefly present definitions and main equations required for describing the structure and evolution of spherically symmetric dissipative fluids. For detail see [3] and the references therein.

We consider spherically symmetric distributions of evolving fluids, which are bounded by a spherical surface Σ. The fluid is assumed to be locally anisotropic (principal stresses unequal) and undergoing dissipation in the form of heat flow (diffusion approximation).

Choosing comoving coordinates, the general interior metric can be written as

$$ds^2 = -A^2dt^2 + B^2dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.1}$$

where A, B, and R are functions of t and r and are assumed positive. We number the coordinates $x^0 = t$, $x^1 = r$, $x^2 = \theta$ and $x^3 = \phi$, where r is a radial comoving coordinate. Observe that A and B are dimensionless, whereas R has the same dimension as r.

The energy-momentum tensor $T_{\alpha\beta}$ of the fluid distribution has the form

$$T_{\alpha\beta} = (\mu + P_\perp)V_\alpha V_\beta + P_\perp g_{\alpha\beta} + (P_r - P_\perp)K_\alpha K_\beta + q_\alpha V_\beta + V_\alpha q_\beta, \tag{2.2}$$

where μ is the energy density, P_r the radial pressure, $P_\perp$ the tangential pressure, q^α the heat flux, V^α the four-velocity of the fluid, and K^α a unit four-vector along the radial direction. These quantities satisfy

$$V^\alpha V_\alpha = -1, \quad V^\alpha q_\alpha = 0, \quad K^\alpha K_\alpha = 1, \quad K^\alpha V_\alpha = 0. \tag{2.3}$$

It will be convenient to express the energy momentum tensor (2.2) in the equivalent (canonical) form

$$T_{\alpha\beta} = \mu V_\alpha V_\beta + P h_{\alpha\beta} + \Pi_{\alpha\beta} + q\left(V_\alpha K_\beta + K_\alpha V_\beta\right), \tag{2.4}$$

with

$$P = \frac{P_r + 2P_\perp}{3}, \qquad h_{\alpha\beta} = g_{\alpha\beta} + V_\alpha V_\beta,$$

$$\Pi_{\alpha\beta} = \Pi\left(K_\alpha K_\beta - \frac{1}{3}h_{\alpha\beta}\right), \qquad \Pi = P_r - P_\perp.$$

In the above, P is the isotropic pressure, and $h_{\alpha\beta}$ is the projector on the hypersurface orthogonal to the four-velocity.

Since we are considering comoving observers, we have

$$V^\alpha = A^{-1}\delta^\alpha_0, \quad q^\alpha = qK^\alpha, \quad K^\alpha = B^{-1}\delta^\alpha_1, \tag{2.5}$$

where $q \equiv \sqrt{q^\mu q_\mu}$ is a function of t and r, measuring the strength of the dissipated energy.

It is worth noticing that we do not explicitly add bulk or shear viscosity to the system because they can be trivially absorbed into the radial and tangential pressures, P_r and $P_\perp$, of the collapsing fluid (in Π). Also we do not explicitly introduce dissipation in the free streaming approximation since it can be absorbed in μ, P_r, and q.

The Einstein equations are given by

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}, \tag{2.6}$$

and theirs non zero components with (2.1) and (2.2) become

$$8\pi T_{00} = 8\pi\mu A^2 = \left(2\frac{\dot{B}}{B} + \frac{\dot{R}}{R}\right)\frac{\dot{R}}{R} - \left(\frac{A}{B}\right)^2\left[2\frac{R''}{R} + \left(\frac{R'}{R}\right)^2 - 2\frac{B'}{B}\frac{R'}{R} - \left(\frac{B}{R}\right)^2\right], \tag{2.7}$$

$$8\pi T_{01} = -8\pi qAB = -2\left(\frac{\dot{R}'}{R} - \frac{\dot{B}}{B}\frac{R'}{R} - \frac{\dot{R}}{R}\frac{A'}{A}\right), \tag{2.8}$$

$$8\pi T_{11} = 8\pi P_r B^2 = -\left(\frac{B}{A}\right)^2\left[2\frac{\ddot{R}}{R} - \left(2\frac{\dot{A}}{A} - \frac{\dot{R}}{R}\right)\frac{\dot{R}}{R}\right] + \left(2\frac{A'}{A} + \frac{R'}{R}\right)\frac{R'}{R} - \left(\frac{B}{R}\right)^2, \tag{2.9}$$

$$\begin{aligned} 8\pi T_{22} = \frac{8\pi}{\sin^2\theta}T_{33} = 8\pi P_\perp R^2 = &-\left(\frac{R}{A}\right)^2\left[\frac{\ddot{B}}{B} + \frac{\ddot{R}}{R} - \frac{\dot{A}}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{R}}{R}\right) + \frac{\dot{B}}{B}\frac{\dot{R}}{R}\right] \\ &+ \left(\frac{R}{B}\right)^2\left[\frac{A''}{A} + \frac{R''}{R} - \frac{A'}{A}\frac{B'}{B} + \left(\frac{A'}{A} - \frac{B'}{B}\right)\frac{R'}{R}\right]. \end{aligned} \tag{2.10}$$

The component (2.8) can be rewritten with (2.15) and (2.17) as

$$4\pi qB = \frac{1}{3}(\Theta - \sigma)' - \sigma\frac{R'}{R}. \tag{2.11}$$

The acceleration a_α and the expansion Θ of the fluid are given by

$$a_\alpha = V_{\alpha;\beta}V^\beta, \quad \Theta = V^\alpha{}_{;\alpha}, \tag{2.12}$$

and its shear $\sigma_{\alpha\beta}$ by

$$\sigma_{\alpha\beta} = V_{(\alpha;\beta)} + a_{(\alpha}V_{\beta)} - \frac{1}{3}\Theta h_{\alpha\beta}. \tag{2.13}$$

From (2.12) we have for the four–acceleration and its scalar a,

$$a_\alpha = aK_\alpha, \quad a = \frac{A'}{AB}, \tag{2.14}$$

and for the expansion

$$\Theta = \frac{1}{A}\left(\frac{\dot{B}}{B} + 2\frac{\dot{R}}{R}\right), \tag{2.15}$$

where the prime stands for r differentiation and the dot stands for differentiation with respect to t.

The shear tensor (2.13) has only one independent non zero component, and may be written as

$$\sigma_{\alpha\beta} = \sigma\left(K_\alpha K_\beta - \frac{h_{\alpha\beta}}{3}\right), \tag{2.16}$$

with

$$\sigma = \frac{1}{A}\left(\frac{\dot{B}}{B} - \frac{\dot{R}}{R}\right). \tag{2.17}$$

Next, the mass function $m(t, r)$ reads

$$m = \frac{R^3}{2}R_{23}{}^{23} = \frac{R}{2}\left[\left(\frac{\dot{R}}{A}\right)^2 - \left(\frac{R'}{B}\right)^2 + 1\right]. \tag{2.18}$$

Introducing the proper time derivative D_T given by

$$D_T = \frac{1}{A}\frac{\partial}{\partial t}, \tag{2.19}$$

we can define the velocity U of the fluid as the variation of the areal radius with respect to proper time, i.e.,

$$U = D_T R, \tag{2.20}$$

where R defines the areal radius of a spherical surface inside the fluid distribution (as measured from its area).

Then (2.18) can be rewritten as

$$E \equiv \frac{R'}{B} = \left(1 + U^2 - \frac{2m}{R}\right)^{1/2}. \tag{2.21}$$

Using (2.21) we can express (2.11) as

$$4\pi q = E\left[\frac{1}{3}D_R(\Theta - \sigma) - \frac{\sigma}{R}\right], \tag{2.22}$$

where D_R denotes the proper radial derivative,

$$D_R = \frac{1}{R'}\frac{\partial}{\partial r}. \tag{2.23}$$

2.1. The complexity factor

The complexity factor is a scalar function intended to describe the degree of complexity of a fluid distribution [2,3]. It is identified with one of the structure scalars [11], more specifically with the scalar defining the trace-free part of the electric Riemann tensor.

Thus, the tensor $Y_{\alpha\beta}$ defining the electric part of the Riemann tensor given by

$$Y_{\alpha\beta} = R_{\alpha\gamma\beta\delta}V^{\gamma}V^{\delta}, \tag{2.24}$$

may be written as

$$Y_{\alpha\beta} = \frac{1}{3}Y_T h_{\alpha\beta} + Y_{TF}(K_{\alpha}K_{\beta} - \frac{1}{3}h_{\alpha\beta}). \tag{2.25}$$

The scalar Y_{TF} describing the trace-free part of $Y_{\alpha\beta}$ is identified with the complexity factor and may be written as (see [2, 3, 11] for details)

$$Y_{TF} = -8\pi\Pi + \frac{4\pi}{R^3}\int_0^r R^3\left(D_R\mu - 3q\frac{U}{RE}\right)R'dr. \tag{2.26}$$

Thus the scalar Y_{TF} may be expressed in terms of the anisotropy of pressure, the density inhomogeneity, and the dissipative variables.

In terms of the metric functions, the scalar Y_{TF} reads

$$Y_{TF} = \frac{1}{A^2}\left[\frac{\ddot{R}}{R} - \frac{\ddot{B}}{B} + \frac{\dot{A}}{A}\left(\frac{\dot{B}}{B} - \frac{\dot{R}}{R}\right)\right] + \frac{1}{B^2}\left[\frac{A''}{A} - \frac{A'}{A}\left(\frac{B'}{B} + \frac{R'}{R}\right)\right]. \tag{2.27}$$

2.2. The exterior spacetime and junction conditions

When considering bounded fluid distributions, we have to satisfy the junction (Darmois) conditions if we wish to avoid the presence of thin shells.

Thus, outside Σ we assume that we have the Vaidya spacetime (i.e., we assume all outgoing radiation is massless), described by

$$ds^2 = -\left[1 - \frac{2M(v)}{\rho}\right]dv^2 - 2d\rho dv + \rho^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.28}$$

where $M(v)$ denotes the total mass, v is the retarded time, and ρ is a null coordinate.

The matching of the full non–adiabatic sphere (including viscosity) to the Vaidya spacetime, on the surface $r = r_{\Sigma}$ = constant, was discussed in [12].

From the continuity of the first differential form, it follows that (see [12] for details),

$$Adt \overset{\Sigma}{=} d\tau, \tag{2.29}$$

$$R \overset{\Sigma}{=} \rho(v), \tag{2.30}$$

and

$$\left(\frac{dv}{d\tau}\right)^{-2} \overset{\Sigma}{=} \left(1 - \frac{2m}{\rho} + 2\frac{d\rho}{dv}\right), \tag{2.31}$$

where τ denotes the proper time measured on Σ.

The continuity of the second differential form produces

$$m(t,r) \overset{\Sigma}{=} M(v), \tag{2.32}$$

and

$$2\left(\frac{\dot{R}'}{R} - \frac{\dot{B}}{B}\frac{R'}{R} - \frac{\dot{R}}{R}\frac{A'}{A}\right) \overset{\Sigma}{=} -\frac{B}{A}\left[2\frac{\ddot{R}}{R} - \left(2\frac{\dot{A}}{A} - \frac{\dot{R}}{R}\right)\frac{\dot{R}}{R}\right] + \frac{A}{B}\left[\left(2\frac{A'}{A} + \frac{R'}{R}\right)\frac{R'}{R} - \left(\frac{B}{R}\right)^2\right], \tag{2.33}$$

where $\overset{\Sigma}{=}$ means that both sides of the equation are evaluated on Σ (observe a misprint in Eq (40) in [12] and a slight difference in notation).

Comparing (2.33) with (2.8) and (2.9), one obtains

$$q \overset{\Sigma}{=} P_r. \tag{2.34}$$

Thus the matching of (2.1) and (2.28) on Σ implies (2.32) and (2.34).

Also, we have

$$q \overset{\Sigma}{=} \frac{L}{4\pi\rho^2}, \tag{2.35}$$

where L_Σ denotes the total luminosity of the sphere as measured on its surface and is given by

$$L \overset{\Sigma}{=} L_\infty\left(1 - \frac{2m}{\rho} + 2\frac{d\rho}{dv}\right)^{-1}, \tag{2.36}$$

and where

$$L_\infty = -\frac{dM}{dv} \overset{\Sigma}{=} -\left[\frac{dm}{dt}\frac{dt}{d\tau}\left(\frac{dv}{d\tau}\right)^{-1}\right], \tag{2.37}$$

is the total luminosity measured by an observer at rest at infinity, which is defined by

$$L_\infty = 4\pi q_\Sigma\left[R^2\left(\frac{R'}{B} + \frac{\dot{R}}{A}\right)^2\right]_\Sigma. \tag{2.38}$$

The boundary redshift z_Σ is given by

$$\frac{dv}{d\tau} \overset{\Sigma}{=} 1 + z, \tag{2.39}$$

with

$$\frac{dv}{d\tau} \overset{\Sigma}{=} \left(\frac{R'}{B} + \frac{\dot{R}}{A}\right)^{-1}. \tag{2.40}$$

Therefore the time of formation of an apparent horizon is given by

$$\left(\frac{R'}{B} + \frac{\dot{R}}{A}\right) \overset{\Sigma}{=} E + U \overset{\Sigma}{=} 0. \tag{2.41}$$

When Darmois conditions are not satisfied, the boundary surface is a thin shell, and we have to assume the Israel conditions [13].

2.3. The transport equation

In order to obtain an expression for the temperature, we need a transport equation. Assuming a causal dissipative theory (e.g., the Israel-Stewart theory [14–16]), the transport equation for the heat flux reads

$$\tilde{\tau}h^{\alpha\beta}V^{\gamma}q_{\beta;\gamma}+q^{\alpha}=-\kappa h^{\alpha\beta}\left(T_{,\beta}+Ta_{\beta}\right)-\frac{1}{2}\kappa T^{2}\left(\frac{\tilde{\tau}V^{\beta}}{\kappa T^{2}}\right)_{;\beta}q^{\alpha}, \tag{2.42}$$

where κ denotes the thermal conductivity, and T and $\tilde{\tau}$ denote temperature and relaxation time, respectively.

In the spherically symmetric case under consideration, the transport equation has only one independent component which may be obtained from (2.42) by contracting with the unit spacelike vector K^{α}. We get

$$\tilde{\tau}V^{\alpha}q_{,\alpha}+q=-\kappa\left(K^{\alpha}T_{,\alpha}+Ta\right)-\frac{1}{2}\kappa T^{2}\left(\frac{\tilde{\tau}V^{\alpha}}{\kappa T^{2}}\right)_{;\alpha}q. \tag{2.43}$$

Sometimes it is possible to simplify the equation above, in the so called truncated transport equation, when the last term in (2.42) may be neglected [17], producing

$$\tilde{\tau}V^{\alpha}q_{,\alpha}+q=-\kappa\left(K^{\alpha}T_{,\alpha}+Ta\right). \tag{2.44}$$

3. The asymmetry factor for isometries

Spacetimes admitting a TKV, satisfy the equation

$$\mathcal{L}_{\chi}g_{\alpha\beta}=0, \tag{3.1}$$

where $\mathcal{L}_{\chi}$ denotes the Lie derivative with respect to the vector field χ, which in our case is time-like, and has the form

$$\chi=\partial_{\mathbf{t}}. \tag{3.2}$$

In this work we shall consider fluid distributions which in general do not admit a TKV, except at a given moment of their evolution. By this we mean the momentarily vanishing of $\mathcal{L}_{\chi}g_{\alpha\beta}$ in the zero-shifting comoving slicing.

Thus let us consider

$$\mathcal{L}_{\chi}g_{\alpha\beta}=H_{\alpha\beta}, \tag{3.3}$$

implying

$$\chi^{\rho}\partial_{\rho}g_{\alpha\beta}+g_{\alpha\rho}\partial_{\beta}\chi^{\rho}+g_{\beta\rho}\partial_{\alpha}\chi^{\rho}=H_{\alpha\beta}, \tag{3.4}$$

where $H_{\alpha\beta}$ is (so far) an arbitrary symmetric tensor. Obviously the vector χ defines a TKV iff $H_{\alpha\beta}=0$.

Using (3.4) one obtains for (2.1) $H_{01}=H_{02}=H_{03}=H_{13}=H_{12}=H_{23}=0$.

Next, let us assume for the tensor $H_{\alpha\beta}$ the form

$$H_{\alpha\beta}=\alpha H_{T}\frac{g_{\alpha\beta}}{4}+\alpha H_{TF}\left(K_{\alpha}K_{\beta}-\frac{1}{3}h_{\alpha\beta}\right), \tag{3.5}$$

where H_T and H_{TF} are two scalar dimensionless functions depending only on t, which define the tensor $H_{\alpha\beta}$. These scalar functions are the asymmetry factors of our problem (below we shall see that in our case one may put one of them to zero, reducing the number of asymmetry factors to one). It describes the deviation of the space-time under consideration from the space-time admitting a TKV. It is worth noticing that it is not the most general form of tensor $H_{\alpha\beta}$; it was chosen so for simplicity.

Then, from the above, the following equations are obtained:

$$\dot{A} = \frac{\alpha H_T}{8} A, \tag{3.6}$$

$$\dot{B} = \frac{\alpha B H_T}{8} + \frac{\alpha H_{TF}, B}{3}, \tag{3.7}$$

$$\dot{R} = \frac{\alpha R H_T}{8} - \frac{\alpha H_{TF} R}{6}, \tag{3.8}$$

where α is a unit constant with dimensions $\frac{1}{length}$.

Integrating (3.6) we obtain

$$A = f(r)\Psi(t), \tag{3.9}$$

where $f(r)$ is an arbitrary function on r and

$$\Psi = e^{\frac{1}{8}\int \alpha H_T dt}. \tag{3.10}$$

Then, reparameterizying t, and holding the vector χ fixed, we may put $\Psi = 1$, implying $H_T = 0$, and thereof the asymmetry factor for our system is given by H_{TF} alone.

Using the above result in (3.6)–(3.8), we obtain

$$A = f(r), \quad B = g(r)e^{\frac{\alpha}{3}\int H_{TF} dt}, \quad \alpha R = h(r)e^{-\frac{\alpha}{6}\int H_{TF} dt}, \tag{3.11}$$

where f, g and h are arbitrary dimensionless functions.

Using these results we shall build up our two models corresponding to the adiabatic and the dissipative cases respectively.

4. Non dissipative case $q = 0$

If we assume that the fluid is non-dissipative, then using (3.11) in (2.8) (with $q = 0$) produces

$$f(r) = \gamma h^3(r), \tag{4.1}$$

where γ is a dimensionless constant.

Let us assume for $H_{TF}(t)$ the form

$$H_{TF} = \alpha(t - t_0), \qquad t_0 = constant. \tag{4.2}$$

Thus our system evolves without admitting a TKV, until $t = t_0$ when the system admits momentarily a TKV, which disappears afterward.

So far the models are defined up to two arbitrary functions of r and some arbitrary constants. In order to specify a model, let us introduce the dimensionless variables

$$x \equiv \frac{r}{r_\Sigma}, \qquad y = \alpha(t - t_0) = H_{TF}. \tag{4.3}$$

It is worth emphasizing that the last equality in the above equation is valid only for the specific choice for H_{TF} given by (4.2). Indeed, y is just a time-like coordinate, whereas H_{TF} is a geometrical variable endowed with a distinct geometrical meaning discussed before.

Next, we shall choose for the free parameters the value

$$\gamma = \alpha r_\Sigma = 1. \tag{4.4}$$

Finally the form of the functions $f(r)$ and $h(r)$ is assumed to be

$$f(r) = g(r) = h^3(r) = \alpha r = x. \tag{4.5}$$

Under the above conditions, the physical variables read

$$\frac{8\pi\mu}{\alpha^2} = \frac{e^{-y^2/3}}{x^2}\left(\frac{1}{x^2} + x^{4/3}e^{y^2/2}\right) - \frac{y^2}{12x^2}, \tag{4.6}$$

$$\frac{8\pi P_r}{\alpha^2} = \frac{7e^{-y^2/3}}{9x^4} - \frac{e^{y^2/6}}{x^{2/3}} - \frac{1}{3x^2}(\frac{y^2}{4} - 1), \tag{4.7}$$

$$\frac{8\pi P_\perp}{\alpha^2} = -\frac{1}{6x^2} - \frac{y^2}{12x^2} - \frac{11e^{-y^2/3}}{9x^4}, \tag{4.8}$$

while the expressions for the complexity factor, the mass function, and the surface redshift become

$$\frac{Y_{TF}}{\alpha^2} = -\frac{4e^{-y^2/3}}{3x^4} - \frac{6 + y^2}{12x^2}, \tag{4.9}$$

$$m\alpha = \frac{x^{1/3}e^{-y^2/12}}{2}\left(\frac{y^2e^{-y^2/6}}{36x^{4/3}} - \frac{e^{-y^2/2}}{9x^{10/3}} + 1\right), \tag{4.10}$$

which evaluated at the boundary surface $x = 1$ reads

$$m_\Sigma\alpha = \frac{e^{-y^2/12}}{2}\left(\frac{y^2e^{-y^2/6}}{36} - \frac{e^{-y^2/2}}{9} + 1\right), \tag{4.11}$$

and

$$z_\Sigma = \frac{6e^{y^2/4}}{2 - ye^{y^2/6}} - 1. \tag{4.12}$$

At this point the following remarks are in order:

- As it follows from (4.6)–(4.9), physical variables are singular at $r = 0$. Accordingly the center of the distribution should be excluded from consideration. However this fact should not bother us since the expansion scalar vanishes and, as it is known in such a case, the center is surrounded by a vacuole inside which we assume a vacuum [18]. Or, in other words, our fluid distribution is restricted to the region within the interval of x [$x \geq x_{min} > 0, 1$], where x_{min} corresponds to the boundary of the vacuole. In such a case, one should in principle impose matching conditions on the inner boundary surface. However as we shall discuss in the last section, we shall not take into account the influence of these matching conditions in the dynamics of our models.
- The Darmois conditions are not satisfied on the boundary surface $r = r_\Sigma$. The radial pressure at the surface vanishes only for $y \approx .47, -.47$.

In Figure 1, we have plotted the evolution of the complexity factor, the areal radius of the boundary surface, and the radial pressure at the boundary surface, whereas in Figure 2 we have plotted the anisotropic factor $8\pi(P_r - P_\perp)$, the mass function evaluated at the boundary, and the surface redshift. These results will be discussed further in the last section.

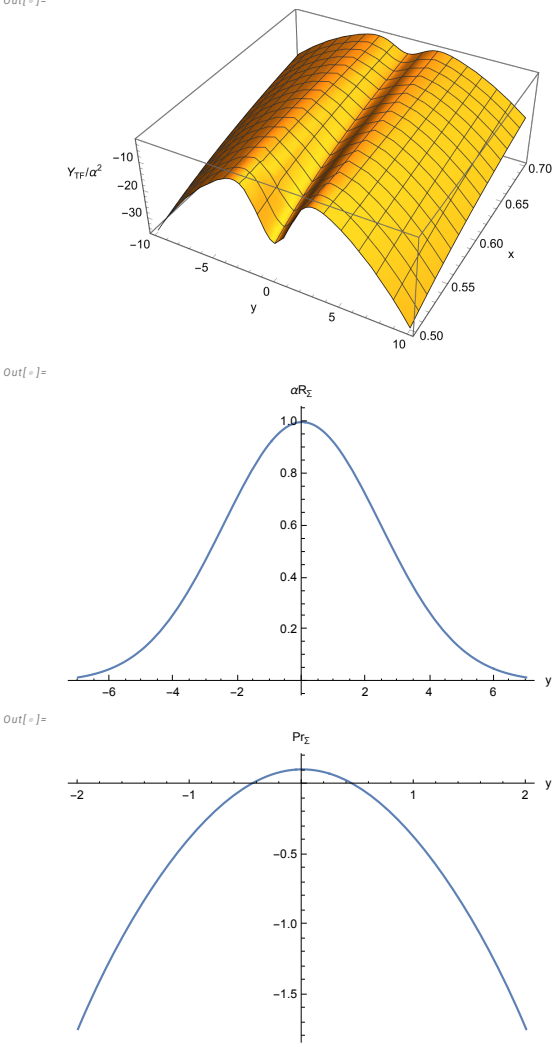

Figure 1. Y_{TF}/α^2 as a function of y and x in the interval $(y, -10, 10)$, $(x, .5, .7)$; αR_Σ in the interval $(y, -7, 7)$; $P_{r\Sigma}$ in the interval $(y, -2, 2)$ for the non-dissipative case.

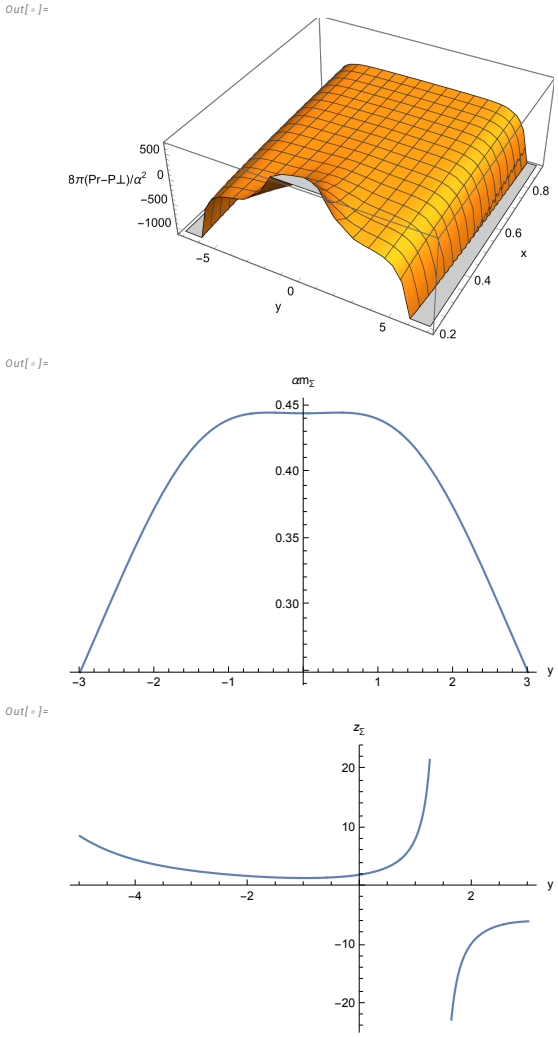

Figure 2. $8\pi(P_r - P_\perp)/\alpha^2$ as a function of y and x in the interval $(y, -7, 7)$, $(x, .2, .9)$; αm_Σ and z_Σ in the interval $(y, -3, 3)$ and $(y, -5, 3)$, respectively, for the non-dissipative case.

5. Dissipative case $q \neq 0$

Let us now consider the dissipative case. Using the form (3.5) of the tensor $H_{\alpha\beta}$, and $H_{TF} = \alpha(t-t_0)$, we obtain as before

$$A = f(r), \quad B = g(r)e^{y^2/6} \quad \alpha R = h(r)e^{-y^2/12}, \tag{5.1}$$

where f, g and h are arbitrary dimensionless functions.

In this case any possible model is determined up to three functions of r.

In order to specify our model, we shall proceed as in the previous case.

Thus, let us introduce the dimensionless variables

$$x \equiv \frac{r}{r_\Sigma}, \qquad y \equiv \alpha(t - t_0) \equiv H_{TF}. \tag{5.2}$$

As in the previous case, the above equation is valid only for the specific choice for H_{TF} given above.

Next, we shall chose for the free parameters the value

$$\alpha r_\Sigma = 1. \tag{5.3}$$

Finally, to fully determine the model we need to specify the functions f, h, and g. Since we assume $q \neq 0$, we have to exclude a relationship like (4.1) any other choice being (in principle) acceptable.

Thus let assume

$$f(r) = g(r) = h(r) = \alpha r = x. \tag{5.4}$$

Under these conditions, the physical variables read

$$8\pi\mu/\alpha^2 = \frac{e^{-y^2/3}}{x^4} - \frac{y^2}{12x^2} + \frac{e^{y^2/6}}{x^2}, \tag{5.5}$$

$$8\pi q/\alpha^2 = -\frac{2ye^{-y^2/6}}{3x^3}, \tag{5.6}$$

$$8\pi P_r/\alpha^2 = \frac{1}{x^2}\left(\frac{1}{3} - \frac{y^2}{12} + \frac{3e^{-y^2/3}}{x^2} - e^{y^2/6}\right), \tag{5.7}$$

$$8\pi P_\perp/\alpha^2 = -\frac{1}{x^2}\left(\frac{1}{6} + \frac{y^2}{12} + \frac{e^{-y^2/3}}{x^2}\right). \tag{5.8}$$

Finally from (2.36) and (2.38), we obtain

$$L_\Sigma = -\frac{ye^{-y^2/3}}{3}, \tag{5.9}$$

and

$$L_\infty = -\frac{ye^{-5y^2/6}}{3}\left(1 + \frac{y^2e^{y^2/3}}{36} - \frac{ye^{y^2/6}}{3}\right), \tag{5.10}$$

whereas the expressions for the complexity factor and the redshift read

$$Y_{TF}/\alpha^2 = -\frac{2e^{-y^2/3}}{x^4} - \frac{1}{2x^2} - \frac{y^2}{12x^2}, \tag{5.11}$$

and

$$z_\Sigma = \frac{6e^{y^2/4}}{6 - ye^{y^2/6}} - 1. \tag{5.12}$$

Finally, using the truncated version of the transport equation (2.44) and (5.6), we may obtain an expression for the temperature. It reads

$$T^* \equiv \tilde{T} - \frac{\tilde{T}_\Sigma}{x} = \frac{y\ln x}{12\pi x} + \frac{\tau^*\left(\frac{1}{x} - 1\right)\left(y^2 - 3\right)}{36\pi x}, \tag{5.13}$$

where $\tau^* = \alpha\tilde{\tau}$ and $\tilde{T} = \frac{\kappa T}{\alpha}$. This choice of variable (T^*) allows us to skip the adoption of a specific value for $\tilde{T}_\Sigma$, which within the context of this study is not available. It should be clear that when dealing with a specific astrophysical phenomenon, such information must be provided.

To highlight the effects derived from the appearance of the TKV on the evolution of the fluid, we have plotted the complexity factor, L_∞, and z_Σ in Figure 3. In Figure 4 we have plotted the temperature (T^*) for $\tau^* = 0$ and $\tau^* = 1$, and the anisotropic factor. All these results together with the ones obtained from the previous case will be discussed next.

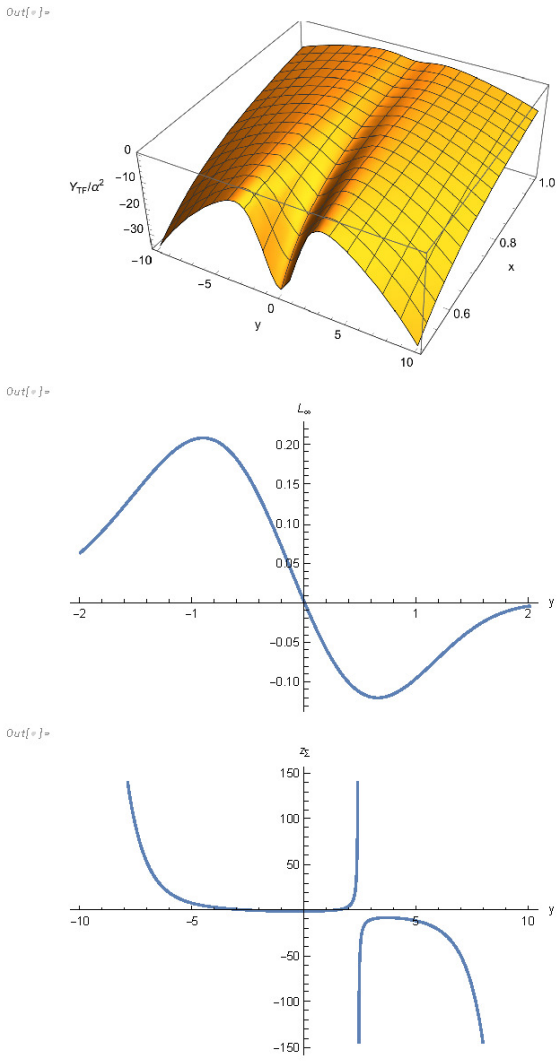

Figure 3. Y_{TF}/α^2 as a function of y, and x in the interval $(y, -10, 10)$, $(x, .5, 1)$; L_∞ and z_Σ in the interval $(y, -2, 2)$, and $(y, -10, 10)$, respectively, for the dissipative case.

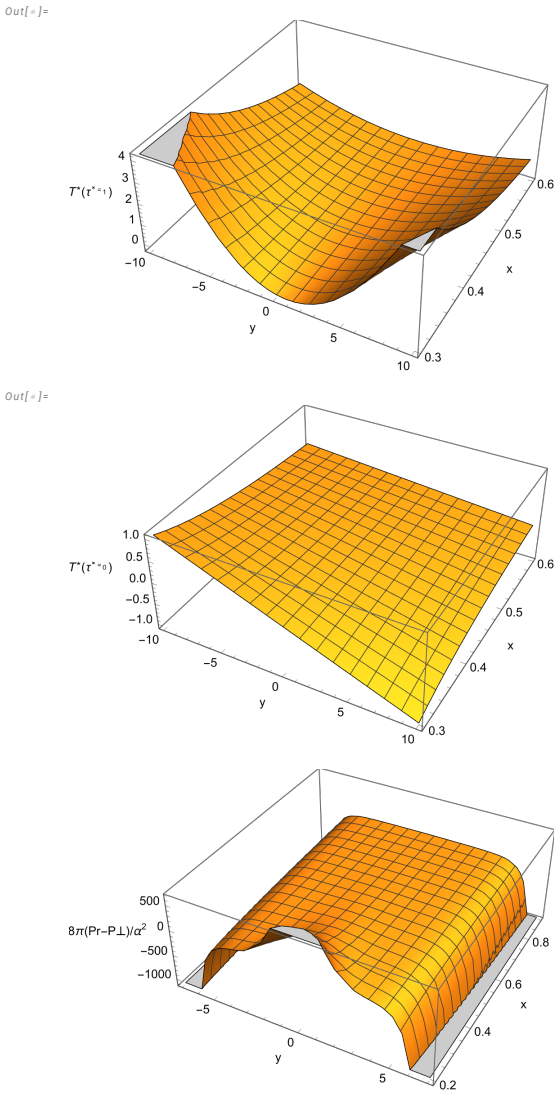

Figure 4. T^* as a function of y and x in the interval $(y, -10, 10)$, $(x, .3, .6)$ for $\tau^* = 1$ and $\tau^* = 0$; $8\pi(P_r - P_\perp)/\alpha^2$ as a function of y and x in the interval $(y, -5, 5)$, $(x, .2, .7)$; for the dissipative case.

6. Discussion and Conclusions

Our goal in this work is twofold. On the one hand we want to bring out the role played by the sudden appearance of a TKV, on some physical aspects of evolving relativistic self-gravitating fluids. In particular we are interested in relating the complexity factor to the instantaneous emergence of a TKV. On the other hand we want to stress the potential relevance that the approach based on the asymmetry factor may have in the modeling of the evolution of self-gravitating systems.

As mentioned before, the asymmetry factor allows us to turn on (off) any type of symmetry at any given point of the evolution. A general formalism derived from the above mentioned concept then applied to study two different models. It is worth emphasizing that these models are not intended to

describe any specific astrophysical scenario, but are presented with the purpose mentioned above.

The first model is assumed to be non-dissipative. It is obtained under the specific choice of the two arbitrary functions defining all adiabatic models admitting instantaneously a TKV. The choice adopted is justified by its simplicity. We shall refer to this issue at the end of this section.

Under the conditions above, we obtain the model described by (4.6)–(4.12). As it is apparent from Figure 1, it represents an expanding fluid sphere for $y < 0$ reaching a maximum at $y = 0$ (the moment of appearance of the TKV), contracting from there on.

As indicated in the graphic for $P_{r\Sigma}$ of Figure 1, Darmois conditions are not satisfied on the boundary surface of the fluid distribution. This is confirmed by the discontinuity of the mass function across the boundary surface as shown in the graphic for αm_Σ in Figure 2. Accordingly we assume that the boundary is a thin shell. All this having been said, it should be clear that the appearance of a thin shell implies that, according to the Israel approach [13], the boundary is endowed with some specific material properties (surface energy-density and tension) whose role in the analysis of our model has not been taken into account (neither for the outer nor finner boundary). It goes without saying that in a model intended to describe a specific astrophysical object, the dynamics of the shell might be a relevant factor, and its role and the nature of its physical origin should be included in the discussion.

There are three remarkable characteristics of this model, illustrated by Figures 1 and 2, which deserve to be emphasized.

- The complexity factor presents a local minimum at $y = 0$. While it is true that such stationary value depends on the specific form of H_{TF}, it is also true that the existence of a stationary value of the complexity factor at $y = 0$ appears also in the case of a conformal Killing vector (CKV) analyzed in [1], thereby suggesting that a possible model independent link might exist between complexity and symmetry.
- The anisotropic factor presents a local maximum at $y = 0$.
- There is a discontinuity of the surface redshift around $y \approx 1.42$. Furthermore the sign of z_Σ changes of sign along the evolution (from blueshift to redshift). If such unexpected behavior is ever observed in astrophysics, then we could try to describe the evolution of the observed system through the asymmetry factor approach.

The second model is dissipative, it was obtained for the choice of the arbitrary functions given in (5.4). As in the previous case, the above mentioned choice obeys to a criterion of simplicity. The full description of the model is provided by Eqs (5.5)–(5.13). As in the previous example, Darmois conditions are not satisfied on the boundary surface (the condition (2.34) is not satisfied identically).

Particular attention deserves the behavior of T^* depicted in Figure 4. In the upper graphic, T^* is plotted for $\tau^* = 1$. It was found that the same behavior (qualitatively) is observed for any value of $\tau^* \neq 0$, namely: T^* decreases as y approaches 0, increasing afterward. However if we assume that the relaxation time vanishes then, as depicted in the corresponding graphic, T^* monotonically decreases after passing through $y = 0$ (notice that, unlike the "physical" temperature T, the variable T^* may be negative). In other words the increasing of T^* for $y > 0$ is a relaxational effect, expected to be observed when the value of the relaxation time is equal to, or larger than, the characteristic time of the evolution.

Thus, in this specific model we are able to detect the appearance of thermal effect directly related to relaxation time and to the vanishing of the asymmetry factor. It is worth noticing that while the quantitative details of such an effect would depend on the transport equation employed, and the boundary temperature, the fact of its appearance, doesn't.

At this point it is worth recalling that although τ is very small for phonon-electron and phonon-phonon interaction at room temperature ($O(10^{-11})$ and $O(10^{-13})$ seconds, respectively [19]), neglecting it is the source of difficulties, and in some cases a bad approximation. Thus, for example, in superfluid helium [20], the relaxation time is of the order of 10^{-3} seconds.

Also, if the heat conductivity is dominated by electrons (as is the case of a neutron star) [21–23], the relaxation time may be in a range as large as $[10^{-4}, 10^{2}]$ seconds, depending on the temperature of the object. Thus for the temperature proposed by Harwit [24] in the degenerate core of aged stars ($T \approx 10^{7}K$), the thermal relaxation time can be as high as 1 second (see [25] and references therein for further discussion on this issue).

Next, as it follows from (5.1), this model (as the previous one) represents an initially expanding sphere reaching a maximum value of R_Σ at $y = 0$ (when the TKV appears), becoming a contracting sphere afterward. The graphic of the luminosity at the boundary surface in Figure 3 indicates that it vanishes at $y = 0$, changing from positive to negative for $y > 0$. As in the previous case the absolute value of the anisotropic factor, and the complexity factor have local minima at $y = 0$ as depicted in Figures 2 and 3. Also z_Σ presents a discontinuity (at $y \approx 2.36$), changing from redshift to blueshift during the evolution.

From the above comments, it is apparent that the instantaneous appearance of a TKV produces important effects on the subsequent evolution of the system.

These effects are of two kinds. On the one hand, from a purely theoretical point of view, the effects on the complexity factor, which bring out a link between complexity and symmetry, the reach of this fact deserves further research. On the other hand some of the above mentioned effects are directly related to observable variables (e.g., luminosity and the gravitational redshift). This paves the way to explain, by resorting to a general approach based on the asymmetry factor, some observational astrophysical or cosmological data, which appears unintelligible in terms of the standard theoretical setup.

We would like to conclude with the following final remarks:

- The asymmetry factor approach seems to be particularly suitable for describing self-gravitating fluid distributions exhibiting an unexpected behavior at some point of its evolution (e.g., a strange evolution of the redshift as the one exhibited in our models).
- In any specific example, whenever it is possible, we should distinguish between model dependent effects and effects associated exclusively to the instantaneous appearance of a symmetry.
- It is of utmost relevance to find out if there is any physical phenomenon behind the appearance (disappearance) of any specific symmetry. In other words, we should be able to identify a physical reason for the vanishing of the asymmetry factor.
- The presented models were not conceived to describe any specific physical scenario but just to bring out the effects produced by the sudden appearance of a TKV. Therefore neither the presence of a thin shell, nor the violation of some energy conditions are, at this point, relevant for the discussion. As mentioned before in a model intended to describe a specific astrophysical object, the physical properties of a thin shell are relevant, and their role and the nature of their physical origin cannot be overlooked. The same comment applies to any eventual violation of energy conditions.
- As mentioned before, the choice for the asymmetry factor is justified by its simplicity. However if we assume that H_{TF} is a well behaving function, we may expand it in power series of $t - t_0$,

implying that, at least, for values of t not far from t_0, the above choice is likely to describe the evolution quite well. Notwithstanding, it is obvious that H_{TF} is a degree of freedom of the system, and different choices may be invoked to describe different scenarios of physical interest. In the same line of arguments, it is worth noticing that other forms of tensor $H_{\alpha\beta}$ may be proposed, which could involve more than one asymmetry factor.

- Our models were obtained under the assumptions (4.5) and (5.4) for the functions $f(r)$, $h(r)$, and $g(r)$, which as mentioned before were adopted for simplicity. In goes without saying that for the treatment of any specific astrophysical problem, the choice of these functions must be dictated by the very nature of the problem under consideration.
- While it is true that specific imprints of the instantaneous appearance of a TKV are clearly model dependent, the very fact that such an appearance drastically changes the subsequent evolution of the system, is not. Besides, for some variables (e.g., the complexity factor, and the anisotropic factor) it seems to exist a persistent imprint for different models and types of symmetries.

Use of Generative-AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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