



*Research article***Unveiling nonlinear wave dynamics: auto-Bäcklund transformation and multiple-distinct analytical solutions to the fourth-order potential Ito equation for water waves****Jing Yang¹, Peng Xu^{2,*}, Feng Shi², Jing-Hua Liu² and Kang-Jia Wang^{2,*}**¹ School of Physics and Telecommunications Engineering, Zhoukou Normal University, Zhoukou 466001, China² School of Physics and Electronic Information Engineering, Henan Polytechnic University, Jiaozuo 454003, China***Correspondence:** Email: xphpu@outlook.com; konka05@163.com.

Abstract: In this study, we focused on the exploration of the Painlevé-type auto-Bäcklund transformation (aBT) and several novel analytical solutions to the fourth-order potential Ito equation describing water wave dynamics. By implementing the truncated Painlevé expansion, the Painlevé-type aBT was rigorously constructed, from which the kink soliton, singular solutions, and rational solutions were systematically derived. Through the Painlevé analysis, the bilinear form of the target equation was explicitly obtained, and the new homoclinic test technique was further adopted to derive the breather-type kink soliton and singular periodic wave solutions. Additionally, the interaction solutions constructed with the cos-cosh ansatz were analyzed in detail. Ultimately, a series of distinct traveling wave solutions were successfully developed by employing four robust analytical approaches, namely the variational method, energy balance method, direct mapping method, and the Kudryashov method. The obtained solutions covered kink-type, anti-kink-type, periodic, singular periodic, and rational forms, which were explicitly expressed via exponential, trigonometric, hyperbolic, arc tangent, and rational functions. The spatiotemporal profiles of some obtained solutions were visualized by selecting appropriate physical parameters to intuitively illustrate their inherent physical properties. Our findings of this work are expected to provide valuable references for theoretical research and contribute to a deeper and more systematic understanding of the fundamental properties and intrinsic mechanisms of the studied equation.

Keywords: Painlevé-type auto-Bäcklund transformation; new homoclinic test technique; analytical solutions; truncated Painlevé expansion

Mathematics Subject Classification: 35A15, 35C07, 35C08

1. Introduction

In natural and social systems, a large number of nonlinear processes are commonly present, and the evolutionary behaviors of such processes cannot be described by simple linear relationships, but is highly related to the core research problems of multiple disciplines [1–3]. In the theoretical system of nonlinear science, partial differential equations play an irreplaceable fundamental role: They can transform cross disciplinary nonlinear phenomena such as reaction diffusion processes in chemistry, non steady state energy transfer in thermal science, field evolution in physics, population dynamics in biology, and price fluctuations in economics into rigorous mathematical expressions, and then establish quantifiable dynamic evolution models [4,5]. Therefore, exploring effective and reliable methods for solving nonlinear partial differential equations has always been a prominent core topic in the field of nonlinear research, and related research results are a necessary foundation for achieving a deep understanding of the underlying mechanisms of nonlinear phenomena [6–8]. So far, mathematicians and physicists have proposed many effective and reliable methods, such as the sub-equation approach [9,10], extended F-expansion approach [11,12], tanh-function approach [13,14], (G'/G) -expansion approach [15], ϕ^6 -model expansion method [16], Darboux transformation approach [17,18], Sardar sub-equation method [19,20], exp-function technique [21], $\exp(-\phi(\xi))$ -expansion scheme [22], trial equation method [23,24], Riccati equation technique [25,26], and extended direct algebraic method [27,28] among many others. In this article, we aim to explore the fourth-order potential Ito equation as [29]

$$\psi_{tt} + \psi_{xxxxt} + \alpha(\psi_x \psi_t)_x + \beta \psi_{xx} = 0, \quad (1.1)$$

which has an important part in modeling the water waves, and α and β represent the non zero real values. In [29], the integrability and multiple soliton solutions are explored. In [30], the Bilinear aBT and Lax Pairs are discussed, and ten different analytical solutions are probed via the three wave method. Herein, we develop the Painlevé-type aBT and obtain some new analytical solutions to Eq (1.1).

The rest of this work is arranged as follows: In Section 2, the Painlevé-type aBT and the bilinear form are constructed, and the kink soliton, singular as well as the rational solutions, are obtained. In Section 3, the singular periodic wave and breather-type kink soliton solutions are attained via the new homoclinic test technique. In Section 4, the cos-cosh type interaction solutions are studied. In Section 5, the diverse traveling wave solutions are researched. Finally, this work is summarized in Section 6.

2. The Painlevé-type aBT, exact solutions, and bilinear form

In this part, we aim to derive the Painlevé-type aBT and bilinear form, and obtain the exact solutions to Eq (1.1).

2.1. The Painlevé-type aBT

We consider the solution of Eq (1.1) with the truncated Laurent series form as

$$\psi(x, t) = f(x, t)^{-s} \sum_{j=0}^s u_j(x, t) f(x, t)^j. \quad (2.1)$$

The leading order of Eq (2.1) is assumed as

$$\psi \sim u_0 f^{-s}. \quad (2.2)$$

Substituting it into Eq (1.1) and balancing the leading terms yields

$$s = 1, \quad u_0 = \frac{6}{\alpha} f_x. \quad (2.3)$$

Then there is

$$\psi = f^{-1} \left(\frac{6f_x}{\alpha} + u_1 f \right). \quad (2.4)$$

Substituting Eq (2.4) into Eq (1.1) and making the coefficients of $f^i(x, y, t)$ as zero yields

$$f^{-3}: \frac{12f_x}{\alpha} [f_t^2 + (\beta + \alpha u_{1t}) f_x^2 - 3f_{xt} f_{xx} + 3f_x f_{xxt} + f_t(\alpha f_x u_{1x} + f_{xxx})] = 0, \quad (2.5)$$

$$f^{-2}: -\frac{6}{\alpha} [f_x f_{tt} + 2\alpha f_x u_{1xt} f_{xt} + \alpha f_x^2 u_{1xt} + 3\beta f_x f_{xx} + 3\alpha u_{1t} f_x f_{xx} - 2f_{xt} f_{xxx} + 4f_x f_{xxxt} + f_t(2f_{xt} + \alpha u_{1x} f_{xx} + \alpha u_{1xx} f_x + f_{xxxx})] = 0, \quad (2.6)$$

$$f^{-1}: \frac{6}{\alpha} [f_{xtt} + \alpha u_{1xt} f_{xx} + \alpha u_{1xx} f_{xt} + \alpha u_{1x} f_{xxt} + \beta f_{xxx} + \alpha u_{1t} f_{xxx} + f_{xxxxt}] = 0, \quad (2.7)$$

$$f^{-0}: u_{1tt} + u_{1xxx} + \alpha(u_{1x} u_{1t})_x + \beta u_{1xx} = 0. \quad (2.8)$$

Equation (2.8) reveals that $u_1(x, y, t)$ is the seed solution of Eq (1.1), so Eqs (2.4)–(2.8) form a Painlevé-type aBT for Eq (1.1).

2.2. The exact solutions

If we take the trivial solution $u_1(x, y, t) = 0$, then Eqs (2.5)–(2.8) are reduced into the following reduced Painlevé-type aBT:

$$\frac{12f_x}{\alpha} [f_t^2 + \beta f_x^2 - 3f_{xt} f_{xx} + 3f_x f_{xxt} + f_t f_{xxx}] = 0, \quad (2.9)$$

$$-\frac{6}{\alpha} [f_x f_{tt} + 3\beta f_x f_{xx} - 2f_{xt} f_{xxx} + 4f_x f_{xxxt} + f_t(2f_{xt} + f_{xxxx})] = 0, \quad (2.10)$$

$$\frac{6}{\alpha} (f_{xtt} + \beta f_{xxx} + f_{xxxxt}) = 0. \quad (2.11)$$

2.2.1. The rational solution

Assuming the solution as

$$f = mx + wt. \quad (2.12)$$

Taking it into Eqs (2.9)–(2.11), we have

$$w = \pm m \sqrt{-\beta}. \quad (2.13)$$

Then the rational solution is found as

$$\psi_{\pm}(x, t) = \frac{12}{\alpha(x \pm \sqrt{-\beta}t)}, \quad (2.14)$$

which is under $\beta < 0$.

2.2.2. The exponential function solution

Assuming the solution is

$$f = 1 + se^{mx+wt}. \quad (2.15)$$

Taking it into Eqs (2.9)–(2.11), we have

$$w = \frac{1}{2}(-m^3 \pm \sqrt{m^6 - 4m^2\beta}). \quad (2.16)$$

Via Eq (2.4), we obtain the exponential function solution as

$$\psi_{\pm}(x, t) = \frac{6mse^{mx + \frac{1}{2}(-m^3 \pm \sqrt{m^6 - 4m^2\beta})t}}{\alpha \left[1 + se^{mx + \frac{1}{2}(-m^3 \pm \sqrt{m^6 - 4m^2\beta})t} \right]}, \quad (2.17)$$

which is under $m^6 - 4m^2\beta > 0$. Then there are:

Set-1: For $s > 0$, the kink soliton solution is found as

$$\psi_{\pm}(x, t) = \frac{3m}{\alpha} \left[1 + \tanh \left(\frac{mx + \frac{1}{2}(-m^3 \pm \sqrt{m^6 - 4m^2\beta})t + \ln s}{2} \right) \right]. \quad (2.18)$$

Set-2: For $s < 0$, the singular solution is found as

$$\psi_{\pm}(x, t) = \frac{3m}{\alpha} \left[1 + \coth \left(\frac{mx + \frac{1}{2}(-m^3 \pm \sqrt{m^6 - 4m^2\beta})t + \ln(-s)}{2} \right) \right]. \quad (2.19)$$

With $m = 1$, $\alpha = 1$, and $\beta = -1$, we draw the outlines of the rational solution provided by Eq (2.14) in Figure 1. Using $\alpha = 1$, $\beta = -1$, $m = 1$, and $s = 1$, the behaviors of Eq (2.18) are displayed in Figure 2. The wave is the kink soliton, which presents a step like gradient contour, that is $\psi_- \rightarrow 6$ for $(x \rightarrow \infty)$ and $\psi_- \rightarrow 0$ for $(x \rightarrow -\infty)$.

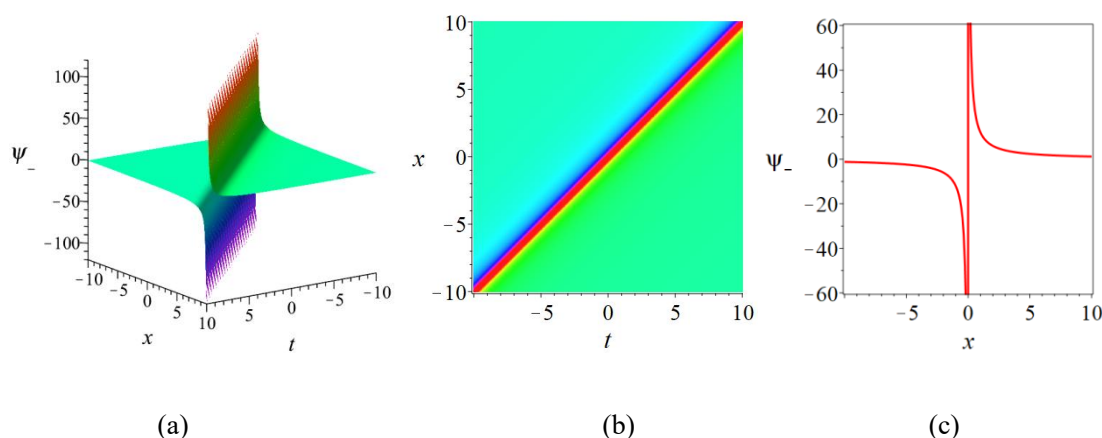


Figure 1. The shapes of Eq (2.14) with $m = 1$, $\alpha = 1$, and $\beta = -1$.

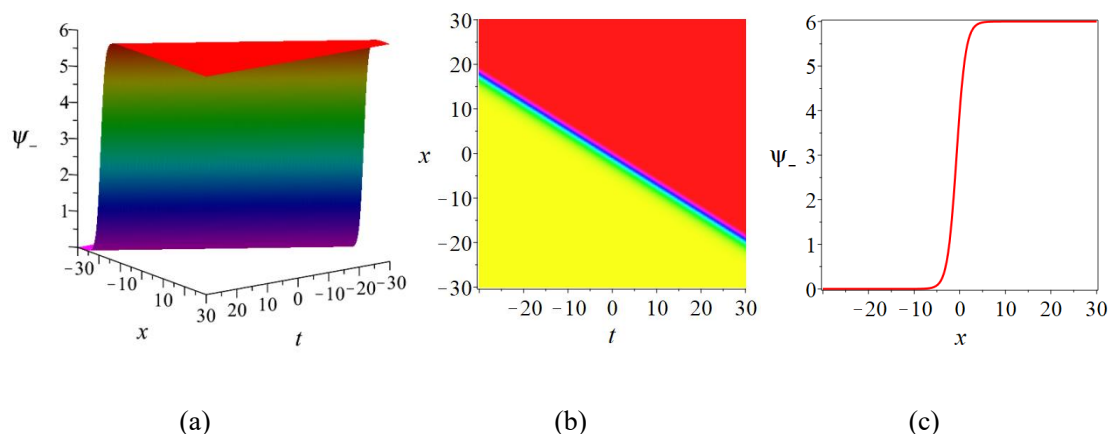


Figure 2. The shapes of Eq (2.18) with $\alpha = 1$, $\beta = -1$, $m = 1$, and $s = 1$.

Figure 3 displays the shapes of Eq (2.19) for $\alpha = 1$, $\beta = -1$, $m = 1$, and $s = -1$. ItThus, the profile is singular since Eq (2.15) has zero values for $s < 0$. Here, we should point out that the subplots (a), (b), and (c) in Figures 1–3 are the 3-D plot, density plot, and the curves, respectively.

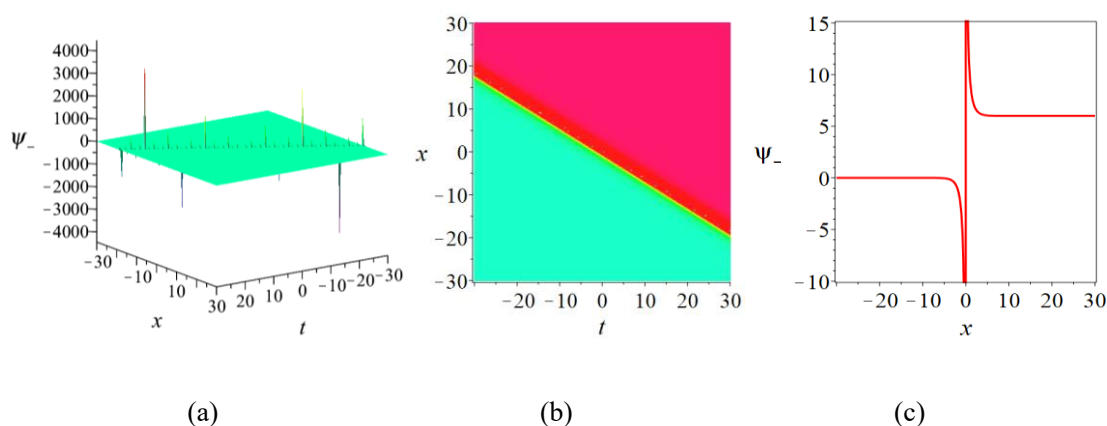


Figure 3. The shapes of Eq (2.19) with $\alpha = 1$, $\beta = -1$, $m = 1$, and $s = -1$.

2.3. The bilinear form

Setting $u_1 = 0$ in Eq (2.4) and taking it into Eq (1.1) yields

$$\frac{6}{\alpha} \frac{\partial}{\partial x} \left(\frac{-f_t^2 + f f_{tt} - \beta f_x^2 + \beta f f_{xx} + 3 f_{xt} f_{xx} - 3 f_x f_{xxt} - f_t f_{xxx} + f f_{xxx}}{f^2} \right) = 0, \quad (2.20)$$

which gives the bilinear form of Eq (1.1) as

$$-f_t^2 + f f_{tt} - \beta f_x^2 + \beta f f_{xx} + 3 f_{xt} f_{xx} - 3 f_x f_{xxt} - f_t f_{xxx} + f f_{xxx} = 0. \quad (2.21)$$

Thus, we get the Hirota bilinear form as

$$(D_x^3 D_t + D_t^2 + \beta D_x^2) g \cdot g = 0, \quad (2.22)$$

where $D_x^m D_t^n$ is

$$D_x^m D_t^n (g \cdot h) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n g(x, t) h(x', t')|_{x=x', t=t'}. \quad (2.23)$$

3. The singular periodic wave and breather-type kink soliton solutions

Using the new homoclinic test technique, we have [31]

$$g = s_5 e^{s_1 x + s_2 t} + s_6 e^{-(s_1 x + s_2 t)} + s_7 \sin(s_3 x + s_4 t), \quad (3.1)$$

where s_i ($i=1,2,\dots,8$) represents the constants. Substituting it into Eq (2.21) and merging the similar items yields the following:

Case 1: $s_4 = 2s_3^3 \pm \sqrt{4s_3^6 - s_3^2 \beta}$, $s_5 = 0$, $s_6 = 0$. Thus, we have

$$f = s_7 \sin \left[s_3 x + \left(2s_3^3 \pm \sqrt{4s_3^6 - s_3^2 \beta} \right) t \right], \quad (3.2)$$

which is under $4s_3^6 - s_3^2 \beta > 0$. Then the singular periodic solution is

$$\psi(x, t) = \frac{6s_3 s_7}{\alpha} \cot \left[s_3 x + \left(2s_3^3 \pm \sqrt{4s_3^6 - s_3^2 \beta} \right) t \right]. \quad (3.3)$$

Case 2: $s_1 = \frac{\beta^{\frac{1}{4}}}{3^{\frac{4}{3}}}$, $s_2 = -\frac{\beta^{\frac{3}{4}}}{3^{\frac{4}{3}}}$, $s_3 = \frac{\beta^{\frac{1}{4}}}{3^{\frac{4}{3}}}$, $s_4 = \frac{\beta^{\frac{3}{4}}}{3^{\frac{4}{3}}}$. Thus, we have

$$f = s_5 e^{\frac{\beta^{\frac{1}{4}}}{3^{\frac{4}{3}}} x - \frac{\beta^{\frac{3}{4}}}{3^{\frac{4}{3}}} t} + s_6 e^{-\left(\frac{\beta^{\frac{1}{4}}}{3^{\frac{4}{3}}} x - \frac{\beta^{\frac{3}{4}}}{3^{\frac{4}{3}}} t \right)} + s_7 \sin \left(\frac{\beta^{\frac{1}{4}}}{3^{\frac{4}{3}}} x + \frac{\beta^{\frac{3}{4}}}{3^{\frac{4}{3}}} t \right). \quad (3.4)$$

Then there is the breather-type kink soliton solution as

$$\psi(x, t) = \frac{6 \left[s_5 \frac{\beta^{\frac{1}{4}}}{3^{\frac{4}{3}}} e^{\frac{\beta^{\frac{1}{4}}}{3^{\frac{4}{3}}} x - \frac{\beta^{\frac{3}{4}}}{3^{\frac{4}{3}}} t} - s_6 \frac{\beta^{\frac{1}{4}}}{3^{\frac{4}{3}}} e^{-\left(\frac{\beta^{\frac{1}{4}}}{3^{\frac{4}{3}}} x - \frac{\beta^{\frac{3}{4}}}{3^{\frac{4}{3}}} t \right)} + s_7 \frac{\beta^{\frac{1}{4}}}{3^{\frac{4}{3}}} \cos \left(\frac{\beta^{\frac{1}{4}}}{3^{\frac{4}{3}}} x + \frac{\beta^{\frac{3}{4}}}{3^{\frac{4}{3}}} t \right) \right]}{\alpha \left[s_5 e^{\frac{\beta^{\frac{1}{4}}}{3^{\frac{4}{3}}} x - \frac{\beta^{\frac{3}{4}}}{3^{\frac{4}{3}}} t} + s_6 e^{-\left(\frac{\beta^{\frac{1}{4}}}{3^{\frac{4}{3}}} x - \frac{\beta^{\frac{3}{4}}}{3^{\frac{4}{3}}} t \right)} + s_7 \sin \left(\frac{\beta^{\frac{1}{4}}}{3^{\frac{4}{3}}} x + \frac{\beta^{\frac{3}{4}}}{3^{\frac{4}{3}}} t \right) \right]}. \quad (3.5)$$

Using $\alpha = 1$, $\beta = 1$, $s_5 = 2$, $s_6 = 1$, and $s_7 = 2$, we plot the performances of Eq (3.5) in Figure 4, where the subplots (a), (b), and (c) represent the 3-D plot, density plot, and the contour, respectively. It is found that the wave is the breather-type kink wave, which is formed by the superposition of breather wave and kink soliton. Thus, the wave possesses a kink topology, while localized periodic oscillations exist within the kink, which is a periodic “contraction-expansion” amplitude oscillation.

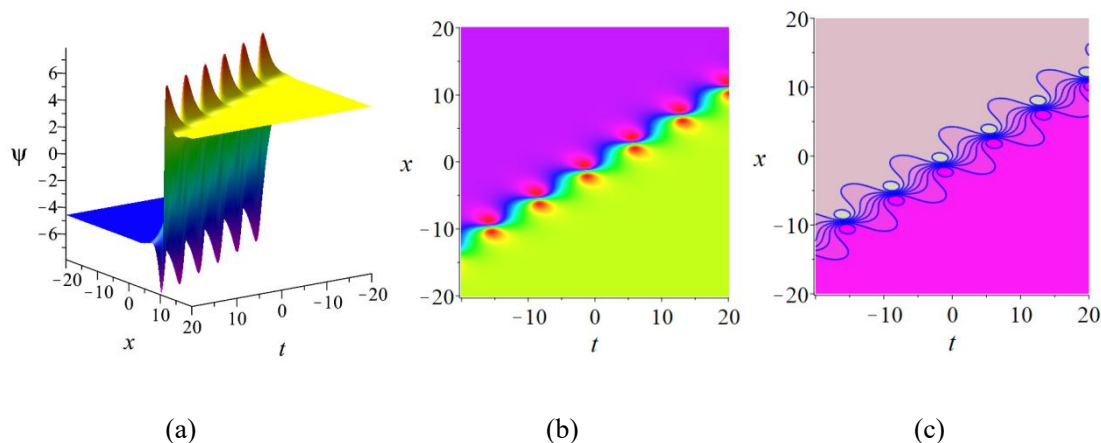


Figure 4. The shapes of breather-type kink soliton solution in Eq (3.5) with $\alpha = 1$, $\beta = 1$, $s_5 = 2$, $s_6 = 1$, and $s_7 = 2$.

4. The interaction solutions

As for the interaction solution, we can set

$$g = \Re_1 \cos(s_1 x + s_2 t) + \Re_2 \cosh(s_3 x + s_4 t) + s_5, \quad (4.1)$$

where s_i ($i=1,2,\dots,5$) and $\Re_i$ ($i=1,2$) represent the constants. Substituting this into Eq (2.21) and merging the similar items yields the following:

Case 1:

$$s_1 = \frac{\beta^{\frac{1}{4}}}{\frac{1}{3^4}}, \quad s_2 = \frac{\beta^{\frac{3}{4}}}{\frac{3}{3^4}}, \quad s_3 = \frac{\beta^{\frac{1}{4}}}{\frac{1}{3^4}}, \quad s_4 = -\frac{\beta^{\frac{3}{4}}}{\frac{3}{3^4}}, \quad s_5 = 0.$$

So there is

$$g = \Re_1 \cos\left(\frac{\beta^{\frac{1}{4}}}{\frac{1}{3^4}} x + \frac{\beta^{\frac{3}{4}}}{\frac{3}{3^4}} t\right) + \Re_2 \cosh\left(\frac{\beta^{\frac{1}{4}}}{\frac{1}{3^4}} x - \frac{\beta^{\frac{3}{4}}}{\frac{3}{3^4}} t\right). \quad (4.2)$$

Then the interaction solution is obtained as

$$\psi(x, t) = \frac{-\frac{6\beta^{\frac{1}{4}}}{\frac{1}{3^4}} \Re_1 \sin\left(\frac{\beta^{\frac{1}{4}}}{\frac{1}{3^4}} x + \frac{\beta^{\frac{3}{4}}}{\frac{3}{3^4}} t\right) + \frac{6\beta^{\frac{1}{4}}}{\frac{1}{3^4}} \Re_2 \sinh\left(\frac{\beta^{\frac{1}{4}}}{\frac{1}{3^4}} x - \frac{\beta^{\frac{3}{4}}}{\frac{3}{3^4}} t\right)}{\alpha \left[\Re_1 \cos\left(\frac{\beta^{\frac{1}{4}}}{\frac{1}{3^4}} x + \frac{\beta^{\frac{3}{4}}}{\frac{3}{3^4}} t\right) + \Re_2 \cosh\left(\frac{\beta^{\frac{1}{4}}}{\frac{1}{3^4}} x - \frac{\beta^{\frac{3}{4}}}{\frac{3}{3^4}} t\right) \right]}. \quad (4.3)$$

Case 2:

$$s_1 = -\frac{\beta^{\frac{1}{4}}}{\frac{1}{3^4}}, \quad s_2 = -\frac{\beta^{\frac{3}{4}}}{\frac{3}{3^4}}, \quad s_3 = \frac{\beta^{\frac{1}{4}}}{\frac{1}{3^4}}, \quad s_4 = -\frac{\beta^{\frac{3}{4}}}{\frac{3}{3^4}}, \quad s_5 = 0.$$

So there is

$$g = \Re_1 \cos\left(-\frac{\beta^{\frac{1}{4}}}{\frac{1}{3^4}} x - \frac{\beta^{\frac{3}{4}}}{\frac{3}{3^4}} t\right) + \Re_2 \cosh\left(\frac{\beta^{\frac{1}{4}}}{\frac{1}{3^4}} x - \frac{\beta^{\frac{3}{4}}}{\frac{3}{3^4}} t\right). \quad (4.4)$$

Then the interaction solution is obtained as

$$\psi(x, t) = \frac{-\frac{6\beta^4}{3^4}\Re_1 \sin\left(\frac{\frac{1}{3^4}x + \frac{\beta^4}{3^4}t\right) + \frac{6\beta^4}{3^4}\Re_2 \sinh\left(\frac{\frac{1}{3^4}x - \frac{\beta^4}{3^4}t\right)}{\alpha \left[\Re_1 \cos\left(\frac{\frac{1}{3^4}x + \frac{\beta^4}{3^4}t\right) + \Re_2 \cosh\left(\frac{\frac{1}{3^4}x - \frac{\beta^4}{3^4}t\right) \right]}, \quad (4.5)$$

which is the same with Eq (4.3).

For $\Re_1 = 1$, $\Re_2 = 2$, $\alpha = 1$, and $\beta = 2$, the performances of Eq (4.3) are plotted in Figure 5, where the subplots (a), (b), and (c) stand for the 3-D plot, density plot, and contour, respectively. We find the wave is also the breather-type kink soliton. This wave describes a topological domain wall propagating between two background states, with local periodic oscillations within the domain wall. The oscillation causes local energy to be confined in space and undergo periodic changes over time, while the twisted structure ensures the topological connection between two different background states. This phenomenon can be explained by the interaction between the cos function and the cosh function.

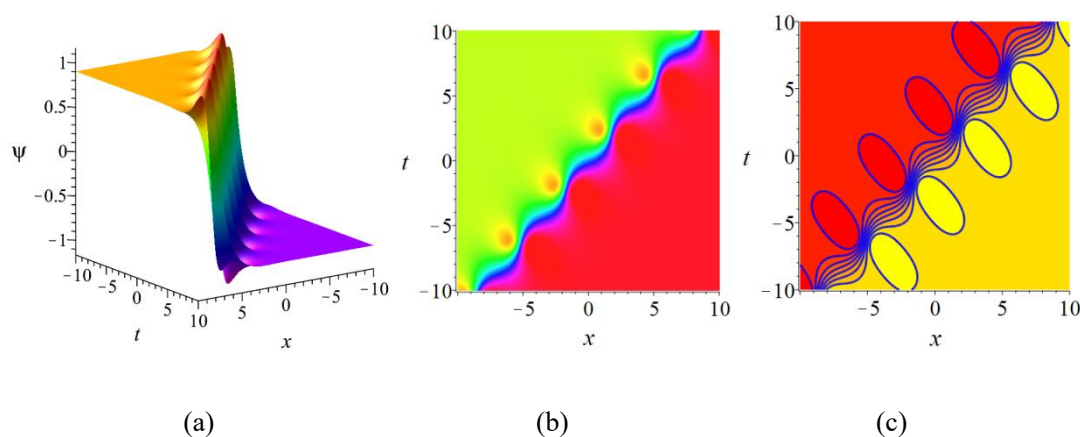


Figure 5. The shapes of interaction solution in Eq (4.3) with $\Re_1 = 1$, $\Re_2 = 2$, $\alpha = 1$, and $\beta = 2$.

5. The distinct traveling wave solutions

Considering the transformation as

$$\psi(x, t) = \theta(\gamma), \quad \gamma = \vartheta x + \varpi t, \quad (5.1)$$

where ϑ and ϖ stand for the non-zero coefficients. According to Eq (5.1), we have

$$\vartheta^3 \varpi \theta^{(4)} + (\varpi^2 + \beta \vartheta^2) \theta'' + \alpha \vartheta^2 \varpi [(\theta')^2]' = 0, \quad (5.2)$$

where $\theta^{(4)} = \frac{d^4 \theta}{d\gamma^4}$, $\theta'' = \frac{d^2 \theta}{d\gamma^2}$, and $\theta' = \frac{d\theta}{d\gamma}$. Integrating the above equation once and ignoring the integral constant yields

$$\vartheta^3 \varpi \theta''' + (\varpi^2 + \beta \vartheta^2) \theta' + \alpha \vartheta^2 \varpi (\theta')^2 = 0. \quad (5.3)$$

5.1. The variational method

Adopting the following form:

$$\theta' = \psi. \quad (5.4)$$

Equation (5.3) becomes

$$\psi'' + \frac{(\varpi^2 + \beta\vartheta^2)}{\vartheta^3\varpi} \psi + \frac{\alpha}{\vartheta} \psi^2 = 0. \quad (5.5)$$

Using the Semi-inverse method [32,33], its variational principle is developed as

$$J(\psi) = \int_0^\infty \left\{ \frac{1}{2} (\psi')^2 - \frac{(\varpi^2 + \beta\vartheta^2)}{2\vartheta^3\varpi} \psi^2 - \frac{\alpha}{3\vartheta} \psi^3 \right\} d\gamma. \quad (5.6)$$

Case 1: Assuming the solution of Eq (5.5) is

$$\psi = \Lambda \operatorname{sech}(\gamma). \quad (5.7)$$

Substituting the above equation into Eq (5.6) yields

$$\begin{aligned} J(\Lambda) &= \int_0^\infty \left\{ \frac{1}{2} [-\Lambda \operatorname{sech}(\gamma) \tanh(\gamma)]^2 - \frac{(\varpi^2 + \beta\vartheta^2)}{2\vartheta^3\varpi} [\Lambda \operatorname{sech}(\gamma)]^2 - \frac{\alpha}{3\vartheta} [\Lambda \operatorname{sech}(\gamma)]^3 \right\} d\gamma \\ &= -\frac{\Lambda^2 [6\varpi^2 + 6\beta\vartheta^2 + \varpi(\Lambda\pi\alpha - 2\vartheta)\vartheta^2]}{12\varpi\vartheta^3}. \end{aligned} \quad (5.8)$$

Computing the stationary condition as

$$\frac{dJ(\Lambda)}{d\Lambda} = 0 \quad (5.9)$$

gives

$$\Lambda \left(\frac{\varpi}{\vartheta^3} + \frac{\beta}{\vartheta\varpi} - \frac{1}{3} \right) + \frac{\Lambda^2 \alpha \pi}{4\vartheta} = 0. \quad (5.10)$$

Solving it yields

$$\Lambda = -\frac{4(3\varpi^2 + 3\beta\vartheta^2 - \varpi\vartheta^3)}{3\pi\alpha\varpi\vartheta^2}. \quad (5.11)$$

So there is

$$\psi = -\frac{4(3\varpi^2 + 3\beta\vartheta^2 - \varpi\vartheta^3)}{3\pi\alpha\varpi\vartheta^2} \operatorname{sech}(\gamma). \quad (5.12)$$

By Eq (5.4), we have

$$\theta = -\frac{8(3\varpi^2 + 3\beta\vartheta^2 - \varpi\vartheta^3)}{3\pi\alpha\varpi\vartheta^2} \arctan \left(\tanh \frac{\gamma}{2} \right). \quad (5.13)$$

The traveling wave solution is

$$\psi(x, t) = -\frac{8(3\varpi^2 + 3\beta\vartheta^2 - \varpi\vartheta^3)}{3\pi\alpha\varpi\vartheta^2} \arctan \left[\tanh \frac{(\vartheta x + \varpi t)}{2} \right]. \quad (5.14)$$

Case 2: We can also assume the solution of Eq (5.5) as

$$\psi = \Lambda \operatorname{sech}^2(\gamma). \quad (5.15)$$

Putting it into Eq (5.6) yields

$$\begin{aligned}
 J(\Lambda) &= \int_0^\infty \left\{ \frac{1}{2} [-2\Lambda \operatorname{sech}(\gamma) \tanh(\gamma)]^2 - \frac{(\varpi^2 + \beta \vartheta^2)}{2\vartheta^3 \varpi} [\Lambda \operatorname{sech}(\gamma)^2]^2 - \frac{\alpha}{3\vartheta} [\Lambda \operatorname{sech}(\gamma)^2]^3 \right\} d\gamma \\
 &= -\frac{\Lambda^2 [15\varpi^2 + 15\beta \vartheta^2 + 4\varpi(\Lambda\alpha - 3\vartheta)\vartheta^2]}{45\varpi \vartheta^3}.
 \end{aligned} \tag{5.16}$$

Computing the stationary condition as

$$\frac{dJ(\Lambda)}{d\Lambda} = 0 \tag{5.17}$$

gives

$$\frac{2\Lambda [5\varpi^2 + 5\beta \vartheta^2 + 4\varpi(\Lambda\alpha - 3\vartheta)\vartheta^2]}{15\varpi \vartheta^3} = 0. \tag{5.18}$$

Solving it yields

$$\Lambda = \frac{4\varpi \vartheta^3 - 5\varpi^2 - 5\beta \vartheta^2}{4\alpha \varpi \vartheta^2}. \tag{5.19}$$

Then there is

$$\Psi = \frac{4\varpi \vartheta^3 - 5\varpi^2 - 5\beta \vartheta^2}{4\alpha \varpi \vartheta^2} \operatorname{sech}^2(\gamma). \tag{5.20}$$

By Eq (5.4), we have

$$\Theta = \frac{4\varpi \vartheta^3 - 5\varpi^2 - 5\beta \vartheta^2}{4\alpha \varpi \vartheta^2} \tanh(\gamma). \tag{5.21}$$

The traveling wave solution is

$$\psi(x, t) = \frac{4\varpi \vartheta^3 - 5\varpi^2 - 5\beta \vartheta^2}{4\alpha \varpi \vartheta^2} \tanh(\vartheta x + \varpi t). \tag{5.22}$$

Figure 6 shows the outlines of Eq (5.22) with $\alpha = -1$, $\beta = 1$, $\vartheta = 1$, and $\varpi = -1$. We find that this soliton is the anti-kink soliton, which displays the step like gradient contour characteristics of the anti-kink soliton, that is $\psi_- \rightarrow -3.5$ for $(x \rightarrow \infty)$ and $\psi_- \rightarrow 3.5$, for $(x \rightarrow -\infty)$. These boundary conditions are opposite to the kink soliton. The subplots (a), (b), and (c) represent the 3-D plot, density plot, and curves, respectively.

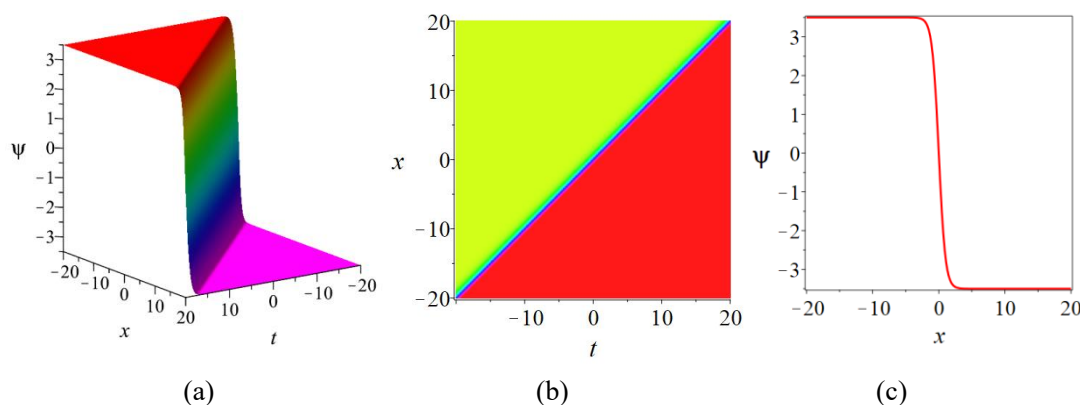


Figure 6. The shapes of Eq (5.22) with $\alpha = -1$, $\beta = 1$, $\vartheta = 1$, and $\varpi = -1$.

5.2. The energy balance method

According to the variational principle in Eq (5.6), we have

$$J(\Psi) = \int_0^\infty \left\{ \frac{1}{2} (\Psi')^2 - \frac{(\varpi^2 + \beta \vartheta^2)}{2\vartheta^3 \varpi} \Psi^2 - \frac{\alpha}{3\vartheta} \Psi^3 \right\} d\gamma = \int_0^\infty \{\mathfrak{R} - \mathfrak{K}\} d\gamma, \quad (5.23)$$

where $\mathfrak{K}$ is potential energy, and $\mathfrak{R}$ stands for the kinetic energy as [34]

$$\mathfrak{R} = \frac{1}{2} (\Psi')^2, \quad (5.24)$$

$$\mathfrak{K} = \frac{(\varpi^2 + \beta \vartheta^2)}{2\vartheta^3 \varpi} \Psi^2 + \frac{\alpha}{3\vartheta} \Psi^3. \quad (5.25)$$

Thus, the Hamiltonian is obtained as

$$h = \mathfrak{R} + \mathfrak{K} = \frac{1}{2} (\Psi')^2 + \frac{(\varpi^2 + \beta \vartheta^2)}{2\vartheta^3 \varpi} \Psi^2 + \frac{\alpha}{3\vartheta} \Psi^3. \quad (5.26)$$

Assume

$$\Psi = \Lambda \cos(\Omega \gamma), \quad \Omega > 0, \quad (5.27)$$

where Λ and Ω represent amplitude and phase, respectively. There are the initial conditions as

$$\Psi(0) = \Lambda, \quad \Psi'(0) = 0. \quad (5.28)$$

Substituting them into Eq (5.26), we get the total energy as

$$h_0 = \frac{(\varpi^2 + \beta \vartheta^2)}{2\vartheta^3 \varpi} \Lambda^2 + \frac{\alpha}{3\vartheta} \Lambda^3. \quad (5.29)$$

Taking Eq (5.27) into Eq (5.26) and using the conservation of energy, there is

$$\begin{aligned} h = \mathfrak{R} + \mathfrak{K} &= \frac{1}{2} [-\Lambda \Omega \sin(\Omega \gamma)]^2 + \frac{(\varpi^2 + \beta \vartheta^2)}{2\vartheta^3 \varpi} [\Lambda \cos(\Omega \gamma)]^2 + \frac{\alpha}{3\vartheta} [\Lambda \cos(\Omega \gamma)]^3 \\ &= \frac{(\varpi^2 + \beta \vartheta^2)}{2\vartheta^3 \varpi} \Lambda^2 + \frac{\alpha}{3\vartheta} \Lambda^3 = h_0. \end{aligned} \quad (5.30)$$

Let [35]

$$\Omega \gamma = \frac{\pi}{4}. \quad (5.31)$$

We have

$$\frac{1}{2} \left(-\frac{\sqrt{2}}{2} \Lambda \Omega \right)^2 + \frac{(\varpi^2 + \beta \vartheta^2)}{2\vartheta^3 \varpi} \left(\frac{\sqrt{2}}{2} \Lambda \right)^2 + \frac{\alpha}{3\vartheta} \left(\frac{\sqrt{2}}{2} \Lambda \right)^3 = \frac{(\varpi^2 + \beta \vartheta^2)}{2\vartheta^3 \varpi} \Lambda^2 + \frac{\alpha}{3\vartheta} \Lambda^3. \quad (5.32)$$

Solving it yields

$$\Omega = \sqrt{\frac{(4-\sqrt{2})\alpha\Lambda}{3\vartheta} + \frac{(\varpi^2 + \beta \vartheta^2)}{\vartheta^3 \varpi}}, \quad (5.33)$$

which is under $\frac{(4-\sqrt{2})\alpha\Lambda}{3\vartheta} + \frac{(\varpi^2 + \beta \vartheta^2)}{\vartheta^3 \varpi} > 0$. Then there is

$$\Psi = \Lambda \cos \left(\sqrt{\frac{(4-\sqrt{2})\alpha\Lambda}{3\vartheta} + \frac{(\varpi^2 + \beta \vartheta^2)}{\vartheta^3 \varpi}} \gamma \right). \quad (5.34)$$

Based on Eq (5.4), we have

$$\theta = \frac{\Lambda}{\sqrt{\frac{(4-\sqrt{2})\alpha\Lambda}{3\vartheta} + \frac{(\varpi^2 + \beta\vartheta^2)}{\vartheta^3\varpi}}} \sin\left(\sqrt{\frac{(4-\sqrt{2})\alpha\Lambda}{3\vartheta} + \frac{(\varpi^2 + \beta\vartheta^2)}{\vartheta^3\varpi}}\gamma\right). \quad (5.35)$$

So there is the periodic solution as

$$\psi(x, t) = \frac{\Lambda}{\sqrt{\frac{(4-\sqrt{2})\alpha\Lambda}{3\vartheta} + \frac{(\varpi^2 + \beta\vartheta^2)}{\vartheta^3\varpi}}} \sin\left[\sqrt{\frac{(4-\sqrt{2})\alpha\Lambda}{3\vartheta} + \frac{(\varpi^2 + \beta\vartheta^2)}{\vartheta^3\varpi}}(\vartheta x + \varpi t)\right]. \quad (5.36)$$

The outlines of Eq (5.36) are unveiled in Figure 7 for $\Lambda = 3$, $\alpha = 1$, $\beta = 1$, $\vartheta = 1$, and $\varpi = -2$, which shows a good periodic wave. The subplots (a), (b), and (c) represent the 3-D plot, density plot, and curves, respectively.

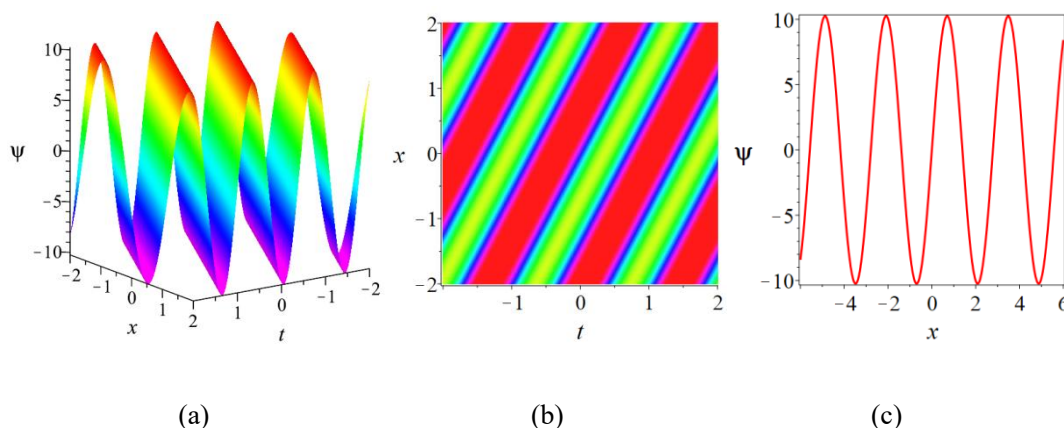


Figure 7. The shapes of Eq (5.36) with $\Lambda = 3$, $\alpha = 1$, $\beta = 1$, $\vartheta = 1$, and $\varpi = -2$.

5.3. The direct mapping method

Based on the unified equation, there is [36]

$$\varphi' = \varepsilon \sqrt{s_0 + s_1\varphi + s_2\varphi^2 + s_3\varphi^3}, \quad (5.37)$$

where $\varphi = \varphi(\gamma)$ and $\varepsilon = \pm 1$. Equation (5.37) owns the different solutions as the following:

Case 1: For $s_1 = s_0 = 0$, $s_2 > 0$, there is the solution as

$$\varphi = -\frac{s_2}{s_3} \operatorname{sech}^2\left(\frac{\sqrt{s_2}}{2}\gamma\right). \quad (5.38)$$

Case 2: For $s_1 = s_0 = 0$, $s_2 < 0$, there is the solution as

$$\varphi = -\frac{s_2}{s_3} \sec^2\left(\frac{\sqrt{-s_2}}{2}\gamma\right). \quad (5.39)$$

Case 3: For $s_1 = s_0 = 0$, $s_2 = 0$, there is a rational solution as

$$\varphi = \frac{4}{s_3\vartheta^2}. \quad (5.40)$$

By Eq (5.5), we have

$$\Psi' = \varepsilon \sqrt{-\frac{(\varpi^2 + \beta\vartheta^2)}{\vartheta^3\varpi}\Psi^2 - \frac{2\alpha}{3\vartheta}\Psi^3}, \quad (5.41)$$

where $\varepsilon = \pm 1$. Mapping it with Eq (5.37), we have

Case 1: For $-\frac{(\varpi^2 + \beta\vartheta^2)}{\vartheta^3\varpi} > 0$, there is

$$\Psi = -\frac{3(\varpi^2 + \beta\vartheta^2)}{2\alpha\vartheta^2\varpi} \operatorname{sech}^2 \left(\frac{1}{2} \sqrt{-\frac{(\varpi^2 + \beta\vartheta^2)}{\vartheta^3\varpi}} \gamma \right). \quad (5.42)$$

So we have

$$\Theta = -\sqrt{-\frac{9(\varpi^2 + \beta\vartheta^2)}{\alpha^2\vartheta\varpi}} \tanh \left(\frac{1}{2} \sqrt{-\frac{(\varpi^2 + \beta\vartheta^2)}{\vartheta^3\varpi}} \gamma \right). \quad (5.43)$$

So the kink type solution is

$$\psi(x, t) = -\sqrt{-\frac{9(\varpi^2 + \beta\vartheta^2)}{\alpha^2\vartheta\varpi}} \tanh \left[\frac{1}{2} \sqrt{-\frac{(\varpi^2 + \beta\vartheta^2)}{\vartheta^3\varpi}} (\vartheta x + \varpi t) \right]. \quad (5.44)$$

Case 2: For $-\frac{(\varpi^2 + \beta\vartheta^2)}{\vartheta^3\varpi} < 0$, there is

$$\Psi = -\frac{3(\varpi^2 + \beta\vartheta^2)}{2\alpha\vartheta^2\varpi} \sec^2 \left(\frac{1}{2} \sqrt{\frac{(\varpi^2 + \beta\vartheta^2)}{\vartheta^3\varpi}} \gamma \right). \quad (5.45)$$

By Eq (5.4), we have

$$\Theta = -\sqrt{\frac{9(\varpi^2 + \beta\vartheta^2)}{\alpha^2\vartheta\varpi}} \tan \left(\frac{1}{2} \sqrt{\frac{(\varpi^2 + \beta\vartheta^2)}{\vartheta^3\varpi}} \gamma \right). \quad (5.46)$$

So the singular periodic solution is

$$\psi(x, t) = -\sqrt{\frac{9(\varpi^2 + \beta\vartheta^2)}{\alpha^2\vartheta\varpi}} \tan \left[\frac{1}{2} \sqrt{\frac{(\varpi^2 + \beta\vartheta^2)}{\vartheta^3\varpi}} (\vartheta x + \varpi t) \right]. \quad (5.47)$$

Case 3: For $-\frac{(\varpi^2 + \beta\vartheta^2)}{\vartheta^3\varpi} = 0$, there is

$$\Psi = -\frac{6\vartheta}{\alpha\gamma^2}. \quad (5.48)$$

By Eq (5.4), we have

$$\Theta = \frac{6\vartheta}{\alpha\gamma}. \quad (5.49)$$

So the rational solution is

$$\Theta = \frac{6\vartheta}{\alpha(\vartheta x + \varpi t)}. \quad (5.50)$$

5.4. The Kudryashov method

According to the Kudryashov method [37,38], we have

$$\eta' = \eta^2 - \eta, \quad (5.51)$$

where $\eta = \eta(\gamma)$ and $\eta' = \frac{d\eta}{d\gamma}$. The solution of Eq (5.51) is

$$\eta = \frac{1}{1+e^\gamma}. \quad (5.52)$$

Supposing the solution of Eq (5.3) is

$$\theta(\gamma) = \sum_{i=0}^k u_i \eta^i(\gamma). \quad (5.53)$$

Inserting it into Eq (5.3) and balancing θ''' and $(\theta')^2$, we have

$$k + 3 = 2k + 2, \quad (5.54)$$

which gives

$$k = 1. \quad (5.55)$$

Thus, Eq (5.53) becomes

$$\theta(\gamma) = u_0 + u_1 \eta(\gamma). \quad (5.56)$$

Putting it into Eq (5.3), we obtain a system of algebraic equations as

$$\begin{aligned} \eta(\gamma): -u_1 \vartheta^3 \varpi - u_1 \varpi^2 - u_1 \vartheta^2 \beta &= 0, \\ \eta^2(\gamma): 7u_1 \vartheta^3 \varpi + u_1 \varpi^2 + \alpha \varpi u_1^2 \vartheta^2 + u_1 \vartheta^2 \beta &= 0 \\ \eta^3(\gamma): -2\alpha u_1^2 \vartheta^2 \varpi - 12\alpha u_1 \vartheta^3 \varpi &= 0, \\ \eta^4(\gamma): 6u_1 \vartheta^3 \varpi + \alpha u_1^2 \vartheta^2 \varpi &= 0. \end{aligned}$$

Solving them yields

$$\varpi = \frac{1}{2}(-\vartheta^3 \pm \sqrt{\vartheta^6 - 4\vartheta^2 \beta}), \quad u_1 = -\frac{6\vartheta}{\alpha}.$$

So there is

$$\theta(\gamma) = u_0 - \frac{6\vartheta}{\alpha(1+e^\gamma)}. \quad (5.57)$$

Then the traveling wave solution is

$$\psi_{\pm}(x, t) = u_0 - \frac{6\vartheta}{\alpha \left(1 + e^{\vartheta x + \frac{1}{2}(-\vartheta^3 \pm \sqrt{\vartheta^6 - 4\vartheta^2 \beta})t} \right)}, \quad (5.58)$$

which is under $\vartheta^6 - 4\vartheta^2 \beta > 0$.

For $\alpha = 1$, $\beta = 1$, $u_0 = 1$, and $\vartheta = 1$, we draw the outlines of Eq (5.58) in Figure 8. Thus, the solution in Eq (5.58) is also the kink soliton, which is $\psi_+ \rightarrow 1$ for $(x \rightarrow \infty)$ and $\psi_+ \rightarrow -5$ for $(x \rightarrow -\infty)$. Moreover, if we use $\alpha = 2$, $\beta = -1$, $u_0 = 1$, and $\vartheta = 1$, the shapes of Eq (5.58) are presented in Figure 9, which is also the kink soliton. Additionally, there is $\psi_+ \rightarrow 1$ for $(x \rightarrow \infty)$ and $\psi_+ \rightarrow -2$ for $(x \rightarrow -\infty)$. We can see that coefficients α and β can effect the wave amplitude and velocity. It should be noted that the subplots (a), (b), and (c) in Figures 8 and 9 represent the 3-D plot, density plot, and curves, respectively.

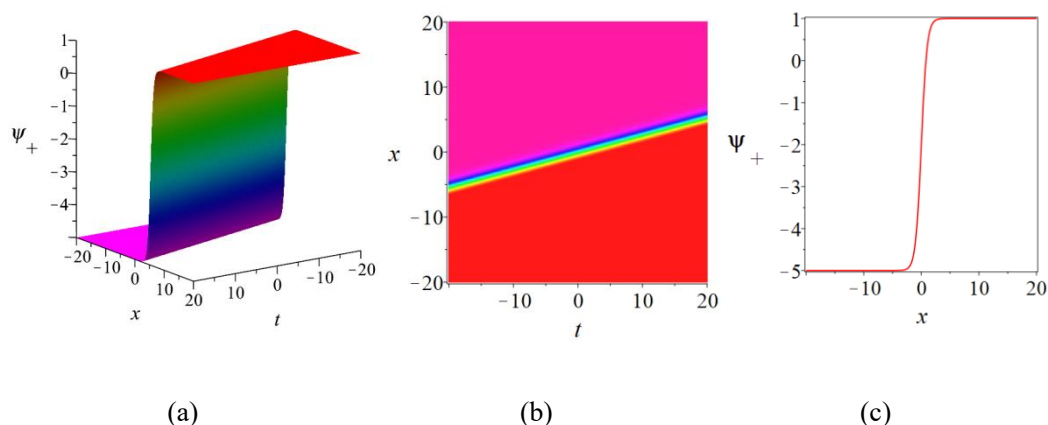


Figure 8. The shapes of Eq (5.58) with $\alpha = 1$, $\beta = 1$, $u_0 = 1$, and $\vartheta = 1$.

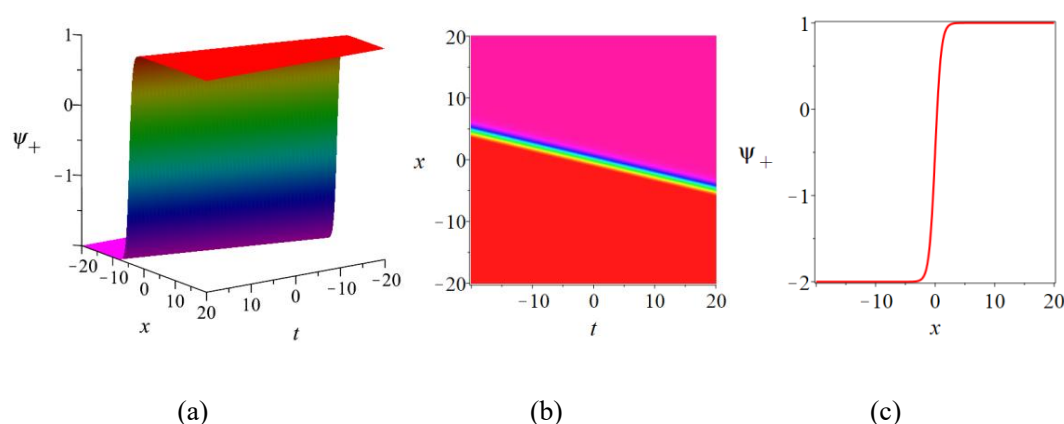


Figure 9. The shapes of Eq (5.58) with $\alpha = 2$, $\beta = -1$, $u_0 = 1$, and $\vartheta = 1$.

6. Conclusions

In this paper, we investigate the fourth-order potential Ito equation, a fundamental model describing the propagation of water waves. First, an auto-Bäcklund transformation of Painlevé-type is systematically constructed using the truncated Laurent series expansion. Based on this transformation, a series of exact wave solutions are derived, including kink soliton, singular, and rational solutions. Subsequently, the Painlevé analysis is performed to establish the bilinear form of the target equation, based on which the breather-type kink soliton and singular periodic wave solutions are successfully derived via the new homoclinic test approach. In addition, the interaction solutions of cos-cosh type are also investigated. To further enrich the solution set of the studied equation, additional abundant traveling wave solutions are explicitly explored through four robust and effective approaches, including the variational method, energy balance method, direct mapping method, and Kudryashov method. By selecting appropriate parameter sets, the dynamical profiles of these solutions are visualized to intuitively reveal their inherent physical properties. As far as we know, compared with the multiple soliton and ten analytical solutions developed in [29,30], the findings reported in this paper represent new discoveries that have not been discussed in any literature, which can contribute to a deeper understanding of the dynamic behaviors of the equation under study.

Although we find some exact solutions based on the bilinear form by introducing different trial functions, due to the limitations of these trial functions, we are unable to obtain the popular research object of complexiton solutions. Thus, this is also our focus future research.

Author contributions

Jing Yang: Software, Validation, Methodology, Software, Writing–review; Peng Xu: Conceptualization, Methodology, Software, Validation, Formal analysis, Writing–review and editing, Visualization; Feng Shi: Software, Validation; Jing-Hua Liu: Conceptualization, Methodology, Software, Writing–original draft preparation, Writing–review; Kang-Jia Wang: Conceptualization, Methodology, Software, Writing–review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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