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Research article

Magnetic quantum number distribution from q-binomial identities

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Abstract: The distribution $\mathscr{P}(M)$ of the total atomic magnetic quantum number M is a fundamental quantity for the computation of atomic structure and spectra. It enables one to deduce the distribution $\mathscr{Q}(J)$ of the total angular momentum J, which determines the sizes of the Hamiltonian blocks. Various techniques have been developed to obtain $\mathscr{P}(M)$, based on recurrence relations, group theory, or q-binomial coefficients (Gaussian polynomials). In this work, we derived new recurrence relations for $\mathscr{P}(M)$ based on identities that have not yet been exploited for this purpose, including a q-analogue of the so-called "Star of David" theorem. We showed that classical combinatorial identities, when extended to Gaussian polynomials and combined with generating-function differentiation, systematically yielded new recurrence relations for the magnetic quantum number distribution. While these new relations may not uniformly surpass existing algorithms in raw computational efficiency, they offer significant theoretical novelty by connecting advanced q-combinatorial theorems to atomic physics. Furthermore, because the recurrence relations involved a distinct stencil over the index set (and thus different subsets of M-values), they provided valuable cross-validation tools to test the consistency and reliability of large-scale calculations.

Keywords: magnetic quantum number distribution; angular momentum; q-binomial coefficients; Gaussian polynomials; Star of David theorem; combinatorial identities; recurrence relations; generating functions; atomic structure; shell model; fermionic systems; q-analogues
Mathematics Subject Classification: 05A10, 05A30, 11B65, 81V35

1. Introduction

The determination of total angular momentum multiplicities has a long and rich history, initially explored by nuclear physicists within the nuclear shell model and later adopted in atomic physics for electronic configurations. For a system of N identical fermions, the key problem is to determine the

allowed total angular momentum values J resulting from the coupling of individual angular momenta. Certain values of J are forbidden due to antisymmetry constraints imposed by the Pauli exclusion principle, while others occur with multiplicities greater than one. The number $\mathscr{Q}(J)$ of levels with total angular momentum J is given by the well-known formula:

$$\mathscr{Q}(J) = \mathscr{P}(M = J) - \mathscr{P}(M = J + 1),$$

where $\mathscr{P}(M)$ denotes the number of states with total magnetic quantum number M. Therefore, determining $\mathscr{P}(M)$ and $\mathscr{Q}(J)$ is of fundamental importance in both nuclear [1–4] and atomic physics [5]. Furthermore, a detailed knowledge of magnetic quantum number distributions and level multiplicities provides the foundational statistical framework required for interpreting sophisticated spectroscopic diagnostics, such as those related to orbital magnetization and circular dichroism sum rules [6, 7], where open-shell configurations play a central role.

Although our present focus lies in the realm of atomic systems, the methods and results are equally valid for other fermionic systems such as protons and neutrons. Hence, we often refer generically to "fermions", particularly when discussing methodologies rooted in nuclear physics. A wide range of approaches has been developed to tackle this problem, including group-theoretical techniques [8, 9], moment methods [10], trace techniques [11], Molien series [12], diagrammatic methods [13], and the use of fractional parentage coefficients [14].

In earlier work, we investigated relativistic configurations (j^N) using generating functions and Gaussian binomial coefficients [15–17] to derive recursion relations, cumulants, and Gram–Charlier and Edgeworth asymptotic expansions for $\mathscr{P}(M)$ [18], subsequently extending the Bethe formula [19]. We also introduced recursive constructions yielding piecewise polynomial closed-form expressions structured by congruence classes [20]. While explicit formulas involving irreducible representations, multinomial coefficients, or hypergeometric functions were previously limited to non-equivalent fermions in single-j orbits [21], the first general explicit formulas for arbitrary complex configurations (with equivalent electrons across multiple subshells) were recently obtained via generating-function integration properties [22].

Despite these advances, finding alternative algebraic pathways to evaluate $\mathscr{P}(M)$ remains highly desirable. The present contribution bridges this gap by introducing a novel framework that systematically yields new recurrence relations for the magnetic quantum number distribution. By extending classical combinatorial identities to Gaussian polynomials, including a q-analogue of the "Star of David" theorem, and combining them with generating-function differentiation, we develop complementary algebraic relations. Although these relations do not necessarily surpass existing algorithms in raw computational speed, each mobilizes a distinct neighborhood of indices (or stencil), thereby providing a valuable cross-validation tool to test the consistency and reliability of large-scale calculations.

It turns out that, for a subshell j^N (which can contain between $N = 0$ and $N = 2j + 1$ fermions), $\mathscr{P}(M)$, which will be denoted by $\mathscr{P}_{2j+1,N}(M)$ in the following to clearly show the dependencies, is actually the coefficient of q^M in

$$q^{-J_{\max}} \begin{bmatrix} 2j+1 \\ N \end{bmatrix}_q = q^{-\frac{N(2j+1-N)}{2}} \begin{bmatrix} 2j+1 \\ N \end{bmatrix}_q,$$

where

$$\left[\begin{array}{c} p \\ r \end{array} \right]_q = \frac{(1-q^p)(1-q^{p-1})\cdots(1-q^{p-r+1})}{(1-q)(1-q^2)\cdots(1-q^r)}$$

represents the q-binomial coefficient (also referred to as "Gaussian polynomial") [23]. The generating function for $\mathscr{P}_{2j+1,N}(M)$ is given by

$$\sum_{n=0}^{2J_{\max}} \mathscr{P}_{2j+1,N}(n-J_{\max})\, q^n = \prod_{p=1}^{N} \frac{q^{2j+2-p}-1}{q^p-1} = \left[\begin{array}{c} 2j+1 \\ N \end{array} \right]_q,$$

and one thus has

$$\mathscr{P}_{2j+1,N}(n-J_{\max}) = \frac{1}{n!}\frac{\partial^n}{\partial q^n} \left[\begin{array}{c} 2j+1 \\ N \end{array} \right]_q \Bigg|_{q=0}, \tag{1.1}$$

where

$$J_{\max} = \frac{N(2j+1-N)}{2}.$$

Equivalently, one can write

$$\mathscr{P}_{2j+1,N}(M) = \frac{1}{(M+J_{\max})!}\frac{\partial^{M+J_{\max}}}{\partial q^{M+J_{\max}}} \left[\begin{array}{c} 2j+1 \\ N \end{array} \right]_q \Bigg|_{q=0}.$$

The discrete distribution $\mathscr{P}_{2j+1,N}(M)$ is even, i.e., $\mathscr{P}_{2j+1,N}(-M) = \mathscr{P}_{2j+1,N}(M)$, the minimum positive value of M is $M_{\min} = 0$ (for an even number of fermions), or $M_{\min} = 1/2$ (for an odd number of fermions), and $M_{\max} = J_{\max}$. One has $\mathscr{P}_{2j+1,N}(M) = 0$ when M falls outside the admissible physical boundaries (i.e., $|M| > M_{\max}$). Except in some simple cases, the distribution $\mathscr{P}_{2j+1,N}(M)$ is not known analytically. For instance, in the case of two fermions in a single-j shell, one has [24]:

$$\mathscr{P}_{2j+1,2}(M) = \left\lfloor \frac{2j+1-|M|}{2} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the integer part of x. In the case of three fermions in a single-j shell, we have the following identity:

$$\mathscr{P}_{2j+1,3}(M=j+1) = \begin{cases} \dfrac{j(j-1)}{3} & \text{if } j \not\equiv 2 \pmod 3, \\ \dfrac{j^2-j+1}{3} & \text{if } j \equiv 2 \pmod 3. \end{cases}$$

An example of a $\mathscr{P}(M)$ distribution is displayed in Figure 1, corresponding to 8 fermions in a $j = 13/2$ shell.

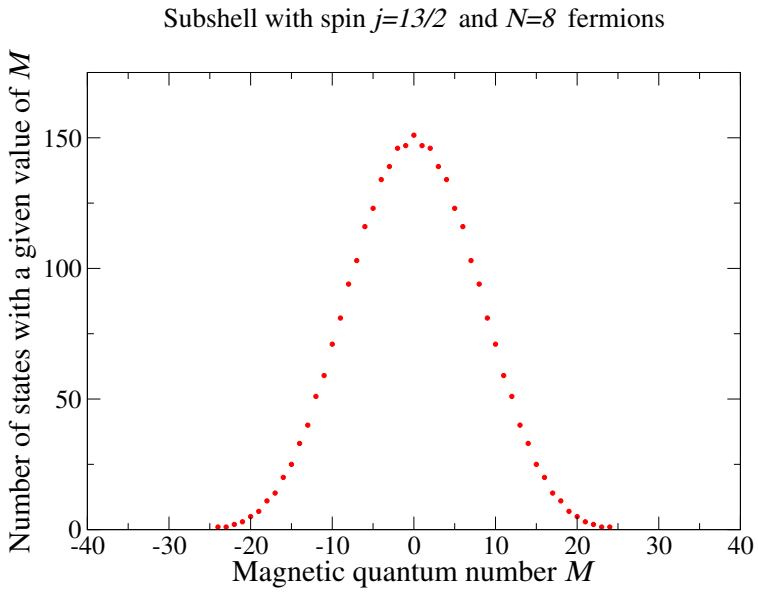

Figure 1. Example of a distribution $\mathscr{P}(M)$ (more precisely $\mathscr{P}_{14,8}(M)$) in the case of $j = 13/2$ and $N = 8$ fermions. One has $M_{\min} = 0$ and the support of the distribution is bounded by $-M_{\min} = -24$ and $M_{\max} = 24$.

In the present study, we take a step forward by deriving new identities for $\mathscr{P}_{2j+1,N}(M)$ based on relations between q-binomial coefficients.

The paper is organized as follows. In Section 2, we recall a classical identity for binomial coefficients, known as the Star of David theorem, and present a general framework allowing its extension to Gaussian (or q-) binomials. In Section 3, we exploit this q-analog to derive new recurrence relations for the magnetic quantum number distribution by combining these identities with differentiation techniques applied to generating functions. Section 4 is devoted to the derivation of additional recurrence relations based on other identities involving q-binomial coefficients, such as summation formulas in the q-Pascal triangle and a recent identity due to He, Shannon, and Shiue. In Section 5, we propose a new theorem based on the q-Vandermonde formula, providing a convolution-like recurrence useful for subshell decomposition. In the conclusion, we suggest possible extensions, particularly regarding the use of Heine's formula and the q-binomial theorem for future combinatorial developments.

2. From binomial to q-binomial coefficients

Because the magnetic quantum number distribution $\mathscr{P}(M)$ is directly generated by coefficients of Gaussian polynomials (or q-binomial coefficients), any algebraic identity satisfied by these polynomials naturally translates into a recurrence relation for $\mathscr{P}(M)$ upon differentiation. To exploit this structural connection, we review the foundational properties of q-binomial coefficients and their identities in the following section.

2.1. *q*-analogues and Gaussian binomial coefficients

We recall here the standard definitions of q-analogues used throughout this work. The q-analogue of a positive integer n is defined by

$$[n]_q = \sum_{i=0}^{n-1} q^i = \frac{1-q^n}{1-q}.$$

The q-factorial is then defined recursively by

$$[n]_q! = [1]_q[2]_q \cdots [n]_q, \qquad [0]_q! = 1.$$

The Gaussian binomial coefficient (or q-binomial coefficient) is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}, \qquad 0 \le k \le n.$$

An equivalent product form, particularly useful in combinatorial and analytical contexts, is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

These coefficients satisfy the symmetry relation

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q,$$

as well as the q-Pascal identity

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q.$$

When $q \to 1$, one recovers the classical binomial coefficients:

$$\lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

From a combinatorial point of view, it is worth mentioning that if q is a prime power, $\begin{bmatrix} n \\ k \end{bmatrix}_q$ counts the number of k-dimensional vector subspaces of $\mathbb{F}_q^n$. The lines are therefore $\begin{bmatrix} n \\ 1 \end{bmatrix}_q$ and the planes $\begin{bmatrix} n \\ 2 \end{bmatrix}_q$. In the projective space $\mathbb{P}^{n-1}(\mathbb{F}_q)$, the points are $\begin{bmatrix} n \\ 1 \end{bmatrix}_q$ and the lines $\begin{bmatrix} n \\ 2 \end{bmatrix}_q$. Thus, the points of $\mathbb{P}^{n-1}(\mathbb{F}_q)$ are the lines of $\mathbb{F}_q^n$ and the projective lines of $\mathbb{P}^{n-1}(\mathbb{F}_q)$ are the planes of $\mathbb{F}_q^n$.

It is worth mentioning that the study of q-analogues has long provided powerful mathematical tools across combinatorics, statistical mechanics, and theoretical physics and is an active field of research. Recent developments involving wavelet-based approximation operators and advanced techniques have further expanded the scope of modern q-analogue methodologies [25], while related studies in q-analogue approximation theory continue to complement and enrich these analytical frameworks [26].

2.2. An identity for usual binomial coefficients

Let us consider the following contiguous binomial coefficients:

$$\mathscr{M}_1 = \binom{n-1}{k-1}, \quad \mathscr{M}_2 = \binom{n}{k-1}, \quad \mathscr{M}_3 = \binom{n+1}{k},$$

and

$$\mathscr{M}_4 = \binom{n+1}{k+1}, \quad \mathscr{M}_5 = \binom{n}{k+1}, \quad \mathscr{M}_6 = \binom{n-1}{k},$$

where $\binom{n}{k} = n!/k!/(n-k)!$ is the usual binomial coefficient. The multiplicative equality

$$\mathscr{M}_1\mathscr{M}_3\mathscr{M}_5 = \mathscr{M}_2\mathscr{M}_4\mathscr{M}_6, \tag{2.1}$$

was found by Hoggatt Jr. and Hansell [27] and corresponds to a configuration called "Hoggatt-Hansell's perfect square hexagon" (see Figure 2). The GCD ("Greatest Common Divisor") counterpart of identity (2.1), namely,

$$GCD(\mathscr{M}_1, \mathscr{M}_3, \mathscr{M}_5) = GCD(\mathscr{M}_2, \mathscr{M}_4, \mathscr{M}_6), \tag{2.2}$$

was found by Gould [28]. These two identities are the first two nontrivial examples of translatable identities of binomial coefficients, which are usually called the "Star of David theorems" [29–32]. Many generalizations of (2.1) and (2.2) have been developed [33, 34]. In particular, Hitotumatu and Sato proved a general Star of David Theorem using the characterization theorem for translatable GCD configurations [32]. The complete characterization of equal product configurations was proved by Sato and Straus and applied to the characterization of perfect k-th power configuration by Gordon, Straus and Sato [35].

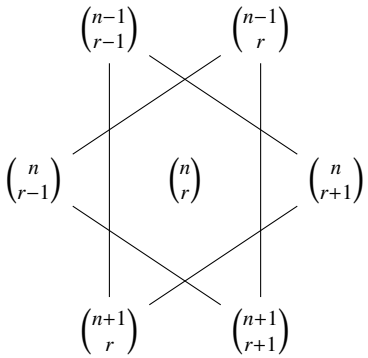

Figure 2. Star of David.

Proof. Let a, b, c, x, y, z be six natural numbers satisfying $x, y, z \leq \min(a, b, c)$ and $(b-a, c-b, a-c) = (y-z, z-x, x-y)$. Let us now count the number of ways to partition a set of size $a+b+c$ into six parts: A (size x), B (size y), C (size z), A' (size $a-x$), B' (size $b-y$), C' (size $c-z$). We start to partition the set into three parts, with respective sizes a, b and c, which can be done in N ways.

In the first method, we cut the part of size a into $A \sqcup A'$: There are $\binom{a}{x}$ ways to do so. There are $\binom{b}{y}$ ways to cut the part of size b in $B \sqcup B'$ and $\binom{c}{z}$ ways to split the part of size c into $C \sqcup C'$. This gives

$$N \times \binom{a}{x}\binom{b}{y}\binom{c}{z} \tag{2.3}$$

for the total number of possibilities.

In the second method, we split the part of size a into $C \sqcup B'$ (since $z + b - y = a$), which makes $\binom{a}{z}$ ways. There are $\binom{b}{x}$ ways to split the part of size b into $A \sqcup C'$ (since $x + c - z = b$), and $\binom{c}{y}$ ways to split the part of size c in $B \sqcup A'$ (because $y + a - x = c$). This yields the following total number of possibilities:

$$N \times \binom{a}{z}\binom{b}{x}\binom{c}{y}. \tag{2.4}$$

We thus have, using Eqs (2.3) and (2.4):

$$\binom{a}{x}\binom{b}{y}\binom{c}{z} = \binom{a}{z}\binom{b}{x}\binom{c}{y}.$$

Let us now set $(a, b, c) = (n-1, n, n+1)$ and $(x, y, z) = (k-1, k+1, k)$, and one checks that $(b-a, c-b, a-c) = (1, 1, -2) = (y-z, z-x, x-y)$. We finally deduce the identity of the Star of David:

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}.$$

□

2.3. Generalization to Gaussian polynomials

We can replace $\binom{n}{k}$ by

$$\binom{n}{k}_{(a_n)} = \frac{f(n)}{f(n-k)f(k)},$$

where

$$f(n) = \coprod_{k=1}^{n} a_k,$$

(a_n) being a sequence of nonzero real numbers. In particular, if (a_n) is the Fibonacci sequence (F_n), $\binom{n}{k}_{(a_n)}$ is the second fibonomial coefficient $\binom{n}{k}_F$; the second Star of David theorem is thus valid in the fibonomial triangle. If (a_n) is the q-analog of n, i.e., $[n]_q = 1 + q + ... + q^{n-1}$, then $\binom{n}{k}_{(a_n)}$ is the Gaussian polynomial

$$\binom{n}{k}_{(a_n)} \equiv \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

The Star of David identity can be extended to q-binomial coefficients.

Proposition 2.1 (Star of David identity for Gaussian binomial coefficients)**.** *For all integers n, k such that the involved coefficients are defined, one has*

$$\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \begin{bmatrix} n \\ k+1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q. \tag{2.5}$$

Proof. We use the product representation of Gaussian binomial coefficients,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{\prod_{i=0}^{k-1}(1-q^{n-i})}{\prod_{i=1}^{k}(1-q^i)}.$$

Each coefficient in Eq (2.5) is a ratio of products of factors of the form $(1-q^m)$. We therefore express both sides of Eq (2.5) as products of such factors. Observe that in the numerator of each Gaussian binomial coefficient, the factors are of the form $(1-q^{n-r})$ for appropriate indices r, while the denominator consists of $(1-q^s)$ with $s \leq k$. When forming the ratio between the left-hand side and the right-hand side of Eq (2.5), each factor $(1-q^m)$ appears with the same total multiplicity in the numerator and denominator. More precisely, for every integer m, the exponent of $(1-q^m)$ in the numerator of the left-hand side coincides with its exponent in the numerator of the right-hand side, and the same holds for denominators. This follows directly from the cancellation pattern induced by the contiguous shifts $(n-1,k-1)$, $(n,k\pm1)$, and $(n+1,k\pm1)$. Hence all factors cancel exactly, and the ratio of both sides is identically equal to 1. This proves the identity. □

To illustrate this cancellation process explicitly, let us consider the case $n = 3$ and $k = 2$. The left-hand side of Eq (2.5) reads:

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \begin{bmatrix} 3 \\ 3 \end{bmatrix}_q \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \left(\frac{1-q^2}{1-q}\right)(1)\left(\frac{(1-q^4)(1-q^3)}{(1-q)(1-q^2)}\right) = \frac{(1-q^3)(1-q^4)}{(1-q)^2},$$

while the right-hand side yields:

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix}_q \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q \begin{bmatrix} 4 \\ 3 \end{bmatrix}_q = (1)\left(\frac{1-q^3}{1-q}\right)\left(\frac{(1-q^4)(1-q^3)(1-q^2)}{(1-q)(1-q^2)(1-q^3)}\right) = \frac{(1-q^3)(1-q^4)}{(1-q)^2}.$$

Both sides thus yield identical expressions after direct factor cancellation, confirming the general mechanism.

It is worth mentioning that generalizations can be obtained with (a_n) a sequence in a commutative multiplicative group. For example, in the group $(\mathbb{Z},+)$, $\binom{n}{k}_{(n)} = k(n-k)$.

3. Deriving new recursion relations

Let us define

$$c_{a,b}(s) = [q^s]\begin{bmatrix} a \\ b \end{bmatrix}_q, \qquad J_{a,b} = \frac{b(a-b)}{2}, \qquad \text{and} \qquad \mathscr{P}_{a,b}(M) = c_{a,b}(M + J_{a,b}), \tag{3.1}$$

with $c_{a,b} = 0$ if $b < 0$, $b > a$, $s < 0$, or $s > b(a-b)$.

In the following, we will use generic integer variables. The connection with angular-momentum theory can be made, for instance, by setting $n = 2j \in \mathbb{Z}_{\geq 0}$, where j is the half-integer one-electron angular momentum, and $k = N$, where N is the number of fermions in the j subshell.

Let us briefly outline the underlying methodological approach. Several subsequent derivations rely on differentiating generating functions or q-binomial identities with respect to the deformation parameter q (followed by setting $q \to 1$). This technique allows us to transition smoothly from

polynomial identities to explicit recurrence relations for state counts and magnetic quantum number distributions $\mathscr{P}(M)$, transforming algebraic relations into differential relations on the associated generating polynomials.

Theorem 3.1. *Let n and k be natural numbers ($n, k \geq 1$). The following relation holds:*

$$\sum_{p_1+p_2+p_3=p} c_{n-1,k-1}(p_1)\, c_{n,k+1}(p_2)\, c_{n+1,k}(p_3) = \sum_{r_1+r_2+r_3=p} c_{n-1,k}(r_1)\, c_{n,k-1}(r_2)\, c_{n+1,k+1}(r_3).$$

The latter identity can be rewritten as

$$\begin{aligned}
&\sum_{p_1+p_2+p_3=p} \mathscr{P}_{n-1,k-1}\left(p_1 - \frac{(k-1)}{2}(n-k)\right) \mathscr{P}_{n,k+1}\left(p_2 - \frac{(k+1)}{2}(n-k-1)\right) \\
&\qquad \times \mathscr{P}_{n+1,k}\left(p_3 - \frac{k}{2}(n+1-k)\right) \\
&= \sum_{r_1+r_2+r_3=p} \mathscr{P}_{n-1,k}\left(r_1 - \frac{k}{2}(n-1-k)\right) \mathscr{P}_{n,k-1}\left(r_2 - \frac{(k-1)}{2}(n+1-k)\right) \\
&\qquad \times \mathscr{P}_{n+1,k+1}\left(r_3 - \frac{(k+1)}{2}(n-k)\right).
\end{aligned}$$

Proof. The Star of David identity for q-binomial coefficients reads

$$\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \begin{bmatrix} n \\ k+1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q. \tag{3.2}$$

Let us differentiate p times the two sides of the latter identity (3.2). For the left-hand side, one gets

$$\begin{aligned}
\frac{\partial^p}{\partial q^p}\left\{ \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \begin{bmatrix} n \\ k+1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ k \end{bmatrix}_q \right\} = \sum_{p_1+p_2+p_3=p} \binom{p}{p_1, p_2, p_3} \frac{\partial^{p_1}}{\partial q^{p_1}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \Bigg|_{q=0} \\
\times \frac{\partial^{p_2}}{\partial q^{p_2}} \begin{bmatrix} n \\ k+1 \end{bmatrix}_q \Bigg|_{q=0} \times \frac{\partial^{p_3}}{\partial q^{p_3}} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q \Bigg|_{q=0},
\end{aligned}$$

where $\binom{p}{p_1,p_2,p_3}$ is the usual multinomial coefficient. This gives us

$$\begin{aligned}
\frac{\partial^p}{\partial q^p}\left\{ \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \begin{bmatrix} n \\ k+1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ k \end{bmatrix}_q \right\} = \sum_{p_1+p_2+p_3=p} \binom{p}{p_1, p_2, p_3} p_1! p_2! p_3!\, c_{n-1,k-1}(p_1)\, c_{n,k+1}(p_2) \\
\times c_{n+1,k}(p_3),
\end{aligned}$$

or

$$\frac{\partial^p}{\partial q^p}\left\{ \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \begin{bmatrix} n \\ k+1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ k \end{bmatrix}_q \right\} = p! \sum_{p_1+p_2+p_3=p} c_{n-1,k-1}(p_1)\, c_{n,k+1}(p_2)\, c_{n+1,k}(p_3). \tag{3.3}$$

Similarly, differentiating p times the right-hand side of Eq (3.2), we get

$$\frac{\partial^p}{\partial q^p}\left\{\begin{bmatrix} n-1 \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q\right\} = \sum_{r_1+r_2+r_3=p} \binom{p}{r_1, r_2, r_3} \frac{\partial^{r_1}}{\partial q^{r_1}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \Bigg|_{q=0} \times \frac{\partial^{r_2}}{\partial q^{r_2}} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \Bigg|_{q=0} \times \frac{\partial^{r_3}}{\partial q^{r_3}} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q \Bigg|_{q=0}$$

yielding

$$\frac{\partial^p}{\partial q^p}\left\{\begin{bmatrix} n-1 \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q\right\} = p! \sum_{r_1+r_2+r_3=p} c_{n-1,k}(r_1)\, c_{n,k-1}(r_2)\, c_{n+1,k+1}(r_3). \tag{3.4}$$

Since Eq (3.2) implies that the left-hand sides of Eqs (3.3) and (3.4) are equal, we obtain

$$\sum_{p_1+p_2+p_3=p} c_{n-1,k-1}(p_1)\, c_{n,k+1}(p_2)\, c_{n+1,k}(p_3) = \sum_{r_1+r_2+r_3=p} c_{n-1,k}(r_1)\, c_{n,k-1}(r_2)\, c_{n+1,k+1}(r_3),$$

which completes the proof. □

4. Deriving new recursion relations from other q-binomial identities

Before proceeding with the formal derivations in this section, let us briefly recall the definitions and physical interpretations of the primary quantities involved. The quantity $\mathscr{P}_{2j+1,N}(M)$ (frequently denoted $\mathscr{P}(M)$) physically represents the exact number of many-electron states possessing a total magnetic quantum number M for N fermions distributed in a subshell of single-electron angular momentum j. Mathematically, these state counts are extracted from the expansions of q-binomial (Gaussian) coefficients (see Eq (3.1)). Specifically, the individual coefficients $c_{n,k}(p)$ appearing in the polynomial expansions,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{p=0}^{k(n-k)} c_{n,k}(p) q^p,$$

quantify the distribution of weights (or partition statistics within a $k \times (n-k)$ grid), bridging the combinatorial structure of q-series with the underlying angular momentum projections.

4.1. Summation in column in the q-Pascal triangle

Theorem 4.1. *Let n, k and r be natural numbers. The following identity holds*

$$\sum_{i=k}^{n} c_{i,k}(r-i+k) = c_{n+1,k+1}(r),$$

or equivalently

$$\sum_{i=k}^{n} \mathscr{P}_{i,k}(r-i+k-J_{i,k}) = \mathscr{P}_{n+1,k+1}(r-J_{n+1,k+1}),$$

or also

$$\sum_{i=k}^{n} \mathscr{P}_{i,k}\left(r - \frac{(k+2)(i-k)}{2}\right) = \mathscr{P}_{n+1,k+1}\left(r - \frac{(k+1)(n-k)}{2}\right).$$

Proof. The summation in column in the q-Pascal triangle takes the form

$$\sum_{i=k}^{n} q^{i-k} \begin{bmatrix} i \\ k \end{bmatrix}_q = \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q .$$

Differentiating r times, dividing by $r!$ and evaluating at $q = 0$, we obtain

$$\frac{1}{r!} \frac{\partial^r}{\partial q^r} \sum_{i=k}^{n} q^{i-k} \begin{bmatrix} i \\ k \end{bmatrix}_q \Bigg|_{q=0} = \frac{1}{r!} \frac{\partial^r}{\partial q^r} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q \Bigg|_{q=0} . \tag{4.1}$$

Applying the Leibniz rule to each term of the summation in the left-hand side of Eq (4.1), we get

$$\frac{1}{r!} \frac{\partial^r}{\partial q^r} \left(q^{i-k} \begin{bmatrix} i \\ k \end{bmatrix}_q \right) = \frac{1}{r!} \sum_{l=0}^{r} \binom{r}{l} \frac{\partial^l}{\partial q^l} q^{i-k} \cdot \frac{\partial^{r-l}}{\partial q^{r-l}} \begin{bmatrix} i \\ k \end{bmatrix}_q .$$

Now, since we have

$$\frac{\partial^l}{\partial q^l} q^{i-k} \Bigg|_{q=0} = \begin{cases} (i-k)! & \text{if } l = i-k, \\ 0 & \text{otherwise,} \end{cases}$$

only the term $l = i - k$ contributes, yielding

$$\frac{1}{r!} \sum_{i=k}^{n} \binom{r}{i-k} (i-k)! \, (r-i+k)! \, c_{i,k} \, (r-i+k) .$$

The right-hand side of Eq (4.1) gives

$$\frac{1}{r!} \frac{\partial^r}{\partial q^r} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q = c_{n+1,k+1} \, (r) ,$$

which completes the proof.

□

Example 4.1. For instance, with $n = 2$, $k = 1$ and $r = 2$, we have, for the left-hand side:

$$\begin{aligned} \sum_{i=1}^{2} c_{i,1}(2-i+1) = \sum_{i=1}^{2} c_{i,1}(3-i) &= c_{1,1}(2) + c_{2,1}(1) \\ &= \left[q^2\right] \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q + [q] \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \\ &= \left[q^2\right] \, (1) + [q] \, (1+q) \\ &= 0 + 1 = 1. \end{aligned}$$

Conversely, the right-hand side gives also

$$\left[q^2\right] \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q = \left[q^2\right] (1 + q + q^2) = 1.$$

This shows that summing over the smaller configurations correctly gives the coefficient of the larger one, providing a simple way to cross-check calculations without a full polynomial expansion.

4.2. Summation in line in the q-Pascal triangle

Theorem 4.2. *Let n and p be natural numbers. The following relations hold:*

$$\sum_{k=0}^{2n}(-1)^k c_{2n,k}(p) = [q^p]\prod_{s=1}^{n}\left(1-q^{2s-1}\right) = \sum_{\substack{S\subseteq\{1,\dots,n\},\\ \sum_{s\in S}(2s-1)=p}}(-1)^{|S|}, \tag{4.2}$$

where S is an arbitrary subset of $\{1,\dots,n\}$ *and*

$$\sum_{k=0}^{2n+1}(-1)^k c_{2n+1,k}(p) = 0. \tag{4.3}$$

The "even-case" formula (4.2) can be recast into

$$\sum_{k=0}^{2n}(-1)^k \mathscr{P}_{2n,k}(p-J_{2n,k}) = \sum_{k=0}^{2n}(-1)^k \mathscr{P}_{2n,k}\left(p-\frac{k(2n-k)}{2}\right) = \sum_{\substack{S\subseteq\{1,\dots,n\},\\ \sum_{s\in S}(2s-1)=p}}(-1)^{|S|},$$

and the "odd-case" formula (4.3) can be rewritten as

$$\sum_{k=0}^{2n+1}(-1)^k \mathscr{P}_{2n+1,k}(p-J_{2n+1,k}) = \sum_{k=0}^{2n+1}(-1)^k \mathscr{P}_{2n+1,k}\left(p-\frac{k(2n+1-k)}{2}\right) = 0.$$

Proof. The summation in line in the q-Pascal triangle reads

$$\sum_{k=0}^{2n}(-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix}_q = (1-q)(1-q^3)\cdots(1-q^{2n-1}) \tag{4.4}$$

and

$$\sum_{k=0}^{2n+1}(-1)^k \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q = 0. \tag{4.5}$$

The first formula (4.4) gives

$$\begin{aligned} p!\sum_{k=0}^{2n}(-1)^k c_{2n,k}(p) &= \frac{\partial^p}{\partial q^p}\left\{\prod_{s=1}^{n}\left(1-q^{2s-1}\right)\right\}\Big|_{q=0} \\ &= \sum_{\substack{u_1,u_2,\cdots,u_n\\ u_1+\cdots+u_n=p}}\prod_{s=1}^{n}\frac{\partial^{u_s}}{\partial q^{u_s}}\left(1-q^{2s-1}\right)\binom{p}{u_1,\cdots,u_n}\Big|_{q=0}, \end{aligned}$$

and we have

$$\frac{d^{u_s}}{dq^{u_s}}(1-q^{2s-1})\Big|_{q=0} = \begin{cases} 1 & u_s = 0, \\ 0 & 0 < u_s < 2s-1, \\ -(2s-1)! & u_s = 2s-1, \\ 0 & u_s > 2s-1. \end{cases}$$

Thus, arbitrary subsets $S \subseteq \{1, \ldots, n\}$ contribute, with the constraint

$$\sum_{s\in S}(2s-1) = p,$$

and the contribution (weight) $(-1)^{|S|}$. For the second formula (4.5), we get easily

$$\sum_{k=0}^{2n+1}(-1)^k c_{2n+1,k}(p) = 0,$$

which completes the proof of Eqs (4.2) and (4.3).

□

Example 4.2. As an example, let us take $n = 2$. We have

$$\begin{aligned}\sum_{k=0}^{4}(-1)^k c_{4,k}(3) &= c_{4,0}(3) - c_{4,1}(3) + c_{4,2}(3) - c_{4,3}(k) + c_{4,4}(3)\\ &= \left[q^3\right]\left(\begin{bmatrix}4\\0\end{bmatrix}_q - \begin{bmatrix}4\\1\end{bmatrix}_q + \begin{bmatrix}4\\2\end{bmatrix}_q - \begin{bmatrix}4\\3\end{bmatrix}_q + \begin{bmatrix}4\\4\end{bmatrix}_q\right)\\ &= \left[q^3\right]\Big(1 - (1+q+q^2+q^3) + (1+q+2q^2+q^3+q^4)\\ &\qquad -(1+q+q^2+q^3) + 1\Big)\\ &= -1+1-1 = -1.\end{aligned}$$

One has also

$$\left[q^3\right]\prod_{s=1}^{2}\left(1-q^{2s-1}\right) = \left[q^3\right]\left((1-q)(1-q^3)\right) = \left[q^3\right]\left(1-q-q^3+q^4\right) = -1$$

as well, and finally the only subset of $\{1, 2, 3, 4\}$ such that

$$\sum_{s\in S}(2s-1) = 3$$

is the singleton $\{2\}$ and thus $|S| = 1$ and $(-1)^{|S|} = -1$, which completes the check.

This illustrates how the alternating coefficient sum simplifies to a subset parity check, providing a direct combinatorial way to verify the result without expanding full polynomials.

4.3. Use of an identity due to He, Shannon, and Shiue

Theorem 4.3. *For all natural numbers p, n, k ($n \geq 2$, $k \geq 1$), one has*

$$c_{n,k}(p) = c_{n-1,k}(p) + c_{n-1,k-1}(p) - c_{n-2,k-1}(p) + c_{n-2,k-1}(p-n+1),$$

or equivalently

$$\begin{aligned}\mathscr{P}_{n,k}\left(p - \frac{k(n-k)}{2}\right) &= \mathscr{P}_{n-1,k}\left(p - \frac{k(n-1-k)}{2}\right) + \mathscr{P}_{n-1,k-1}\left(p - \frac{(k-1)(n-k)}{2}\right)\\ &\quad - \mathscr{P}_{n-2,k-1}\left(p - \frac{(k-1)(n-k-1)}{2}\right) + \mathscr{P}_{n-2,k-1}\left(p - \frac{n-1+k(n-k)}{2}\right).\end{aligned}$$

Proof. We start from the identity due to He, Shannon, and Shiue [36]:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q - (1-q^{n-1}) \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_q .$$

We now take the p-th derivative with respect to q and evaluate at $q = 0$. Let us start with the left-hand side. We obtain

$$\frac{1}{p!}\frac{d^p}{dq^p} \begin{bmatrix} n \\ k \end{bmatrix}_q \Bigg|_{q=0} = c_{n,k}(p).$$

Let us handle the first two terms of the right-hand side. Similarly, one has

$$\frac{1}{p!}\frac{d^p}{dq^p} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \Bigg|_{q=0} = c_{n-1,k}(p),$$

and *mutatis mutandis*

$$\frac{1}{p!}\frac{d^p}{dq^p} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \Bigg|_{q=0} = c_{n-1,k-1}(p).$$

By the Leibniz rule, we get

$$\frac{1}{p!}\frac{d^p}{dq^p} \left[(1-q^{n-1}) \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_q \right] = \frac{1}{p!} \sum_{l=0}^{p} \binom{p}{l} \frac{d^l}{dq^l}(1-q^{n-1}) \cdot \frac{d^{p-l}}{dq^{p-l}} \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_q .$$

We also have

$$\frac{d^l}{dq^l}(1-q^{n-1})\Big|_{q=0} = \begin{cases} 1 & \text{if } l = 0, \\ 0 & \text{if } 0 < l < n-1, \\ -(n-1)! & \text{if } l = n-1, \\ 0 & \text{if } l > n-1, \end{cases}$$

since the derivative of order l of q^{n-1} is

$$\frac{d^l}{dq^l} q^{n-1} = \frac{(n-1)!}{(n-1-l)!} q^{n-1-l}.$$

Thus, only the terms $l = 0$ and $l = n-1$ contribute at $q = 0$; the former gives $c_{n-2,k-1}(p)$ and the latter

$$-\frac{1}{p!}\binom{p}{n-1}(n-1)! \frac{d^{p-(n-1)}}{dq^{p-(n-1)}} \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_q \Bigg|_{q=0},$$

or also

$$-\frac{1}{p!}\binom{p}{n-1}(n-1)!\,(p-(n-1))! c_{n-2,k-1}(p-n+1) = -c_{n-2,k-1}(p-n+1).$$

Combining all contributions, we obtain the claimed identity.

□

Example 4.3. As an example, let us take $n = 3$, $k = 1$ and $p = 2$. We have, for the left-hand side:

$$c_{3,1}(2) = \left[q^2\right] \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q = \left[q^2\right]\left(1 + q + q^2\right) = 1.$$

For the right-hand side, we get

$$\begin{aligned} &c_{2,1}(2) + c_{2,0}(2) - c_{1,0}(2) + c_{1,0}(2 - 3 + 1) \\ &= c_{2,1}(2) + c_{2,0}(2) - c_{1,0}(2) + c_{1,0}(0) \\ &= \left[q^2\right]\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q + \begin{bmatrix} 2 \\ 0 \end{bmatrix}_q - \begin{bmatrix} 1 \\ 0 \end{bmatrix}_q\right) + \left[q^0\right]\begin{bmatrix} 1 \\ 0 \end{bmatrix}_q \\ &= \left[q^2\right](1 + q + 1 - 1) + \left[q^0\right](1) \\ &= 0 + 1 = 1. \end{aligned}$$

This confirms that the recurrence relation correctly relates the coefficient to lower-order terms, providing a quick way to verify values without full polynomial expansions.

5. A theorem using the q-Vandermonde formula

The q-Vandermonde identity, which is the q-analogue of the Chu-Vandermonde identity, provides a way to partition a system of fermions into two subsystems. It is expressed as:

$$\begin{bmatrix} n + m \\ r \end{bmatrix}_q = \sum_{k=0}^{r} q^{k(n-r+k)} \begin{bmatrix} n \\ r - k \end{bmatrix}_q \begin{bmatrix} m \\ k \end{bmatrix}_q. \tag{5.1}$$

This identity is particularly useful for decomposing a large shell into smaller, more manageable subshells.

Theorem 5.1. *Let n, m, r, and p be natural numbers. The following identity holds*

$$c_{n+m,r}(p) = \sum_{k=0}^{r} \sum_{\substack{u,v\geq 0, \\ u+v=p-k(n-r+k)}} c_{n,r-k}(u)\, c_{m,k}(v), \tag{5.2}$$

where the range $k = 0, \ldots, r$ is acceptable only with constraints; one must have $p, u, v \geq 0$, $k \geq r - n$, and $k \leq m$. As a consequence, the magnetic quantum number distribution $\mathscr{P}$ satisfies the following convolution-like recurrence:

$$\begin{aligned} \mathscr{P}_{n+m,r}\left(p - \frac{r(n+m-r)}{2}\right) = \sum_{k=0}^{r} \sum_{\substack{u,v\geq 0, \\ u+v=p-k(n-r+k)}} &\mathscr{P}_{n,r-k}\left(u - \frac{(r-k)(n-r+k)}{2}\right) \\ &\times \mathscr{P}_{m,k}\left(v - \frac{k(m-k)}{2}\right). \end{aligned}$$

Proof. Applying the operator $\frac{1}{p!}\frac{\partial^p}{\partial q^p}\ldots\Big|_{q=0}$ to both sides of Eq (5.1), setting $K_0 = k(n-r+k)$ and using the Leibniz rule for the product of three functions $f(q) = q^{K_0}$, $g(q) = \begin{bmatrix} n \\ r-k \end{bmatrix}_q$, and $h(q) = \begin{bmatrix} m \\ k \end{bmatrix}_q$, we note that $\frac{\partial^l}{\partial q^l} q^{K_0}\Big|_{q=0}$ is nonzero (and equal to $K_0!$) if, and only if, $l = K_0$. Therefore, the triple sum over the indices of the Leibniz rule collapses to a double sum over u and v such that $u+v+K_0 = p$. Dividing by $p!$ and using the Definition (1.1) for each q-binomial coefficient yields:

$$\begin{aligned}
\frac{1}{p!}\frac{\partial^p}{\partial q^p}\begin{bmatrix} n+m \\ r \end{bmatrix}_q\Bigg|_{q=0} &= \sum_{k=0}^{r} \sum_{\substack{u,v\geq 0,\\ u+v=p-K_0}} \frac{1}{p!}\binom{p}{K_0,u,v} K_0!\, \frac{\partial^u}{\partial q^u}\begin{bmatrix} n \\ r-k \end{bmatrix}_q\Bigg|_{q=0} \times \frac{\partial^v}{\partial q^v}\begin{bmatrix} m \\ k \end{bmatrix}_q\Bigg|_{q=0} \\
&= \sum_{k=0}^{r} \sum_{\substack{u,v\geq 0,\\ u+v=p-K_0}} \frac{1}{u!}\frac{\partial^u}{\partial q^u}\begin{bmatrix} n \\ r-k \end{bmatrix}_q\Bigg|_{q=0} \times \frac{1}{v!}\frac{\partial^v}{\partial q^v}\begin{bmatrix} m \\ k \end{bmatrix}_q\Bigg|_{q=0} \\
&= \sum_{k=0}^{r} \sum_{\substack{u,v\geq 0,\\ u+v=p-k(n-r+k)}} \frac{1}{u!}\frac{\partial^u}{\partial q^u}\begin{bmatrix} n \\ r-k \end{bmatrix}_q\Bigg|_{q=0} \times \frac{1}{v!}\frac{\partial^v}{\partial q^v}\begin{bmatrix} m \\ k \end{bmatrix}_q\Bigg|_{q=0},
\end{aligned}$$

which leads to the expected identity:

$$c_{n+m,r}(p) = \sum_{k=0}^{r} \sum_{\substack{u,v\geq 0,\\ u+v=p-k(n-r+k)}} c_{n,r-k}(u)\, c_{m,k}(v).$$

In terms of the distribution $\mathscr{P}$, this gives

$$\begin{aligned}
\mathscr{P}_{n+m,r}\left(p - \frac{r(n+m-r)}{2}\right) = \sum_{k=0}^{r} \sum_{\substack{u,v\geq 0,\\ u+v=p-k(n-r+k)}} &\mathscr{P}_{n,r-k}\left(u - \frac{(r-k)(n-r+k)}{2}\right) \\
&\times \mathscr{P}_{m,k}\left(v - \frac{k(m-k)}{2}\right).
\end{aligned}$$

□

The relation (5.2) is a direct transcription to angular-momentum theory of the q-Vandermonde identity. It may be efficient if coefficients $c_{i,j}$ are known (and tabulated) for small values of i and j, or can be used as an identity in order to check the consistency of some complex and intensive computations.

Standard evaluations of q-binomial coefficients via full polynomial multiplication scale quadratically with the maximum degree. By avoiding the explicit construction of high-degree Gaussian polynomials, which can become memory-intensive for large atomic subshells, the recurrence relations proposed in this work allow state counts and distribution values to be computed incrementally. This approach reduces storage overhead and minimizes the risk of intermediate coefficient overflow.

6. Conclusions

The distribution $\mathscr{P}(M)$ of the total atomic magnetic quantum number M is a fundamental quantity for the computation of atomic structure and spectra. It enables one to deduce the distribution $\mathscr{Q}(J)$ of

total angular momentum J, defining the sizes of the blocks of the Hamiltonian. Different techniques were provided in the past to obtain $\mathscr{P}(M)$, based on induction relations, group theory or q-binomial coefficients (or Gaussian polynomials). In this work, we proposed new recurrence relations for $\mathscr{P}(M)$, based on identities which have not been exploited for that purpose yet, such as a generalization to q-binomial coefficients of the so-called "Star of David" theorem. We also used summation formulas in the q-Pascal triangle and recent identities due to He, Shannon, and Shiue, as well as a theorem based on the q-Vandermonde formula. While these formulas are structurally related to coefficient extraction in known q-binomial identities, translating them into explicit recurrence relations provides a clear, practical prescription for evaluating $\mathscr{P}(M)$ step-by-step. The main advantage is not necessarily a drastic reduction in raw computational complexity, but rather a more direct and physically transparent link to the magnetic quantum number distributions, bypassing the need to perform the full algebraic extraction at each step.

In the future, we plan to derive further recurrence relations using the q-binomial theorem

$$\prod_{i=1}^{n}(1+q^{i-1}z)=\sum_{k=0}^{n}q^{k(k-1)/2}z^k\begin{bmatrix}n\\k\end{bmatrix}_q$$

as well as Heine's formula, the q-analogue of the negative binomial theorem:

$$\frac{1}{\prod_{i=1}^{n}(1-q^{i-1}z)}=\sum_{j=0}^{\infty}\begin{bmatrix}n+j-1\\j\end{bmatrix}_q z^j.$$

It would also be worth investigating the potential usefulness of the Star of David theorem for multinomials [37], especially in order to calculate the number of atomic configurations, as well as studying congruence properties of q-binomial coefficients [38], already observed in the distribution of angular momentum in atomic and nuclear physics [24].

Beyond their combinatorial and algebraic significance, the recurrence relations and evaluation frameworks developed in this work hold practical utility for computational atomic physics and spectroscopy. By enabling the rapid, incremental calculation of magnetic quantum number distributions and state counts without heavy polynomial expansions or direct factorial evaluations, these tools can support the evaluation of level densities and partition functions.

Summary of the notations

For the convenience of the reader, the principal symbols and notations used throughout this work are summarized below:

N: Number of fermions in the subshell.

j: Single-electron angular momentum (half-integer).

J: Total angular momentum quantum number.

$J_{\max}$: Maximum magnetic quantum number for N fermions in a single-j shell, equal to $N(2j+1-N)/2$.

M: Total magnetic quantum number.

$\mathscr{P}(M)$: Number of states with a total magnetic quantum number M.

$\mathscr{P}_{2j+1,N}(M)$: Number of states with a total magnetic quantum number M for N fermions in a single-j shell.

$\mathscr{Q}(J)$: Number of levels with total angular momentum J.

q: Deformation parameter in q-calculus.

$[n]_q$: q-analogue of the integer n, defined as $[n]_q = (1 - q^n)/(1 - q)$.

$\begin{bmatrix} n \\ k \end{bmatrix}_q$: Gaussian binomial coefficient (q-binomial coefficient).

$c_{n,k}(p)$: Coefficient of q^p in the expansion of the Gaussian polynomial $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

Use of Generative-AI tools declaration

The author declares that no generative Artificial Intelligence (AI) tools were used.

Conflict of interest

I declares no conflict of interest.

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