



Research article**Limit theorems for a resonant interacting elephant random walk with random amplitudes****Rafik Aguech* and Mohamed Abdelkader**

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Abstract: We study two interacting elephant random walks. The second has memory $\alpha_2 = 2p_2 - 1 \in (-1, 1]$: It repeats a uniformly sampled past step with probability p_2 and reverses it otherwise. With probability β , the first copies a past step of the second, multiplies it by an independent amplitude of mean m_L and second moment v , and repeats or reverses it with memory $\alpha_1 = 2p_1 - 1$; otherwise, it does the same with its own past and memory $\alpha_3 = 2p_3 - 1$. Its effective memory is $\gamma = \alpha_3(1 - \beta)$, and the mean coupling is $d = \beta\alpha_1 m_L$. In the resonant regime $\gamma = \alpha_2$, and under the moment assumptions stated in Theorem 3.1, the first position is driven by the source $d S_n^{(2)}/n$. If $\gamma > 1/2$ and $d \neq 0$, $S_n^{(1)}/(n^\gamma \log n)$ converges almost surely to dL , where L is the limit of the second walk. If $\gamma = 1/2$ and $d \neq 0$, $S_n^{(1)}/\sqrt{n(\log n)^3}$ converges in law to $N(0, d^2/3)$. If $-1 < \gamma < 1/2$ and $\mathbb{E}[|L_1|^{2+\delta}] < \infty$ for some $\delta > 0$, the $\sqrt{n}$ fluctuations are jointly Gaussian, and the variance of the first coordinate is $v/(1 - 2\gamma) + 2d^2/(1 - 2\gamma)^3$. If $d = 0$ and $\gamma = 1/2$, under a moment of order $2 + \delta$, the scale returns to $\sqrt{n \log n}$, with variance v . The case $\gamma \neq \alpha_2$ is left to a companion manuscript, which is not used in the proofs.

Keywords: elephant random walk; interacting random walks; random amplitudes; resonance; Jordan block; martingales; central limit theorem; law of the iterated logarithm

Mathematics Subject Classification: 60F05, 60F15, 60G42, 60G50, 60K35, 82C41

1. Introduction

The elephant random walk of Schütz and Trimper [1] is a one-dimensional walk with complete memory: At each step, the walker chooses one of its past increments uniformly and then either repeats it or inverts it. A phase transition is already present in this simple model. If the memory parameter is smaller than $1/2$, the walk is diffusive; if it is larger than $1/2$, the walk is super-diffusive. Martingale methods, as in Bercu [2] and Coletti, Gava, and Schütz [3], make this quite transparent. Among

the many extensions, Dedecker, Fan, Hu and Merlevède, [4] allowed the step size to be random, and Aguech and Qin [5] considered two elephants that copy one another. Reinforced processes are surveyed by Pemantle [6]; see also Metzler, Jeon, Cherstvy, and Barkai [7] and Masó-Puigdellosas, Campos, and Méndez [8] for related questions of anomalous diffusion.

A different interaction, through a pair of generalized Pólya urns, was studied in [9]. That paper did not treat random amplitudes, and every estimate used below is proved in Section 5. The nonresonant regime is left to the companion [10]. The reduction used there divides by $\gamma - \alpha_2$, and therefore stops on the resonant line $\gamma = \alpha_2$ studied here.

On that line, the conditional mean is the Jordan block

$$B = \begin{pmatrix} a & d \\ 0 & a \end{pmatrix}, \quad a = \alpha_2 = \gamma, \quad d = \beta\alpha_1 m_L.$$

If $d \neq 0$, then $B = aI + N$ with $N^2 = 0$ and $N \neq 0$. A block of size two produces the scales $\sqrt{n}(\log n)^{3/2}$ and $n^a \log n$, as in [11, Theorem 2.1] and in Theorems 12 and 14 of [12]. For random amplitudes, the innovation matrix is $\text{diag}(v, 1)$, and the variances are those of Theorem 3.1.

The endpoint $\alpha_2 = -1$ makes the normalizing product vanish and $\Gamma(1 + \alpha_2)$ undefined, so every statement assumes $\alpha_2 \in (-1, 1]$.

The paper is organized as follows. Section 2 recalls the model. The main results are stated in Section 3. Section 4 reports numerical checks designed for the corrected scales. Proofs are collected in Section 5.

2. The model

Let $(p_1, p_2, p_3, \beta) \in [0, 1]^4$, and set $\alpha_i = 2p_i - 1$ for $i = 1, 2, 3$ so that each α_i lies in $[-1, 1]$. The parameter $\gamma = \alpha_3(1 - \beta)$ is the effective memory of the first elephant. We consider independent families of random variables as follows:

- $(\xi_n^{(i)})_{n \geq 2}$ are identical and independently distributed (i.i.d.) Bernoulli random variables with parameter p_i , for $i = 1, 2, 3$;
- $(\eta_n)_{n \geq 2}$ are i.i.d. Bernoulli random variables with parameter β ;
- $(u_n^{(i)})_{n \geq 1}$ and $(v_n)_{n \geq 1}$ are uniformly distributed on $\{1, \dots, n\}$;
- $(L_n)_{n \geq 1}$ are i.i.d., with mean m_L and $\mathbb{E}[L_1^2] = v = \sigma_L^2 + m_L^2 < \infty$.

All these sequences are independent of one another. No fourth moment of L_1 is required. The superdiffusive limit, and the critical limit with $d \neq 0$ use only $\mathbb{E}[L_1^2] < \infty$. Wherever the noise of the first elephant is of leading order, the central limit theorem assumes $\mathbb{E}[|L_1|^{2+\delta}] < \infty$ for some $\delta > 0$, and a law of the iterated logarithm driven by that noise assumes that L_1 is almost surely bounded.

The positions are $S_0^{(i)} = 0$ and $S_n^{(i)} = \sum_{k=1}^n X_k^{(i)}$ for $i = 1, 2$. Let $\varepsilon_1^{(1)}$ and $\varepsilon_1^{(2)}$ be independent Rademacher random variables, independent of L_1 and of every sequence introduced above, and set

$$X_1^{(2)} = \varepsilon_1^{(2)}, \quad X_1^{(1)} = \varepsilon_1^{(1)} L_1.$$

Thus, $\mathbb{E}[(X_1^{(1)})^2] = v$, and $\mathbb{E}[X_1^{(1)}] = 0$. For $n \geq 1$, the increments are given by

$$X_{n+1}^{(2)} = (2\xi_{n+1}^{(2)} - 1) X_{u_n^{(1)}}^{(2)}, \quad (2.1)$$

$$X_{n+1}^{(1)} = \eta_{n+1}(2\xi_{n+1}^{(1)} - 1)L_{n+1}X_{u_n^{(2)}}^{(2)} + (1 - \eta_{n+1})(2\xi_{n+1}^{(3)} - 1)X_{v_n}^{(1)}.$$

Thus, the second elephant is a classical elephant random walk of memory α_2 , with steps in $\{\pm 1\}$. The first elephant copies the second one with probability β after multiplication by the fresh amplitude L_{n+1} and otherwise copies itself. Let

$$\mathcal{F}_n = \sigma(X_1^{(1)}, \dots, X_n^{(1)}, X_1^{(2)}, \dots, X_n^{(2)}).$$

Taking conditional expectations in (2.1) gives

$$\begin{aligned}\mathbb{E}[X_{n+1}^{(1)} | \mathcal{F}_n] &= \gamma \frac{S_n^{(1)}}{n} + \beta\alpha_1 m_L \frac{S_n^{(2)}}{n}, \\ \mathbb{E}[X_{n+1}^{(2)} | \mathcal{F}_n] &= \alpha_2 \frac{S_n^{(2)}}{n}.\end{aligned}\tag{2.2}$$

In matrix form, with $a = \alpha_2$ and $d = \beta\alpha_1 m_L$, resonance $\gamma = a$ means

$$\mathbb{E}[S_{n+1} - S_n | \mathcal{F}_n] = \frac{1}{n} \mathbf{B} S_n, \quad \mathbf{B} = \begin{pmatrix} a & d \\ 0 & a \end{pmatrix}, \quad S_n = \begin{pmatrix} S_n^{(1)} \\ S_n^{(2)} \end{pmatrix}.\tag{2.3}$$

For an arbitrary constant c , the difference $A_n = S_n^{(1)} - c S_n^{(2)}$ satisfies

$$\mathbb{E}[A_{n+1} - A_n | \mathcal{F}_n] = \frac{a A_n + d S_n^{(2)}}{n}.\tag{2.4}$$

The choice $c = d/a$, available only for $a \neq 0$, does not delete the forcing term $d S_n^{(2)}/n$.

3. Main results

Throughout this section, we assume $\beta \in (0, 1)$ and $\gamma = \alpha_2 = a$, with $a \in (-1, 1]$. The value $a = -1$ is excluded. The value $a = 0$ is included; it is not singular. The ratio $c = d/a$ is not used in the statements. Set $d = \beta\alpha_1 m_L$ and $v = \sigma_L^2 + m_L^2$ (See Table 1 for the meanings of these parameters). When $a > 1/2$, Bercu's theorem for the classical elephant walk gives $n^{-a} S_n^{(2)} \rightarrow L$ almost surely, for a finite, nondegenerate random variable L ; the hypotheses are recorded in Lemma 5.4.

Table 1. Notation used throughout the paper.

Symbol	Meaning
p_1, p_2, p_3	Probabilities of repeating a copied step
$\alpha_i = 2p_i - 1$	Memory parameters, $i = 1, 2, 3$
β	Probability that the first elephant copies the second one
L_n	i.i.d. amplitudes, mean m_L , second moment v
$\gamma = \alpha_3(1 - \beta)$	Effective memory of the first elephant
$a = \alpha_2 = \gamma$	Common resonant memory
$d = \beta\alpha_1 m_L$	Mean coupling; the Jordan block is nontrivial when $d \neq 0$
$\mathcal{F}_n$	History of both walkers up to time n

Theorem 3.1. Consider the walk $(S_n^{(1)}, S_n^{(2)})_{n \geq 0}$ of Section 2, with $\beta \in (0, 1)$, $\gamma = \alpha_2 = a \in (-1, 1]$, and $d = \beta\alpha_1 m_L$ (See Table 1).

(1) If $a > 1/2$ and $d \neq 0$, and if $\mathbb{E}[L_1^2] < \infty$, then

$$\frac{S_n^{(1)}}{n^a \log n} \xrightarrow{\text{a.s.}} dL.$$

If $a > 1/2$ and $d = 0$, then $S_n^{(1)}/(n^a \log n) \rightarrow 0$ almost surely, and

$$\frac{S_n^{(1)}}{n^a} \xrightarrow{\text{a.s.}} \frac{S_1^{(1)} + Y_\infty^{(1)}}{\Gamma(1+a)},$$

where $Y_\infty^{(1)}$ is the almost sure and L^2 limit of the martingale (5.4). If $v > 0$, this limit is non-degenerate.

(2) If $a = 1/2$ and $d \neq 0$, and if $\mathbb{E}[L_1^2] < \infty$, then

$$\frac{S_n^{(1)}}{\sqrt{n(\log n)^3}} \xrightarrow{d} \mathcal{N}\left(0, \frac{d^2}{3}\right).$$

If in addition, $\mathbb{E}[|L_1|^{2+\delta}] < \infty$ for some $\delta > 0$, then

$$\limsup_{n \rightarrow \infty} \frac{|S_n^{(1)}|}{\sqrt{2n(\log n)^3 \log \log \log n}} = \frac{|d|}{\sqrt{3}} \quad \text{a.s.}$$

If $a = 1/2$ and $d = 0$, and if $\mathbb{E}[|L_1|^{2+\delta}] < \infty$ for some $\delta > 0$, then

$$\frac{S_n^{(1)}}{\sqrt{n \log n}} \xrightarrow{d} \mathcal{N}(0, v).$$

If, moreover, L_1 is almost surely bounded,

$$\limsup_{n \rightarrow \infty} \frac{|S_n^{(1)}|}{\sqrt{2n \log n \log \log \log n}} = \sqrt{v} \quad \text{a.s.}$$

(3) If $-1 < a < 1/2$, including $a = 0$, and if $\mathbb{E}[|L_1|^{2+\delta}] < \infty$ for some $\delta > 0$, set $r = 1 - 2a > 0$. Then,

$$\frac{1}{\sqrt{n}} \begin{pmatrix} S_n^{(1)} \\ S_n^{(2)} \end{pmatrix} \xrightarrow{d} \mathcal{N}(0, \Sigma),$$

where

$$\Sigma = \begin{pmatrix} \frac{v}{r} + \frac{2d^2}{r^3} & \frac{d}{r^2} \\ \frac{d}{r^2} & \frac{1}{r} \end{pmatrix}. \quad (3.1)$$

If L_1 is almost surely bounded,

$$\limsup_{n \rightarrow \infty} \frac{|S_n^{(1)}|}{\sqrt{2n \log \log n}} = \sqrt{\Sigma_{11}} \quad \text{a.s.}$$

At $a = 0$, one has $\Sigma_{11} = v + 2d^2$. The determinant equals $v/r^2 + d^2/r^4$, so Σ is positive definite if and only if $v > 0$. If $v = 0$, then $L_1 = 0$ almost surely, $d = 0$, and $\Sigma = \text{diag}(0, 1/r)$.

Remark 3.2. The factor $\Gamma(3/2)$ cancels between the quadratic variation and the normalizing product. For the classical critical walk $L_n \equiv 1$, $d = 0$, and $\nu = 1$, Bercu's constant is 1, whereas $\Gamma(3/2) = \sqrt{\pi}/2$. The diffusive law of the iterated logarithm is read from the single quadratic variation $\Sigma_{11}n$. Adding the marginal constants $\sqrt{\nu/r}$ and $|d|\sqrt{2/r^3}$ overestimates it.

4. Numerical illustrations

In the numerical experiments, each random walk consists of $N = 10,000$ steps, and $M = 1000$ independent trajectories are simulated. The amplitudes L_n follow an exponential distribution with rate parameter 1. Three regimes are considered: superdiffusive, with $a = 0.7$, $d = 0.3$, and seed 20260; critical, with $a = 0.5$, $d = 0.5$, and seed 20260; and diffusive, with $a = 0.2$, $d = 0.8$, and seed 20260. In all cases, the parameters satisfy $\gamma = \alpha_2 = a$. These simulations are used to compare the asymptotic behavior of the interacting elephant random walk across the three regimes. The parameter settings are summarized in Table 2.

Table 2. Parameter values used in the corrected experiments. In every row, $\gamma = \alpha_2$ and $L_n \equiv 1$.

Case	p_1	p_2	p_3	β	a	d	Seed
Superdiffusive	1	0.85	1	0.3	0.7	0.3	20260
Critical	1	0.75	1	0.5	0.5	0.5	20260
Diffusive	1	0.6	1	0.8	0.2	0.8	20260

Table 3 provides numerical evidence for the critical-regime results in Theorem 3.1(ii). For $d = 0.5$, the normalized second moment $\mathbb{E}_M[(S_n^{(1)})^2]/[n(\log n)^3]$ decreases from 0.162 to 0.125 as n increases to 10,000, consistent with the predicted normalization and theoretical limit $d^2/3 = 1/12$. In contrast, normalization by $n \log n$ yields increasing values, reflecting the additional logarithmic factor in the variance. For $d = 0$, the normalized second moment $\mathbb{E}_M[(S_n^{(1)})^2]/(n \log n)$ remains close to its theoretical limit $\nu = 2$, reaching 2.187 at $n = 10,000$, whereas normalization by $n(\log n)^3$ yields decreasing values toward zero. These observations highlight the importance of selecting the appropriate normalization in each critical case. Although some discrepancies remain, the numerical results are consistent with the distinct asymptotic behaviors predicted by Theorem 3.1(ii).

Table 3. Critical second moments for $\nu = 2$, $M = 10^3$, and seed 20260, comparing $d \neq 0$ ($d = 0.5$) and $d = 0$. For $d \neq 0$, the limit of the third column is $d^2/3 = 1/12$.

n	$d = 1/2$		$d = 0$	
	$\frac{\mathbb{E}_M[(S_n^{(1)})^2]}{(n \log n)}$	$\frac{\mathbb{E}_M[(S_n^{(1)})^2]}{(n(\log n)^3)}$	$\frac{\mathbb{E}_M[(S_n^{(1)})^2]}{(n \log n)}$	$\frac{\mathbb{E}_M[(S_n^{(1)})^2]}{(n(\log n)^3)}$
500	6.248	0.162	2.254	0.056
1,000	7.090	0.149	2.201	0.044
2,000	8.047	0.139	2.213	0.036
4,000	9.094	0.132	2.207	0.031
6,000	9.755	0.129	2.214	0.027
8,000	10.222	0.127	2.211	0.026
10,000	10.369	0.125	2.187	0.025

Table 4 reports the empirical variances and covariance of the random walks $\frac{1}{\sqrt{n}}(S_n^{(1)}, S_n^{(2)})$, for different values of n . These numerical results illustrate Point (3) of Theorem 3.1, which establishes the convergence in distribution of this vector to a centered bivariate normal distribution with covariance matrix Σ . The empirical covariance estimates exhibit moderate fluctuations as n increases, while remaining within relatively narrow ranges. This behavior provides preliminary numerical support for the Gaussian limit theorem and its associated covariance structure. A more direct assessment of the theoretical prediction can be obtained by comparing the empirical covariance estimates with the corresponding entries of Σ for the parameter values used in the simulations. In the diffusive run, $N = 5000$, and $M = 1000$. Here, $r = 0.6$, $d = 0.8$, $v = 2$, so the elements of the theoretical covariance matrix are given by:

$$\Sigma_{11} = \frac{1}{0.6} + \frac{2(0.8)^2}{(0.6)^3} = 9.259, \quad \Sigma_{12} = 2.222, \quad \Sigma_{22} = 1.667.$$

Table 4. The empirical covariance matrix of $\frac{1}{\sqrt{n}}(S_n^{(1)}, S_n^{(2)})$ for different values of n .

n	$\text{Var}(S_n^{(1)})$	$\text{Var}(S_n^{(2)})$	$\text{Cov}(S_n^{(1)}, S_n^{(2)})$
1000	8.313	1.636	1.975
2000	8.484	1.637	2.031
4000	8.637	1.572	1.984
6000	8.837	1.588	1.924
8000	8.875	1.525	1.990
10000	8.772	1.481	1.960

Figure 1 illustrates the asymptotic regimes described in Theorem 3.1. In the superdiffusive case $a = 0.7$, the first panel shows the growth of $S_n^{(1)}/n^a$, consistent with the almost sure convergence to a finite random limit. For the critical case $a = 0.5$ with $d \neq 0$, the empirical normalized second moment $\mathbb{E}[(S_n^{(1)})^2]/[n(\log n)^3]$ approaches the theoretical value $d^2/3 = 1/12$, as shown in the upper-middle panel. The upper-right panel displays the corresponding unnormalized second moment, which increases with n . When $a = 1/2$ and $d = 0$, the empirical normalized second moment $\mathbb{E}[(S_n^{(1)})^2]/(n \log n)$ remains close to the theoretical value $v = 2$, whereas normalization by $n(\log n)^3$ yields a quantity tending towards zero, as predicted by the theorem. Finally, in the diffusive regime $a = 0.2$, the empirical density of the normalized first coordinate $S_n^{(1)}/\sqrt{n}$ closely matches the corresponding Gaussian density, supporting the stated central limit theorem. In the diffusive run, $N = 5000$, and $M = 1000$.

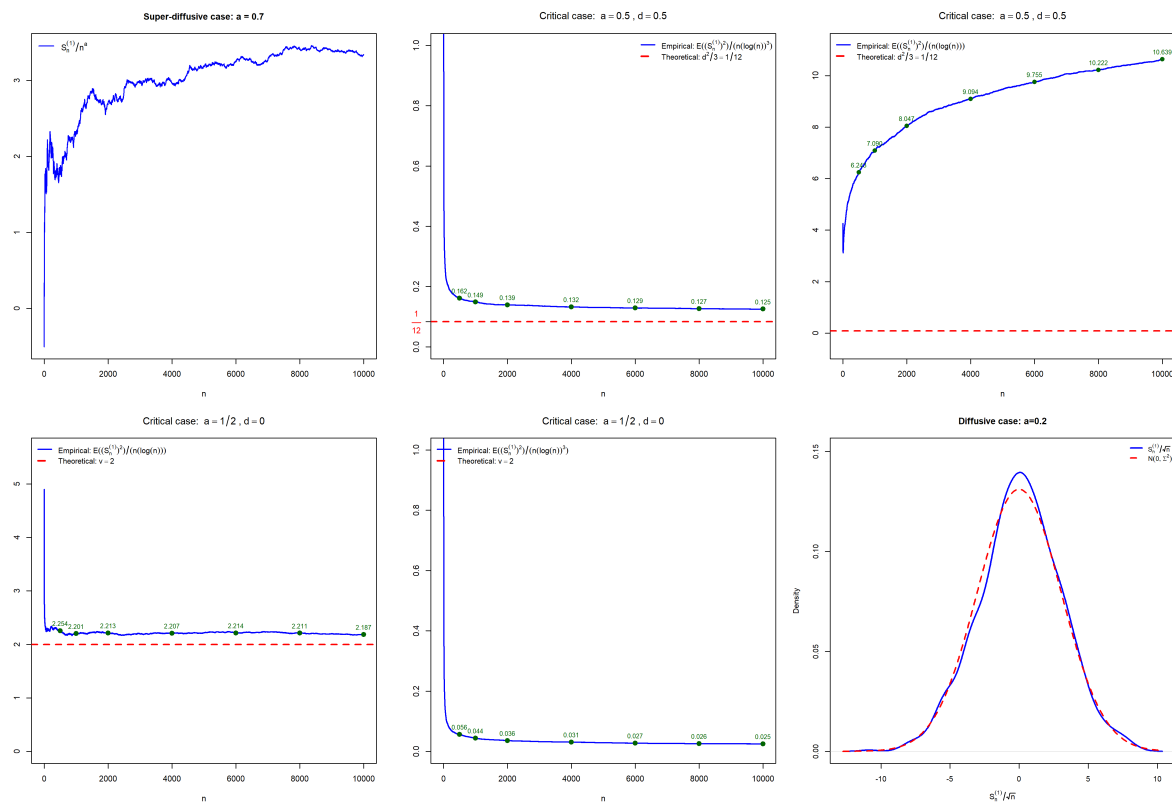


Figure 1. Numerical illustration of the asymptotic regimes in Theorem 3.1: Superdiffusive, critical, and diffusive cases when $N = 10,000$, and $M = 1000$.

5. Proofs

5.1. Preliminaries

Equation (2.2) is the one-step recursion. Summing it from the first step, and keeping the initial positions, gives

$$\begin{aligned} S_n^{(1)} &= S_1^{(1)} + a \sum_{k=1}^{n-1} \frac{S_k^{(1)}}{k} + d \sum_{k=1}^{n-1} \frac{S_k^{(2)}}{k} + M_n^{(1)}, \\ S_n^{(2)} &= S_1^{(2)} + a \sum_{k=1}^{n-1} \frac{S_k^{(2)}}{k} + M_n^{(2)}, \end{aligned} \quad (5.1)$$

where $M_1^{(i)} = 0$, and

$$M_n^{(i)} = \sum_{k=1}^{n-1} (X_{k+1}^{(i)} - \mathbb{E}[X_{k+1}^{(i)} | \mathcal{F}_k]).$$

For $n = 1$, both sums are empty, and $M_1^{(i)} = 0$, so both sides equal $S_1^{(i)}$.

For $a > -1$, define $\phi_1 = 1$ and

$$\phi_n = \prod_{k=1}^{n-1} \left(1 + \frac{a}{k}\right) = \frac{\Gamma(n+a)}{\Gamma(n)\Gamma(1+a)}, \quad n \geq 2. \quad (5.2)$$

Then, $\phi_n \sim n^a/\Gamma(1+a)$. Iterating the conditional means yields

$$\begin{aligned} S_n^{(2)} &= \phi_n(S_1^{(2)} + Y_n^{(2)}), \\ S_n^{(1)} &= \phi_n(S_1^{(1)} + Y_n^{(1)}) + d\phi_n \sum_{j=1}^{n-1} \frac{S_1^{(2)} + Y_j^{(2)}}{j+a}, \end{aligned} \quad (5.3)$$

where

$$Y_n^{(i)} = \sum_{j=1}^{n-1} \frac{\Delta M_{j+1}^{(i)}}{\phi_{j+1}}, \quad \Delta M_{j+1}^{(i)} = M_{j+1}^{(i)} - M_j^{(i)}. \quad (5.4)$$

Each $(Y_n^{(i)})$ is a martingale.

Lemma 5.1. Write $Q_n = n^{-1} \sum_{i=1}^n (X_i^{(1)})^2$. Then,

$$\mathbb{E}[(X_{n+1}^{(2)})^2 \mid \mathcal{F}_n] = 1, \quad (5.5)$$

$$\mathbb{E}[(X_{n+1}^{(1)})^2 \mid \mathcal{F}_n] = \beta v + (1-\beta)Q_n, \quad (5.6)$$

$$\mathbb{E}[X_{n+1}^{(1)} X_{n+1}^{(2)} \mid \mathcal{F}_n] = ad \left(\frac{S_n^{(2)}}{n} \right)^2 + a^2 \frac{S_n^{(1)} S_n^{(2)}}{n^2}. \quad (5.7)$$

Consequently, the conditional covariance $\mathbb{E}[\Delta M_{n+1}^{(1)} \Delta M_{n+1}^{(2)} \mid \mathcal{F}_n]$ vanishes for every n , not only in the limit: The conditional second-moment matrix of the innovations is exactly diagonal. On the event $\{S_n/n \rightarrow 0\}$ and $\{Q_n \rightarrow v\}$, its diagonal entries converge to v and 1.

Proof. The second elephant selects a past step of size ± 1 and multiplies it by an independent sign of mean a , so its conditional second moment equals 1. The first elephant selects the second walk with probability β and then multiplies by a fresh amplitude, which contributes βv , or else selects one of its own past steps uniformly, which contributes $(1-\beta)Q_n$. For the cross moment, independence of $(\eta_{n+1}, \xi_{n+1}^{(1)}, \xi_{n+1}^{(2)}, \xi_{n+1}^{(3)}, L_{n+1})$ from $\mathcal{F}_n$ gives

$$\mathbb{E}[X_{n+1}^{(1)} X_{n+1}^{(2)} \mid \mathcal{F}_n] = \beta \alpha_1 m_L \cdot a \cdot \left(\frac{S_n^{(2)}}{n} \right)^2 + (1-\beta) \alpha_3 \cdot a \cdot \frac{S_n^{(1)} S_n^{(2)}}{n}.$$

Resonance $(1-\beta)\alpha_3 = a$ and the definition of d give (5.7). The product of the conditional means is the same expression, so $\mathbb{E}[\Delta M_{n+1}^{(1)} \Delta M_{n+1}^{(2)} \mid \mathcal{F}_n] = 0$ for every n . On $\{S_n/n \rightarrow 0\}$ and $\{Q_n \rightarrow v\}$, the conditional variances tend to v and 1. $\square$

Remark 5.2. If $a \neq 0$ and $c = d/a$, the conditional covariance between $\Delta M_{n+1}^{(1)} - c \Delta M_{n+1}^{(2)}$ and $\Delta M_{n+1}^{(2)}$ tends to $-c$ on $\{S_n^{(2)}/n \rightarrow 0\}$. This only records the subtraction of $c \Delta M_{n+1}^{(2)}$. The drift (2.4) still contains the source $d S_n^{(2)}/n$, which produces the term $2d^2/r^3$ in Σ_{11} and the critical scale $\sqrt{n(\log n)^3}$.

Lemma 5.3. Assume $\beta \in (0, 1)$ and $\mathbb{E}[|L_1|^2] < \infty$ for some $\delta \in (0, 2]$. Then, $Q_n \rightarrow v$ almost surely, and $\sup_n \mathbb{E}[|X_n^{(1)}|^2] \leq \mathbb{E}[|L_1|^2]$.

Proof. From (2.1), we have

$$\mathbb{E}[|X_{n+1}^{(1)}|^2] \leq \beta \mathbb{E}[|L_1|^2] + (1 - \beta) \mathbb{E}[|X_{v_n}^{(1)}|^2],$$

then,

$$\sup_n \mathbb{E}[|X_n^{(1)}|^2] \leq \beta \mathbb{E}[|L_1|^2] + (1 - \beta) \sup_n \mathbb{E}[|X_n^{(1)}|^2].$$

Then

$$\sup_n \mathbb{E}[|X_n^{(1)}|^2] \leq \mathbb{E}[|L_1|^2].$$

Set $\delta_n = Q_n - v$ and $\Delta_{n+1} = Y_{n+1} - \mu_n$, where $Y_n = (X_n^{(1)})^2$ and $\mu_n = \mathbb{E}[Y_{n+1} | \mathcal{F}_n] = \beta v + (1 - \beta)Q_n$. Let $\pi_1 = 1$ and $\pi_n = \prod_{k=2}^n (1 - \beta/k)$ for $n \geq 2$, so that $\pi_n \sim \kappa n^{-\beta}$ with $\kappa = 1/\Gamma(2 - \beta)$. Then,

$$\delta_n = \pi_n \delta_1 + \sum_{j=1}^{n-1} \lambda_{n,j} \Delta_{j+1}, \quad \lambda_{n,j} = \frac{\pi_n}{(j+1)\pi_{j+1}}.$$

The convergence $n^\beta \pi_n \rightarrow \kappa$ gives a finite constant C such that $\lambda_{n,j} \leq C j^{\beta-1} n^{-\beta}$ for every $n > j \geq 1$. Because $\mathbb{E}[Q_n] \leq v$, and $S_n = nQ_n$ is nondecreasing, $\mathbb{P}(Q_{2^k} > (k+1)^2) \leq v/(k+1)^2$. Thus, $Q_{2^k} = O(k^2)$ almost surely. For $2^k \leq n \leq 2^{k+1}$, $Q_n \leq 2Q_{2^{k+1}}$, so $Q_n = O((\log n)^2)$ almost surely, and likewise, $\mu_n = O((\log n)^2)$. In particular, $\mu_n \leq n$ for every large n . Also, $L_n^2/n \rightarrow 0$ almost surely; hence, $\max_{m \leq n} L_m^2 = o(n)$, and $\max_{i \leq n} Y_i = o(n)$. Define $\tau(1) = 1$, and $\tau(n+1) = n+1$ if $\eta_{n+1} = 1$ and $\tau(n+1) = \tau(v_n)$ if $\eta_{n+1} = 0$. Then, $Y_n = L_{\tau(n)}^2$, and

$$\sum_{i: \tau(i)=m} \frac{1}{i^2} \leq \sum_{i \geq m} \frac{1}{i^2} \leq \frac{C}{m^2}.$$

Therefore, $\sum_i Y_i^2/i^2 \leq C \sum_m L_m^4/m^2$. The series on the right converges almost surely: $\sum_m \mathbb{P}(L_m^2 > m) < \infty$, and

$$\sum_{m \geq 1} m^{-2} \mathbb{E}[L_1^4 \mathbf{1}_{\{L_1^2 \leq m\}}] \leq C \mathbb{E}[L_1^2] < \infty.$$

Hence, $\sum_j j^{-3} \sum_{i=1}^j Y_i^2 < \infty$ almost surely. For every large j ,

$$\mathbb{E}[\Delta_{j+1}^2 \mathbf{1}_{\{|\Delta_{j+1}| \leq j+1\}} | \mathcal{F}_j] \leq 2\beta \mathbb{E}[L_1^4 \mathbf{1}_{\{L_1^2 \leq 2(j+1)\}}] + 2(1 - \beta) j^{-1} \sum_{i=1}^j Y_i^2 \mathbf{1}_{\{Y_i \leq 2(j+1)\}} + 2\mu_j^2.$$

Dividing by $(j+1)^2$ and summing gives a finite sum almost surely. Set

$$c_{j+1} = \mathbb{E}[\Delta_{j+1} \mathbf{1}_{\{|\Delta_{j+1}| > j+1\}} | \mathcal{F}_j], \quad \Delta_{j+1}^* = \Delta_{j+1} \mathbf{1}_{\{|\Delta_{j+1}| \leq j+1\}} + c_{j+1}.$$

For every large j , one has $\mu_j \leq j/2$ and $\max_{i \leq j} Y_i \leq j/2$, so a copied step cannot satisfy $|\Delta_{j+1}| > j+1$ and $0 \leq c_{j+1} \leq \beta \mathbb{E}[L_1^2 \mathbf{1}_{\{L_1^2 > j\}}]$. The deterministic majorant is square-summable against j^{-2} . Thus,

$$\sum_{j \geq 1} \frac{\mathbb{E}[(\Delta_{j+1}^*)^2 | \mathcal{F}_j]}{(j+1)^2} < \infty$$

almost surely, and the martingale $H_n^* = \sum_{j=1}^{n-1} \Delta_{j+1}^*/(j+1)$ converges almost surely to a finite limit H_∞^* . Summation by parts gives

$$\sum_{j=1}^{n-1} \lambda_{n,j} \Delta_{j+1}^* = H_n^* - \pi_n \sum_{j=2}^{n-1} H_j^* \left(\frac{1}{\pi_{j+1}} - \frac{1}{\pi_j} \right).$$

The coefficients $1/\pi_{j+1} - 1/\pi_j$ are positive and telescope. Writing $H_j^* = H_\infty^* + \varepsilon_j$ with $\varepsilon_j \rightarrow 0$, the head of the sum is multiplied by π_n , and the tail is at most a constant multiple of $\sup_{j \geq J} |\varepsilon_j|$. Hence, $\sum_{j=1}^{n-1} \lambda_{n,j} \Delta_{j+1}^* \rightarrow 0$ almost surely. On the same event, $\mathbb{P}(|\Delta_{j+1}| > j+1 \mid \mathcal{F}_j) \leq \beta \mathbb{P}(L_1^2 > j)$ for every large j . The conditional Borel–Cantelli lemma gives $|\Delta_{j+1}| > j+1$ for only finitely many j . Because $\lambda_{n,j} \rightarrow 0$ for each fixed j ,

$$\sum_{j=1}^{n-1} \lambda_{n,j} \Delta_{j+1} \mathbf{1}_{\{|\Delta_{j+1}| > j+1\}} \rightarrow 0$$

almost surely. Finally $u_j = \mathbb{E}[L_1^2 \mathbf{1}_{\{L_1^2 > j\}}]$ decreases to zero, and

$$\sum_{j=1}^{n-1} \lambda_{n,j} u_{j+1} \leq C n^{-\beta} \sum_{j=1}^{n-1} j^{\beta-1} u_{j+1} \rightarrow 0,$$

so $\sum_j \lambda_{n,j} c_{j+1} \rightarrow 0$ almost surely. Therefore, $\sum_j \lambda_{n,j} \Delta_{j+1} \rightarrow 0$. Combined with $\pi_n \rightarrow 0$, this proves $\delta_n \rightarrow 0$ almost surely. $\square$

Lemma 5.4. *The second coordinate $(S_n^{(2)})$ is a classical elephant random walk of memory $a \in (-1, 1]$: its steps lie in $\{\pm 1\}$ and $\mathbb{E}[X_{n+1}^{(2)} \mid \mathcal{F}_n] = a S_n^{(2)}/n$. The following statements from [2] are used for $(S_n^{(2)})$ only. If $a \in (1/2, 1]$, then $S_n^{(2)}/n^a \rightarrow L$ almost surely, and L is nondegenerate [2, Theorem 3.7]. If $a = 1/2$, then $S_n^{(2)}/\sqrt{n \log n} \rightarrow \mathcal{N}(0, 1)$ [2, Theorem 3.6], and*

$$\limsup_{n \rightarrow \infty} \frac{|S_n^{(2)}|}{\sqrt{2n \log n \log \log n}} = 1 \quad a.s.$$

[2, Theorem 3.5]. If $-1 < a < 1/2$, then $S_n^{(2)}/\sqrt{n} \rightarrow \mathcal{N}(0, 1/(1-2a))$ [2, Theorem 3.3], and

$$\limsup_{n \rightarrow \infty} \frac{|S_n^{(2)}|}{\sqrt{2n \log \log n}} = \frac{1}{\sqrt{1-2a}} \quad a.s.$$

[2, Theorem 3.2]. They do not transfer to $S^{(1)}$: Its steps are not of size ± 1 , and the drift is not one-dimensional when $d \neq 0$.

Proof. The first step satisfies $|X_1^{(2)}| = 1$. If $|X_k^{(2)}| = 1$ for every $k \leq n$, then $X_{n+1}^{(2)}$ has absolute value 1, because it is a past step of the second elephant multiplied by ± 1 . Thus, every step lies in $\{\pm 1\}$. The sign $2\xi_{n+1}^{(2)} - 1$ has mean $a = 2p_2 - 1$ and is independent of $\mathcal{F}_n$, and the index $u_n^{(1)}$ is uniform on $\{1, \dots, n\}$ and independent of $\mathcal{F}_n$. Hence,

$$\mathbb{E}[X_{n+1}^{(2)} \mid \mathcal{F}_n] = a \cdot \frac{1}{n} \sum_{k=1}^n X_k^{(2)} = a \frac{S_n^{(2)}}{n}.$$

This is the defining relation of Bercu's walk [2, Equation (2.2)]. The memory parameter of that walk is $p_2 = (1 + a)/2$, and the first step is symmetric. For $a \in (-1, 1]$, one has $p_2 \in (0, 1]$, so the three regimes above are $p_2 \in (3/4, 1]$, $p_2 = 3/4$, and $p_2 \in (0, 3/4)$. In each regime, $1 - 2a = 3 - 4p_2$, which converts Bercu's variance $1/(3 - 4p_2)$ and his iterated-logarithm constant $1/\sqrt{3 - 4p_2}$ into the expressions stated in the lemma. $\square$

We also record the form of Stout's theorem that is applied below.

Lemma 5.5. *Let $(N_n)_{n \geq 0}$ be a square-integrable martingale with $N_0 = 0$ and predictable quadratic variation $s_n^2 = \langle N \rangle_n$. Suppose that $s_n^2 \rightarrow \infty$ that*

$$\frac{|\Delta N_n| \sqrt{\log \log s_n^2}}{s_n} \leq \varepsilon_n \quad a.s.$$

for a deterministic sequence $\varepsilon_n \rightarrow 0$. Then,

$$\limsup_{n \rightarrow \infty} \frac{N_n}{\sqrt{2s_n^2 \log \log s_n^2}} = 1 \quad a.s.,$$

and the corresponding $\liminf$ equals -1 [13]. The limit runs over all times n . Along a sub-sequence the upper bound persists, but the matching lower bound does not follow. In each application below, the increment condition is checked for that martingale. When L_1 is almost surely bounded, the increments are of size at most $(\log n)/\phi_n$ or $1/\phi_n$, and the comparison uses the quadratic variation of the martingale at hand.

Lemma 5.6. *Let $(\eta_k, \mathcal{F}_k)_{k \geq 1}$ be martingale differences with $|\eta_k| \leq Ck^{-1/2}$ and $\mathbb{E}[\eta_{k+1}^2 | \mathcal{F}_k] \sim \tau k^{-1}$ almost surely, for constants $C < \infty$ and $\tau > 0$. Set*

$$Z_n = \sum_{k=1}^{n-1} \log\left(\frac{n}{k}\right) \eta_{k+1}, \quad v_n = \sum_{k=1}^{n-1} \log\left(\frac{n}{k}\right)^2 \mathbb{E}[\eta_{k+1}^2 | \mathcal{F}_k].$$

Then, $v_n \sim \tau(\log n)^3/3$ almost surely, and

$$\limsup_{n \rightarrow \infty} \frac{|Z_n|}{\sqrt{2v_n \log \log v_n}} = 1 \quad a.s.$$

In particular, $\log \log v_n \sim \log \log \log n$.

Proof. The substitution $u = \log(n/x)$ gives

$$\sum_{k < n} \frac{\log(n/k)^2}{k} \sim \int_1^n \frac{\log(n/x)^2}{x} dx = \frac{(\log n)^3}{3}.$$

The hypothesis $\mathbb{E}[\eta_{k+1}^2 | \mathcal{F}_k] \sim \tau k^{-1}$ therefore yields $v_n \sim v_n^*$ almost surely, where $v_n^* = \tau(\log n)^3/3$. In particular, $\log \log v_n \sim \log \log \log n$. Write $\lambda_n^* = \sqrt{2v_n^* \log \log v_n^*}$ and $\lambda_n = \sqrt{2v_n \log \log v_n}$. The ratio λ_n/λ_n^* tends to 1 almost surely, so it is enough to prove the limsup with λ_n^* .

For each fixed n , the sums $Z_{n,m} = \sum_{k \leq m} \log(n/k) \eta_{k+1}$ form a martingale in m . Their increments are bounded by $K_n = C \log n$. Because λ_n^* is of order $(\log n)^{3/2}(\log \log \log n)^{1/2}$, one has $K_n \lambda_n^*/v_n^* \rightarrow 0$.

Freedman's inequality [14] gives, for a martingale with increments bounded by K and predictable variation at most b ,

$$\Pr(|S| \geq a, \langle S \rangle \leq b) \leq 2 \exp\left(-\frac{a^2}{2(b + Ka)}\right).$$

Fix $\varepsilon \in (0, 1)$ and $\theta \in (1, 1 + \varepsilon)$, and set $n_j = \lfloor \exp(\theta^j) \rfloor$. Then, $\log v_{n_j}^* = 3j \log \theta + O(1)$, so $\log \log v_{n_j}^* \sim \log j$. Apply Freedman's inequality to the row at time n_j with threshold $a = (1 + \varepsilon)\lambda_{n_j}^*$ and bound $b = (1 + \varepsilon)v_{n_j}^*$.

Expanding the two definitions,

$$a^2 = (1 + \varepsilon)^2 \cdot 2v_{n_j}^* \log \log v_{n_j}^*, \quad 2b = 2(1 + \varepsilon)v_{n_j}^*,$$

and therefore,

$$\frac{a^2}{2b} = (1 + \varepsilon) \log \log v_{n_j}^*.$$

The denominator in Freedman's exponent is $2(b + K_{n_j}a)$, and $K_{n_j}a/b \rightarrow 0$. Because $\log \log v_{n_j}^* = \log j + O(1)$,

$$\frac{a^2}{2(b + K_{n_j}a)} = \frac{a^2/(2b)}{1 + K_{n_j}a/b} \geq \left(1 + \frac{\varepsilon}{2}\right) \log j$$

for every large j . Hence,

$$\mathbb{P}(|Z_{n_j}| > (1 + \varepsilon)\lambda_{n_j}^*, v_{n_j} \leq (1 + \varepsilon)v_{n_j}^*) \leq 2j^{-(1+\varepsilon/2)}$$

for every large j . The series converges. Borel–Cantelli gives only finitely many such joint events, almost surely. Combined with $v_n/v_n^* \rightarrow 1$, this yields

$$\limsup_{j \rightarrow \infty} \frac{|Z_{n_j}|}{\lambda_{n_j}^*} \leq 1 + \varepsilon \quad \text{a.s.}$$

For $n_j \leq n \leq n_{j+1}$, split the sum at n_j :

$$Z_n = Z_{n_j} + \log(n/n_j) M_{n_j} + R_{n,j},$$

where $M_n = \sum_{k < n} \eta_{k+1}$ and $R_{n,j} = \sum_{n_j \leq k < n} \log(n/k) \eta_{k+1}$. Here, $\log(n/n_j) \leq (\theta - 1) \log n_j$. The martingale M has the predictable variation $\langle M \rangle_n \sim \tau \log n \rightarrow \infty$. Its increments satisfy $|\eta_n| \leq Cn^{-1/2}$. Since $\langle M \rangle_n \sim \tau \log n$,

$$\frac{|\eta_n| \sqrt{\log \log \langle M \rangle_n}}{\sqrt{\langle M \rangle_n}} \leq C \sqrt{\frac{\log \log \log n}{n \log n}} \rightarrow 0.$$

Lemma 5.5 therefore applies to the whole sequence (M_n) and gives, for every large n ,

$$|M_n| \leq C \sqrt{\log n \cdot \log \log \log n} \quad \text{a.s.} \quad (5.8)$$

Multiplying by the weight,

$$\log(n/n_j) |M_{n_j}| \leq (\theta - 1) \log n \cdot C \sqrt{\log n \cdot \log \log \log n} = C(\theta - 1) (\log n)^{3/2} \sqrt{\log \log \log n}.$$

On the other hand $\lambda_n^* \sim \sqrt{2\tau/3} (\log n)^{3/2} \sqrt{\log \log \log n}$, so the ratio of the product to λ_n^* is bounded by a constant multiple of $\theta - 1$. The middle term is at most $C(\theta - 1)\lambda_n^*$.

The weights $\log(n/k)$ decrease from $\log(n/n_j)$ to 0 on the range $n_j \leq k < n$. Summation by parts therefore gives

$$|R_{n,j}| \leq \sup_{n_j \leq m \leq n} |M_m - M_{n_j}| \cdot \log(n/n_j).$$

The bound (5.8) holds at every large time, so

$$\sup_{n_j \leq m \leq n} |M_m - M_{n_j}| \leq C \sqrt{\log n \cdot \log \log \log n}.$$

Multiplication by $\log(n/n_j) \leq (\theta - 1) \log n$ again produces at most $C(\theta - 1)\lambda_n^*$, by the same cancellation. Hence,

$$\limsup_{n \rightarrow \infty} \frac{|Z_n|}{\lambda_n^*} \leq 1 + \varepsilon + C(\theta - 1) \quad \text{a.s.}$$

Letting $\theta \downarrow 1$ and then $\varepsilon \downarrow 0$ gives the upper bound 1. In particular, $|Z_n| \leq 2\lambda_n^*$ for every large n , almost surely.

For the lower bound, keep $n_j = \lfloor \exp(\theta^j) \rfloor$, but now, let $\theta > 1$ be large. Write I_j for the sum of $\log(n_j/k) \eta_{k+1}$ over $n_{j-1} \leq k < n_j$. These weights are deterministic. The predictable variation of the block is

$$b_j = \sum_{n_{j-1} \leq k < n_j} \log(n_j/k)^2 \mathbb{E}[\eta_{k+1}^2 | \mathcal{F}_k].$$

The same substitution as above yields

$$\sum_{n_{j-1} \leq k < n_j} \frac{\log(n_j/k)^2}{k} = \frac{1}{3} (\log(n_j/n_{j-1}))^3 + O((\log n_j)^2),$$

and $\log(n_j/n_{j-1}) = (1 - 1/\theta) \log n_j + O(1)$. Therefore,

$$b_j \sim b_j^* := \tau \left(1 - \frac{1}{\theta}\right)^3 \frac{(\log n_j)^3}{3} = \left(1 - \frac{1}{\theta}\right)^3 v_{n_j}^* \quad \text{a.s.} \quad (5.9)$$

The lower bound is read on the blocks I_j , normalized by their own variation b_j^* .

Let $\rho_k = k \mathbb{E}[\eta_{k+1}^2 | \mathcal{F}_k] / \tau$, so $\rho_k \rightarrow 1$ almost surely. On $\{\rho_k \geq 1/2\}$, set

$$\xi_{k+1} = \log(n_{j(k)}/k) \frac{\eta_{k+1}}{\sqrt{\rho_k}}.$$

On $\{\rho_k < 1/2\}$, which occurs for finitely many k almost surely, substitute a fresh, conditionally centered random variable with the same deterministic bound and with conditional variance $\log(n_{j(k)}/k)^2 \tau/k$. Then, identically,

$$\mathbb{E}[\xi_{k+1}^2 | \mathcal{F}_k] = \log(n_{j(k)}/k)^2 \frac{\tau}{k}, \quad |\xi_{k+1}| \leq C \log n_{j(k)} k^{-1/2}.$$

The block sum $I_j^0 = \sum_{n_{j-1} \leq k < n_j} \xi_{k+1}$ has the deterministic predictable variation

$$b_j^0 = \sum_{n_{j-1} \leq k < n_j} \log(n_j/k)^2 \frac{\tau}{k}.$$

The integral comparison already used for b_j gives $b_j^0 \sim b_j^*$, so $\sqrt{b_j^0 \log \log b_j^0} \sim \sqrt{b_j^* \log \log b_j^*}$. Because $\rho_k \rightarrow 1$ almost surely, for every $\eta > 0$, there exists a finite random index after which the conditional variance of $\log(n_j/k) \eta_{k+1} (1 - \rho_k^{-1/2})$ is at most η times $\log(n_j/k)^2 \tau/k$. Once n_{j-1} has passed that index, the predictable variation e_j of $I_j - I_j^0$ satisfies $e_j \leq \eta b_j^0$. In particular, $e_j = o(b_j^*)$ almost surely. Fix $\delta > 0$, and set $a_j = \delta \sqrt{2b_j^* \log \log b_j^*}$ and $\eta = \delta^2/8$. On the block, the increments of the error are at most $K_j = C \log n_j \cdot n_{j-1}^{-1/2}$ and $K_j a_j / b_j^* \rightarrow 0$. For every large j , one has $b_j^0 \leq 2b_j^*$, and hence $e_j \leq \eta b_j^0$ implies $e_j \leq 2\eta b_j^*$. Therefore, for every large j ,

$$\mathbb{P}(|I_j - I_j^0| \geq a_j, e_j \leq \eta b_j^0) \leq 2 \exp\left(-\frac{a_j^2}{2(2\eta b_j^* + K_j a_j)}\right) \leq C j^{-2}.$$

The series converges, so Borel–Cantelli gives only finitely many such joint events. The event $\{e_j > \eta b_j^0\}$ itself occurs only finitely often, almost surely. Hence, $|I_j - I_j^0| \leq a_j$ for every large j , almost surely. As $\delta > 0$ is arbitrary,

$$|I_j - I_j^0| = o(\sqrt{b_j^* \log \log b_j^*}) \quad \text{a.s.} \quad (5.10)$$

Consequently, I_j and I_j^0 have the same limsup on this scale.

Normalize the summands of I_j^0 by $\sqrt{b_j^0}$, and write X_k for the normalized variables on the block. Under $\mathbb{P}(\cdot \mid \mathcal{F}_{n_{j-1}})$, the sequence (X_k) is a martingale difference sequence for $(\mathcal{F}_k)$. The conditional variances are the deterministic numbers $\log(n_j/k)^2 \tau / (k b_j^0)$ and sum to 1, so Condition (A2) of [15] holds with $\delta = 0$. Moreover, $|X_k| \leq \gamma_j$ holds for every outcome, with the deterministic bound $\gamma_j \leq C \exp(-\theta^{j-1}/2)$. Thus, $\gamma_j \leq 1/2$ for every large j , and the bounded increments give Condition (A1) of that paper with $\rho = 1$ and $\epsilon = \gamma_j$:

$$\mathbb{E}[|X_k|^3 \mid \mathcal{F}_{k-1}] \leq \gamma_j \mathbb{E}[X_k^2 \mid \mathcal{F}_{k-1}].$$

Both conditions are pathwise and do not depend on $\mathcal{F}_{n_{j-1}}$, so they hold under the conditional probability. Part (2) of [15, Theorem 2.1] therefore applies to this conditional martingale and yields

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}(I_j^0 / \sqrt{b_j^0} \leq x \mid \mathcal{F}_{n_{j-1}}) - \Phi(x) \right| \leq c \gamma_j |\log \gamma_j|.$$

For large j , the right-hand side is at most $\exp(-\theta^{j-1}/4)$, which is $o(j^{-2})$. Here, $\log b_j^* = 3j \log \theta + O(1)$; hence, $\log \log b_j^* = \log j + O(1)$. The Gaussian tail at height $(1 - \epsilon) \sqrt{2 \log \log b_j^*}$ is therefore at least $c_\epsilon j^{-(1-\epsilon)^2} (\log j)^{-1/2}$. Subtracting the Berry–Esseen error leaves at least half of that tail. Because $b_j^0 \sim b_j^*$, for every large j , the level $(1 - \epsilon) \sqrt{2b_j^* \log \log b_j^*}$ lies below $(1 - \epsilon/2) \sqrt{2b_j^0 \log \log b_j^0}$. The conditional probability of the higher level is at least a constant multiple of $j^{-(1-\epsilon/2)}$. For each fixed $\epsilon \in (0, 1)$ and every large j ,

$$\mathbb{P}\left(I_j^0 \geq (1 - \epsilon) \sqrt{2b_j^* \log \log b_j^*} \mid \mathcal{F}_{n_{j-1}}\right) \geq c_\epsilon j^{-(1-\epsilon/2)}.$$

The series of these conditional probabilities diverges. Lévy's conditional Borel–Cantelli lemma yields

$$\limsup_{j \rightarrow \infty} \frac{I_j^0}{\sqrt{2b_j^* \log \log b_j^*}} \geq 1 - \varepsilon \quad \text{a.s.}$$

Letting $\varepsilon \downarrow 0$, and transferring the negligible error $I_j - I_j^0$, gives

$$\limsup_{j \rightarrow \infty} \frac{|I_j|}{\sqrt{2b_j^* \log \log b_j^*}} \geq 1 \quad \text{a.s.}$$

Equation (5.9) gives $b_j^* = (1 - 1/\theta)^3 v_{n_j}^*$. Because $\log \log b_j^* \sim \log \log v_{n_j}^*$, the block scale and the global scale are related by

$$\frac{\sqrt{2b_j^* \log \log b_j^*}}{\lambda_{n_j}^*} = \frac{\sqrt{2b_j^* \log \log b_j^*}}{\sqrt{2v_{n_j}^* \log \log v_{n_j}^*}} \rightarrow \left(1 - \frac{1}{\theta}\right)^{3/2}. \quad (5.11)$$

Therefore,

$$\limsup_{j \rightarrow \infty} \frac{|I_j|}{\lambda_{n_j}^*} \geq \left(1 - \frac{1}{\theta}\right)^{3/2} \quad \text{a.s.}$$

The target sum decomposes exactly as

$$Z_{n_j} = I_j + \log(n_j/n_{j-1}) M_{n_{j-1}} + Z_{n_{j-1}}.$$

The upper bound already proved yields $|Z_{n_{j-1}}| \leq 2\lambda_{n_{j-1}}^*$ for every large j , and $\lambda_{n_{j-1}}^*/\lambda_{n_j}^* \sim \theta^{-3/2}$, so the carried term $Z_{n_{j-1}}$ is at most $C\theta^{-3/2}\lambda_{n_j}^*$. The bound (5.8) holds at every large time. With $\log n_{j-1} = \theta^{-1} \log n_j$ and $\log(n_j/n_{j-1}) = (1 - 1/\theta) \log n_j + O(1)$, the product is a constant multiple of $(1 - 1/\theta)\theta^{-1/2}(\log n_j)^{3/2} \sqrt{\log \log \log n_j}$. Dividing by $\lambda_{n_j}^*$ cancels $(\log n_j)^{3/2} \sqrt{\log \log \log n_j}$, so

$$|\log(n_j/n_{j-1}) M_{n_{j-1}}| \leq C\left(1 - \frac{1}{\theta}\right)\theta^{-1/2} \lambda_{n_j}^*$$

for every large j . Subtracting them from I_j and inserting (5.11) gives

$$\limsup_{j \rightarrow \infty} \frac{|Z_{n_j}|}{\lambda_{n_j}^*} \geq \left(1 - \frac{1}{\theta}\right)^{3/2} - C\left(1 - \frac{1}{\theta}\right)\theta^{-1/2} - C\theta^{-3/2} \quad \text{a.s.}$$

As $\theta \rightarrow \infty$, one has $(1 - 1/\theta)^{3/2} \rightarrow 1$, and $(1 - 1/\theta)\theta^{-1/2} \rightarrow 0$ and $\theta^{-3/2} \rightarrow 0$. The right-hand side tends to 1. This is the lower bound. $\square$

5.2. The superdiffusive case $a > 1/2$

Assume that $a \in (1/2, 1]$ and that $\mathbb{E}[L_1^2] < \infty$. By (5.3) and Lemma 5.4,

$$\frac{S_n^{(1)}}{n^a \log n} = \frac{\phi_n(S_1^{(1)} + Y_n^{(1)})}{n^a \log n} + d \frac{\phi_n}{n^a \log n} \sum_{j=1}^{n-1} \frac{S_j^{(2)}}{\phi_j(j+a)}. \quad (5.12)$$

The martingale $Y^{(1)}$ satisfies

$$\mathbb{E}[\langle Y^{(1)} \rangle_\infty] = \sum_{j=1}^{\infty} \frac{\mathbb{E}[(\Delta M_{j+1}^{(1)})^2]}{\phi_{j+1}^2} \leq C \sum_{j=1}^{\infty} j^{-2a} < \infty.$$

Hence, $\langle Y^{(1)} \rangle_\infty < \infty$ almost surely, and $Y_n^{(1)}$ converges almost surely to a finite limit [16]. Both $\phi_n S_1^{(1)}$ and $\phi_n Y_n^{(1)}$ are $o(n^a \log n)$ almost surely.

It remains to treat the source. Set $\rho_m = \Gamma(m+a)/(\Gamma(m)m^a)$, so that $\rho_m \rightarrow 1$ and $\phi_m = \rho_m m^a / \Gamma(1+a)$. Define

$$b_j = \frac{S_j^{(2)}}{j^a} \cdot \frac{j}{j+a} \cdot \frac{1}{\rho_j}. \quad (5.13)$$

A direct substitution of ϕ_j gives the identity

$$\phi_n \sum_{j=1}^{n-1} \frac{S_j^{(2)}}{\phi_j(j+a)} = \rho_n n^a \sum_{j=1}^{n-1} \frac{b_j}{j}.$$

Because $S_j^{(2)}/j^a \rightarrow L$ almost surely, one has $b_j \rightarrow L$ almost surely.

If $b_j \rightarrow L$ almost surely, then,

$$\sum_{j=1}^{n-1} \frac{b_j}{j} = L \log n + o(\log n) \quad \text{a.s.} \quad (5.14)$$

Combining this evaluation with (5.12) gives $S_n^{(1)}/(n^a \log n) \rightarrow dL$ almost surely when $d \neq 0$. If $d = 0$, the source is absent, so $S_n^{(1)}/(n^a \log n) \rightarrow 0$. At the same time, $S_n^{(1)}/\phi_n = S_1^{(1)} + Y_n^{(1)}$ converges almost surely and in L^2 to $S_1^{(1)} + Y_\infty^{(1)}$. Because $\phi_n \sim n^a / \Gamma(1+a)$,

$$\frac{S_n^{(1)}}{n^a} \xrightarrow{\text{a.s.}} \frac{S_1^{(1)} + Y_\infty^{(1)}}{\Gamma(1+a)}.$$

Because $(Y_n^{(1)})$ is a martingale with $Y_1^{(1)} = 0$, and $S_1^{(1)}$ is $\mathcal{F}_1$ -measurable,

$$\mathbb{E}[(S_1^{(1)} + Y_\infty^{(1)})^2] = \mathbb{E}[(S_1^{(1)})^2] + \mathbb{E}[\langle Y^{(1)} \rangle_\infty] \geq \nu.$$

If $\nu > 0$, the limit is nondegenerate.

5.3. The critical case, $a = 1/2$

Now, $a = 1/2$, so $c_\phi := 1/\Gamma(3/2) = 2/\sqrt{\pi}$, and $\phi_n \sim c_\phi \sqrt{n}$. For an arbitrary constant c , (2.4) at $a = 1/2$ reads

$$\mathbb{E}[A_{n+1} - A_n \mid \mathcal{F}_n] = \frac{1}{2n} S_n^{(1)} + \left(d - \frac{c}{2}\right) \frac{S_n^{(2)}}{n}.$$

The choice $c = 2d$ cancels the second summand and leaves a drift in $S_n^{(1)}$. Substituting $S_n^{(1)} = A_n + c S_n^{(2)}$ restores the source $d S_n^{(2)}/n$, so A_n is not a critical elephant walk.

The correct analysis starts from (5.3). The term $\phi_n Y_n^{(1)}$ is negligible on the scale $\sqrt{n(\log n)^3}$ as soon as $d \neq 0$. Because $\phi_j^2 \sim c_\phi^2 j$,

$$\mathbb{E}[(\phi_n Y_n^{(1)})^2] = \phi_n^2 \sum_{j < n} \frac{\mathbb{E}[(\Delta M_{j+1}^{(1)})^2]}{\phi_{j+1}^2} = O(n \log n) = o(n(\log n)^3).$$

The initial term $\phi_n S_1^{(1)}$ is of order $\sqrt{n}$. The leading contribution is therefore

$$T_n = d \phi_n \sum_{j=1}^{n-1} \frac{S_j^{(2)}}{\phi_j(j+1/2)} = d \phi_n \sum_{j=1}^{n-1} \frac{S_1^{(2)} + Y_j^{(2)}}{j+1/2}.$$

The sum of $S_1^{(2)}/(j+1/2)$ is a harmonic sum of order $\log n$, and $\phi_n \log n = o(\sqrt{n}(\log n)^{3/2})$. Write $\xi_{j+1} = \Delta M_{j+1}^{(2)}$ and $M_n = Y_n^{(2)} = \sum_{k < n} \xi_{k+1}/\phi_{k+1}$. Then,

$$\sum_{j=1}^{n-1} \frac{M_j}{j+1/2} = \sum_{k=1}^{n-2} \frac{\xi_{k+1}}{\phi_{k+1}} \sum_{j=k+1}^{n-1} \frac{1}{j+1/2}. \quad (5.15)$$

The inner sum equals $\log(n/k) + \varepsilon_n - \varepsilon_k$, where $\varepsilon_m \rightarrow 0$. The error produced by ε is of lower order than $(\log n)^{3/2}$ in L^2 because its quadratic variation is $O(\log n)$. Set

$$Z_n = \sum_{k=1}^{n-1} \log\left(\frac{n}{k}\right) \frac{\xi_{k+1}}{\phi_{k+1}}, \quad H_n = \sum_{k=1}^{n-1} \log(k+1) \frac{\xi_{k+1}}{\phi_{k+1}}.$$

The process (M_n, H_n) is a square-integrable martingale, and $Z_n = (\log n) M_n - H_n + R_n$ with a remainder of strictly smaller order. Because $|\xi_k| \leq 2$, and $S_k^{(2)}/k \rightarrow 0$ almost surely, $\mathbb{E}[\xi_{k+1}^2 | \mathcal{F}_k] \rightarrow 1$ almost surely. Therefore,

$$\begin{aligned} \langle M \rangle_n &\sim c_\phi^{-2} \log n, \\ \langle H \rangle_n &\sim c_\phi^{-2} \cdot \frac{1}{3} (\log n)^3, \\ \langle M, H \rangle_n &\sim c_\phi^{-2} \cdot \frac{1}{2} (\log n)^2. \end{aligned} \quad (5.16)$$

The quadratic variation of the combination $(\log n)M_n - H_n$ is

$$\begin{aligned} (\log n)^2 \langle M \rangle_n + \langle H \rangle_n - 2(\log n) \langle M, H \rangle_n &\sim c_\phi^{-2} \left((\log n)^3 + \frac{1}{3} (\log n)^3 - (\log n)^3 \right) \\ &= c_\phi^{-2} \cdot \frac{1}{3} (\log n)^3. \end{aligned}$$

Multiplying by $(d\phi_n)^2$ and using $\phi_n \sim c_\phi \sqrt{n}$ gives

$$(d\phi_n)^2 \cdot c_\phi^{-2} \cdot \frac{1}{3} (\log n)^3 \sim \frac{d^2}{3} n (\log n)^3.$$

After division by $\sqrt{n}(\log n)^{3/2}$, every increment of the row is of order $(\log(n/k))/(\sqrt{k}(\log n)^{3/2})$ and tends to 0 uniformly in k . For each fixed n , the partial sums

$$Z_{n,m} = \sum_{k=1}^m \log\left(\frac{n}{k}\right) \frac{\xi_{k+1}}{\phi_{k+1}}, \quad m < n,$$

form a martingale in m , and $Z_{n,n-1}$ differs from Z_n by the negligible remainder already removed. The sequence (Z_n) itself is not a martingale, because the weights depend on n . The martingale central limit theorem is applied to this array, in the form of [16, Chapter 3]: The predictable variation of the row is asymptotic to $c_\phi^{-2}(\log n)^3/3$, and the maximal increment tends to 0. It yields $T_n/\sqrt{n(\log n)^3} \rightarrow \mathcal{N}(0, d^2/3)$. Adding the negligible term $\phi_n Y_n^{(1)}$ proves the critical central limit theorem for $d \neq 0$.

Lemma 5.6 applies to $\eta_{k+1} = \xi_{k+1}/\phi_{k+1}$. These differences satisfy $|\eta_k| \leq Ck^{-1/2}$ and $\mathbb{E}[\eta_{k+1}^2 | \mathcal{F}_k] \sim c_\phi^{-2}k^{-1}$, as $|\xi_k| \leq 2$, and $\phi_k \sim c_\phi k^{1/2}$, so

$$\limsup_{n \rightarrow \infty} \frac{|Z_n|}{\sqrt{2 \cdot c_\phi^{-2} \cdot \frac{1}{3}(\log n)^3 \log \log \log n}} = 1 \quad \text{a.s.}$$

Multiplication by $d\phi_n \sim d c_\phi \sqrt{n}$ cancels c_ϕ and produces the constant $|d|/\sqrt{3}$. Under $\mathbb{E}[|L_1|^{2+\delta}] < \infty$, Lemma 5.3 gives $\langle Y^{(1)} \rangle_n = O(\log n)$ almost surely. Kronecker's lemma, applied to $\Delta M_{j+1}^{(1)}/(\phi_{j+1} b_j)$ with $b_j = (\log j)^{(1+\eta)/2}$ and $\eta > 0$, yields $Y_n^{(1)} = o(b_n)$ almost surely, so $\phi_n Y_n^{(1)}$ is negligible on the scale $\sqrt{n(\log n)^3 \log \log \log n}$.

If $d = 0$, the source T_n is absent, and $S_n^{(1)} = \phi_n(S_1^{(1)} + Y_n^{(1)})$. Lemmas 5.3 and 5.1 give $\mathbb{E}[(\Delta M_{n+1}^{(1)})^2 | \mathcal{F}_n] \rightarrow v$ almost surely, so

$$\langle Y^{(1)} \rangle_n \sim v c_\phi^{-2} \log n.$$

Lindeberg's condition for $Y^{(1)}$ uses the same moment. Normalize by $s_n \sim c_\phi^{-1} \sqrt{v \log n}$. Then, $\phi_k \sim c_\phi k^{1/2}$, so

$$\mathbb{E}[|\Delta Y_{k+1}^{(1)}|^{2+\delta}] \leq C k^{-(1+\delta/2)} \mathbb{E}[|X_{k+1}^{(1)}|^{2+\delta}],$$

and Lemma 5.3 bounds the expectation on the right. The series $\sum_k k^{-1-\delta/2}$ converges; hence, the Lindeberg sum is of order $s_n^{-2-\delta}$ and tends to 0. The martingale central limit theorem yields $S_n^{(1)}/\sqrt{n \log n} \rightarrow \mathcal{N}(0, v)$: The product $\phi_n \cdot c_\phi^{-1}$ tends to $\sqrt{n}$, and the factor c_ϕ cancels. If L_1 is almost surely bounded, then $\langle Y^{(1)} \rangle_n \rightarrow \infty$, and $|\Delta Y_n^{(1)}| \leq Cn^{-1/2}$. With $s_n^2 = \langle Y^{(1)} \rangle_n \sim v c_\phi^{-2} \log n$,

$$\frac{|\Delta Y_n^{(1)}| \sqrt{\log \log s_n^2}}{s_n} \leq C n^{-1/2} \left(\frac{\log \log \log n}{\log n} \right)^{1/2} \rightarrow 0.$$

Lemma 5.5 applied directly to $Y^{(1)}$ therefore gives

$$\limsup_{n \rightarrow \infty} \frac{|S_n^{(1)}|}{\sqrt{2n \log n \log \log n}} = \sqrt{v} \quad \text{a.s.}$$

The second-moment recursions give an independent check of the constants. With $u_n = \mathbb{E}[(S_n^{(2)})^2]$, $p_n = \mathbb{E}[S_n^{(1)} S_n^{(2)}]$, and $w_n = \mathbb{E}[(S_n^{(1)})^2]$, the conditional calculation of Section 5.1 is exact at $a = 1/2$: The quadratic drift terms of the second elephant cancel, and

$$\begin{aligned} u_{n+1} &= \left(1 + \frac{1}{n}\right) u_n + 1, \\ p_{n+1} &= \left(1 + \frac{1}{n} + \frac{1}{4n^2}\right) p_n + \left(\frac{d}{n} + \frac{d}{2n^2}\right) u_n, \\ w_{n+1} &= \left(1 + \frac{1}{n}\right) w_n + \frac{2d}{n} p_n + \beta v + (1 - \beta) \mathbb{E}[Q_n]. \end{aligned}$$

Taking expectations in (5.1) and using $\mathbb{E}[\Delta_{n+1}] = 0$ gives $\mathbb{E}[Q_n] - v = \pi_n(\mathbb{E}[Q_1] - v) \rightarrow 0$, so the last summand tends to v . This uses only $\mathbb{E}[L_1^2] < \infty$ and $\beta \in (0, 1)$. Hence, $u_n \sim n \log n$, $p_n \sim (d/2)n(\log n)^2$, and, when $d \neq 0$, $w_n \sim (d^2/3)n(\log n)^3$. When $d = 0$, the source term in w_n drops out, and $w_n \sim vn \log n$. If $L_n \equiv 1$, then $Q_n = 1$, and $v = 1$, so the recursion for w_n is closed and deterministic.

5.4. The diffusive case, $-1 < a < 1/2$

Assume that $-1 < a < 1/2$ and that $\mathbb{E}[|L_1|^{2+\delta}] < \infty$ for some $\delta \in (0, 2]$, and set $r = 1 - 2a$. There is again a source. From (2.4),

$$A_n = A_1 + a \sum_{k=1}^{n-1} \frac{A_k}{k} + d \sum_{k=1}^{n-1} \frac{S_k^{(2)}}{k} + \widetilde{M}_n$$

for any fixed c , whether or not $c = d/a$. The sum involving $S^{(2)}$ is not zero. We therefore work with (5.3), which remains valid at $a = 0$, where $\phi_n = 1$.

Lemma 5.3 gives $Q_n \rightarrow v$ almost surely. Lemma 5.4 gives $S_n^{(2)}/n \rightarrow 0$ almost surely. On almost every path, Q_n converges, so $\mathbb{E}[(\Delta M_{n+1}^{(1)})^2 | \mathcal{F}_n] \leq \beta v + (1 - \beta)Q_n$ is eventually bounded. Thus, $\langle Y^{(1)} \rangle_n = O(n^r)$ almost surely. Kronecker's lemma yields $Y_n^{(1)} = o(n^{r/2} \log n)$, and hence, $\phi_n Y_n^{(1)} = o(n^{1/2} \log n)$. For the source, the bound $S_j^{(2)} = O(\sqrt{j \log \log j})$ and $\phi_j \sim j^a/\Gamma(1+a)$ give

$$\left| \phi_n \sum_{j < n} \frac{S_j^{(2)}}{\phi_j(j+a)} \right| = O(\sqrt{n \log \log n}),$$

because $a < 1/2$ makes $\sum_{j < n} j^{-a-1/2}$ of order $n^{1/2-a}$, and multiplication by ϕ_n restores a factor n^a . Therefore $S_n^{(1)} = o(n)$ almost surely, and Lemma 5.1 yields the innovation limit $\text{diag}(v, 1)$.

The two contributions to $S_n^{(1)}$ are $\phi_n Y_n^{(1)}$ and the source

$$T_n = d \phi_n \sum_{j=1}^{n-1} \frac{S_1^{(2)} + Y_j^{(2)}}{j+a}.$$

Because $\phi_j \sim c_a j^a$ with $c_a = 1/\Gamma(1+a)$, write $\sigma_k^{(i)} = \mathbb{E}[(\Delta M_{k+1}^{(i)})^2 | \mathcal{F}_k]$. Because $Q_n \rightarrow v$, and $S_n/n \rightarrow 0$, lemma 5.1 gives $\sigma_k^{(1)} \rightarrow v$ and $\sigma_k^{(2)} \rightarrow 1$ almost surely.

The weights $w_k = k^{-2a}$ are positive, and $W_n = \sum_{k < n} w_k \rightarrow \infty$, because $r > 0$. Any sequence $c_k \rightarrow L$ therefore satisfies $\sum_{k < n} c_k w_k / W_n \rightarrow L$. Applied to $\sigma_k^{(1)}$, this is

$$\langle Y^{(1)} \rangle_n = \sum_{k < n} \frac{\sigma_k^{(1)}}{\phi_{k+1}^2} \sim \frac{v}{c_a^2} \sum_{j < n} j^{-2a} \sim \frac{v}{c_a^2} \frac{n^r}{r}.$$

Hence, $\mathbb{E}[(\phi_n Y_n^{(1)})^2] \sim (v/r)n$. Expanding $Y_j^{(2)}$ and using $\sum_{j=k+1}^{n-1} (j+a)^{-1} = \log(n/k) + O(k^{-1})$ transfers the source onto the increments $\Delta M^{(2)}$, with deterministic weights $\log(n/k)/\phi_{k+1}$. For these weights, fix $\eta > 0$ and a finite index, after which $|\sigma_k^{(2)} - 1| < \eta$. The sum over the finite initial range is $O((\log n)^2)$ and is negligible against n^r . The remaining sum is at most η times $\sum_{k < n} \log(n/k)^2 k^{-2a}$, whose integral is the profile below. Because η is arbitrary,

$$\sum_{k < n} \frac{\log(n/k)^2}{\phi_{k+1}^2} \sigma_k^{(2)} \sim c_a^{-2} \int_1^n \frac{\log(n/x)^2}{x^{2a}} dx = c_a^{-2} n^r \int_0^1 (-\log u)^2 u^{-2a} du.$$

The integral equals $2/r^3$. Multiplying by $(d\phi_n)^2$ produces the variance $(2d^2/r^3)n$. The same splitting, with the weight $\log(n/k)k^{-2a}$, gives the covariance computation below. Lemma 5.1 gives $\mathbb{E}[\Delta M_{k+1}^{(1)} \Delta M_{k+1}^{(2)} | \mathcal{F}_k] = 0$ for every k , so the predictable covariation of $Y^{(1)}$ and $Y^{(2)}$ is identically zero. Adding the two variances gives Σ_{11} . This does not make the limiting coordinates uncorrelated. The covariance comes from the source and from $S_n^{(2)} = \phi_n(S_1^{(2)} + Y_n^{(2)})$ sharing $Y^{(2)}$:

$$\mathbb{E}[S_n^{(2)} T_n] \sim d \phi_n^2 \sum_{k < n} \frac{\log(n/k)}{\phi_{k+1}^2} \sim d n^{2a} \cdot n^r \int_0^1 (-\log u) u^{-2a} du = \frac{d}{r^2} n.$$

The contribution of $\phi_n Y_n^{(1)}$ to this expectation is zero. Lemma 5.4 gives the remaining entry, $\Sigma_{22} = 1/r$.

An equivalent and fully explicit description of Σ is the stochastic-approximation integral. With $\mathbf{N} = \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix}$, one has $\mathbf{B} = a\mathbf{I} + \mathbf{N}$ and $\mathbf{N}^2 = 0$, and hence,

$$\mathbf{x}^{-\mathbf{B}} = x^{-a} \begin{pmatrix} 1 & -d \log x \\ 0 & 1 \end{pmatrix}, \quad 0 < x \leq 1.$$

If $\mathbf{Q} = \text{diag}(v, 1)$, then

$$\Sigma = \int_0^1 x^{-B} \mathbf{Q} (x^{-B})^\top dx,$$

which evaluates to (3.1). At $a = 0$, $r = 1$ and $\Sigma_{11} = v + 2d^2$.

Asymptotic normality follows from the martingale central limit theorem on the triangular array below, after the weights $1/(j+a)$ have been transferred onto the increments.

The diffusive limit is read on the triangular array whose row at time n has the increments

$$\begin{aligned} \zeta_{n,k+1} &= n^{-1/2} \frac{\phi_n}{\phi_{k+1}} \left(\Delta M_{k+1}^{(1)} + d \log\left(\frac{n}{k}\right) \Delta M_{k+1}^{(2)} \right), \\ \psi_{n,k+1} &= n^{-1/2} \frac{\phi_n}{\phi_{k+1}} \Delta M_{k+1}^{(2)}, \end{aligned}$$

for $1 \leq k < n$. Set

$$Z_n = \phi_n Y_n^{(1)} + d \phi_n \sum_{k < n} \log\left(\frac{n}{k}\right) \frac{\Delta M_{k+1}^{(2)}}{\phi_{k+1}}.$$

The sum of the $\zeta_{n,k}$ equals $Z_n / \sqrt{n}$, and the sum of the $\psi_{n,k}$ is the martingale part of $S_n^{(2)} / \sqrt{n}$. Both weights are deterministic. Because $\phi_n / \phi_{k+1} \sim (n/k)^a$,

$$|\zeta_{n,k+1}| \leq C n^{-1/2} \left(\frac{n}{k}\right)^a \left(1 + \log \frac{n}{k}\right) (|\Delta M_{k+1}^{(1)}| + |\Delta M_{k+1}^{(2)}|),$$

whereas $|\psi_{n,k+1}|$ carries only the factor $n^{-1/2}(n/k)^a$. The second innovation is bounded by 2. Lemma 5.3 gives $\sup_k \mathbb{E}[|X_k^{(1)}|^{2+\delta}] < \infty$, so with $p = 2 + \delta$,

$$\sum_{k < n} \mathbb{E}[|\zeta_{n,k+1}|^p] \leq C n^{-p/2} \sum_{k < n} \left(\frac{n}{k}\right)^{ap} \left(1 + \log \frac{n}{k}\right)^p. \quad (5.17)$$

The same bound holds for $\psi_{n,k+1}$, without the logarithmic factor. The three ranges of ap are as follows.

If $ap < 1$, then $\sum_{k < n} k^{-ap} \leq Cn^{1-ap}$, and the factor n^{ap} in front of that sum cancels it:

$$\sum_{k < n} \left(\frac{n}{k}\right)^{ap} \left(1 + \log \frac{n}{k}\right)^p \leq C n (\log n)^p.$$

Multiplication by $n^{-p/2} = n^{-(1+\delta/2)}$ leaves a constant multiple of $n^{-\delta/2}(\log n)^p$, which tends to 0.

If $ap = 1$, the change of variables $u = \log(n/k)$ gives

$$\sum_{k < n} k^{-1} \left(1 + \log \frac{n}{k}\right)^p = O((\log n)^{p+1}),$$

and the prefactor $n^{ap} = n$ produces

$$n^{-p/2} \cdot n \cdot O((\log n)^{p+1}) = O(n^{-\delta/2}(\log n)^{p+1}),$$

which tends to 0.

If $ap > 1$, the series $\sum_k k^{-ap}$ converges, and $(1 + \log(n/k))^p \leq C(\log n)^p$, so the inner sum of (5.17) is at most $Cn^{ap}(\log n)^p$. Therefore,

$$n^{-p/2} \cdot n^{ap}(\log n)^p = n^{p(a-1/2)}(\log n)^p,$$

which tends to 0 because $a < 1/2$.

In all three cases,

$$\sum_{k < n} \mathbb{E}[|\zeta_{n,k+1}|^2 \mathbf{1}_{\{|\zeta_{n,k+1}| > \varepsilon\}}] \leq \varepsilon^{-\delta} \sum_{k < n} \mathbb{E}[|\zeta_{n,k+1}|^p] \rightarrow 0,$$

and the same limit holds for $\psi_{n,k+1}$. The conditional Lindeberg sums have expectations tending to 0, and hence tend to 0 in probability. The predictable covariance matrix of the row converges to Σ by the weighted averaging already used for Σ_{11} , Σ_{12} , and Σ_{22} . The martingale central limit theorem in [16, Chapter 3] gives the Gaussian limit $\mathcal{N}(0, \Sigma)$.

If $v = 0$, then $L_1 = 0$ almost surely; hence, $d = 0$, $\Sigma_{11} = 0$ and $S_n^{(1)} = 0$. The arguments below assume $\Sigma_{11} > 0$, which forces $v > 0$.

For the law of the iterated logarithm, assume in addition that L_1 is almost surely bounded and that $\Sigma_{11} > 0$. The term $\phi_n Y_n^{(1)}$ is a deterministic multiple of the martingale $Y^{(1)}$. The source is not. With the expansion $\sum_{j=k+1}^{n-1} (j+a)^{-1} = \log(n/k) + O(k^{-1})$ already used for the variance, set

$$U_n = \sum_{k < n} \frac{\log k}{\phi_{k+1}} \Delta M_{k+1}^{(2)}.$$

Then, U_n is a martingale, and

$$\sum_{k < n} \frac{\log(n/k)}{\phi_{k+1}} \Delta M_{k+1}^{(2)} = (\log n) Y_n^{(2)} - U_n.$$

The summand $\phi_n Y_n^{(1)}$ is driven by $\Delta M^{(1)}$, and the source is driven by $\Delta M^{(2)}$. These innovations are conditionally uncorrelated for every n , by Lemma 5.1, so the predictable variation of the sum is exactly the sum of the variations already computed:

$$\phi_n^2 \langle Y^{(1)} \rangle_n \sim \frac{v}{r} n, \quad (d\phi_n)^2 \cdot c_a^{-2} n^r \cdot \frac{2}{r^3} \sim \frac{2d^2}{r^3} n.$$

By Lemma 5.1, for every k , $\mathbb{E}[\Delta M_{k+1}^{(1)} \Delta M_{k+1}^{(2)} \mid \mathcal{F}_k] = 0$. Hence, the array which represents $S_n^{(1)}$, up to an error $o(\sqrt{n \log \log n})$, has

$$s_n^2 \sim \Sigma_{11} n = \left(\frac{v}{r} + \frac{2d^2}{r^3}\right)n.$$

A generic summand is of order $(n/k)^a(1 + \log(n/k))$. If $a \geq 0$, this is at most $Cn^a \log n$. If $a < 0$, then $(n/k)^a \leq 1$, so the summand is at most $C \log n$. In both cases, it is $o(\sqrt{n/\log \log n})$. Write $\eta_{k+1}^{(2)} = \Delta M_{k+1}^{(2)} / \phi_{k+1}$ and

$$Z_n = \phi_n Y_n^{(1)} + d \phi_n \sum_{k < n} \log\left(\frac{n}{k}\right) \eta_{k+1}^{(2)}.$$

The expansion $\sum_{j=k+1}^{n-1} (j+a)^{-1} = \log(n/k) + O(k^{-1})$ shows that $S_n^{(1)} - Z_n$ is the sum of the initial terms, of order $n^a \log n$ and of a martingale whose predictable variation is $O(n^{2a})$. Both are $o(\sqrt{n \log \log n})$ because $a < 1/2$. Thus, $S_n^{(1)}$ and Z_n have the same limsup on the scale $\sqrt{2\Sigma_{11}n \log \log n}$. Let $\lambda_n^* = \sqrt{2\Sigma_{11}n \log \log n}$. The conditional variances tend to v and 1, so the predictable variation v_n of the row defining Z_n satisfies $v_n \sim \Sigma_{11}n$ almost surely, and $\lambda_n/\lambda_n^* \rightarrow 1$ if $\lambda_n = \sqrt{2v_n \log \log v_n}$. For the upper bound, fix $\varepsilon \in (0, 1)$ and $\theta \in (1, 1 + \varepsilon)$, and set $n_j = \lfloor \theta^j \rfloor$. A summand is $o(\lambda_n^*)$ uniformly in k , as checked above, so $K_n a_n / b_n \rightarrow 0$ for the row at time n_j . Set $a = (1 + \varepsilon)\lambda_{n_j}^*$ and $b = (1 + \varepsilon)\Sigma_{11}n_j$. Then,

$$\frac{a^2}{2b} = (1 + \varepsilon) \log \log n_j,$$

because $\lambda_{n_j}^* = \sqrt{2\Sigma_{11}n_j \log \log n_j}$. Here, $\log \log n_j = \log j + O(1)$, so

$$\frac{a^2}{2(b + K_{n_j}a)} \geq \left(1 + \frac{\varepsilon}{2}\right) \log j$$

for every large j . Freedman's inequality therefore produces probabilities of order $j^{-(1+\varepsilon/2)}$. Hence,

$$\limsup_{j \rightarrow \infty} \frac{|Z_{n_j}|}{\lambda_{n_j}^*} \leq 1 + \varepsilon \quad \text{a.s.}$$

For $n_j \leq n \leq n_{j+1}$, the weights split exactly:

$$Z_n = \frac{\phi_n}{\phi_{n_j}} Z_{n_j} + d \phi_n \log\left(\frac{n}{n_j}\right) Y_{n_j}^{(2)} + R_{n,j},$$

where $R_{n,j}$ collects the summands with $k \geq n_j$. Here, $\phi_n/\phi_{n_j} = (n/n_j)^a(1 + o(1))$, so

$$\frac{\phi_n}{\phi_{n_j}} \frac{\lambda_{n_j}^*}{\lambda_n^*} = \left(\frac{n}{n_j}\right)^{a-1/2} (1 + o(1)) \leq \theta^{a-1/2} (1 + o(1)).$$

The middle term is at most a constant multiple of $\theta^{a-1/2} \log \theta \lambda_n^*$. The martingale $Y^{(2)}$ has predictable variation of order n^r , with $r = 1 - 2a > 0$, so $s_n = \sqrt{\langle Y^{(2)} \rangle_n}$ is of order $n^{r/2}$ and tends to infinity. Its increments are of order n^{-a} . Because $a + r/2 = 1/2$,

$$\frac{|\Delta Y_n^{(2)}| \sqrt{\log \log s_n^2}}{s_n} = O(n^{-1/2} (\log \log n)^{1/2}) \rightarrow 0.$$

Lemma 5.5 therefore applies to $Y^{(2)}$ and bounds $\phi_{n_j} Y_{n_j}^{(2)}$ by a multiple of $\sqrt{n_j \log \log n_j}$. The fresh sum $R_{n,j}$, evaluated with weights frozen at the right endpoint n_{j+1} , is a martingale whose predictable variation is at most a constant multiple of $(\theta - 1)n_j$. Freedman's maximal inequality bounds it by a constant multiple of $\sqrt{\theta - 1} \lambda_n^*$. Therefore,

$$\limsup_{n \rightarrow \infty} \frac{|Z_n|}{\lambda_n^*} \leq 1 + \varepsilon + C \sqrt{\theta - 1} \quad \text{a.s.}$$

Letting $\theta \downarrow 1$ and then $\varepsilon \downarrow 0$ gives the upper bound 1. In particular, $|Z_n| \leq 2\lambda_n^*$ for every large n , almost surely.

For the lower bound, let $\theta > 1$ be large, with the same grid $n_j = \lfloor \theta^j \rfloor$. On the block $n_{j-1} \leq k < n_j$, freeze the coefficients at the horizon n_j . The corresponding increment is the martingale difference

$$\xi_{k+1} = \phi_{n_j} \frac{\Delta M_{k+1}^{(1)}}{\phi_{k+1}} + d \phi_{n_j} \log\left(\frac{n_j}{k}\right) \frac{\Delta M_{k+1}^{(2)}}{\phi_{k+1}},$$

and I_j is its sum over the block. Lemma 5.1 makes the two innovations conditionally orthogonal, and both conditional variances converge, to v and to 1. With $\phi_n \sim c_a n^a$ and $2a + r = 1$, the predictable variation of the block is asymptotic to $\kappa(\theta) \Sigma_{11} n_j$, where

$$\kappa(\theta) = \frac{1}{\Sigma_{11}} \left(\frac{v}{r} (1 - \theta^{-r}) + d^2 \int_{\theta^{-1}}^1 (-\log u)^2 u^{-2a} du \right). \quad (5.18)$$

The integral from 0 to 1 equals $2/r^3$, so $\kappa(\theta) \rightarrow 1$ as $\theta \rightarrow \infty$. Because L_1 is almost surely bounded, $|\Delta M^{(1)}|$ and $|\Delta M^{(2)}|$ are bounded by a deterministic constant. On the block, one has $n_j/k \leq \theta$; hence, $|\xi_{k+1}| \leq C\theta^{|a|}(1 + \log \theta)$. Because $\Sigma_{11} > 0$, one has $v > 0$, and the deterministic profile

$$\alpha_{j,k} = \left(\frac{\phi_{n_j}}{\phi_{k+1}} \right)^2 \left(v + d^2 \log\left(\frac{n_j}{k}\right)^2 \right)$$

is positive for every k of the block. Conditional orthogonality gives

$$\mathbb{E}[\xi_{k+1}^2 \mid \mathcal{F}_k] = \left(\frac{\phi_{n_j}}{\phi_{k+1}} \right)^2 \left(\sigma_k^{(1)} + d^2 \log\left(\frac{n_j}{k}\right)^2 \sigma_k^{(2)} \right),$$

where $\sigma_k^{(i)} = \mathbb{E}[(\Delta M_{k+1}^{(i)})^2 \mid \mathcal{F}_k]$. Set $\rho_{j,k} = \mathbb{E}[\xi_{k+1}^2 \mid \mathcal{F}_k] / \alpha_{j,k}$. Lemma 5.1 gives $\sigma_k^{(1)} \rightarrow v$ and $\sigma_k^{(2)} \rightarrow 1$, so $\rho_{j,k} \rightarrow 1$ as $k \rightarrow \infty$. On $\{\rho_{j,k} \geq 1/2\}$, define $\xi_{k+1}^0 = \xi_{k+1} / \sqrt{\rho_{j,k}}$. On $\{\rho_{j,k} < 1/2\}$, which occurs for finitely many k almost surely, substitute a fresh, conditionally centered random variable with conditional variance $\alpha_{j,k}$ and with the deterministic bound $C\theta^{|a|}(1 + \log \theta)$. Then, identically,

$$\mathbb{E}[(\xi_{k+1}^0)^2 \mid \mathcal{F}_k] = \alpha_{j,k}, \quad |\xi_{k+1}^0| \leq C\theta^{|a|}(1 + \log \theta).$$

The block sum $I_j^0 = \sum \xi_{k+1}^0$ therefore has deterministic predictable variation $b_j^0 = \sum_{n_{j-1} \leq k < n_j} \alpha_{j,k}$. The change of variables $u = k/n_j$, together with $\phi_{n_j}/\phi_{k+1} \sim (n_j/k)^a$ and $2a + r = 1$, gives

$$b_j^0 \sim n_j \left(\frac{v}{r} (1 - \theta^{-r}) + d^2 \int_{\theta^{-1}}^1 (-\log u)^2 u^{-2a} du \right) = \kappa(\theta) \Sigma_{11} n_j.$$

Write $b_j^* = \kappa(\theta)\Sigma_{11}n_j$, so $b_j^0 \sim b_j^*$. In particular,

$$\frac{\sqrt{b_j^* \log \log b_j^*}}{\sqrt{n_j \log \log n_j}} \rightarrow \sqrt{\kappa(\theta)\Sigma_{11}}.$$

An error which is $o(\sqrt{b_j^* \log \log b_j^*})$ is therefore $o(\sqrt{n_j \log \log n_j})$.

In the event that $\rho_{j,k} \geq 1/2$, the error increment is $\xi_{k+1}(1 - \rho_{j,k}^{-1/2})$, with conditional variance $\alpha_{j,k}(\sqrt{\rho_{j,k}} - 1)^2$. Because $\rho_{j,k} \rightarrow 1$, for every $\eta > 0$, there is a finite, random index after which this conditional variance is at most $\eta\alpha_{j,k}$. Once n_{j-1} has passed that index, the predictable variation e_j of $I_j - I_j^0$ satisfies $e_j \leq \eta b_j^0$. Fix $\delta > 0$, set $\eta = \delta^2/8$ and $a_j = \delta\sqrt{2b_j^* \log \log b_j^*}$, and write $K = C\theta^{|a|}(1 + \log \theta)$ for the uniform bound on the error increments. Then, $Ka_j/b_j^* \rightarrow 0$. For every large j , $b_j^0 \leq 2b_j^*$, and Freedman's inequality gives

$$\mathbb{P}(|I_j - I_j^0| \geq a_j, e_j \leq \eta b_j^0) \leq 2 \exp\left(-\frac{a_j^2}{2(2\eta b_j^* + Ka_j)}\right) \leq Cj^{-2}.$$

The series converges. The event $\{e_j > \eta b_j^0\}$ occurs only finitely often, so Borel–Cantelli yields $|I_j - I_j^0| \leq a_j$ for every large j almost surely. As $\delta > 0$ is arbitrary,

$$|I_j - I_j^0| = o(\sqrt{n_j \log \log n_j}) = o(\sqrt{b_j^* \log \log b_j^*}) \quad \text{a.s.}$$

The limsup of I_j and of I_j^0 coincide on this scale.

After division by $\sqrt{b_j^0}$, write $X_k = \xi_{k+1}^0 / \sqrt{b_j^0}$ for $n_{j-1} \leq k < n_j$. For every large j , one has $\rho_{j,k} \geq 1/2$ throughout the block, so the auxiliary substitution is idle, and $\xi_{k+1}^0 = \xi_{k+1} / \sqrt{\rho_{j,k}}$ with the factor $\rho_{j,k}^{-1/2}$ measurable with respect to $\mathcal{F}_k$. Thus, X_k is $\mathcal{F}_{k+1}$ -measurable, $\mathbb{E}[X_k | \mathcal{F}_k] = 0$, and $\mathbb{E}[X_k^2 | \mathcal{F}_k] = \alpha_{j,k}/b_j^0$. These three identities hold almost surely, and hence, they hold $\mathbb{P}(\cdot | \mathcal{F}_{n_{j-1}})$ almost surely. Under that conditional probability, (X_k) is a martingale difference sequence for the filtration $(\mathcal{F}_k)_{k \geq n_{j-1}}$. The conditional variances $\alpha_{j,k}/b_j^0$ are deterministic and sum to 1, which is Condition (A2) of [15] with $\delta = 0$. The bound $|X_k| \leq \gamma_j$ holds for every outcome, with $\gamma_j \leq C_\theta \theta^{-j/2} \leq 1/2$ for every large j , and it is Condition (A1) of that paper with $\rho = 1$ and $\epsilon = \gamma_j$. Part (2) of [15, Theorem 2.1] applied to this martingale on the probability space $(\Omega, \mathcal{F}, \mathbb{P}(\cdot | \mathcal{F}_{n_{j-1}}))$, therefore bounds the conditional distribution function of $I_j^0 / \sqrt{b_j^0}$ and gives a Kolmogorov distance at most $c\gamma_j |\log \gamma_j|$. This quantity is $o(j^{-2})$. Because $b_j^0 \sim b_j^*$, the same distance holds on the scale $\sqrt{b_j^*}$. Because $\log \log b_j^* = \log j + O(1)$, it is smaller than the Gaussian tail $j^{-(1-\epsilon)^2}(\log j)^{-1/2}$. Lévy's conditional Borel–Cantelli lemma therefore yields

$$\limsup_{j \rightarrow \infty} \frac{|I_j|}{\sqrt{2b_j^* \log \log b_j^*}} \geq 1 \quad \text{a.s.}$$

The block scale and the global scale satisfy

$$\frac{\sqrt{2b_j^* \log \log b_j^*}}{\lambda_{n_j}^*} \rightarrow \sqrt{\kappa(\theta)}, \quad (5.19)$$

because $\lambda_{n_j}^* = \sqrt{2\Sigma_{11}n_j \log \log n_j}$, and $\log \log b_j^* \sim \log \log n_j$. Hence,

$$\limsup_{j \rightarrow \infty} \frac{|I_j|}{\lambda_{n_j}^*} \geq \sqrt{k(\theta)} \quad \text{a.s.}$$

The same decomposition as in the upper bound gives, up to the $o(1)$ in $\phi_n \sim c_a n^a$,

$$Z_{n_j} = \frac{\phi_{n_j}}{\phi_{n_{j-1}}} Z_{n_{j-1}} + d \phi_{n_j} \log \theta Y_{n_{j-1}}^{(2)} + I_j.$$

The upper bound $|Z_n| \leq 2\lambda_n^*$ and $\phi_{n_j}/\phi_{n_{j-1}} \sim \theta^a$ give

$$\left| \frac{\phi_{n_j}}{\phi_{n_{j-1}}} Z_{n_{j-1}} \right| \leq C \theta^{a-1/2} \lambda_{n_j}^*$$

for every large j . The ratio displayed above for $Y^{(2)}$ is $O(n^{-1/2}(\log \log n)^{1/2})$ and tends to 0, so Lemma 5.5 bounds $\phi_{n_{j-1}} Y_{n_{j-1}}^{(2)}$ by a constant multiple of $\sqrt{n_{j-1} \log \log n_{j-1}}$. Multiplication by $\phi_{n_j}/\phi_{n_{j-1}}$ and by $\log \theta$ produces a second carried term of order $\theta^{a-1/2} \log \theta \lambda_{n_j}^*$. Because $a < 1/2$, both factors tend to 0 as $\theta \rightarrow \infty$, whereas $\sqrt{k(\theta)} \rightarrow 1$ by (5.18). Subtracting the carried terms from I_j and inserting (5.19) shows that the limsup of $|Z_{n_j}|/\lambda_{n_j}^*$ is at least a quantity which tends to 1 as $\theta \rightarrow \infty$. Hence,

$$\limsup_{n \rightarrow \infty} \frac{|Z_n|}{\sqrt{2\Sigma_{11}n \log \log n}} = 1 \quad \text{a.s.}$$

Equivalently, the limsup of $|S_n^{(1)}|$ equals $\sqrt{\Sigma_{11}}$ on the scale $\sqrt{2n \log \log n}$. When $d = 0$, the source is absent, and Lemma 5.5 applies directly to $Y^{(1)}$: Here s_n^2 is of order n^{1-2a} and tends to infinity, the increments are of order n^{-a} , and

$$\frac{|\Delta Y_n^{(1)}| \sqrt{\log \log s_n^2}}{s_n} = O(n^{-1/2}(\log \log n)^{1/2}) \rightarrow 0,$$

because $a + (1 - 2a)/2 = 1/2$. Adding $\sqrt{v/r}$ and $|d| \sqrt{2/r^3}$ separately exceeds $\sqrt{\Sigma_{11}}$ as soon as $d \neq 0$ and $v > 0$.

6. Conclusions

On the resonant line $\gamma = \alpha_2 = a \in (-1, 1]$, a nonzero coupling produces one extra logarithm: $n^a \log n$ for $a > 1/2$, $\sqrt{n}(\log n)^3$ for $a = 1/2$, and $\sqrt{n}$ for $a < 1/2$, with variance $v/(1 - 2a) + 2d^2/(1 - 2a)^3$. If $d = 0$, the critical window returns to $\sqrt{n \log n}$. These scales are the Jordan asymptotics of [11, 12]; the variances and the laws of the iterated logarithm are those computed above for random amplitudes.

Several questions remain open. The diffusive central limit theorem is proved under a moment of order $2 + \delta$. With only two moments of L_1 , the Lindeberg truncation above does not close, and whether that conclusion still holds is left open. If the amplitudes are correlated in time, both Q and Σ change. Extending the block from two walkers to a longer Jordan chain would replace $(\log n)^{3/2}$ by $(\log n)^{v-1/2}$, as in [12].

Author contributions

Mohamed Abdelkader: Conceptualization, methodology, software, formal analysis, writing – original draft. Rafik Aguech: Conceptualization, software, formal analysis, writing – original draft. Both authors have read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have used Artificial Intelligence (AI) tools in the creation of this article. A large language model was used for language editing and bibliographic formatting. The mathematical results were obtained and checked by the authors.

Data availability

The figures and tables in Section 4 were produced by the script `simulate_revision.py`. The seeds are those of Table 2. The amplitudes in those experiments are constant and equal to one.

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Conflict of interest

The authors declare no conflicts of interest.

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