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Research article

Unit-inverted Dagum distribution: statistical properties, estimation, and applications

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Abstract: This paper proposes the unit-inverted Dagum distribution (UIDD), a novel three-parameter probability model for analyzing bounded lifetime and reliability data. The UIDD is developed through a transformation of the inverted Dagum distribution onto the unit interval $(0, 1)$, yielding a flexible model capable of representing diverse distributional shapes and accommodating varying degrees of skewness and tail behavior. The mathematical and statistical properties of the proposed distribution are systematically investigated, including its survival and hazard rate functions, moments, quantile function, order statistics, mean residual life, mean inactivity time, vitality function, and Rényi and Tsallis entropies. Maximum likelihood estimation is employed to estimate the unknown parameters under both complete and right-censored sampling schemes. The finite-sample behavior of the resulting estimators is examined through an extensive Monte Carlo simulation study, with particular attention to their bias and mean squared error. The simulation findings reveal an overall reduction in both measures as the sample size increases, indicating improved estimation accuracy. The practical performance of the UIDD is assessed using a real right-censored lifetime dataset and benchmarked against several established probability distributions. The empirical results demonstrate that the proposed model provides the best fit among the competing distributions according to the selected information criteria and goodness-of-fit measures. These findings establish the UIDD as a promising and versatile alternative for modeling bounded lifetime observations, with potential applications to other unit-interval data, including proportions, rates, and normalized measurements.

Keywords: unit distribution; inverted Dagum distribution; maximum likelihood estimation; Dagum distribution; right-censored data
Mathematics Subject Classification: 62E15, 62E10

1. Introduction

Developing new probability distributions plays a crucial role in the advancement of statistical theory and the improvement of data modeling techniques across a wide range of disciplines, including engineering, economics, and environmental sciences. Although traditional distributions such as the normal, exponential, and Weibull distributions form the foundation of statistical analysis, they are often insufficient for representing the complex characteristics of real-world data, including heavy-tailed behavior, multimodality, and nonmonotonic hazard functions such as bathtub-shaped patterns. By extending or combining existing distributions, researchers can introduce additional shape parameters that enhance model flexibility, leading to better data fitting and more accurate predictions. Consequently, these new distributions help connect theoretical statistical models with complex practical applications, offering more effective tools for reliability studies, risk evaluation, and informed decision-making.

Proposing new unit distributions is vital for modeling bounded data, such as rates, indices, and proportions, where classical models like the beta distribution may require computationally intensive numerical procedures for some distributional characteristics and may not provide sufficient flexibility for certain empirical shapes. By transforming standard lifetime distributions onto the $[0, 1]$ interval, researchers can introduce novel frameworks that offer both tractable mathematical properties and highly flexible hazard rate shapes. These new models provide precise, computationally efficient tools that accurately capture the skewness and tail behaviors of bounded empirical phenomena without violating natural boundary constraints. In particular, unit distributions constructed from flexible lifetime models can inherit useful shape characteristics while providing additional flexibility for bounded observations. This is especially relevant in reliability and survival analysis, where normalized lifetimes, proportions, failure probabilities, and other bounded measurements naturally occur. The authors of [1] proposed Lomax–Bilal distribution based on the Bilal-G family of distributions.

Al-Omari et al. [2] suggested unit two parameters Mirra distribution. Aidi et al. [3] proposed power Zeghdoudi distribution with applications to real right-censored data. Wang et al. [4] considered the problem of modeling cure fractions in interval-censored competing risks data. Nassr et al. [5] investigated the reliability analysis of inverted exponentiated Rayleigh parameters in terms of progressive hybrid censoring data and applied it in medical data. Yu et al. [6] considered the estimation of the parameters of the Weibull distribution using an analytical solution for interval-censored data. The Dagum distributions were considered by [7] in analyzing competing risks with progressively type-II censored data. Hassan et al. [8] considered censoring for modeling engineering and medical data based on an extension of the XShanker distribution under. Statistical analysis of alpha-power exponential distribution is considered by [9] using unified hybrid censored data.

Gündüz and Korkmaz [10] suggested a new unit distribution based on the unbounded Johnson distribution. Their model demonstrated the usefulness of transformation-based constructions for obtaining flexible probability models on the unit interval. Ragab et al. [11] proposed unit-extended exponential distribution with some applications. The development of this model further illustrates the growing interest in constructing unit distributions from well-established continuous lifetime distributions. Bashiru et al. [12] introduced the unit Zeghdoudi distribution. More recently, additional

unit distributions have been proposed to increase the range of density and hazard-rate shapes that can be represented on bounded domains. Elbatal and Aryal [13] suggested transmuted Dagum distribution. The authors in [14] suggested unit new two-parameter Aradhana distribution. The authors in [15] proposed a novel unit exponential regression model with estimation method with multicollinearity. Dey et al. [16] proposed Dagum distribution with some properties and estimation.

Osi et al. [17] suggested the inverted Dagum distribution (IDD) with a cumulative distrributiin function (CDF) given by:

$$G(y;\alpha,\theta,\lambda) = 1 - (\lambda y^{\alpha} + 1)^{-\theta} \tag{1.1}$$

with the corresponding probability density function (PDF) given by

$$g(y;\alpha,\theta,\lambda) = \alpha\theta\lambda y^{\alpha-1}\,(\lambda y^{\alpha} + 1)^{-\theta-1}, \quad x > 0, \alpha > 0, \theta > 0, \lambda > 0. \tag{1.2}$$

The Dagum distribution is a flexible, three-parameter probability distribution that has been widely used for modeling income, wealth, survival, and reliability data. Owing to its ability to accommodate positively skewed data with heavy tails, it has received considerable attention in economics, actuarial science, engineering, and biomedical research. The distribution can exhibit a variety of density and hazard rate shapes depending on its parameter values, making it suitable for analyzing heterogeneous lifetime data. Over the years, several extensions and generalizations of the Dagum distribution have been proposed to improve its flexibility and broaden its range of applications. Among these, the IDD has emerged as an effective alternative for modeling lifetime data with decreasing or nonmonotone hazard rate functions.

Although these developments provide useful alternatives for bounded and lifetime data, an important gap remains in the construction of a unit-valued counterpart of the IDD. In particular, there is a need for a bounded model that combines the three-parameter flexibility of the inverted Dagum family with tractable reliability characteristics and estimation procedures for both complete and censored observations. Existing unit distributions do not directly provide this combination of flexibility and the specific hazard-rate behavior inherited from the inverted Dagum structure.

Motivated by the increasing demand for probability models defined on bounded intervals, this paper introduces the unit-inverted Dagum distribution (UIDD), which extends the IDD to the unit interval while preserving its desirable flexibility and providing an effective model for bounded lifetime and reliability data.

The main novelty of the proposed UIDD lies in its ability to transfer the flexible three-parameter structure of the inverted Dagum model to the unit interval, thereby providing a unified model for bounded lifetime and reliability data. In addition to deriving its fundamental distributional properties, we investigate several reliability measures, including the survival and hazard rate functions, mean residual life, mean inactivity time, and vitality functions, together with moments, the quantile function, order statistics, and Rényi and Tsallis entropies. Importantly, the practical performance of the UIDD is assessed quantitatively through both simulation and real-data analyses. The simulation study examines the finite-sample bias and mean squared error (MSE) of the maximum likelihood estimators under different sample sizes and parameter configurations, and the empirical study compares the UIDD with several competing models using established goodness-of-fit and information criteria.

The empirical results show that the proposed model can provide substantial improvements in goodness of fit over the competing distributions considered in the analysis. In particular, the UIDD

produces lower values of the selected information criteria and goodness-of-fit statistics for the considered censored lifetime dataset, demonstrating its ability to capture the observed bounded lifetime behavior. Moreover, the simulation results indicate that the bias and MSE of the maximum likelihood estimators decrease as the sample size increases, supporting the reliability of the proposed estimation procedure. These quantitative findings demonstrate that the proposed model is not merely a theoretical extension but provides a practical and flexible alternative for modeling bounded data.

Accordingly, the principal contributions of this study can be summarized as follows: (i) introducing a new three-parameter unit distribution derived from the inverted Dagum model; (ii) establishing a comprehensive set of distributional, reliability, and information-theoretic properties; (iii) developing maximum likelihood estimation for both complete and right-censored samples; (iv) evaluating the finite-sample behavior of the estimators through Monte Carlo simulation; and (v) quantitatively comparing the proposed model with existing competitors using real censored lifetime data.

The remainder of the paper is organized as follows. Section 2 introduces the proposed UIDD distribution, including its probability density function and its fundamental characteristics. Section 3 presents several statistical properties of the distribution, including the reliability functions, moments, quantile function, Tsallis and Rényi entropies, the distribution of order statistics, the mean residual life function, the mean inactivity time, and the vitality function. Section 4 discusses the estimation of the model parameters using the maximum likelihood estimation (MLE) method for both complete and right-censored data. A simulation study is presented in Section 5 to evaluate the performance of the proposed estimators. Section 6 illustrates the applicability of the proposed distribution by analyzing censored real-life datasets and comparing its performance with existing models using various goodness-of-fit criteria. Finally, Section 7 concludes the paper with a summary of the main findings and suggestions for future research.

2. The proposed distribution

In this section, we introduce the CDF and PDF of the suggested UIDD with illustrated plots. Let $X = e^{-Y}$, and then, $Y = -\ln X$ for $X \in (0, 1]$. Using the CDF given in Eq (1.1), the suggested unit IDD has the CDF defined as

$$F(x;\alpha,\theta,\lambda) = \begin{cases} 0, & x \le 0, \\ (1+\lambda(-\ln x)^{\alpha})^{-\theta}, & 0 < x \le 1, \\ 1, & x > 1, \end{cases} \tag{2.1}$$

with a PDF given as:

$$f(x;\alpha,\theta,\lambda) = \begin{cases} \dfrac{\theta\lambda\alpha}{x}(-\ln x)^{\alpha-1}(1+\lambda(-\ln x)^{\alpha})^{-(\theta+1)}, & 0 < x < 1, \\ 0, & \text{otherwise}, \end{cases} \tag{2.2}$$

for $\alpha > 0, \lambda > 0, \theta > 0$. Figure 1 presented the PDF and CDF plots of the distribution for some parameter values. The plot shows that the UIDD has many shapes, including decreasing, increasing, and increasing-decreasing.

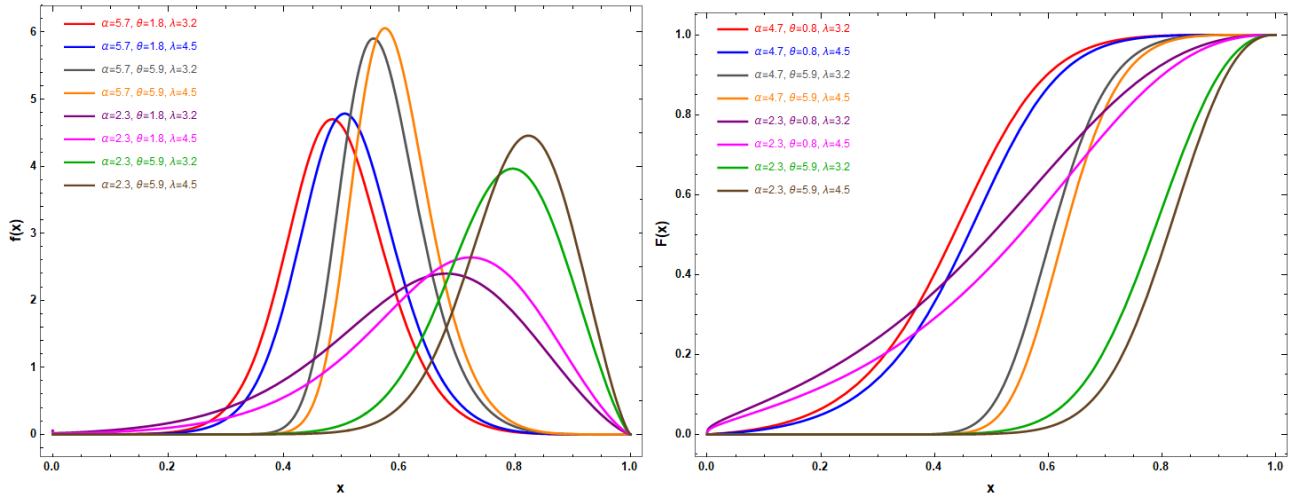


Figure 1. The PDF and CDF plots of the UIDD for some parameter values.

3. Properties of the UIDD

In this section, the main statistical properties of the distribution are discussed, such as the reliability functions, moments, quantile function, Tsallis and Rényi entropies, order statistics, mean residual life function, mean inactivity time, and vitality function.

3.1. Reliability functions

Survival function: In reliability and survival analysis, the survival function $S(x)$ quantifies the likelihood that a lifetime variable X persists beyond a specified time x. It is expressed for the UIDD as

$$SF(x, \alpha, \theta, \lambda) = 1 - (\lambda(-\ln(x))^{\alpha} + 1)^{-\theta}. \tag{3.1}$$

In Figure 2, the survival function (SF) plots of the UIDD for some values of λ, θ, and α are provided.

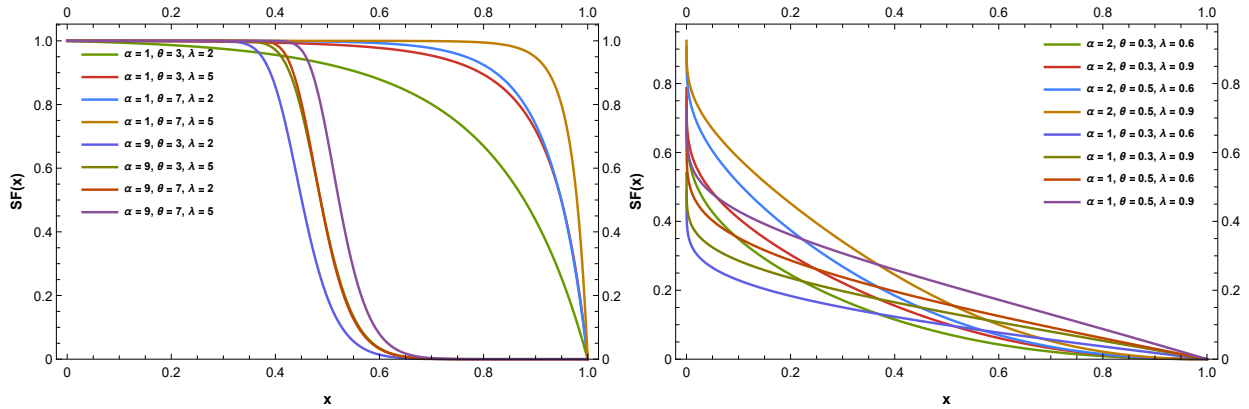


Figure 2. The SF plots of the UIDD for some parameter values.

The hazard rate function (HrF) is a key characteristic used to support model selection, and for the

UIDD, it is expressed as

$$HrF(x,\alpha,\lambda,b) = \frac{\alpha\theta\lambda(-\ln(x))^{\alpha-1}\left(1+\lambda(-\ln x)^{\alpha}\right)^{-(\theta+1)}}{x - x\left(\lambda(-\ln(x))^{\alpha}+1\right)^{-\theta}}.$$

Figure 3 shows the hazard rate function plots of the UIDD for some values of α, θ, and λ.

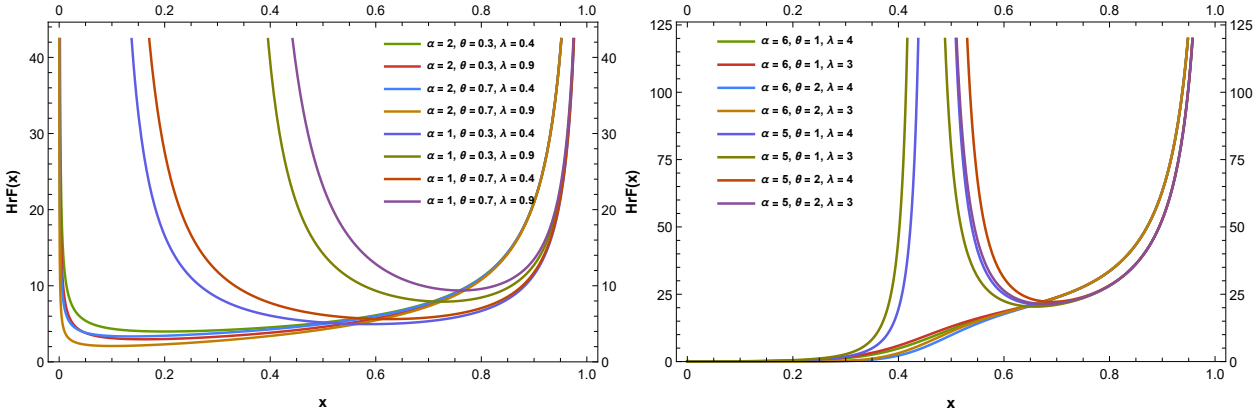


Figure 3. The HrF plots of the UIDD for some parameter values.

The reversed hazard rate function (RHF) of the UIDD is defined as:

$$RHF(x,\alpha,\theta,\lambda) = \frac{\alpha\theta\lambda(-\ln x)^{\alpha-1}}{x\left[1+\lambda(-\ln x)^{\alpha}\right]}.$$

Figure 4 displays the reversed hazard rate function plots of the UIDD for some choices of α, θ, and λ.

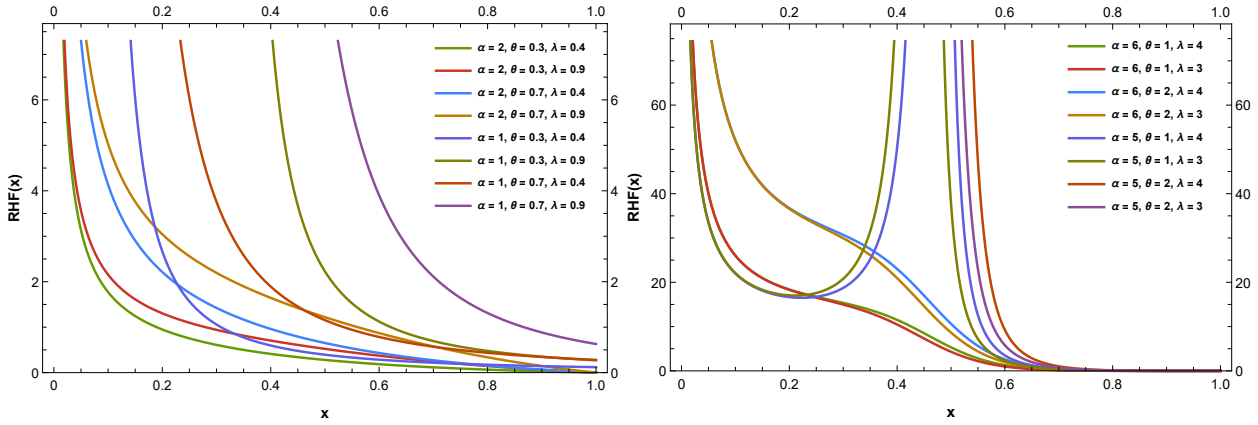


Figure 4. The RHF plots of the UIDD for some parameter values.

3.2. Quantile function and moments

For a random variable X with cumulative distribution function $F(x) = P(X \leq x)$, the quantile function $Q(p)$ is defined as the generalized inverse of F:

$$Q(p) = F^{-1}(p) = \inf\{x \in \mathbb{R} : F(x) \geq p\}, \quad p \in (0,1).$$

Given the CDF of the UIDD in Eq (2.1), the quantile function $Q(u) = F^{-1}(u)$ is as follows:

$$Q(u) = F^{-1}(u) = \exp\left[-\left(\frac{u^{-1/\theta} - 1}{\lambda}\right)^{1/\alpha}\right], \qquad 0 < u < 1.$$

Theorem 1. *Let X be a random variable with PDF in Eq* (2.2)*. Then, for r > 0, the rth moment of X is given by*

$$\mu'_r = E(X^r) = \theta\lambda\alpha \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\theta + 1 + k)}{k!\,\Gamma(\theta + 1)} \lambda^k \frac{\Gamma(\alpha(k+1))}{r^{\alpha(k+1)}}. \tag{3.2}$$

Proof. By definition,

$$\mu'_r = E(X^r) = \int_0^1 x^r f(x;\alpha,\theta,\lambda)\,dx = \theta\lambda\alpha \int_0^1 x^{r-1} (-\ln x)^{\alpha-1} (1 + \lambda(-\ln x)^{\alpha})^{-(\theta+1)}\,dx. \tag{3.3}$$

Substituting $y = -\ln x$, so that $x = e^{-y}$ and $dx = -e^{-y}\,dy$, transforms the range $x \in (0,1)$ into $y \in (0,\infty)$, and becuse $x^{r-1}dx = -e^{-ry}\,dy$, we have

$$\mu'_r = \theta\lambda\alpha \int_0^{\infty} e^{-ry}\, y^{\alpha-1} (1 + \lambda y^{\alpha})^{-(\theta+1)}\,dy. \tag{3.4}$$

Applying the generalized binomial series

$$(1 + \lambda y^{\alpha})^{-(\theta+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k(\theta + 1)_k}{k!} \lambda^k y^{\alpha k}, \text{ where } (\theta + 1)_k = \frac{\Gamma(\theta + 1 + k)}{\Gamma(\theta + 1)},$$

we obtain

$$\mu'_r = \theta\lambda\alpha \sum_{k=0}^{\infty} \frac{(-1)^k(\theta + 1)_k}{k!} \lambda^k \int_0^{\infty} e^{-ry}\, y^{\alpha(k+1)-1}\,dy.$$

Because $r > 0$ and $\alpha(k + 1) > 0$ for every $k \geq 0$,

$$\int_0^{\infty} e^{-ry}\, y^{\alpha(k+1)-1}\,dy = \frac{\Gamma(\alpha(k+1))}{r^{\alpha(k+1)}}$$

completes the proof. □

Now, to obtain the moment generating function of the UIDD, let $U = -\ln X$, $X = e^{-U}$, $U > 0$. Then, the PDF of U is

$$f_U(u) = \alpha\theta\lambda\, u^{\alpha-1} (1 + \lambda u^{\alpha})^{-(1+\theta)}, \qquad u > 0.$$

Let

$$Z = \lambda U^{\alpha} \implies U = \left(\frac{Z}{\lambda}\right)^{1/\alpha}.$$

It follows that

$$f_Z(z) = \theta(1+z)^{-(1+\theta)}, \qquad z > 0,$$

which is the PDF of a Lomax distribution. Therefore, the moment generating function of X is

$$\begin{aligned} M_X(t) &= \mathbb{E}\left(e^{tX}\right) \\ &= \alpha\theta\lambda \int_0^\infty e^{te^{-u}} u^{\alpha-1} \left(1+\lambda u^\alpha\right)^{-(1+\theta)} \, du \\ &= \theta \int_0^\infty \frac{\exp\left\{t \exp\left[-(z/\lambda)^{1/\alpha}\right]\right\}}{(1+z)^{1+\theta}} \, dz. \end{aligned} \tag{3.5}$$

Because $0 < X \leq 1$, for every finite $t \in \mathbb{R}$,

$$\sum_{n=0}^{\infty} \frac{|t|^n X^n}{n!} \leq \sum_{n=0}^{\infty} \frac{|t|^n}{n!} = e^{|t|}.$$

Thus, the Taylor series of e^{tX} is absolutely dominated by the integrable constant $e^{|t|}$. Consequently, by the dominated convergence theorem, the expectation and the infinite sum may be interchanged. Using

$$e^{tx} = \sum_{n=0}^{\infty} \frac{t^n}{n!} x^n,$$

we obtain

$$\begin{aligned} M_X(t) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathbb{E}\left(X^n\right) \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \theta \int_0^\infty (1+z)^{-(1+\theta)} \exp\left[-n(z/\lambda)^{1/\alpha}\right] \, dz. \end{aligned} \tag{3.6}$$

For $n = 0$, the corresponding term is

$$\theta \int_0^\infty (1+z)^{-(1+\theta)} \, dz = 1.$$

Therefore, the zeroth term of the MGF is equal to one, as required by the identity $M_X(0) = 1$.

For the special case $\alpha = 1$ and $n \geq 1$, we have

$$\theta \int_0^\infty e^{-nz/\lambda} (1+z)^{-(1+\theta)} \, dz = \theta e^{n/\lambda} \left(\frac{n}{\lambda}\right)^\theta \Gamma\left(-\theta, \frac{n}{\lambda}\right),$$

where $\Gamma(s, x)$ denotes the upper incomplete gamma function. Hence,

$$M_X(t) = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} \theta e^{n/\lambda} \left(\frac{n}{\lambda}\right)^\theta \Gamma\left(-\theta, \frac{n}{\lambda}\right)$$

for every finite $t \in \mathbb{R}$. For general $\alpha > 0$, the moment generating function can be expressed as:

$$M_X(t) = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} \theta \int_0^\infty (1+z)^{-(1+\theta)} \exp\left[-n(z/\lambda)^{1/\alpha}\right] \, dz,$$

for $\alpha > 0$ and every finite $t \in \mathbb{R}$.

3.3. Entropy measures

The Rényi entropy [18] is a generalization of the Shannon entropy that introduces a parameter η to control the weight given to different probabilities, and it is defined as:

$$H_\eta(X) = \frac{1}{1-\eta}\log\left(\int_{-\infty}^{\infty}[f(x)]^\eta\,dx\right), \eta > 0, \eta \neq 1.$$

Theorem 2. *Let X be a random variable following the UIDD with parameters $\alpha, \theta, \lambda > 0$. Then, for $0 < \eta < 1$, the Rényi entropy of X is given by*

$$H_\eta(X) = \frac{1}{1-\eta}\log\left\{\frac{(\theta\lambda\alpha)^\eta}{\Gamma(c_\eta)\beta_\eta^{\nu_\eta}} \times H_{2,1}^{1,2}\left[\frac{\lambda}{\beta_\eta^\alpha}\,\middle|\,\begin{matrix}(1-c_\eta,1),\ (1-\nu_\eta,\alpha)\\ (0,1)\end{matrix}\right]\right\}, \tag{3.7}$$

where $H_{p,q}^{m,n}[\cdot]$ denotes the Fox H-function, $\beta_\eta = 1-\eta,\ \nu_\eta = 1+\eta(\alpha-1),\ c_\eta = \eta(\theta+1)$.

Proof. By definition, the Rényi entropy of order η is

$$H_\eta(X) = \frac{1}{1-\eta}\log\left(\int_0^1[f(x;\alpha,\theta,\lambda)]^\eta\,dx\right), \qquad \eta > 0, \quad \eta \neq 1. \tag{3.8}$$

Hence,

$$I(\eta) = \int_0^1 (\theta\lambda\alpha)^\eta x^{-\eta}(-\ln x)^{\eta(\alpha-1)} \times (1+\lambda(-\ln x)^\alpha)^{-\eta(\theta+1)}\,dx.$$

Applying the transformation $y = -\ln x,\ x = e^{-y},\ dx = -e^{-y}\,dy$, the interval $0 < x < 1$ is transformed into $0 < y < \infty$. Therefore,

$$I(\eta) = (\theta\lambda\alpha)^\eta \int_0^\infty e^{-(1-\eta)y} y^{\eta(\alpha-1)} \times (1+\lambda y^\alpha)^{-\eta(\theta+1)}\,dy. \tag{3.9}$$

For $0 < \eta < 1$, the integral in Eq (3.9) is finite. Indeed, as $y \to 0^+$, its integrand behaves as $y^{\eta(\alpha-1)}$, which is integrable because $\eta(\alpha-1) > -1$. Moreover, as $y \to \infty$, the factor $e^{-(1-\eta)y}$ guarantees exponential decay because $1-\eta > 0$. Using $\beta_\eta = 1-\eta,\ \nu_\eta = 1+\eta(\alpha-1),\ c_\eta = \eta(\theta+1)$, we can express Eq (3.9) as

$$I(\eta) = (\theta\lambda\alpha)^\eta \int_0^\infty e^{-\beta_\eta y} y^{\nu_\eta - 1}\,(1+\lambda y^\alpha)^{-c_\eta}\,dy. \tag{3.10}$$

To evaluate this integral, consider the Mellin–Barnes representation

$$(1+z)^{-c} = \frac{1}{2\pi i\,\Gamma(c)}\int_{\mathcal{L}}\Gamma(s)\Gamma(c-s)z^{-s}\,ds, \tag{3.11}$$

where $c > 0$ and the contour $\mathcal{L}$ is chosen such that $0 < \Re(s) < c$. Taking $z = \lambda y^\alpha$ and $c = c_\eta$, we obtain

$$(1+\lambda y^\alpha)^{-c_\eta} = \frac{1}{2\pi i\,\Gamma(c_\eta)}\int_{\mathcal{L}}\Gamma(s)\Gamma(c_\eta - s)\lambda^{-s}y^{-\alpha s}\,ds. \tag{3.12}$$

Substitution of Eq (3.12) into Eq (3.10) gives

$$I(\eta) = \frac{(\theta\lambda\alpha)^{\eta}}{2\pi i\,\Gamma(c_\eta)} \int_{\mathcal{L}} \Gamma(s)\Gamma(c_\eta - s)\lambda^{-s} \times \left[\int_0^{\infty} e^{-\beta_\eta y} y^{\nu_\eta - \alpha s - 1}\, dy\right] ds. \tag{3.13}$$

The interchange of the integrals is justified by absolute convergence when the contour is chosen such that

$$0 < \Re(s) < \min\left\{c_\eta, \frac{\nu_\eta}{\alpha}\right\}.$$

The inner integral in Eq (3.13) is the gamma integral

$$\int_0^{\infty} e^{-\beta_\eta y} y^{\nu_\eta - \alpha s - 1}\, dy = \beta_\eta^{-(\nu_\eta - \alpha s)}\Gamma(\nu_\eta - \alpha s). \tag{3.14}$$

It follows that

$$I(\eta) = \frac{(\theta\lambda\alpha)^{\eta}}{\Gamma(c_\eta)\beta_\eta^{\nu_\eta}} \frac{1}{2\pi i} \int_{\mathcal{L}} \Gamma(s)\Gamma(c_\eta - s) \times \Gamma(\nu_\eta - \alpha s)\left(\frac{\lambda}{\beta_\eta^{\alpha}}\right)^{-s} ds. \tag{3.15}$$

By the Mellin–Barnes definition of the Fox H-function,

$$\frac{1}{2\pi i} \int_{\mathcal{L}} \Gamma(s)\Gamma(c_\eta - s)\Gamma(\nu_\eta - \alpha s) z^{-s}\, ds = H_{2,1}^{1,2}\left[z \,\middle|\, \begin{matrix} (1 - c_\eta, 1),\ (1 - \nu_\eta, \alpha) \\ (0, 1) \end{matrix}\right].$$

Consequently,

$$I(\eta) = \frac{(\theta\lambda\alpha)^{\eta}}{\Gamma(c_\eta)\beta_\eta^{\nu_\eta}} \times H_{2,1}^{1,2}\left[\frac{\lambda}{\beta_\eta^{\alpha}} \,\middle|\, \begin{matrix} (1 - c_\eta, 1),\ (1 - \nu_\eta, \alpha) \\ (0, 1) \end{matrix}\right]. \tag{3.16}$$

Finally, substituting Eq (3.16) into Eq (3.8) completes the proof. □

Theorem 3. *Let X be a random variable having the UIDD probability density function given by Eq (2.2), with $\alpha, \theta, \lambda > 0$. Define $\beta_\varrho = 1 - \varrho$, $\nu_\varrho = 1 + \varrho(\alpha - 1)$, $c_\varrho = \varrho(\theta + 1)$. Then, for $0 < \varrho < 1$, the Tsallis entropy of X is given by*

$$H_\varrho(X) = \frac{1}{\varrho - 1}\left\{1 - \frac{(\theta\lambda\alpha)^{\varrho}}{\Gamma(c_\varrho)\beta_\varrho^{\nu_\varrho}} \times H_{2,1}^{1,2}\left[\frac{\lambda}{\beta_\varrho^{\alpha}} \,\middle|\, \begin{matrix} (1 - c_\varrho, 1),\ (1 - \nu_\varrho, \alpha) \\ (0, 1) \end{matrix}\right]\right\}, \tag{3.17}$$

where $H_{p,q}^{m,n}[\cdot]$ denotes the Fox H-function.

Proof. By definition, the Tsallis entropy of order ϱ is

$$H_\varrho(X) = \frac{1}{\varrho - 1}\left[1 - \int_0^1 [f(x; \alpha, \theta, \lambda)]^{\varrho}\, dx\right], \qquad \varrho > 0, \quad \varrho \neq 1,$$

and following the steps of the proof of Theorem 2, the proof directly follows. □

3.4. Order statistics

The CDF and PDF of the kth order statistic $X_{(k)}$ from a random sample $X_1, X_2, \ldots, X_n$ with a common CDF $G(x)$ are given by:

$$G_{X_{(k)}}(x) = \sum_{j=k}^{n} \binom{n}{j} [G(x)]^j [1 - G(x)]^{n-j} =_2 F_1\left(1, k-n; k; -\frac{1}{(\lambda(-\ln(x))^\alpha + 1)^\theta - 1}\right)$$
$$\times \frac{n!\left((\lambda(-\ln(x))^\alpha + 1)^{-\theta}\right)^k \left(1 - (\lambda(-\ln(x))^\alpha + 1)^{-\theta}\right)^{n-k}}{(k-1)!(n-k)!} \tag{3.18}$$

and

$$g_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!}[G(x)]^{k-1}[1 - G(x)]^{n-k} g(x)$$
$$= \frac{\alpha\theta\lambda n!(-\ln(x))^{\alpha-1}\left((\lambda(-\ln(x))^\alpha + 1)^{-\theta}\right)^{j-1}\left(1 - (\lambda(-\ln(x))^\alpha + 1)^{-\theta}\right)^{n-j}}{x\,(\lambda \ln^\alpha(x) + 1)^{\theta+1}\,\Gamma(j)\Gamma(n-j+1)}. \tag{3.19}$$

The PDF of the minimum-and-maximum-order statistics are given by:

$$g_{X_{(1)}}(x) = \frac{\alpha\theta\lambda n!(-\ln(x))^{\alpha-1}\left(1 - (\lambda(-\ln(x))^\alpha + 1)^{-\theta}\right)^{n-1}}{x\,(\lambda \ln^\alpha(x) + 1)^{\theta+1}\,\Gamma(n)},$$
$$g_{X_{(n)}}(x) = \frac{\alpha\theta\lambda n!(-\ln(x))^{\alpha-1}\left((\lambda(-\ln(x))^\alpha + 1)^{-\theta}\right)^{n-1}}{x\,(\lambda \ln^\alpha(x) + 1)^{\theta+1}\,\Gamma(n)}.$$

3.5. Mean residual life function, mean inactivity time, and vitality function

Let $y_t = -\ln t$. The lower and upper incomplete gamma functions are defined, respectively, by

$$\gamma(s, z) = \int_0^z u^{s-1} e^{-u}\, du \quad \text{and} \quad \Gamma(s, z) = \int_z^\infty u^{s-1} e^{-u}\, du.$$

For $t \in (0, 1)$, the mean residual life (MRL) function, the mean inactivity time (MIT) function, and the vitality function (VF) are defined, respectively, as

$$MRL(t) = E\,(X - t \mid X > t) = \frac{1}{1 - F(t)} \int_t^1 [1 - F(x)]\, dx,$$

$$MIT(t) = E\,(t - X \mid X \le t) = t - \frac{1}{F(t)} \int_0^t x f(x)\, dx,$$

$$VF(t) = E\,(X \mid X \ge t) = \frac{1}{1 - F(t)} \int_t^1 x f(x)\, dx.$$

Theorem 4. *Let $y_t = -\ln t$. For $t \in (0, 1)$, the MRL, MIT, and VF are given by*

$$MRL(t) = \frac{\theta\lambda\alpha}{1 - (1 + \lambda y_t^\alpha)^{-\theta}} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\theta + 1 + k)}{k!\,\Gamma(\theta + 1)} \lambda^k \gamma(\alpha(k+1), y_t) - t, \tag{3.20}$$

$$MIT(t) = t - \frac{\theta\lambda\alpha}{(1+\lambda y_t^{\alpha})^{-\theta}} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\theta+1+k)}{k!\,\Gamma(\theta+1)} \lambda^k \Gamma(\alpha(k+1), y_t), \tag{3.21}$$

$$VF(t) = \frac{\theta\lambda\alpha}{1-(1+\lambda y_t^{\alpha})^{-\theta}} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\theta+1+k)}{k!\,\Gamma(\theta+1)} \lambda^k \gamma(\alpha(k+1), y_t). \tag{3.22}$$

Proof. Using the transformation $y = -\ln x$, we have $x = e^{-y}$ and $dx = -e^{-y}\,dy$. Because this transformation is decreasing, the integration limits are reversed. In particular, $x \in (t,1) \iff y \in (0, y_t)$, whereas $x \in (0,t) \iff y \in (y_t, \infty)$. Therefore, the integrals associated with the MRL and VF produce the lower incomplete gamma function $\gamma(s, y_t)$, whereas the integral associated with the MIT produces the upper incomplete gamma function $\Gamma(s, y_t)$. Finally, because the distribution is continuous,

$$VF(t) = E(X \mid X \geq t) = E(X \mid X > t), \text{and hence } MRL(t) = VF(t) - t.$$

This completes the proof. □

4. Estimation of parameters

This section addresses parameter estimation for both complete and censored data.

4.1. Parameter identifiability

Before proceeding with the estimation procedure, it is important to establish the identifiability of the parameters of the proposed UIDD. Let the parameter vector be defined as

$$\boldsymbol{\Theta} = (\alpha, \theta, \lambda)^{\top}, \qquad \alpha > 0,\ \theta > 0,\ \lambda > 0.$$

The probability density function of the UIDD is given by

$$f(x; \alpha, \theta, \lambda) = \frac{\alpha\theta\lambda}{x}(-\ln x)^{\alpha-1}\left[1+\lambda(-\ln x)^{\alpha}\right]^{-(\theta+1)}, \qquad 0 < x < 1.$$

To establish identifiability, suppose that two parameter vectors

$$\boldsymbol{\Theta}_1 = (\alpha_1, \theta_1, \lambda_1)^{\top} \quad \text{and} \quad \boldsymbol{\Theta}_2 = (\alpha_2, \theta_2, \lambda_2)^{\top}$$

generate the same probability density function, that is,

$$f(x; \alpha_1, \theta_1, \lambda_1) = f(x; \alpha_2, \theta_2, \lambda_2), \qquad 0 < x < 1.$$

Using the transformation $y = -\ln x$, where $y > 0$, the corresponding density can be expressed as

$$g(y; \alpha, \theta, \lambda) = \alpha\theta\lambda y^{\alpha-1}\left(1+\lambda y^{\alpha}\right)^{-(\theta+1)}.$$

The parameters α, θ, and λ have distinct effects on the distributional form. In particular, α determines the power behavior through the term $y^{\alpha-1}$, θ controls the asymptotic decay rate, and λ determines the scaling of the term $1+\lambda y^{\alpha}$. Therefore, equality of the two density functions for all $y > 0$ implies

$$\alpha_1 = \alpha_2, \qquad \theta_1 = \theta_2, \qquad \lambda_1 = \lambda_2.$$

Hence, the parameter vector $(\alpha, \theta, \lambda)$ is identifiable over the parameter space

$$\mathcal{P} = \{(\alpha, \theta, \lambda) : \alpha > 0,\ \theta > 0,\ \lambda > 0\}.$$

In addition, local identifiability can be examined numerically through the observed information matrix evaluated at the maximum likelihood estimates. A nonsingular observed information matrix provides further evidence that the three parameters are locally distinguishable and that the likelihood function is locally well-behaved. This identifiability result provides the theoretical basis for the subsequent MLE and inferential procedures.

4.2. MLE for complete data

The likelihood function for a random variable X is given by the joint PDF of the observed data. If $X_1, X_2, ..., X_n$ are independent and identically distributed (i.i.d.) observations from the given PDF in Eq (2.2), then, the likelihood function for a sample of size n is

$$L(\Theta) = \prod_{i=1}^{n} f(x_i; \Theta).$$

The log-likelihood function, $l(\theta, \lambda, \alpha)$, is

$$\begin{aligned} l(\Theta) = \ln L(\Theta) &= \sum_{i=1}^{n} \ln f(x_i; \Theta) \\ &= n\ln(\theta) + n\ln(\lambda) + n\ln(\alpha) - \sum_{i=1}^{n} \ln(x_i) + (\alpha - 1)\sum_{i=1}^{n} \ln(-\ln x_i) \\ &- (\theta + 1)\sum_{i=1}^{n} \ln(1 + \lambda(-\ln x_i)^{\alpha}). \end{aligned}$$

We find the MLEs by taking the partial derivatives of the log-likelihood with respect to each parameter and setting them to zero. The partial derivative with respect to θ, λ, and α, respectively, are:

$$\begin{aligned} \frac{\partial l}{\partial \theta} &= \frac{n}{\theta} - \sum_{i=1}^{n} \ln(1 + \lambda(-\ln x_i)^{\alpha}), \\ \frac{\partial l}{\partial \lambda} &= \frac{n}{\lambda} - (\theta + 1)\sum_{i=1}^{n} \frac{(-\ln x_i)^{\alpha}}{1 + \lambda(-\ln x_i)^{\alpha}} \end{aligned}$$

and

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln(-\ln x_i) - (\theta + 1)\sum_{i=1}^{n} \frac{\lambda(-\ln x_i)^{\alpha}\ln(-\ln x_i)}{1 + \lambda(-\ln x_i)^{\alpha}}.$$

Setting this to zero and solving for $\hat{\theta}$, $\hat{\lambda}$, and α gives:

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} \ln(1 + \hat{\lambda}(-\ln x_i)^{\hat{\alpha}})},$$

$$\frac{n}{\hat{\lambda}} = (\hat{\theta}+1)\sum_{i=1}^{n}\frac{(-\ln x_i)^{\hat{\alpha}}}{1+\hat{\lambda}(-\ln x_i)^{\hat{\alpha}}},$$

$$\frac{n}{\hat{\alpha}} + \sum_{i=1}^{n}\ln(-\ln x_i) = (\hat{\theta}+1)\sum_{i=1}^{n}\frac{\hat{\lambda}(-\ln x_i)^{\hat{\alpha}}\ln(-\ln x_i)}{1+\hat{\lambda}(-\ln x_i)^{\hat{\alpha}}}.$$

The MLEs for θ, λ, and α are the solutions to this system of three nonlinear equations. Because these equations are interdependent and complex, they do not have a simple closed-form solution. To find the specific numerical values of the MLEs for a given dataset, numerical methods such as the Newton Raphson method or other optimization algorithms are used.

4.3. MLE for right censored data

Consider $X = (X_1, X_2, ..., X_n)^{\mathbf{T}}$ as a sample from the UIDD characterized by the parameter vector $\Theta = (\alpha, \lambda, \theta)^{\mathbf{T}}$, which may include right-censored data with a specified censoring duration τ. Each X_i can be expressed as $X_i = (x_i, \delta_i)$ where

$$\delta_i = \begin{cases} 0, & \text{if } x_i \text{ is a censoring time,} \\ 1, & \text{if } x_i \text{ is a failure time.} \end{cases}$$

The censoring is presumed to be noninformative, thus, the likelihood function can be expressed as

$$L_n(\Theta) = \prod_{i=1}^{n}\left[f(x_i,\Theta)\right]^{\delta_i}\left[S(x_i,\Theta)\right]^{1-\delta_i}.$$

The log-likelihood function $\ell_n(\Psi)$ is equivalent to

$$\ell_n(\Psi) = \sum_{i=1}^{n}\delta_i \ln f(x_i,\Theta) + \sum_{i=1}^{n}(1-\delta_i)\ln S(x_i,\Theta).$$

By employing the identical parameters, we may deduce

$$\ell_n(\theta) = \sum_{i=1}^{n}\delta_i\begin{bmatrix}\ln(\alpha\theta\lambda) - \ln(x_i) + (\alpha-1)\ln\left[-\ln(x_i)\right] \\ -(\theta+1)\ln\left(1+\lambda\left[-\ln(x_i)\right]^{\alpha}\right)\end{bmatrix} + \sum_{i=1}^{n}(1-\delta_i)\ln\left[1-\left(1+\lambda\left[-\ln(x_i)\right]^{\alpha}\right)^{-\theta}\right].$$

The maximum likelihood estimator $\hat{\Theta} = (\hat{\alpha}, \hat{\lambda}, \hat{\theta})^T$ for the unknown parameter $\Theta = (\alpha, \lambda, \theta)^T$ is generated from the subsequent nonlinear score equations. We present $\varpi_i = \left[-\ln(x_i)\right]$, where

$$\frac{\partial \ell_n}{\partial \alpha} = \sum_{i=1}^{n}\delta_i\left[\frac{1}{\alpha} + \ln(\omega_i) - \lambda(\theta+1)\frac{\omega_i^{\alpha}\ln(\omega_i)}{1+\lambda\omega_i^{\alpha}}\right] + \lambda\theta\sum_{i=1}^{n}(1-\delta_i)\frac{\omega_i^{\alpha}\ln\omega_i\left(1+\lambda\omega_i^{\alpha}\right)^{-(\theta+1)}}{1-\left(1+\lambda\omega_i^{\alpha}\right)^{-\theta}},$$

$$\frac{\partial \ell_n}{\partial \lambda} = \sum_{i=1}^{n}\delta_i\left[\frac{1}{\lambda} - \frac{(\theta+1)\omega_i^{\alpha}}{1+\lambda\omega_i^{\alpha}}\right] + \theta\sum_{i=1}^{n}(1-\delta_i)\frac{\omega_i^{\alpha}(1+\lambda\omega_i^{\alpha})^{-(\theta+1)}}{1-\left(1+\lambda\omega_i^{\alpha}\right)^{-\theta}},$$

$$\frac{\partial \ell_n}{\partial \theta} = \sum_{i=1}^{n}\delta_i\left[\frac{1}{\theta} - \ln(1+\lambda\omega_i^{\alpha})\right] + \sum_{i=1}^{n}(1-\delta_i)\frac{\left(1+\lambda\omega_i^{\alpha}\right)^{-\theta}\ln\left(1+\lambda\omega_i^{\alpha}\right)}{1-\left(1+\lambda\omega_i^{\alpha}\right)^{-\theta}}.$$

To compute the maximum likelihood estimators $\hat{\alpha}$, $\hat{\lambda}$, and $\hat{\theta}$ in the whole data scenario, we employ numerical techniques such as the Newton Raphson method, the Monte Carlo method, or the block blast(BB)-solve package.

5. Test statistic for right-censored data

The modified chi-square goodness-of-fit test, based on the methodology of [19], is a crucial statistical instrument for assessing model adequacy in the presence of right-censoring in data. In contrast to traditional chi-square tests, it integrates the Kaplan–Meier estimator to effectively address inadequate observations, rendering it particularly pertinent in survival and reliability studies. Its significance resides in enabling researchers to verify distributional assumptions in the presence of censoring, a frequent occurrence in medical, engineering, and geological contexts. Numerous writers have implemented this test, including [20] for the unit-generalized inverse Weibull distribution, and [21] for censored lifespan data, among others.

To verify the null hypothesis

$$H_0\colon P(X_i \le x \mid H_0) = F_0(x;\Theta),\ x \ge 0,\quad \boldsymbol{\Theta} = (\Theta_1, ..., \Theta_s)^T,\ \Theta \subset R^s,$$

let $X_1, ..., X_n$ be n i.i.d. random variables categorized into r classes I_j where $r \succ s$. When the data are subject to right-censoring and the parameter vector $\boldsymbol{\Theta}$ is unknown, the authors introduced a statistical test Y^2 defined by

$$Y^2 = \sum_{j=1}^{r} \frac{(D_j - b_j)^2}{D_j} + Q,$$

where D_j and b_j are the observed and expected failure periods corresponding to the grouping intervals I_j, and the quadratic form Q is defined as

$$Q = L^T \widehat{O}^{-1} L,\quad \widehat{K}_j = D_j/n,\quad D_j = \sum_{i:X_i \in I_j} \delta_i,\quad L = (L_1, ..., L_s)^T,\quad \widehat{O} = [\widehat{o}_{ll'}]_{s\times s},$$

$$\widehat{o}_{ll'} = \widehat{i}_{ll'} - \sum_{j=1}^{r} \widehat{P}_{lj}\widehat{P}_{l'j}\widehat{K}_j^{-1},\quad \widehat{P}_{lj} = \frac{1}{n}\sum_{i:x_i \in I_j} \delta_i \frac{\partial}{\partial\Theta} \ln h(x_i, \widehat{\boldsymbol{\Theta}}),$$

$$\widehat{i}_{ll'} = \frac{1}{n}\sum_{i=1}^{n} \delta_i \frac{\partial \ln h(x_i, \widehat{\boldsymbol{\Theta}})}{\partial \boldsymbol{\Theta}_l} \frac{\partial \ln h(x_i, \widehat{\boldsymbol{\Theta}})}{\partial \boldsymbol{\Theta}_{l'}},\quad \widehat{L}_l = \sum_{j=1}^{r} \widehat{P}_{lj}\widehat{K}_j^{-1} T_j,$$

$$l, l' = 1, 2, ..., s,\quad T_j = \frac{1}{\sqrt{n}}(D_j - b_j),\quad j = 1, 2, ..., r.$$

5.1. Goodness-of-fit test for UIDD

We denote $X_1, X_2, ..., X_n$ as a right-censored sample, where each observation is expressed as $x_i = \min(X_i, C_i)$ for $i = 1, 2, ..., n$ with X_i representing failure times and C_i indicating censoring times. Let τ be of finite duration, and let the observed data be categorized into $r > s$ subintervals

$$I_j = \left(a_{j-1}, a_j\right]$$

within the interval $[0, \tau]$. To evaluate the null hypothesis H_0 that this sample is in accordance with the UIDD model, we must first determine the predicted limit intervals a_j, which are defined by

$$\hat{a}_j = H^{-1}\left(\frac{E_j - \sum_{l=1}^{i-1} H\left(x_l, \widehat{\mathbf{\Theta}}\right)}{n-i+1}, \widehat{\mathbf{\Theta}}\right), \quad \hat{a}_r = \max\left(X_{(n),}\tau\right).$$

$H\left(x_l, \widehat{\mathbf{\Theta}}\right)$ denotes the cumulative hazard rate function of UIDD, and $\widehat{\mathbf{\Theta}}$ represents the maximum likelihood parameter estimators for ungrouped data. We subsequently acquire the observed and anticipated failure periods D_j and b_j, where $b_j = E_r/r$ for every j, with

$$E_r = \sum_{i=1}^{n} H\left(x_i, \widehat{\mathbf{\Theta}}\right).$$

Regarding the UIDD distribution, we obtain

$$E_j = \frac{-j}{k-1} \sum_{i=1}^{n} \ln\left[1 - \left(1 + \lambda\left[-\ln(x_i)\right]^{\alpha}\right)^{-\theta}\right], \quad j = 1, 2, ...r-1.$$

5.2. Quadratic form Q

To compute the quadratic form Q of the statistic Y_n^2, and because its distribution is independent of the parameters, we can utilize the estimated matrices $\hat{\boldsymbol{L}}$, $\hat{P}$, and the estimated information matrix $\hat{I}$. The components of $\hat{P}$ are delineated by

$$\widehat{P}_{lj} = \frac{1}{n} \sum_{i:x_i \in I_j}^{n} \delta_i \frac{\partial \ln h(x_i; \widehat{\mathbf{\Theta}})}{\partial \mathbf{\Theta}_l}$$

and acquired as follows:

$$\hat{C}_{1j} = \sum_{i:x_i \in I_j}^{n} \delta_i \left(\frac{1}{\alpha} - \ln(\omega_i) - \lambda(\theta-1)\frac{\omega_i^{\alpha}\ln(\omega_i)}{1+\lambda\omega_i^{\alpha}} + \frac{\lambda\theta\omega_i^{\alpha}\ln\omega_i\left(1+\lambda\omega_i^{\alpha}\right)^{-(\theta+1)}}{1-\left(1+\lambda\omega_i^{\alpha}\right)^{\theta}}\right),$$

$$\hat{C}_{2j} = \sum_{i:x_i \in I_j}^{n} \delta_i \left(\frac{1}{\lambda} + \frac{(\theta-1)\omega_i^{\alpha}}{1+\lambda\omega_i^{\alpha}} + \frac{\theta\omega_i^{\alpha}(1+\lambda\omega_i^{\alpha})^{-(\theta-1)}}{1-\left(1+\lambda\omega_i^{\alpha}\right)^{-\theta}}\right),$$

$$\hat{C}_{3j} = \sum_{i:x_i \in I_j}^{n} \delta_i \left(\frac{1}{\theta} - \ln(1+\lambda\omega_i^{\alpha}) + \frac{\left(1+\lambda\omega_i^{\alpha}\right)^{-\theta}\ln\left(1+\lambda\omega_i^{\alpha}\right)}{1-\left(1+\lambda\omega_i^{\alpha}\right)^{-\theta}}\right)$$

and

$$\hat{L}_l = \sum_{j=1}^{r} \hat{P}_{lj} K_j^{-1} T_j, \; l = 1, 2, ..., m, \; j = 1, 2, ..., r.$$

SEWe calculated the statistic Y^2 for the UIDD distribution with unknown parameters, approximated using the maximum likelihood technique, while the data are censored on the right. This statistic adheres to a chi-squared distribution with r degrees of freedom.

$$Y_n^2\left(\widehat{\Theta}\right) = \sum_{j=1}^{r} \frac{\left(D_j - b_j\right)^2}{D_j} + \hat{L}^T \left[\hat{\imath}_{ll'} - \sum_{j=1}^{r} \hat{P}_{lj}\hat{P}_{l'j}\hat{K}_j^{-1}\right]^{-1} \hat{L}.$$

6. Simulations

In this section, some simulations are conducted to investigate the power of the estimators under MLE and MLE for right-censored data.

6.1. MLE

For each sample size $n \in \{10, 25, 50, 130, 350, 500\}$, one repeatedly generates $N = 10,000$ random samples; estimate the parameters $\left(\hat{\alpha}, \hat{\lambda}, \hat{\theta}\right)$ via MLE, and then study the bias, MSE, and confidence interval (CI) of the estimators as functions of n.

For each n:

(1) Generate $10,000$ i.i.d. samples from the UIDD ($\alpha = 4, \theta = 2, \lambda = 7$) distribution.

(2) For each replicate, compute the MLE $\left(\hat{\alpha}, \hat{\lambda}, \hat{\theta}\right)$.

(3) Compute empirical averages. Also, find

$$Bias\left(\hat{\Theta}\right) \approx \hat{\mathbf{\Theta}} - \mathbf{\Theta},\ MSE\left(\hat{\mathbf{\Theta}}\right) = \frac{1}{10,000} \sum_{i=1}^{10,000} (\hat{\mathbf{\Theta}} - \mathbf{\Theta})^2,\ \text{where } \mathbf{\Theta} = \alpha, \lambda, \theta,$$

and

$$\text{MSE}(\widehat{\mathbf{\Theta}}) = \widehat{\text{Var}}(\widehat{\mathbf{\Theta}}) + \text{Bias}(\widehat{\mathbf{\Theta}})^2,$$

and the CI is defined as:

$$\hat{\mathbf{\Theta}} \pm 1.96 * \widehat{SE}_{\text{emp}},$$

where

$$\widehat{SE}_{\text{emp}} = \sqrt{\text{Var}(\widehat{\mathbf{\Theta}})}.$$

The simulation results presented in Table 1 for various sample sizes $n = 10, 25, 50, 130, 350, 500$ demonstrate the finite-sample performance of the MLEs for the parameters $\widehat{\alpha}$, $\widehat{\lambda}$, and $\widehat{\theta}$

Table 1. The MLEs of $\widehat{\alpha}$, $\widehat{\lambda}$, $\widehat{\theta}$, and the 95% CI under different sample sizes.

Parameter	n	Estimate	Bias	MSE	95% CI
α	10	4.0345	0.0345	0.0202	(3.7643, 4.3047)
	25	4.0315	0.0315	0.0185	(3.7722, 4.2908)
	50	4.0267	0.0267	0.0164	(3.7812, 4.2722)
	130	4.0179	0.0179	0.0134	(3.7937, 4.2421)
	350	4.0092	0.0092	0.0115	(3.7998, 4.2186)
	500	4.0042	0.0042	0.0052	(3.8631, 4.1453)
λ	10	7.0536	0.0536	0.0262	(6.7542, 7.3530)
	25	7.0352	0.0352	0.0121	(6.8309, 7.2395)
	50	7.0237	0.0237	0.0095	(6.8384, 7.2090)
	130	7.0184	0.0184	0.0076	(6.8514, 7.1854)
	350	7.0075	0.0075	0.0042	(6.8813, 7.1337)
	500	7.0024	0.0024	0.0019	(6.9171, 7.0877)
θ	10	1.9756	-0.0244	0.0189	(1.7104, 2.2408)
	25	1.9794	-0.0206	0.0154	(1.7395, 2.2193)
	50	1.9823	-0.0177	0.0105	(1.7845, 2.1801)
	130	1.9856	-0.0144	0.0068	(1.8265, 2.1447)
	350	1.9902	-0.0098	0.0036	(1.8742, 2.1062)
	500	1.9975	-0.0025	0.0018	(1.9145, 2.0805)

Based on Table 1, the following remarks can be concluded:

- For all three parameters, the estimated values ($\widehat{\alpha}$, $\widehat{\lambda}$, and $\widehat{\theta}$) move progressively closer to their corresponding true values as the sample size increases, indicating that the estimators are asymptotically unbiased.
- The absolute bias decreases substantially with increasing sample size. At $n = 10$, the bias is relatively large, whereas by $n = 350$ and $n = 500$, it becomes almost negligible for all parameters.
- The MSE consistently decreases as the sample size increases, demonstrating that the estimators become increasingly accurate and stable. This reduction in MSE confirms the improved efficiency of the MLEs for larger samples.
- The 95% confidence intervals gradually narrow as n increases, reflecting greater precision in parameter estimation. At $n = 500$, all confidence intervals are tightly concentrated around the true parameter values ($\alpha = 4$, $\lambda = 7$, and $\theta = 2$), indicating excellent estimation precision.
- Among the three parameters, the estimator of λ exhibits the smallest MSE across all sample sizes, and the estimators of α and θ also display clear improvements in both bias and precision as the sample size increases.

In Figure 5 are the convergence plots of the three estimators for more illustration. They provide convergence graphs of the estimators versus the true parameters with 95% CI.

Figure 5 provides a graphical assessment of the finite-sample performance and convergence behavior of the maximum likelihood estimators of α, λ, and θ. The graphical results are consistent

with the numerical findings reported in Table 1. For all three parameters, the estimated values remain close to their respective true values, $\alpha = 4$, $\lambda = 7$, and $\theta = 2$, even for relatively small sample sizes, and the agreement becomes more pronounced as the sample size increases.

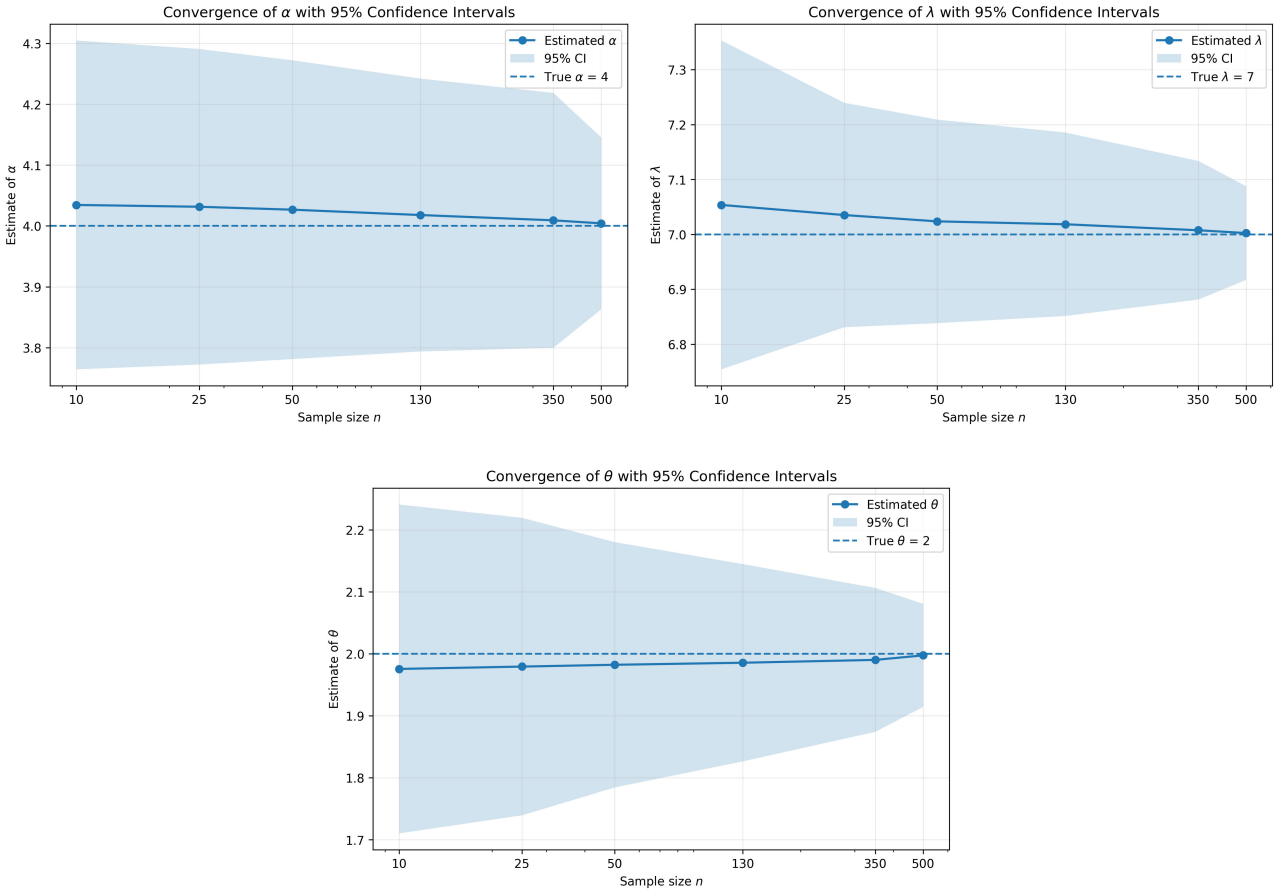


Figure 5. Plot convergence graphs of the estimators vs. the true parameters with a 95% CI.

More specifically, for α, the bias decreases from 0.0345 at $n = 10$ to 0.0042 at $n = 500$, whereas the MSE decreases from 0.0202 to 0.0052. A similar pattern is observed for λ, for which the bias decreases from 0.0536 to 0.0024, and the MSE decreases from 0.0262 to 0.0019. For θ, the bias decreases from -0.0244 to -0.0025, accompanied by a reduction in the MSE from 0.0189 to 0.0018. These results indicate that the estimation error gradually decreases as the sample size increases.

The 95% confidence intervals shown in Figure 5 also become narrower with increasing n, reflecting the corresponding reduction in the estimated sampling variability. For example, the confidence interval for α changes from $(3.7643, 4.3047)$ at $n = 10$ to $(3.8631, 4.1453)$ at $n = 500$. Similarly, for λ, the interval decreases from $(6.7542, 7.3530)$ to $(6.9171, 7.0877)$, whereas for θ, it decreases from $(1.7104, 2.2408)$ to $(1.9145, 2.0805)$. Importantly, the true parameter values remain within the corresponding confidence intervals throughout the considered sample sizes.

6.2. MLE for right-censored data

We generated $N = 10,000$ right-censored samples with different sizes

$$n = 10, 25, 50, 130, 350, 500$$

from the UIDD with parameters $\alpha = 2.5$, $\lambda = 4$, and $\theta = 1.5$. Using R statistical software and the Barzilai–Borwein (BB) algorithm [23], we calculate the maximum likelihood estimators of the unknown parameters and their MSE. The results are given in Table 2.

Table 2. The MLEs of α, λ, θ, and 95% CI under different sample sizes.

Parameter	n	Estimate	Bias	MSE	SE	95% CI
α	10	2.5835	0.0835	0.0752	0.2688	(2.0567, 3.1103)
	25	2.5762	0.0762	0.0532	0.2239	(2.1375, 3.0149)
	50	2.5678	0.0678	0.0212	0.1324	(2.3082, 2.8274)
	130	2.5326	0.0326	0.0164	0.1261	(2.2854, 2.7798)
	350	2.5139	0.0139	0.0089	0.0934	(2.3309, 2.6969)
	500	2.5056	0.0056	0.0067	0.0818	(2.3453, 2.6659)
λ	10	3.7569	0.2431	0.0784	0.1697	(3.4242, 4.0896)
	25	3.8230	0.1770	0.0432	0.1255	(3.5770, 4.0690)
	50	3.8952	0.1048	0.0325	0.1467	(3.6079, 4.1825)
	130	3.9536	0.0464	0.0105	0.0868	(3.7835, 4.1237)
	350	3.9765	0.0235	0.0075	0.0852	(3.8095, 4.1435)
	500	3.9985	0.0015	0.0032	0.0565	(3.8877, 4.1093)
θ	10	1.4625	0.0375	0.0452	0.2115	(1.0480, 1.8770)
	25	1.4783	0.0217	0.0316	0.1768	(1.1318, 1.8255)
	50	1.4852	0.0148	0.0248	0.1566	(1.1782, 1.7922)
	130	1.4920	0.0080	0.0102	0.1006	(1.2948, 1.6892)
	350	1.4956	0.0044	0.0076	0.0871	(1.3250, 1.6662)
	500	1.4978	0.0022	0.0025	0.0498	(1.4002, 1.5954)

Table 2 summarizes the finite-sample performance of the MLEs for the parameters α, λ, and θ. The simulation results provide the following:

- The estimated values of α, λ, and θ converge steadily to their corresponding true parameter values ($\alpha = 2.5$, $\lambda = 4$, and $\theta = 1.5$) as the sample size increases, demonstrating the consistency of the MLEs.
- The absolute bias decreases monotonically with increasing sample size for all three parameters. Aithoughmoderate bias is observed for small samples ($n = 10$ and $n = 25$), it becomes negligible for $n \geq 130$ and is virtually zero at $n = 500$, indicating that the estimators are asymptotically unbiased.
- The MSE decreases substantially as the sample size increases, confirming that the estimators become increasingly accurate and efficient. This reduction in MSE illustrates the improved stability and reliability of the parameter estimates for larger samples.

- The 95% confidence intervals become progressively narrower as the sample size increases, reflecting a significant gain in estimation precision. At $n = 500$, the confidence intervals are tightly concentrated around the true parameter values, indicating a high degree of statistical precision.
- Among the three parameters, the estimator of θ exhibits the smallest bias for all sample sizes, whereas the estimator of λ shows the largest initial bias for small samples but converges rapidly as the sample size increases. Overall, all three estimators demonstrate excellent finite-sample performance and very strong asymptotic behavior.

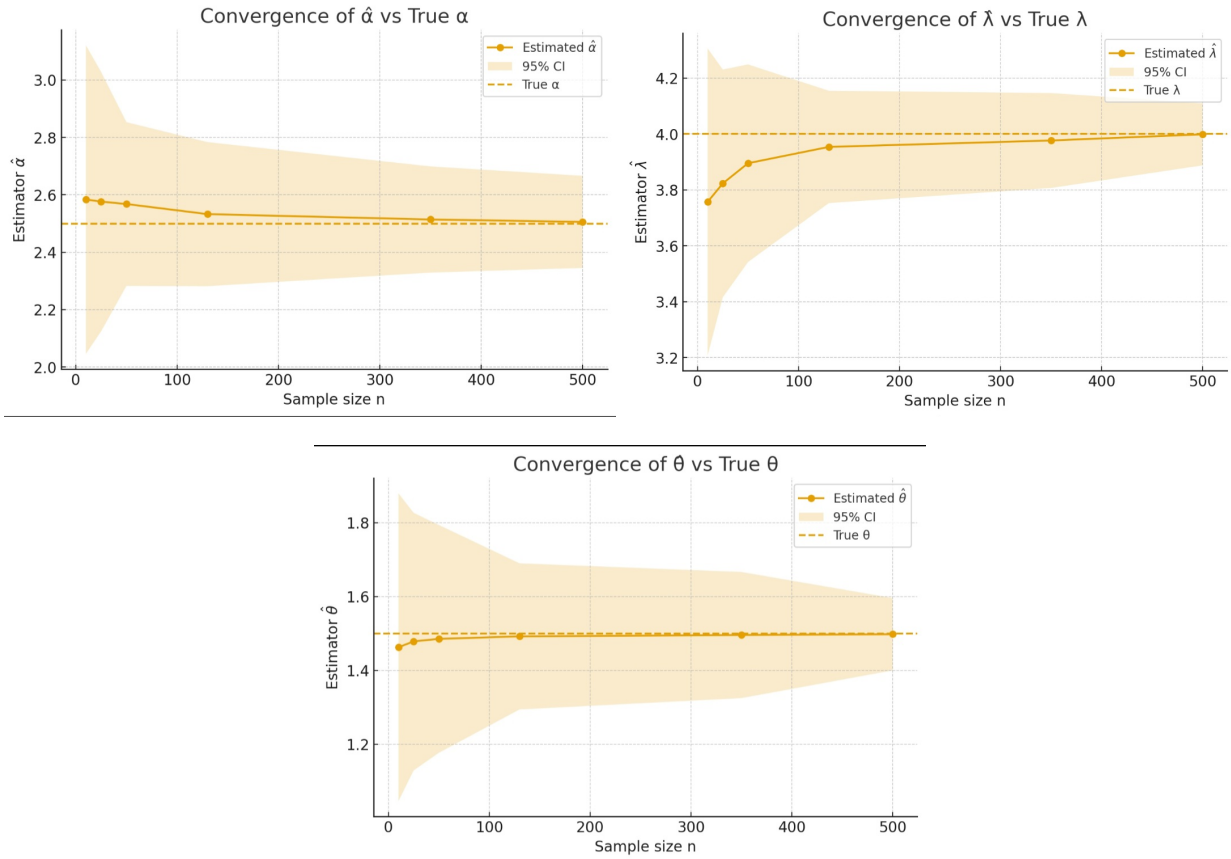


Figure 6. Plot convergence graphs of the estimators vs. the true parameters with 95% CI.

Figure 6 leads to the following remarks:

(1) For α: The estimated values $\hat{\alpha}$ (orange curve) start above the true value (2.5) when the sample size is small ($n = 10$), but they progressively decrease and stabilize very close to the true value as n increases. The 95% confidence intervals (shaded area) are wide for small n, showing large uncertainty. As n grows, the intervals shrink significantly, indicating improved precision. The true α (red dashed line) lies within all confidence intervals, confirming that the estimator is reliable. The convergence is smooth: Bias and MSE decrease with n, which is typical of a consistent estimator.

(2) For λ: The marker line shows the estimated $\hat{\lambda}$ as n increases. The shaded band is the 95% CI at each n. The dashed horizontal line is the true $\lambda = 4$. As n grows, $\hat{\lambda}$ approaches 4, and the CI narrows, showing consistency and increasing precision.

(3) For θ: The orange markers track the estimated values of θ as the sample size grows. The shaded region shows the 95% confidence intervals. The dashed horizontal line is the true parameter $\theta = 1.5$. You can see that the estimator converges quickly to 1.5, and the confidence intervals shrink, indicating better precision with larger n.

7. Real data application

To illustrate the applicability of the proposed distribution, a real dataset was analyzed.

7.1. Complete data

We assume that the times-to-failure for a type of turbocharger studied by [24] follow the UGE distribution. The observed failure times are given below, using the same method applied to the initial data to obtain values in the range [0, 1]:

$$4.8, 6.5, 7.0, 7.3, 7.7, 8.0, 8.4, 1.6, 3.5, 3.0, 4.6, 2.0, 3.9, 5.0,$$
$$5.6, 6.1, 6.5, 7.1, 7.3, 7.8, 8.1, 8.4, 2.6, 4.5, 5.1, 5.8, 6.3, 6.7,$$
$$7.3, 7.7, 7.9, 8.3, 8.5, 5.3, 6.0, 8.7, 8.8, 5.4, 6.0, 9.0.$$

To assess the performance of the UIDD distribution, we compare it with several classical models using goodness-of-fit measures.

To ensure a consistent comparison with the proposed UIDD model, the observations were transformed to the unit interval $(0, 1)$ using the min–max normalization prior to parameter estimation. In this analysis, all competing models considered in Table 3 are unit-valued distributions, namely the unit Burr-III, Kumaraswamy, and unit generalized inverse Weibull distributions. Consequently, all models are fitted to the same normalized observations and have the same support $(0, 1)$.

Table 3. Goodness-of-fit measures for unit-valued competing models.

Model	Parameter	$-\ell$	AIC	BIC	A	W	K-S	P-V
UIDD	(0.567,0.935,0.124)	63.847	133.694	138.760	0.1256	0.0767	0.0567	0.9746
Unit Burr-III	(1.6732,3.6239)	69.179	142.358	145.735	0.1393	0.0812	0.0634	0.9687
Kumaraswamy	(1.9369, 1.2890)	69.586	143.172	146.549	0.1438	0.0865	0.0678	0.9654
Unit generalized inverse Weibull	(1.743, 0.278, 0.789)	64.221	134.442	139.508	0.1287	0.0799	0.0598	0.9702

The results show that the UIDD yields the smallest values of $-\ell$, Anderson–Darling (A), Cramér–von Mises (W), and the Kolmogorov–Smirnov statistic, while also providing a high p-value. These results indicate a favorable empirical fit of the UIDD for the considered dataset.

7.2. Censored data analysis

We consider the following data consisting of 47 pieces of strength of a certain type of braided cord that had been weathered for a specified length of time, taken from [25]. The observed right-censored strength-values are given below:

0.0826*, 0.1947*, 0.2419*, 0.2802, 0.3894*, 0.4395, 0.4454*,
0.4631, 0.5044*, 0.6814, 0.6873, 0.7080, 0.7404, 0.7463, 0.7522, 0.7522, 0.7581,
0.7640, 0.7640, 0.7758, 0.7994, 0.7994, 0.8083, 0.8083, 0.8083, 0.8201, 0.8230,
0.8230, 0.8348, 0.8466, 0.8584, 0.8614, 0.8643, 0.8761, 0.8879, 0.8997, 0.8997,
0.9056, 0.9174, 0.9204, 0.9322, 0.9528, 0.9528, 0.9558, 0.9646, 0.9882, 1.0000.

The observed failure times are shown above, using the same method applied to the raw data to obtain values within the range [0, 1].

Using BB-solve software, we compute the maximum likelihood estimators of the unknown parameters

$$\widehat{\Theta} = \left(\hat{\alpha}, \hat{\lambda}, \hat{\theta}\right)^T = (2.145, 1.218, 0.956)^T .$$

Then, we group the observations into $r = 5$ intervals I_j. To calculate the criteria test Y_n^2 , we need the following results in Table 4.

Table 4. Estimated values of a_j, b_j, D_j, and the probabilities $\hat{P}_{1j}$, $\hat{P}_{2j}$, and $\hat{P}_{3j}$.

Parameter	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
$\hat{a}_j$	0.456	0.755	0.8250	0.9067	1.000
D_j	7	9	12	10	9
$\hat{P}_{1j}$	0.042	0.165	0.357	0.295	0.226
$\hat{P}_{2j}$	0.056	0.497	0.667	0.336	0.289
$\hat{P}_{3j}$	0.098	0.583	0.621	0.478	0.312
b_j	0.215	0.215	0.215	0.215	0.215

We therefore obtain the value of Y_n^2 as:

$$Y_n^2 = X^2 + Q = 2.6534 + 3.2265 = 5.8799.$$

As the value of $Y_n^2 = 5.8799$ is less than the critical value $\chi_5^2 = 11.0705$ (for significance level $\alpha = 0.05$), there is no evidence against the adequacy of the UIDD model for this data set censored with rights.

Table 5 summarizes the goodness-of-fit results for the right-censored braided-cord strength data. For this analysis, the observations were transformed to the unit interval prior to model fitting, and all competing distributions considered in the table are defined on $(0, 1)$. Therefore, the comparison is conducted under a common support using the same normalized observations and censoring scheme.

Table 5. Goodness-of-fit statistics based on the data.

Model	Parameters	Y^2	$-LL$	AIC	BIC	HQIC
UIDD	(2.145, 1.218, 0.956)	5.8799	42.8856	91.7712	96.9842	93.6819
Unit Burr-III	(0.9356, 1.6782)	5.9357	43.9657	91.9314	95.4067	93.2052
Kumaraswamy	(2.8401, 3.8316)	6.6048	46.1023	96.2046	99.6799	97.4784
Unit generalized inverse Weibull	(1.875, 0.136, 1.059)	6.1987	44.0241	94.0482	99.2612	95.9589

The UIDD model gives

$$Y_n^2 = 5.8799, \quad -\ell = 42.8856, \quad AIC = 91.7712, \quad BIC = 96.9842, \quad \text{and} \quad HQIC = 93.6819.$$

These values are lower than those obtained for the unit Burr-III, Kumaraswamy, and unit generalized inverse Weibull models for the corresponding criteria. In particular, the modified chi-square statistic confirms that the UIDD is compatible with the observed right-censored data at the 5% significance level. The results provide empirical support for the applicability of the proposed model to this dataset, although the conclusions remain specific to the data and competing models considered.

8. Conclusions

This paper introduced the UIDD, a new three-parameter probability model for analyzing bounded lifetime and reliability data. The proposed distribution extends the IDD and offers greater flexibility in modeling a variety of distributional shapes commonly encountered in reliability and survival studies. Several important statistical properties of the UIDD were derived, including its reliability functions, moments, quantile function, order statistics, mean residual life function, mean inactivity time, vitality function, and Rényi and Tsallis entropies. Parameter estimation was carried out using the MLE method for both complete and right-censored samples. The Monte Carlo simulation showed that the estimators perform satisfactorily, with their bias and MSE generally decreasing as the sample size increases. The practical applicability of the proposed distribution was demonstrated using a real right-censored lifetime dataset. The comparative analysis showed that the UIDD provides a better fit than the competing models considered, highlighting its flexibility and effectiveness for modeling bounded lifetime data. Despite these advantages, the UIDD has some limitations. As a three-parameter model, its estimation may be more computationally demanding than simpler unit distributions, particularly for small samples and heavily censored data. In addition, the current study considers MLE and a limited number of real-data applications; therefore, the observed improvement in fit should not be regarded as universal for all bounded datasets. Overall, the theoretical developments, simulation results, and real-data application indicate that the UIDD is a useful addition to the family of unit distributions.

Future research could address these limitations by developing Bayesian or robust estimation methods, UIDD-based regression models, and extensions to progressive and adaptive censoring schemes. Further applications to larger and more diverse bounded datasets, as well as multivariate extensions, would provide additional evidence on the model's applicability and limitations.

Author contributions

Khaoula Aidi: conceptualization, formal analysis, methodology, validation, software, writing–review and editing; Amer Al-Omari: conceptualization, validation, software, methodology, supervision, writing–review and editing, formal analysis; Ghadah Alomani: conceptualization, writing—review and editing, formal analysis, resources, validation, methodology, funding acquisition. All authors have read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare no conflicts of interest to report regarding the present study.

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