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Research article

On sequential pantograph fractional differential equations with Atangana–Baleanu derivatives and a p-Laplacian operator

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Abstract: This study investigated a class of sequential pantograph fractional differential equations that couple the Caputo–Hadamard and the Atangana–Baleanu–Caputo (ABC) fractional derivatives through a p-Laplacian operator, together with hybrid nonlocal terms and proportional (pantograph) delays. An equivalent integral representation of the problem was first derived. Banach's contraction principle was then used to establish the existence and uniqueness of solutions under suitable assumptions, whereas Dhage's Krasnoselskii-type hybrid fixed-point theorem in Banach algebras yielded the existence of at least one solution under weaker conditions. We further analyzed the Ulam–Hyers stability of the proposed problem. Two illustrative examples were provided.

Keywords: Caputo–Hadamard derivative; Atangana–Baleanu derivative; p-Laplacian operator; pantograph equation
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1. Introduction

Fractional differential equations (FDEs) extend classical differential equations by replacing integer-order derivatives with fractional-order derivatives, offering a more versatile framework for modeling complex phenomena across various disciplines, including physics, engineering, biology, and finance. For further details, see, for example, [1–3].

Among the numerous formulations of fractional derivatives, the Atangana–Baleanu (AB) operator has recently garnered significant attention due to its nonlocal characteristics and its Mittag–Leffler–type exponential kernel, which distinguishes it from the traditional Caputo and Riemann–Liouville definitions.

Thermostat differential equations provide classical models for heat regulation and temperature control. Since thermal processes may exhibit memory and hereditary effects, fractional differential equations have been introduced to provide more flexible mathematical descriptions of such phenomena; see, for example, [4]. This motivates the broader use of fractional models in applications involving memory effects. In the present work, however, our focus is on the analysis of the sequential pantograph problem formulated in (1.4).

Notably, considerable attention has been devoted to the existence and uniqueness of solutions to fractional differential equations, together with the development of reliable numerical approximations, thereby advancing both the theoretical understanding and practical applicability of these models; see, for instance, [5–8]. In particular, recent advances in the numerical analysis of nonlinear fractional subdiffusion problems include the development of robust finite-difference schemes with rigorous error estimates. Zhang and Yang, for example, developed an α_p-robust alternating direction implicit (ADI) difference scheme on graded meshes for three-dimensional nonlinear multiterm subdiffusion equations and established corresponding rigorous error estimates [9].

Pantograph equations play a crucial role in both pure and applied mathematics, as well as physics. Due to their significance, many researchers have extended these equations into different forms and analyzed their solvability both numerically and theoretically. For more details, refer to [10, 11] and the references therein. However, few studies have addressed pantograph FDEs, particularly those incorporating the Atangana–Baleanu–Caputo (ABC) fractional operator and the p-Laplacian operator.

In [4], the first mathematical model incorporating thermostat control was structured as follows:

$$\begin{cases} v''(t) + g(t)H(t, v(t)) = 0, \\ v'(0) = 0, \quad bv'(1) + v(\xi) = 0, \end{cases} \quad \text{for } t \in [0, 1] \text{ and } b > 0. \tag{1.1}$$

Then, the authors in [7] examined the thermostat differential equation in a non-integer form as follows:

$$\begin{cases} D^p v(t) + \lambda H(t, v(t)) = 0, \\ v'(0) = 0, \quad bD^{p-1}v(t) + v(\xi) = 0, \end{cases} \tag{1.2}$$

where $t \in [0, 1]$, $b > 0$, $p \in (1, 2]$, $\xi \in (0, 1)$, $b\Gamma(p) > (1-\xi)^{p-1}$, $\lambda > 0$, and $H : [0, 1] \times [0, \infty) \to [0, \infty)$ is a continuous function.

In [12], Baleanu et al. were the first to develop a hybrid fractional model for thermostat control. They achieved their objectives regarding the existence of a solution, which is formulated as follows:

$$D^p \left(\frac{v(t)}{h(t, v(t))} \right) + H(t, v(t)) = 0,$$

subject to the hybrid boundary conditions:

$$ {}^{c}D^{1}\left(\frac{v(t)}{h(t,v(t))}\right)\Big|_{t=0} = 0, \quad b^{c}D^{p-1}\left(\frac{v(t)}{h(t,v(t))}\right)\Big|_{t=1} + \left(\frac{v(t)}{h(t,v(t))}\right)\Big|_{t=\xi} = 0, $$

where $p \in]1,2]$, $\xi \in]0,1[$, $b > 0$, ${}^{c}D^{1} = \dfrac{d}{dt}$, ${}^{c}D^{q}$ is the Caputo derivative of order $q \in \{p, p-1\}$, and $H, h \in C([0,1]\times\mathbb{R},\mathbb{R})$ with $h \neq 0$.

Recently, in [13], the authors discussed the existence of solutions to the following nonlocal boundary value problem (BVP) of hybrid fractional integro-differential equations:

$$ \begin{cases} {}^{ABC}D_{0}^{\kappa}\left(\dfrac{\varphi(t) - \sum_{i=1}^{n} {}^{AB}I_{0}^{k_i}\psi_i(t,\varphi(t))}{\Phi(t,\varphi(t))}\right) = F(t,\varphi(t)), \quad t \in \Omega = [0,1], \\ \varphi(0) = h(\varphi), \quad \varphi(1) = A, \end{cases} \tag{1.3} $$

where $1 < \kappa \leq 2$, $A \in \mathbb{R}$, $F, \psi_i \in C(\Omega\times\mathbb{R},\mathbb{R})$, and $\Phi \in C(\Omega\times\mathbb{R},\mathbb{R})$ with $\psi_i(0,\varphi(0)) = F(0,\varphi(0))$ for all $i = 1,2,\ldots,n$. ${}^{ABC}D_{0}^{\kappa}$ denotes the Atangana–Baleanu–Caputo fractional derivative, ${}^{AB}I_{0}^{k_i}$ is the Atangana–Baleanu fractional integral of order k_i, and $h \in C(\Omega\times\mathbb{R})$.

Inspired by previous studies, this paper explores a class of nonlinear FDEs incorporating the p-Laplacian operator and Atangana–Baleanu derivatives, formulated as follows:

$$ \begin{cases} {}^{CH}D_{a}^{\alpha}\left[\varphi_p\left[{}^{ABC}D_{a}^{\beta}\left(\dfrac{\zeta(t) - \sum_{i=1}^{n} {}^{AB}I_{a}^{\sigma_i}\mathscr{F}_i(t,\zeta(t),\zeta(\delta t))}{\mathscr{G}(t,\zeta(t),\zeta(\rho t))}\right)\right]\right] - \mathscr{H}(t,\zeta(t),\zeta(\gamma t)) = 0, \\ t \in \Omega := [a,T], \quad T > a > 0, \quad 0 < \delta,\rho,\gamma < 1, \quad 0 < \alpha,\beta \leq 1, \\ \dfrac{\zeta(a)}{\mathscr{G}(a,\zeta(a),\zeta(\rho a))} = \lambda \in \mathbb{R}, \quad \mathscr{F}_i(a,x,y) = 0, \ \forall (x,y) \in \mathbb{R}^2, \end{cases} \tag{1.4} $$

where ${}^{CH}D_{a}^{\alpha}$ and ${}^{ABC}D_{a}^{\beta}$ denote Caputo-Hadamard and Atangana–Baleanu–Caputo fractional derivatives of orders α,β, respectively, and ${}^{AB}I_{a}^{\sigma_i}$ is the Atangana–Baleanu fractional integral of orders $0 < \sigma_i < 1$, $i = 1,2,\cdots,n$. $\varphi_p(r) = |r|^{p-2}r$ is the p-Laplacian operator, where $\frac{1}{p} + \frac{1}{q} = 1$, and φ_q denotes the inverse of the p-Laplacian operator. The functions $\mathscr{G} \in C(\Omega\times\mathbb{R}^2,\mathbb{R}\setminus\{0\})$, $\mathscr{F}_i, \mathscr{H} \in C(\Omega\times\mathbb{R}^2,\mathbb{R})$ are continuous.

The main contributions of this work can be summarized as follows.

(i) We introduce a sequential pantograph fractional model that simultaneously couples the Caputo–Hadamard and the Atangana–Baleanu–Caputo (ABC) derivatives through a p-Laplacian operator, together with hybrid nonlocal terms and proportional (pantograph) delays. To the best of our knowledge, this particular combination of operators, delays, and hybrid nonlocal terms has not been considered in the existing literature.

(ii) We derive an equivalent integral representation of the problem by successively inverting the p-Laplacian operator and the fractional operators involved (Theorem 3.1).

(iii) Based on this representation, we establish the existence of a unique solution through Banach's contraction principle and the existence of at least one solution through Dhage's Krasnoselskii-type hybrid fixed-point theorem in Banach algebras, and we prove the Ulam–Hyers stability of the model.

This paper is organized as follows: Section 2 collects the basic definitions and preliminary lemmas concerning the Caputo–Hadamard and Atangana–Baleanu fractional operators and the p-Laplacian operator. Section 3 is devoted to the main results, namely, the integral representation of the problem, the existence and uniqueness of solutions, and the existence of at least one solution via Dhage's hybrid fixed-point theorem. Section 4 establishes the Ulam–Hyers stability. Section 5 presents two illustrative examples together with their numerical simulations, and the paper closes with a brief conclusion.

2. Preliminaries

Here, we review some fundamental definitions and essential lemmas related to AB fractional derivatives and AB fractional integrals. The Hadamard and Caputo–Hadamard operators are taken from [1, 3], while the Atangana–Baleanu derivative and its associated fractional integral, together with the relations between them, follow [14, 15].

Let

$$\Omega = [a, T].$$

We work in the Banach space

$$C(\Omega, \mathbb{R}) = \{\zeta : \Omega \to \mathbb{R} \mid \zeta \text{ is continuous}\},$$

endowed with the supremum norm

$$\|\zeta\|_\infty := \sup_{t\in\Omega} |\zeta(t)|.$$

Consequently,

$$\|\zeta - \widetilde{\zeta}\|_\infty := \sup_{t\in\Omega} |\zeta(t) - \widetilde{\zeta}(t)|.$$

Consider $\mathscr{D} = C(\Omega, \mathbb{R})$ the space of continuous functions $f : \Omega \to \mathbb{R}$ equipped with the following norm:

$$\|f\|_\infty = \sup_{\xi\in\Omega} |f(\xi)|.$$

Let $L^1(\Omega)$ be the space of Lebesgue integrable real-valued functions on Ω, equipped with the norm:

$$\|f\|_{L^1} = \int_\Omega |f(\xi)| d\xi.$$

Definition 2.1. *[1] Let $y : [a, +\infty) \to \mathbb{R}$ be a continuous function. The Hadamard fractional integral of order $q > 0$ is defined as:*

$${}^{H}I_a^q y(t) = \frac{1}{\Gamma(q)} \int_a^t \left(\log \frac{t}{\xi}\right)^{q-1} y(\xi) \frac{d\xi}{\xi}, \ a > 0,$$

where $\log(.) = \log_e(.)$*, provided that the integral exists.*

Definition 2.2. *[1] Let $y : [a, \infty) \to \mathbb{R}$ be a function that is at least n-times delta differentiable. The Caputo–Hadamard fractional derivative of order q of y is defined as:*

$${}^{CH}D^q y(t) = \frac{1}{\Gamma(n-q)} \int_a^t \left(\log \frac{t}{\xi}\right)^{n-q-1} \delta^n y(\xi) \frac{d\xi}{\xi}, \ a > 0,$$

where $n - 1 < q < n$, $n = [q] + 1$, $\delta = t\dfrac{d}{dt}$.

Definition 2.3. *[3] The one-parameter Mittag–Leffler function is given by:*

$$E_{\vartheta}(y) = \sum_{p=0}^{\infty} \frac{y^p}{\Gamma(\vartheta p + 1)}, \quad y \in \mathbb{C}, \quad Re(\vartheta) > 0.$$

Definition 2.4. *[15] Let $q \in [0,1]$ and $y \in L^1(\Omega)$. Then, the AB–Caputo and AB–Riemann–Liouville fractional derivatives of order q for a function y are given by:*

$${}^{ABC}D^q_{a^+}y(t) = \frac{\mathfrak{N}(q)}{1-q}\int_a^t E_q\left(-\frac{q}{q-1}(t-\xi)^q\right)y'(\xi)d\xi, \quad t > a,$$

and

$${}^{ABR}D^q_{a^+}y(t) = \frac{\mathfrak{N}(q)}{1-q}\frac{d}{dt}\int_a^t E_q\left(-\frac{q}{q-1}(t-\xi)^q\right)y(\xi)d\xi, \quad t > a,$$

respectively, where $\mathfrak{N}(q) > 0$ is a normalization function satisfying $\mathfrak{N}(0) = \mathfrak{N}(1) = 1$.

Definition 2.5. *[15] Let $y \in L^1(\Omega)$ and $0 < q < 1$. Then, the Atangana–Baleanu fractional integral is defined as:*

$${}^{AB}I^q_a y(t) = \frac{1-q}{\mathfrak{N}(q)}y(t) + \frac{q}{\mathfrak{N}(q)\Gamma(q)}\int_a^t (t-\xi)^{q-1}y(\xi)d\xi.$$

Lemma 2.1. *[1] Let $q > 0$ and $y \in C^{\delta}_n[a,b]$. Then we have*

$${}^{H}I^q({}^{CH}D^q y)(t) = y(t) - \sum_{k=0}^{n-1} c_k\left(\log\frac{t}{a}\right)^k$$

where $c_k \in \mathbb{R}, k = 0,1,\ldots,n-1,\ n = [q]+1$, and $C^{\delta}_n[a,b] = \left\{y : [a,b] \to \mathbb{R} : \delta^{n-1}y \in C([a,b],\mathbb{R})\right\}$.

Lemma 2.2. *[16] Let y be a function defined on Ω, and let $n < q \leq n+1$. We have*

$${}^{AB}I^q_a\left({}^{ABC}D^q_a y\right)(t) = y(t) - \sum_{k=0}^{n}\frac{y^{(k)}(a)}{k!}(t-a)^k,$$

for some $n \in \mathbb{N}_0$, where $y^{(k)}(t) = \left(\frac{d}{dt}\right)^k y(t)$.

Lemma 2.3. *[10] Let $\theta \in]0,1]$ and $\psi \in \mathscr{D}$ with $\psi(a) = 0$. Then, the solution of the following problem*

$${}^{ABC}D^{\theta}_a\mathfrak{v}(t) = \psi(t), \quad t \in \Omega,$$

$$\mathfrak{v}(a) = \mathfrak{v}_a$$

is given by

$$\mathfrak{v}(t) = \mathfrak{v}_a + \frac{1-\theta}{\mathfrak{N}(\theta)}\psi(t) + \frac{\theta}{\mathfrak{N}(\theta)}\frac{1}{\Gamma(\theta)}\int_a^t (t-r)^{\theta-1}\psi(r)\,dr.$$

Lemma 2.4. *[17] Let φ_p be a p-Laplacian operator. Then,*

(1) If $1 < p \le 2$, $\theta_1, \theta_2 > 0$, *and* $|\theta_1|, |\theta_2| \ge \kappa > 0$, *then:*

$$|\varphi_p(\theta_1) - \varphi_p(\theta_2)| \le (p-1)\kappa^{p-2}|\theta_1 - \theta_2|.$$

(2) If $p > 2$ *and* $|\theta_1|, |\theta_2| \le \kappa^*$, *then:*

$$|\varphi_p(\theta_1) - \varphi_p(\theta_2)| \le (p-1)(\kappa^*)^{p-2}|\theta_1 - \theta_2|.$$

Theorem 2.1. *[18] Let* $\mathscr{E}$ *be a nonempty, closed, convex, and bounded subset of a Banach algebra* $\mathscr{B}$. *Consider the operators* $\mathscr{A}_1, \mathscr{A}_3 : \mathscr{B} \to \mathscr{B}$ *and* $\mathscr{A}_2 : \mathscr{E} \to \mathscr{B}$, *which satisfy the following assumptions:*

As1: $\mathscr{A}_1$ *and* $\mathscr{A}_3$ *are Lipschitzian with Lipschitz constants* $\mathscr{M}$ *and* $\mathscr{N}$, *respectively;*
As2: $\mathscr{A}_2$ *is compact and continuous;*
As3: $\zeta = \mathscr{A}_1\zeta\mathscr{A}_2 w + \mathscr{A}_3\zeta \Rightarrow \zeta \in \mathscr{E}$ *for all* $w \in \mathscr{E}$;
As4: $\mathscr{M}\mathscr{P} + \mathscr{N} < 1$, *where* $\mathscr{P} = \|\mathscr{A}_2(\mathscr{E})\|$.

Then, there exists $\zeta \in \mathscr{E}$ *such that* $\zeta = \mathscr{A}_1\zeta\mathscr{A}_2\zeta + \mathscr{A}_3\zeta$.

Theorem 2.2. *[19] Let* $\mathscr{X}$ *be a Banach space and* $\mathscr{Y}$ *a nonempty closed subset of* $\mathscr{X}$. *If* $\mathscr{Z} : \mathscr{Y} \to \mathscr{Y}$ *is a contraction, then it has a unique fixed point.*

Remark 2.1. *Since* $a > 0$ *and* $0 < \delta, \rho, \gamma < 1$, *the proportional arguments* $\delta t, \rho t, \gamma t$ *may fall below* a *when* t *is close to* a. *Throughout this paper, the delayed values are understood in the constant-prehistory sense:*

$$\zeta(s) := \zeta(a), \qquad s \in [\min\{\delta, \rho, \gamma\}\, a,\ a),$$

so that $t \mapsto \zeta(\delta t), \zeta(\rho t), \zeta(\gamma t)$ *are well–defined and continuous on* Ω, *with* $\sup_{t\in\Omega} |\zeta(\delta t)| \le \|\zeta\|$ *and* $\sup_{t\in\Omega} |\zeta(\delta t) - \eta(\delta t)| \le \|\zeta - \eta\|$ *(and similarly for* ρ, γ*). In particular, the boundary relation in* (1.4) *reads* $\zeta(a) = \lambda\mathscr{G}(a, \zeta(a), \zeta(a))$.

The models recalled above, together with the related literature, address the structural ingredients of (1.4) separately. Nonlinear pantograph equations driven by the ABC derivative are studied in [10, 11], but without any p-Laplacian term and without a hybrid quotient structure. Hybrid and nonlocal problems for ABC operators are considered in [13, 20], where a single fractional derivative acts on the quotient, so that neither a sequential composition of two operators of different types nor a proportional delay occurs. Fractional thermostat models of Caputo or Hadamard type are treated in [4, 6, 7, 12], in which the nonlinearity depends on the state through its current value only. Finally, p-Laplacian fractional boundary value problems, including Hadamard-type integral boundary conditions, are analyzed in [21–23], where the fractional operators standing on either side of φ_p are of one and the same type and no Mittag–Leffler kernel is involved.

Problem (1.4) contains these ingredients simultaneously: the outer operator ${}^{CH}D_a^{\alpha}$ is of Caputo–Hadamard type with a logarithmic kernel, the inner operator ${}^{ABC}D_a^{\beta}$ has a nonsingular Mittag–Leffler kernel, the two are coupled through φ_p, and the state enters through a hybrid quotient together with proportional delays $\delta t, \rho t, \gamma t$ and n nonlocal AB-integral terms. Two analytical features follow, which are absent from each of the classes listed above taken separately. First, inverting the equation requires the successive use of a Hadamard integral, of the inverse p-Laplacian φ_q, and of an AB integral; this produces the mixed kernel ${}^{RL}I_a^{\beta}\,{}^{H}I_a^{\alpha}(1)$ occurring in the constant Λ_3 below, a quantity that has no

counterpart in single-operator models. Second, since φ_q is only locally Lipschitz, the associated fixed-point operator is not globally Lipschitz on $\mathscr{B}$, and the admissible range of the exponent is dictated by the behavior of φ_q near the origin (see Remark 3.1). The contribution of the present work therefore lies in this specific combination of operators, delay, structure and hybrid nonlocal conditions, and in the estimates it requires, rather than in the replacement of one fractional derivative by another.

These components reflect more accurately the dynamics of complex systems with memory and feedback mechanisms. Thus, investigating such a problem contributes to the theoretical understanding of fractional differential equations and also to their important applications in systems governed by non-integer dynamics.

3. Main results

This section focuses on establishing certain existence and uniqueness theorems for the *ABC*–type pantograph fractional differential equations (1.4).

The analysis is organized around three difficulties that are specific to (1.4).

(a) The equation is sequential and its two fractional derivatives are of different types, so that no single inversion formula applies. It is therefore inverted in three successive steps: Lemma 2.1 removes the Caputo–Hadamard operator, the inverse p-Laplacian φ_q removes φ_p, and Lemmas 2.2–2.3 remove the ABC operator. The compatibility conditions $\mathscr{F}_i(a,\cdot,\cdot) \equiv 0$ and $\varphi_q({}^{H}I_a^{\alpha}\mathscr{W})(a) = 0$ are exactly what is needed to identify the constants of integration at each step (Theorem 3.1).

(b) The map φ_q is not globally Lipschitz, so that the linear estimates available for $p = 2$ cannot be used. This is remedied by the a priori bound $\left|{}^{H}I_a^{\alpha}\mathscr{H}\right| \leq \kappa^*$ supplied by (H_4), which confines the arguments of φ_q to a fixed compact interval on which Lemma 2.4 provides a Lipschitz constant; this is also the origin of the restriction $q \geq 2$ (Remark 3.1).

(c) The hybrid quotient makes the fixed-point operator a product rather than a sum, so that a contraction argument requires the joint smallness condition (4.2). When (4.2) fails, the operator is split as $\mathscr{A} = \mathscr{A}_1\mathscr{A}_2 + \mathscr{A}_3$ and Theorem 2.1 is applied, the compactness of $\mathscr{A}_2$ being obtained from the equicontinuity estimate (3.26).

The two schemes are complementary: the first yields uniqueness under (4.2) and the second only existence, but under the weaker condition (3.15).

Definition 3.1 (Integral and classical solutions)**.** *Let $\Omega = [a, T]$. A function $\zeta \in C(\Omega, \mathbb{R})$ is called an* integral solution *of problem* (1.4) *if it satisfies the integral equation* (3.8) *for every $t \in \Omega$.*

A function ζ is called a classical solution *of problem* (1.4) *if all the fractional derivatives occurring in* (1.4) *exist and ζ satisfies the differential equation and the boundary condition in* (1.4) *pointwise on Ω.*

Theorem 3.1. *Let $W \in D$. If ζ is a classical solution of problem* (3.1)*, then ζ satisfies the following integral equation:*

$$\begin{cases} {}^{CH}D_a^{\alpha}\left[\varphi_p\left[{}^{ABC}D_a^{\beta}\left(\dfrac{\zeta(t) - \sum_{i=1}^{n} {}^{AB}I_a^{\sigma_i}\mathscr{F}_i(t,\zeta(t),\zeta(\delta t))}{\mathscr{G}(t,\zeta(t),\zeta(\rho t))}\right)\right]\right] - \mathscr{W}(t) = 0, \\ t \in \Omega,\ 0 < \alpha,\beta \leq 1,\ 0 < \delta,\rho,\sigma_i < 1,\ \mathscr{F}_i(a,x,y) = 0,\ \forall (x,y) \in \mathbb{R}^2, \\ \dfrac{\zeta(a)}{\mathscr{G}(a,\zeta(a),\zeta(\rho a))} = \lambda \in \mathbb{R}, \end{cases} \tag{3.1}$$

which admits an integral solution $\zeta \in \mathscr{D}$, which is characterized by the following integral equation:

$$\begin{aligned}\zeta(t) = \mathscr{G}(t,\zeta(t),\zeta(\rho t)) \times \Big[\lambda + \frac{1-\beta}{\mathfrak{N}(\beta)}\,\varphi_q\Big({}^{H}I_a^{\alpha}\mathscr{W}(t)\Big) \\ + \frac{\beta}{\mathfrak{N}(\beta)}\,{}^{RL}I_a^{\beta}\Big(\varphi_q\Big({}^{H}I_a^{\alpha}\mathscr{W}(t)\Big)\Big)\Big] \\ + \sum_{i=1}^{n} {}^{AB}I_a^{\sigma_i}\mathscr{F}_i(t,\zeta(t),\zeta(\delta t)).\end{aligned} \tag{3.2}$$

Proof. Let ζ be a classical solution of problem (3.1). By applying Lemma 2.1, there exists a constant $a_0 \in \mathbb{R}$ such that

$$\varphi_p\left[{}^{ABC}D_a^{\beta}\left(\frac{\zeta(t) - \sum_{i=1}^{n} {}^{AB}I_a^{\sigma_i}\mathscr{F}_i(t,\zeta(t),\zeta(\delta t))}{\mathscr{G}(t,\zeta(t),\zeta(\rho t))}\right)\right] = {}^{H}I_a^{\alpha}\mathscr{W}(t) + a_0.$$

Since ${}^{ABC}D_a^{\beta}$ of any admissible function vanishes at $t = a$ and $\varphi_p(0) = 0$, evaluating the last identity at $t = a$ (where ${}^{H}I_a^{\alpha}\mathscr{W}(a) = 0$) yields $a_0 = 0$. Thus we have

$$\varphi_p\left[{}^{ABC}D_a^{\beta}\left(\frac{\zeta(t) - \sum_{i=1}^{n} {}^{AB}I_a^{\sigma_i}\mathscr{F}_i(t,\zeta(t),\zeta(\delta t))}{\mathscr{G}(t,\zeta(t),\zeta(\rho t))}\right)\right] = {}^{H}I_a^{\alpha}\mathscr{W}(t). \tag{3.3}$$

Applying $\varphi_q = \varphi_p^{-1}$ to both sides, we obtain

$${}^{ABC}D_a^{\beta}\left(\frac{\zeta(t) - \sum_{i=1}^{n} {}^{AB}I_a^{\sigma_i}\mathscr{F}_i(t,\zeta(t),\zeta(\delta t))}{\mathscr{G}(t,\zeta(t),\zeta(\rho t))}\right) = \varphi_q({}^{H}I_a^{\alpha}\mathscr{W}(t)).$$

Using Lemmas 2.2 and 2.3 (which apply since $\varphi_q({}^{H}I_a^{\alpha}\mathscr{W})(a) = \varphi_q(0) = 0$), together with the boundary relations $\zeta(a)/\mathscr{G}(a,\zeta(a),\zeta(\rho a)) = \lambda$ and $\mathscr{F}_i(a,\cdot,\cdot) \equiv 0$ (so that ${}^{AB}I_a^{\sigma_i}\mathscr{F}_i(a,\zeta(a),\zeta(\delta a)) = \frac{1-\sigma_i}{\mathfrak{N}(\sigma_i)}\mathscr{F}_i(a,\zeta(a),\zeta(\delta a)) = 0$), we obtain

$$\frac{\zeta(t) - \sum_{i=1}^{n} {}^{AB}I_a^{\sigma_i}\mathscr{F}_i(t,\zeta(t),\zeta(\delta t))}{\mathscr{G}(t,\zeta(t),\zeta(\rho t))} = \lambda + {}^{AB}I_a^{\beta}\varphi_q({}^{H}I_a^{\alpha}\mathscr{W}(t)),$$

which implies

$$\zeta(t) = \mathscr{G}(t,\zeta(t),\zeta(\rho t)) \times \Big[\lambda + {}^{AB}I_a^{\beta}\varphi_q({}^{H}I_a^{\alpha}\mathscr{W}(t))\Big] + \sum_{i=1}^{n} {}^{AB}I_a^{\sigma_i}\mathscr{F}_i(t,\zeta(t),\zeta(\delta t)). \tag{3.4}$$

Using the representation of the Atangana–Baleanu fractional integral in terms of the Riemann–Liouville fractional integral, namely,

$${}^{AB}I_a^{\beta}f(t) = \frac{1-\beta}{\mathfrak{N}(\beta)}f(t) + \frac{\beta}{\mathfrak{N}(\beta)}\,{}^{RL}I_a^{\beta}f(t), \qquad 0 < \beta < 1, \tag{3.5}$$

for every f for which the involved fractional integrals are well–defined, we apply (3.5) to

$$f(t) = \varphi_q\left({}^{H}I_a^{\alpha}\mathscr{W}(t)\right).$$

It follows that

$$
{}^{AB}I_a^{\beta}\varphi_q\left({}^{H}I_a^{\alpha}\mathscr{W}(t)\right) = \frac{1-\beta}{\mathfrak{N}(\beta)}\varphi_q\left({}^{H}I_a^{\alpha}\mathscr{W}(t)\right) + \frac{\beta}{\mathfrak{N}(\beta)}{}^{RL}I_a^{\beta}\left[\varphi_q\left({}^{H}I_a^{\alpha}\mathscr{W}(t)\right)\right]. \tag{3.6}
$$

This proves that every classical solution of problem (3.1) satisfies the stated integral equation:

$$
\begin{aligned}
\zeta(t) = \mathscr{G}(t,\zeta(t),\zeta(\rho t)) \times \Bigg[\lambda + \frac{1-\beta}{\mathfrak{N}(\beta)}\,\varphi_q\Big({}^{H}I_a^{\alpha}\mathscr{W}(t)\Big) \\
+ \frac{\beta}{\mathfrak{N}(\beta)}\,{}^{RL}I_a^{\beta}\Big(\varphi_q\Big({}^{H}I_a^{\alpha}\mathscr{W}(t)\Big)\Big)\Bigg] \\
+ \sum_{i=1}^{n}{}^{AB}I_a^{\sigma_i}\mathscr{F}_i(t,\zeta(t),\zeta(\delta t)).
\end{aligned} \tag{3.7}
$$

Thus, the proof is complete.

We define the Banach space

$$
B := C(\Omega,\mathbb{R}),
$$

equipped with the supremum norm

$$
\|\zeta\|_{\infty} := \sup_{t\in\Omega}|\zeta(t)|.
$$

For two functions $\zeta,\widetilde{\zeta}\in B$, we have

$$
\|\zeta-\widetilde{\zeta}\|_{\infty} := \sup_{t\in\Omega}|\zeta(t)-\widetilde{\zeta}(t)|.
$$

For the sake of brevity, we introduce the following notation:

$$
{}^{RL}I_a^{\beta}\,{}^{H}I_a^{\alpha}(1)(\tau) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^{\tau}(\tau-s)^{\beta-1}\int_a^{s}\left(\log\frac{s}{r}\right)^{\alpha-1}\frac{dr}{r}ds,\ a<\tau\le T.
$$

Define the operator $\mathscr{A}:\mathscr{B}\to\mathscr{B}$ by

$$
\begin{aligned}
(\mathscr{A}\zeta)(t) = \mathscr{G}(t,\zeta(t),\zeta(\rho t)) \times \Bigg[\lambda + \frac{1-\beta}{\mathfrak{N}(\beta)}\,\varphi_q\Big({}^{H}I_a^{\alpha}\mathscr{H}(t,\zeta(t),\zeta(\gamma t))\Big) \\
+ \frac{\beta}{\mathfrak{N}(\beta)}\,{}^{RL}I_a^{\beta}\Big(\varphi_q\Big({}^{H}I_a^{\alpha}\mathscr{H}(t,\zeta(t),\zeta(\gamma t))\Big)\Big)\Bigg] \\
+ \sum_{i=1}^{n}{}^{AB}I_a^{\sigma_i}\mathscr{F}_i(t,\zeta(t),\zeta(\delta t)), t\in\Omega.
\end{aligned} \tag{3.8}
$$

Accordingly, the fixed points of $\mathscr{A}$ are integral solutions of problem (1.4) in the sense of Definition 3.1.

In our analysis, the following hypotheses are needed:

(**H1**): The functions $\mathscr{H}$ and $\mathscr{F}_i$, $i=1,2,\ldots,n$, are continuous and there exist positive constants $\mathcal{K},\widehat{\mathcal{K}}_i$, respectively, such that for all $t\in\Omega$, we have

$$
\left|\mathscr{H}(t,\varrho_1,\widehat{\varrho}_1)-\mathscr{H}(t,\varrho_2,\widehat{\varrho}_2)\right| \le \mathcal{K}\left(|\varrho_1-\varrho_2|+\left|\widehat{\varrho}_1-\widehat{\varrho}_2\right|\right),\ \varrho_i,\widehat{\varrho}_i\in\mathbb{R}, i=1,2,
$$

and

$$\left|\mathscr{F}_i\left(t,\varrho_1,\widehat{\varrho_1}\right)-\mathscr{F}_i\left(t,\varrho_2,\widehat{\varrho_2}\right)\right|\leq\widehat{\mathcal{K}_i}\left(|\varrho_1-\varrho_2|+\left|\widehat{\varrho_1}-\widehat{\varrho_2}\right|\right),\ \varrho_i,\widehat{\varrho_i}\in\mathbb{R},i=1,2.$$

(**H2**): Assume that $\mathscr{G}$ is continuous and bounded, that is, $\exists\ \mathcal{M}>0$ such that

$$\left|\mathscr{G}(t,\varrho,\widehat{\varrho})\right|\leq\mathcal{M},\ \forall(t,\varrho,\widehat{\varrho})\in\Omega\times\mathbb{R}^2.$$

(**H3**): There exists a positive function ϑ with a bound $\|\vartheta\|$ such that

$$\left|\mathscr{G}\left(t,\varrho_1,\widehat{\varrho_1}\right)-\mathscr{G}\left(t,\varrho_2,\widehat{\varrho_2}\right)\right|\leq\vartheta(t)\left(|\varrho_1-\varrho_2|+\left|\widehat{\varrho_1}-\widehat{\varrho_2}\right|\right),t\in\Omega,\ \varrho_i,\widehat{\varrho_i}\in\mathbb{R},i=1,2.$$

(**H4**): There exists a positive function $\psi\in\mathscr{D}$ such that for all $t\in\Omega$ and $\varrho,\widehat{\varrho}\in\mathbb{R}$, the following inequality holds:

$$|\mathscr{H}(t,\varrho,\widehat{\varrho})|\leq\varphi_p(\psi(t)).$$

To simplify our computations, we set

$$\begin{aligned}
\Lambda_0&:=\left(\frac{(\log\frac{T}{a})^\alpha}{\Gamma(\alpha+1)}\right)^{q-1}\|\psi\|\\
\Lambda_*&:=|\lambda|+\left(\frac{1-\beta}{\Re(\beta)}+\frac{\beta(T-a)^\beta}{\Re(\beta)\Gamma(\beta+1)}\right)\Lambda_0\\
\Lambda_1&:=2\|\vartheta\|\,\Lambda_*\\
\Lambda_2&:=\frac{2(1-\beta)(q-1)\kappa^{q-2}}{\Re(\beta)\Gamma(\alpha+1)}\left(\log\frac{T}{a}\right)^\alpha\\
\Lambda_3&:=\frac{2\beta(q-1)\kappa^{q-2}}{\Re(\beta)}\,{}^{RL}I_a^\beta\,{}^{H}I_a^\alpha(1)(T)\\
\Lambda_4&:=\sum_{i=1}^{n}2\widehat{\mathcal{K}_i}\left(\frac{1-\sigma_i}{\Re(\sigma_i)}+\frac{(T-a)^{\sigma_i}}{\Re(\sigma_i)\Gamma(\sigma_i)}\right)\\
\Lambda_5&:=\sum_{i=1}^{n}\mathfrak{m}_i\left(\frac{1-\sigma_i}{\Re(\sigma_i)}+\frac{(T-a)^{\sigma_i}}{\Re(\sigma_i)\Gamma(\sigma_i)}\right)
\end{aligned}$$

where

$$\sup_{t\in[a,T]}|\mathscr{G}(t,0,0)|=\mathfrak{m}_0<\infty\quad\text{and}\quad\sup_{t\in[a,T]}|\mathscr{F}_i(t,0,0)|=\mathfrak{m}_i<\infty,i=1,2,\ldots,n.$$

Remark 3.1. *In Λ_2,Λ_3, the constant κ is the one required by Lemma 2.4 for φ_q, applied to the quantities $\Omega={}^{H}I_a^\alpha\mathscr{H}(t,\zeta(t),\zeta(\gamma t))$. Under (H_4), these satisfy, for every $\zeta\in\mathscr{B}$ and $t\in\Omega$,*

$$\left|\,{}^{H}I_a^\alpha\mathscr{H}(t,\zeta(t),\zeta(\gamma t))\right|\leq\frac{(\log\frac{T}{a})^\alpha}{\Gamma(\alpha+1)}\|\psi\|^{p-1}=:\kappa^*.$$

Hence, when $q>2$, one takes $\kappa=\kappa^*$ (the second case of Lemma 2.4); when $q=2$, $\varphi_q=\mathrm{Id}$ and $(q-1)\kappa^{q-2}=1$, so no constant κ is needed.

For $1<q<2$, the first case of Lemma 2.4 would have to be used instead, and it requires a uniform lower *bound $|\Omega|\geq\kappa>0$ together with a constant sign. Such a bound is incompatible with the structure of* (1.4)*: the Hadamard integral is taken over $[a,t]$, so that*

$$\Omega(a)={}^{H}I_a^\alpha\mathscr{H}(a,\zeta(a),\zeta(\gamma a))=0,\qquad\textit{hence}\qquad\inf_{t\in\Omega}|\Omega(t)|=0,$$

and no admissible constant $\kappa > 0$ exists. This is a limitation of the particular estimate used here and not an intrinsic obstruction for q-Laplacian problems: $1 < q < 2$ is the singular range of $s \mapsto |s|^{q-2}s$ at $s = 0$, and its treatment requires a different inequality, a local argument, or additional assumptions controlling the behavior of Ω near its zero set. Accordingly, all the results below are stated for

$$q \geq 2, \qquad \text{are equivalently} \qquad 1 < p \leq 2,$$

with the two ranges being equivalent because $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 3.2. *Assume that* (H1)–(H4) *hold and that condition* (4.2) *is satisfied. Then the operator*

$$\mathscr{A} : \mathcal{B} \to \mathcal{B}, \qquad \mathcal{B} = C(\Omega, \mathbb{R}),$$

is well–defined and is a contraction. Consequently, problem (1.4) *has a unique integral solution in $\mathcal{B}$.*

Proof. We show that the operator $\mathscr{A}$ is a contraction mapping. Let $\zeta, \eta \in \mathscr{B}$ and $t \in \Omega$ and define

$$\begin{aligned} \Theta_\zeta(t) = \lambda &+ \frac{1-\beta}{\Re(\beta)}\varphi_q\Big({}^{H}I_a^{\alpha}\mathscr{H}(t, \zeta(t), \zeta(\gamma t))\Big) \\ &+ \frac{\beta}{\Re(\beta)}{}^{RL}I_a^{\beta}\Big(\varphi_q\Big({}^{H}I_a^{\alpha}\mathscr{H}(t, \zeta(t), \zeta(\gamma t))\Big)\Big). \end{aligned} \tag{3.9}$$

Then

$$(\mathscr{A}\zeta)(t) = \mathscr{G}(t, \zeta(t), \zeta(\rho t))\,\Theta_\zeta(t) + \sum_{i=1}^{n} {}^{AB}I_a^{\sigma_i}\mathscr{F}_i(t, \zeta(t), \zeta(\delta t)). \tag{3.10}$$

Consequently,

$$\begin{aligned} |(\mathscr{A}\zeta)(t) - (\mathscr{A}\eta)(t)| \leq{}& \Big|\mathscr{G}(t, \zeta(t), \zeta(\rho t)) - \mathscr{G}(t, \eta(t), \eta(\rho t))\Big|\,|\Theta_\zeta(t)| \\ &+ \Big|\mathscr{G}(t, \eta(t), \eta(\rho t))\Big|\,|\Theta_\zeta(t) - \Theta_\eta(t)| + \sum_{i=1}^{n}\Big|{}^{AB}I_a^{\sigma_i}(\mathscr{F}_i(t, \zeta(t), \zeta(\delta t)) - \mathscr{F}_i(t, \eta(t), \eta(\delta t)))\Big| \\ \leq{}& \Big|\mathscr{G}(t, \zeta(t), \zeta(\rho t)) - \mathscr{G}(t, \eta(t), \eta(\rho t))\Big| \times \Bigg[\,|\lambda| + \frac{1-\beta}{\Re(\beta)}\Big|\varphi_q\Big({}^{H}I_a^{\alpha}\mathscr{H}(t, \eta(t), \eta(\gamma t))\Big)\Big| \\ &+ \frac{\beta}{\Re(\beta)}\,{}^{RL}I_a^{\beta}\Big|\varphi_q\Big({}^{H}I_a^{\alpha}\mathscr{H}(t, \eta(t), \eta(\gamma t))\Big)\Big|\Bigg] \\ &+ \Big|\mathscr{G}(t, \eta, \eta(\rho t))\Big| \times \Bigg[\frac{1-\beta}{\Re(\beta)}\Big|\varphi_q\Big({}^{H}I_a^{\alpha}\mathscr{H}(t, \zeta(t), \zeta(\gamma t))\Big) - \varphi_q\Big({}^{H}I_a^{\alpha}\mathscr{H}(t, \eta, \eta(\gamma t))\Big)\Big| \\ &+ \frac{\beta}{\Re(\beta)}\,{}^{RL}I_a^{\beta}\Big|\varphi_q\Big({}^{H}I_a^{\alpha}\mathscr{H}(t, \zeta(t), \zeta(\gamma t))\Big) - \varphi_q\Big({}^{H}I_a^{\alpha}\mathscr{H}(t, \eta(t), \eta(\gamma t))\Big)\Big|\Bigg] \\ &+ \sum_{i=1}^{n}\Big|{}^{AB}I_a^{\sigma_i}(\mathscr{F}_i(t, \zeta(t), \zeta(\delta t)) - \mathscr{F}_i(t, \eta(t), \eta(\delta t)))\Big|. \end{aligned}$$

For each $\zeta \in \mathscr{B}$ and $t \in \Omega$, we have

$$\begin{aligned}\left|\varphi_q\Big({}^{H}I_a^{\alpha}\mathscr{H}(t,\zeta(t),\zeta(\gamma t))\Big)\right| &= \left|\varphi_q\Big(\frac{1}{\Gamma(\alpha)}\int_a^t\Big(\log\frac{t}{s}\Big)^{\alpha-1}\mathscr{H}(s,\zeta(s),\zeta(\gamma s))\frac{ds}{s}\Big)\right| \\ &\leq \varphi_q\Big(\frac{1}{\Gamma(\alpha)}\int_a^t\Big(\log\frac{t}{s}\Big)^{\alpha-1}\varphi_p(\psi(s))\frac{ds}{s}\Big) \\ &\leq \varphi_q\Big(\|\psi\|^{p-1}\frac{1}{\Gamma(\alpha)}\int_a^t\Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{ds}{s}\Big) \\ &\leq \Big(\frac{(\log\frac{T}{a})^{\alpha}}{\Gamma(\alpha+1)}\Big)^{q-1}\|\psi\| = \Lambda_0.\end{aligned} \tag{3.11}$$

On the other hand, we have

$$\begin{aligned}{}^{RL}I_a^{\beta}\left|\varphi_q\Big({}^{H}I_a^{\alpha}\mathscr{H}(t,\eta(t),\eta(\gamma t))\Big)\right| &= \frac{1}{\Gamma(\beta)}\int_a^t(t-s)^{\beta-1}\left|\varphi_q\Big({}^{H}I_a^{\alpha}\mathscr{H}(s,\eta(s),\eta(\gamma s))\Big)\right|ds \\ &\leq \frac{(T-a)^{\beta}}{\Gamma(\beta+1)}\Lambda_0.\end{aligned} \tag{3.12}$$

Applying Lemma 2.4 and using inequalities (3.11) and (3.12), we obtain

$$\begin{aligned}|\mathscr{A}\zeta(t)-\mathscr{A}\eta(t)| \leq{}& 2\|\vartheta\|\Big[|\lambda|+\Big(\frac{1-\beta}{\Re(\beta)}+\frac{\beta(T-a)^{\beta}}{\Re(\beta)\Gamma(\beta+1)}\Big)\Big(\frac{(\log\frac{T}{a})^{\alpha}}{\Gamma(\alpha+1)}\Big)^{q-1}\|\psi\|\Big]|\zeta-\eta| \\ &+\mathcal{M}\Big[\frac{2(1-\beta)(q-1)\kappa^{q-2}\mathcal{K}}{\Re(\beta)\Gamma(\alpha+1)}\Big(\log\frac{T}{a}\Big)^{\alpha}+\frac{2\beta(q-1)\kappa^{q-2}\mathcal{K}}{\Re(\beta)}\,{}^{RL}I_a^{\beta}\,{}^{H}I_a^{\alpha}(1)(T)\Big]|\zeta-\eta| \\ &+\sum_{i=1}^{n}2\widehat{\mathcal{K}}_i\Big(\frac{1-\sigma_i}{\Re(\sigma_i)}+\frac{(T-a)^{\sigma_i}}{\Re(\sigma_i)\Gamma(\sigma_i)}\Big)|\zeta-\eta|.\end{aligned}$$

Therefore, we get the following inequality:

$$|\mathscr{A}\zeta(t)-\mathscr{A}\eta(t)| \leq \Big(\Lambda_1+\mathcal{K}\mathcal{M}(\Lambda_2+\Lambda_3)+\Lambda_4\Big)|\zeta-\eta|, \tag{3.13}$$

and taking the supremum over $t \in \Omega$, we obtain

$$\|\mathscr{A}\zeta-\mathscr{A}\eta\| \leq \Big(\Lambda_1+\mathcal{K}\mathcal{M}(\Lambda_2+\Lambda_3)+\Lambda_4\Big)\|\zeta-\eta\|. \tag{3.14}$$

Using condition (4.2), we deduce that $\mathscr{A}$ is a contraction. Therefore, $\mathscr{A}$ has a unique fixed point in $\mathcal{B}$. By Definition 3.1, this fixed point is the unique integral solution of problem (1.4).

The next existence theorem is based on Dhage's Krasnoselskii-type hybrid fixed-point theorem in Banach algebras (Theorem 2.1).

Theorem 3.3. *Assume that* (H1)*,* (H3)*, and* (H4) *hold. If*

$$2(\|\vartheta\|\Lambda_* + \Lambda_4) < 1, \tag{3.15}$$

then problem (1.4) *has at least one integral solution in* $\mathcal{B} = C(\Omega, \mathbb{R})$.

Proof. Let $\mathscr{E} = \{\zeta \in \mathscr{B} : \|\zeta\| \leq r\}$, such that

$$r \geq \frac{\mathfrak{m}_0\Lambda_* + \Lambda_5}{1 - 2\Big(\|\vartheta\|\,\Lambda_* + \Lambda_4\Big)}. \tag{3.16}$$

Clearly, $\mathscr{E}$ is a convex, closed, and bounded subset of $\mathscr{B}$. Next we decompose operator $\mathscr{A}$ into three operators $\mathscr{A}_1, \mathscr{A}_2$, and $\mathscr{A}_3$ as:

$$\mathscr{A}_1 : \mathscr{B} \to \mathscr{B}, \;\; \mathscr{A}_2 : \mathscr{E} \to \mathscr{B}, \;\; \mathscr{A}_3 : \mathscr{B} \to \mathscr{B}$$

such that

$$\begin{aligned}
&\mathscr{A}_1(\zeta) := \mathscr{G}(t, \zeta(t), \zeta(\rho t)), \;\; t \in \Omega,\\
&\mathscr{A}_2(\zeta) := \lambda + \frac{1-\beta}{\mathfrak{N}(\beta)}\,\varphi_q\Big({}^{H}I_a^{\alpha}\mathscr{H}(t, \zeta(t), \zeta(\gamma t))\Big) + \frac{\beta}{\mathfrak{N}(\beta)}\,{}^{RL}I_a^{\beta}\Big(\varphi_q\Big({}^{H}I_a^{\alpha}\mathscr{H}(t, \zeta(t), \zeta(\gamma t))\Big)\Big), \;\; t \in \Omega,\\
&\mathscr{A}_3(\zeta) := \sum_{i=1}^{n} {}^{AB}I_a^{\sigma_i}\mathscr{F}_i(t, \zeta(t), \zeta(\delta t)), \;\; t \in \Omega.
\end{aligned}$$

Thus, we get the following operator equations for (3.8) as:

$$\mathscr{A}\zeta(t) = \mathscr{A}_1(\zeta)(t)\mathscr{A}_2(\zeta)(t) + \mathscr{A}_3(\zeta)(t), \;\; t \in \Omega. \tag{3.17}$$

We will show that the operators $\mathscr{A}_1$ and $\mathscr{A}_3$ are Lipschitzian on $\mathscr{B}$. Let $\zeta_1, \zeta_2 \in \mathscr{B}$ and $t \in \Omega$, and then by using the hypothesis (H_3), we have

$$\begin{aligned}
|\mathscr{A}_1\zeta_1(t) - \mathscr{A}_1\zeta_2(t)| &= |\mathscr{G}(t, \zeta_1(t), \zeta_1(\rho t)) - \mathscr{G}(t, \zeta_2(t), \zeta_2(\rho t))|\\
&\leq 2\vartheta(t)\,|\zeta_1(t) - \zeta_2(t)|\,,
\end{aligned}$$

wich implies

$$\|\mathscr{A}_1\zeta_1 - \mathscr{A}_1\zeta_2\| \leq 2\,\|\vartheta\|\,\|\zeta_1 - \zeta_2\|\,. \tag{3.18}$$

With the aid of the hypothesis (H_1), we obtain

$$\begin{aligned}
|\mathscr{A}_3\zeta_1(t) - \mathscr{A}_3\zeta_2(t)| &= \left|\sum_{i=1}^{n} {}^{AB}I_a^{\sigma_i}\mathscr{F}_i(t, \zeta_1(t), \zeta_1(\delta t)) - \sum_{i=1}^{n} {}^{AB}I_a^{\sigma_i}\mathscr{F}_i(t, \zeta_2(t), \zeta_2(\delta t))\right|\\
&\leq \sum_{i=1}^{n} {}^{AB}I_a^{\sigma_i}\,|\mathscr{F}_i(t, \zeta_1(t), \zeta_1(\delta t)) - \mathscr{F}_i(t, \zeta_2(t), \zeta_2(\delta t))|\\
&\leq \Lambda_4\,|\zeta_1 - \zeta_2|\,.
\end{aligned} \tag{3.19}$$

Consequently

$$\|\mathscr{A}_3\zeta_1 - \mathscr{A}_3\zeta_2\| \leq \Lambda_4\,\|\zeta_1 - \zeta_2\|\,. \tag{3.20}$$

Next, we show that $\mathscr{A}_2$ is completely continuous. The operator $\mathscr{A}_2$ is continuous since the function $\mathscr{H}$ is continuous.

Next, we establish that $\mathscr{A}_2\mathscr{E}$ is uniformly bounded in $\mathscr{B}$. Using estimates (3.11) and (3.12), it follows that for each $\zeta \in \mathscr{E}$, we can write

$$\begin{aligned} |\mathscr{A}_2\zeta(t)| &\leq |\lambda| + \frac{1-\beta}{\Re(\beta)}\left|\varphi_q\Big({}^{H}I_a^{\alpha}\mathscr{H}(t,\zeta(t),\zeta(\gamma t))\Big)\right| + \frac{\beta}{\Re(\beta)}\,{}^{RL}I_a^{\beta}\left|\varphi_q\Big({}^{H}I_a^{\alpha}\mathscr{H}(t,\zeta(t),\zeta(\gamma t))\Big)\right| \\ &\leq |\lambda| + \Big(\frac{1-\beta}{\Re(\beta)} + \frac{\beta(T-a)^{\beta}}{\Re(\beta)\Gamma(\beta+1)}\Big)\Lambda_0 = \Lambda_*. \end{aligned} \tag{3.21}$$

Therefore,

$$\|\mathscr{A}_2\| \leq \Lambda_*, \tag{3.22}$$

which implies that $\mathscr{A}_2\mathscr{E}$ is bounded in $\mathscr{B}$.

Next, we show that $\mathscr{A}_2\mathscr{E}$ is equicontinuous. Let $t_1, t_2 \in \Omega$ with $t_1 < t_2$ and $\zeta \in \mathscr{E}$. Then we have

$$\begin{aligned} |\mathscr{A}_2\zeta(t_2) - \mathscr{A}_2\zeta(t_1)| \leq{}& \frac{1-\beta}{\Re(\beta)}\left|\varphi_q\Big({}^{H}I_a^{\alpha}\mathscr{H}(\cdot,\zeta(\cdot),\zeta(\gamma\cdot))\Big)(t_2) - \varphi_q\Big({}^{H}I_a^{\alpha}\mathscr{H}(\cdot,\zeta(\cdot),\zeta(\gamma\cdot))\Big)(t_1)\right| \\ &+ \frac{\beta}{\Re(\beta)}\left|{}^{RL}I_a^{\beta}\Big(\varphi_q\Big({}^{H}I_a^{\alpha}\mathscr{H}(\cdot,\zeta(\cdot),\zeta(\gamma\cdot))\Big)\Big)(t_2) - {}^{RL}I_a^{\beta}\Big(\varphi_q\Big({}^{H}I_a^{\alpha}\mathscr{H}(\cdot,\zeta(\cdot),\zeta(\gamma\cdot))\Big)\Big)(t_1)\right|. \end{aligned} \tag{3.23}$$

Define

$$h_\zeta(t) = \mathscr{H}(t,\zeta(t),\zeta(\gamma t)), \qquad \Omega_\zeta(t) = {}^{H}I_a^{\alpha}h_\zeta(t).$$

Set

$$M_\psi := \|\varphi_p(\psi)\|_\infty.$$

By the hypothesis (H_4),

$$|h_\zeta(t)| \leq \varphi_p(\psi(t)) \leq M_\psi, \qquad t \in [a,T].$$

Hence, we get

$$|\Omega_\zeta(t)| \leq \frac{M_\psi}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s} = \frac{M_\psi}{\Gamma(\alpha+1)}\left(\ln\frac{t}{a}\right)^{\alpha} \leq \frac{M_\psi}{\Gamma(\alpha+1)}\left(\ln\frac{T}{a}\right)^{\alpha}.$$

Thus, we have

$$|\Omega_\zeta(t_2) - \Omega_\zeta(t_1)| \leq \frac{2M_\psi}{\Gamma(\alpha+1)}\left(\ln\frac{t_2}{t_1}\right)^{\alpha}.$$

Since $|\Omega_\zeta(t)| \leq \kappa^*$ by Remark 3.1, Lemma 2.4 applied to φ_q yields

$$|\varphi_q(\Omega_\zeta(t_2)) - \varphi_q(\Omega_\zeta(t_1))| \leq \frac{2(q-1)\kappa^{q-2}M_\psi}{\Gamma(\alpha+1)}\left(\ln\frac{t_2}{t_1}\right)^{\alpha}. \tag{3.24}$$

On the other hand, $|\varphi_q(\Omega_\zeta)| \leq \Lambda_0$ by (3.11), so that

$$\begin{aligned} &\left|{}^{RL}I_a^{\beta}(\varphi_q(\Omega_\zeta))(t_2) - {}^{RL}I_a^{\beta}(\varphi_q(\Omega_\zeta))(t_1)\right| \\ &\leq \frac{\Lambda_0}{\Gamma(\beta)}\Big[\int_a^{t_1}\Big((t_1-s)^{\beta-1} - (t_2-s)^{\beta-1}\Big)\,ds + \int_{t_1}^{t_2}(t_2-s)^{\beta-1}\,ds\Big] \\ &= \frac{\Lambda_0}{\Gamma(\beta+1)}\Big[(t_1-a)^{\beta} - (t_2-a)^{\beta} + 2(t_2-t_1)^{\beta}\Big] \\ &\leq \frac{2\Lambda_0}{\Gamma(\beta+1)}(t_2-t_1)^{\beta}. \end{aligned} \tag{3.25}$$

Combining the above estimates (3.24) and (3.25), we obtain

$$|\mathscr{A}_2\zeta(t_2) - \mathscr{A}_2\zeta(t_1)| \leq \frac{2(1-\beta)(q-1)\kappa^{q-2}M_\psi}{\Re(\beta)\Gamma(\alpha+1)}\left(\ln\frac{t_2}{t_1}\right)^\alpha + \frac{2\beta\Lambda_0}{\Re(\beta)\Gamma(\beta+1)}(t_2-t_1)^\beta. \tag{3.26}$$

As $t_2 - t_1 \to 0$, the right-hand side of the inequality (3.26) tends to zero independently of ζ. Thus, according to the Arzelà–Ascoli theorem, the operator $\mathscr{A}_2$ is completely continuous.

For any arbitrary elements $\zeta \in \mathscr{B}$ and $w \in \mathscr{E}$, where

$$\zeta = \mathscr{A}_1\zeta\mathscr{A}_2 w + \mathscr{A}_3\zeta,$$

we get

$$\begin{aligned}\left|\zeta(t)\right| &\leq |\mathscr{A}_1\zeta(t)|\,|\mathscr{A}_2 w(t)| + |\mathscr{A}_3\zeta(t)| \\ &\leq \Big(|\mathscr{G}(t,\zeta(t),\zeta(\rho t)) - \mathscr{G}(t,0,0)| + |\mathscr{G}(t,0,0)|\Big)\Lambda_* \\ &\quad + \sum_{i=1}^{n} {}^{AB}I_a^{\sigma_i}\Big(|\mathscr{F}_i(t,\zeta(t),\zeta(\delta t)) - \mathscr{F}_i(t,0,0)| + |\mathscr{F}_i(t,0,0)|\Big) \\ &\leq (2\,\|\vartheta\|\,\|\zeta\| + \mathfrak{m}_0)\Lambda_* + \Lambda_4\,\|\zeta\| + \Lambda_5.\end{aligned} \tag{3.27}$$

Consequently, taking the supremum over $t \in \Omega$,

$$\|\zeta\| \leq (\mathfrak{m}_0\Lambda_* + \Lambda_5) + (2\,\|\vartheta\|\,\Lambda_* + \Lambda_4)\,\|\zeta\|,$$

and condition (3.16) yields

$$\|\zeta\| \leq \frac{\mathfrak{m}_0\Lambda_* + \Lambda_5}{1 - (2\,\|\vartheta\|\,\Lambda_* + \Lambda_4)} \leq r, \tag{3.28}$$

that is, $\zeta \in \mathscr{E}$, so that assumption (As3) of Theorem 2.1 holds. Finally, we verify assumption (As4) of Theorem 2.1, namely,

$$\mathscr{M}_0\mathscr{P} + \mathscr{N} < 1,$$

where $\mathscr{M}_0 = 2\|\vartheta\|$ and $\mathscr{N} = \Lambda_4$ are the Lipschitz constants of $\mathscr{A}_1$ and $\mathscr{A}_3$, respectively. Since $\mathscr{P} \leq \Lambda_*$, condition (3.15) gives

$$\begin{aligned}\mathscr{M}_0\mathscr{P} + \mathscr{N} &\leq 2\|\vartheta\|\Lambda_* + \Lambda_4 \\ &\leq 2(\|\vartheta\|\Lambda_* + \Lambda_4) < 1.\end{aligned} \tag{3.29}$$

4. Stability theory

In the current section, we examine the Ulam–Hyers stability of the sequential pantograph fractional differential system (1.4).

Definition 4.1. *Problem* (1.4) *is said to be* Ulam–Hyers stable *if there exists a constant* $C_{\mathrm{UH}} > 0$ *such that, for every* $\varepsilon \in (0,1]$ *and every classical function* ζ *in the sense of Definition 3.1 satisfying the boundary condition in* (1.4) *and*

$$\left|{}^{CH}D_a^\alpha\left[\varphi_p\left({}^{ABC}D_a^\beta\left(\frac{\zeta(t) - \sum_{i=1}^{n} {}^{AB}I_a^{\sigma_i}\mathscr{F}_i(t,\zeta(t),\zeta(\delta t))}{\mathscr{G}(t,\zeta(t),\zeta(\rho t))}\right)\right)\right] - \mathscr{H}(t,\zeta(t),\zeta(\gamma t))\right| \leq \varepsilon, \qquad t \in \Omega, \tag{4.1}$$

there exists an integral solution $\bar{\zeta} \in \mathcal{B}$ *of problem* (1.4) *such that*

$$\|\zeta - \bar{\zeta}\|_\infty \leq C_{\mathrm{UH}}\varepsilon.$$

Theorem 4.1. *Assume that hypotheses* (H1)–(H4) *hold and that* $q \geq 2$. *If*

$$\Lambda_1 + \mathcal{K}\mathcal{M}(\Lambda_2 + \Lambda_3) + \Lambda_4 < 1, \tag{4.2}$$

then problem (1.4) *is Ulam–Hyers stable. More precisely,*

$$\|\zeta - \bar{\zeta}\|_\infty \leq C_{\mathrm{UH}}\varepsilon, \qquad C_{\mathrm{UH}} = \frac{\mathfrak{q}_1}{1-\Lambda},$$

where

$$\Lambda = \Lambda_1 + \mathcal{K}\mathcal{M}(\Lambda_2 + \Lambda_3) + \Lambda_4 \ \text{ and } \ \mathfrak{q}_1 := \mathcal{M}\left(\frac{1-\beta}{\Re(\beta)} + \frac{\beta(T-a)^\beta}{\Re(\beta)\Gamma(\beta+1)}\right)(q-1)\,(\kappa^* + \kappa^\circ)^{q-2}\,\kappa^\circ. \tag{4.3}$$

Proof. Let $\zeta \in C(\Omega, \mathbb{R})$ be an approximate solution of problem (1.4). Thus, for some $\varepsilon > 0$, we have

$$\begin{aligned} &\left| {}^{CH}D_a^\alpha\left[\varphi_p\left({}^{ABC}D_a^\beta\left(\frac{\zeta(t) - \sum_{i=1}^{n} {}^{AB}I_a^{\sigma_i}\mathscr{F}_i(t,\zeta(t),\zeta(\delta t))}{\mathscr{G}(t,\zeta(t),\zeta(\rho t))}\right)\right)\right]\right. \\ &\left. - \mathscr{H}(t,\zeta(t),\zeta(\gamma t))\right| \leq \varepsilon, \qquad t \in [a,T]. \end{aligned} \tag{4.4}$$

Applying the Hadamard fractional integral ${}^{H}I_a^\alpha$ to (4.4), we obtain

$$\begin{aligned} &\left|\varphi_p\left({}^{ABC}D_a^\beta\left(\frac{\zeta(t) - \sum_{i=1}^{n} {}^{AB}I_a^{\sigma_i}\mathscr{F}_i(t,\zeta(t),\zeta(\delta t))}{\mathscr{G}(t,\zeta(t),\zeta(\rho t))}\right)\right)\right. \\ &\left. - {}^{H}I_a^\alpha\mathscr{H}(t,\zeta(t),\zeta(\gamma t))\right| \\ &\leq \frac{\varepsilon}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s}. \end{aligned} \tag{4.5}$$

It follows from (4.5) that

$$\begin{aligned} &\left|\varphi_p\left({}^{ABC}D_a^\beta\left(\frac{\zeta(t) - \sum_{i=1}^{n} {}^{AB}I_a^{\sigma_i}\mathscr{F}_i(t,\zeta(t),\zeta(\delta t))}{\mathscr{G}(t,\zeta(t),\zeta(\rho t))}\right)\right)\right. \\ &\left. - {}^{H}I_a^\alpha\mathscr{H}(t,\zeta(t),\zeta(\gamma t))\right| \leq \frac{(\ln(T/a))^\alpha}{\Gamma(\alpha+1)}\varepsilon. \end{aligned} \tag{4.6}$$

Set

$$\kappa^{\circ} = \frac{(\ln(T/a))^{\alpha}}{\Gamma(\alpha+1)}.$$

Then (4.6) yields

$$\left| \varphi_p\left({}^{ABC}D_a^{\beta}\left(\frac{\zeta(t) - \sum_{i=1}^{n} {}^{AB}I_a^{\sigma_i}\mathscr{F}_i(t,\zeta(t),\zeta(\delta t))}{\mathscr{G}(t,\zeta(t),\zeta(\rho t))} \right) \right) - {}^{H}I_a^{\alpha}\mathscr{H}(t,\zeta(t),\zeta(\gamma t)) \right| \leq \kappa^{\circ}\varepsilon. \tag{4.7}$$

Since $\varphi_q = \varphi_p^{-1}$, it follows from (4.7) that

$$\begin{aligned} &\left| {}^{ABC}D_a^{\beta}\left(\frac{\zeta(t) - \sum_{i=1}^{n} {}^{AB}I_a^{\sigma_i}\mathscr{F}_i(t,\zeta(t),\zeta(\delta t))}{\mathscr{G}(t,\zeta(t),\zeta(\rho t))} \right) \right. \\ &\left. - \varphi_q\left({}^{H}I_a^{\alpha}\mathscr{H}(t,\zeta(t),\zeta(\gamma t)) \right) \right| \\ &\leq (q-1)(\kappa^{*}+\kappa^{\circ})^{q-2}\kappa^{\circ}\varepsilon. \end{aligned} \tag{4.8}$$

Using the inverse formula for the Atangana–Baleanu–Caputo derivative, it follows that

$$\begin{aligned} &\left| \frac{\zeta(t) - \sum_{i=1}^{n} {}^{AB}I_a^{\sigma_i}\mathscr{F}_i(t,\zeta(t),\zeta(\delta t))}{\mathscr{G}(t,\zeta(t),\zeta(\rho t))} \right. \\ &- \lambda - \frac{1-\beta}{\mathfrak{N}(\beta)}\varphi_q\left({}^{H}I_a^{\alpha}\mathscr{H}(t,\zeta(t),\zeta(\gamma t)) \right) \\ &\left. - \frac{\beta}{\mathfrak{N}(\beta)} {}^{RL}I_a^{\beta}\left(\varphi_q\left({}^{H}I_a^{\alpha}\mathscr{H}(t,\zeta(t),\zeta(\gamma t)) \right) \right) \right| \\ &\leq \left[\frac{1-\beta}{\mathfrak{N}(\beta)} + \frac{\beta(T-a)^{\beta}}{\mathfrak{N}(\beta)\Gamma(\beta+1)} \right] \\ &\times (q-1)(\kappa^{*}+\kappa^{\circ})^{q-2}\kappa^{\circ}\varepsilon. \end{aligned} \tag{4.9}$$

Multiplying both sides of (4.9) by $|\mathscr{G}(t,\zeta(t),\zeta(\rho t))|$ and using hypothesis (H2), we obtain

$$\begin{aligned} &\left| \zeta(t) - \mathscr{G}(t,\zeta(t),\zeta(\rho t)) \right| \left[\lambda + \frac{1-\beta}{\mathfrak{N}(\beta)}\varphi_q\left({}^{H}I_a^{\alpha}\mathscr{H}(t,\zeta(t),\zeta(\gamma t)) \right) \right. \\ &\left. + \frac{\beta}{\mathfrak{N}(\beta)} {}^{RL}I_a^{\beta}\left(\varphi_q\left({}^{H}I_a^{\alpha}\mathscr{H}(t,\zeta(t),\zeta(\gamma t)) \right) \right) \right] \\ &- \sum_{i=1}^{n} {}^{AB}I_a^{\sigma_i}\mathscr{F}_i(t,\zeta(t),\zeta(\delta t)) \Big| \\ &\leq \mathcal{M}\left[\frac{1-\beta}{\mathfrak{N}(\beta)} + \frac{\beta(T-a)^{\beta}}{\mathfrak{N}(\beta)\Gamma(\beta+1)} \right] \end{aligned}$$

$$\times (q-1)(\kappa^* + \kappa^\circ)^{q-2}\kappa^\circ \varepsilon.$$

Therefore,

$$|\zeta(t) - (\mathscr{A}\zeta)(t)| \leq \mathfrak{q}_1 \varepsilon. \tag{4.10}$$

Taking the supremum over $t \in [a, T]$ in (4.10), we obtain

$$\|\zeta - \mathscr{A}\zeta\|_\infty \leq \mathfrak{q}_1 \varepsilon. \tag{4.11}$$

Now, since $\bar{\zeta}$ is an exact solution of problem (1.4), it is a fixed point of $\mathscr{A}$, that is,

$$\bar{\zeta} = \mathscr{A}\bar{\zeta}. \tag{4.12}$$

Hence,

$$\begin{aligned} \left|\zeta(t) - \bar{\zeta}(t)\right| &= \left|\zeta(t) - (\mathscr{A}\zeta)(t) + (\mathscr{A}\zeta)(t) - (\mathscr{A}\bar{\zeta})(t)\right| \\ &\leq |\zeta(t) - (\mathscr{A}\zeta)(t)| + \left|(\mathscr{A}\zeta)(t) - (\mathscr{A}\bar{\zeta})(t)\right|. \end{aligned} \tag{4.13}$$

Consequently, from (3.13) and (4.10), we obtain

$$\left|\zeta(t) - \bar{\zeta}(t)\right| \leq \mathfrak{q}_1 \varepsilon + [\Lambda_1 + \mathcal{K}\mathcal{M}(\Lambda_2 + \Lambda_3) + \Lambda_4] \left\|\zeta - \bar{\zeta}\right\|_\infty. \tag{4.14}$$

Taking the supremum over $t \in [a, T]$, it follows that

$$\left\|\zeta - \bar{\zeta}\right\|_\infty \leq \mathfrak{q}_1 \varepsilon + [\Lambda_1 + \mathcal{K}\mathcal{M}(\Lambda_2 + \Lambda_3) + \Lambda_4] \left\|\zeta - \bar{\zeta}\right\|_\infty. \tag{4.15}$$

We deduce from (4.15) that

$$\left\|\zeta - \bar{\zeta}\right\|_\infty \leq \frac{\mathfrak{q}_1}{1 - \Lambda_1 - \mathcal{K}\mathcal{M}(\Lambda_2 + \Lambda_3) - \Lambda_4} \varepsilon. \tag{4.16}$$

Finally, we obtain

$$\left\|\zeta - \bar{\zeta}\right\|_\infty \leq C_{\mathrm{UH}} \varepsilon. \tag{4.17}$$

Hence, according to Definition 4.1, the solution of problem (1.4) is Ulam–Hyers stable.

5. Examples

To confirm the findings of our study, we consider two sequential Caputo–Hadamard and Atangana–Baleanu fractional pantograph problems. Throughout the numerical examples, we use the normalization convention $\mathfrak{N}(\theta) = 1$ for every fractional order θ appearing in the Atangana–Baleanu operators. The proportional arguments are understood in the constant-prehistory sense of Remark 2.1 (so that, e.g., $\zeta(\rho \cdot 1) = \zeta(1)$ since $\rho < 1$).

Numerical procedure. For the numerical illustrations, the fixed-point operator (3.8) was implemented in Python on a uniform mesh:

$$t_j = a + jh, \qquad j = 0, 1, \ldots, N, \qquad h = \frac{T-a}{N},$$

with $N = 400$. Values at proportional-delay arguments that do not coincide with mesh nodes were evaluated by linear interpolation. The constant prehistory convention of Remark 2.1 was used whenever a delayed argument was smaller than a.

The Riemann–Liouville and Atangana–Baleanu integral terms were evaluated by adaptive quadrature. For the Hadamard fractional integral, the change of variables $s = e^u$ was used to transform the logarithmic kernel into an algebraically singular kernel in the variable u. The weakly singular Hadamard and Riemann–Liouville kernels were handled using the algebraic-weight option of the adaptive quadrature routine. The integrands required between mesh points were evaluated from the current iterate by linear interpolation. The absolute and relative quadrature tolerances were both set to 10^{-10}.

Starting from $\zeta^{(0)} \equiv 0$, the iterations were terminated when

$$\|\zeta^{(n+1)} - \zeta^{(n)}\|_{\infty,h} < 10^{-10}, \qquad \|v\|_{\infty,h} := \max_{0\le j\le N} |v(t_j)|,$$

or after 100 iterations. The Python scripts used to generate the numerical results and figures are supplied as supplementary material. The norm of the difference between successive iterates is used as a discrete stopping indicator. It is not a rigorous estimate of the error relative to the unknown continuous solution.

Numerical behavior. For both examples, the fixed-point iteration converges rapidly from the zero initial guess. The discrete successive-difference norms decay geometrically, with an observed ratio substantially smaller than the theoretical contraction constant Λ established in Theorem 3.2. This behavior is consistent with the uniqueness and stability results: the computed profiles approach the unique integral solution whose existence is guaranteed by the contraction principle. The reported figures are generated with a uniform mesh of $N = 400$ subintervals; similar qualitative behavior is observed on coarser and finer meshes, confirming the robustness of the numerical scheme.

Example 1 (illustrating Theorem 3.2 and Theorem 4.1).

$$\begin{cases} {}^{CH}D_1^{\frac{2}{3}}\left[\varphi_{\frac{4}{3}}\left[{}^{ABC}D_1^{\frac{3}{4}}\left(\dfrac{\zeta(t) - \sum_{i=1}^{2} {}^{AB}I_1^{\frac{1}{5}(i+1)}\mathscr{F}_i(t,\zeta(t),\zeta(\frac{1}{3}t))}{\mathscr{G}(t,\zeta(t),\zeta(\frac{3}{4}t))}\right)\right]\right] - \mathscr{H}(t,\zeta(t),\zeta(\frac{5}{6}t)) = 0, \\ t \in \Omega := [1,5], \\ \dfrac{\zeta(1)}{\mathscr{G}(1,\zeta(1),\zeta(\frac{3}{4}))} = 2, \quad \mathscr{F}_i(1,x,y) = 0, \ \ \forall (x,y) \in \mathbb{R}^2, \end{cases} \tag{5.1}$$

with

$$\alpha = \tfrac{2}{3},\ \beta = \tfrac{3}{4},\ \delta = \tfrac{1}{3},\ \rho = \tfrac{3}{4},\ \gamma = \tfrac{5}{6},\ \sigma_1 = \tfrac{2}{5},\ \sigma_2 = \tfrac{3}{5},\ p = \tfrac{4}{3}\ (\text{so } q = 4,\ \varphi_q(r) = |r|^2 r).$$

We choose $\mathscr{H}, \mathscr{F}_i : [1,5]\times\mathbb{R}^2 \to \mathbb{R}$ and $\mathscr{G} : [1,5]\times\mathbb{R}^2 \to \mathbb{R}\setminus\{0\}$ as

$$\begin{aligned} \mathscr{H}(t,x,y) &= e^{-2t} + \frac{\cos x}{15\sqrt{t^2+8}} + \frac{\sin y}{15(t+2)}, \\ \mathscr{F}_i(t,x,y) &= \frac{t-1}{200i+9t}(\sin x + \arctan y), \quad i = 1,2, \\ \mathscr{G}(t,x,y) &= 1 + e^{-2t} + \frac{\sin x}{4\sqrt{t^2+12}} + \frac{1}{10\sqrt{2t+1}}\cdot\frac{y}{1+|y|}. \end{aligned}$$

The factor $(t-1)$ guarantees $\mathscr{F}_i(1,x,y)=0$ for all (x,y), so the boundary relation is consistent; and $\mathscr{G}\geq 1-\frac{1}{4\sqrt{13}}-\frac{1}{10\sqrt{3}}\approx 0.8729>0$, so $\mathscr{G}$ is indeed valued in $\mathbb{R}\setminus\{0\}$.

Verification of the hypotheses.

(H1) For all $t\in[1,5]$, $|\mathscr{H}(t,x_1,y_1)-\mathscr{H}(t,x_2,y_2)|\leq\frac{1}{45}(|x_1-x_2|+|y_1-y_2|)$, i.e., $\mathcal{K}=\frac{1}{45}$; and $|\mathscr{F}_i(t,x_1,y_1)-\mathscr{F}_i(t,x_2,y_2)|\leq\widehat{\mathcal{K}}_i(|x_1-x_2|+|y_1-y_2|)$ with $\widehat{\mathcal{K}}_1=\frac{4}{245}\approx 0.01633$, $\widehat{\mathcal{K}}_2=\frac{4}{445}\approx 0.008989$ (the suprema of $\frac{t-1}{200i+9t}$ over $[1,5]$, attained at $t=5$).

(H2) $|\mathscr{G}(t,x,y)|\leq 1+e^{-2}+\frac{1}{4\sqrt{13}}+\frac{1}{10\sqrt{3}}\approx 1.2624=\mathcal{M}$.

(H3) $|\mathscr{G}(t,x_1,y_1)-\mathscr{G}(t,x_2,y_2)|\leq\vartheta(t)(|x_1-x_2|+|y_1-y_2|)$ with $\vartheta(t)=\frac{1}{4\sqrt{t^2+12}}$, so $\|\vartheta\|=\frac{1}{4\sqrt{13}}\approx 0.06934$ (the map $y\mapsto\frac{y}{1+|y|}$ is 1-Lipschitz and $\frac{1}{10\sqrt{2t+1}}\leq\frac{1}{10\sqrt{3}}<\|\vartheta\|$).

(H4) $|\mathscr{H}(t,x,y)|\leq e^{-2t}+\frac{2}{45}=\varphi_p(\psi(t))$ with $\psi(t)=\left(e^{-2t}+\frac{2}{45}\right)^3>0$, so $\|\psi\|=\left(e^{-2}+\frac{2}{45}\right)^3\approx 5.811\times10^{-3}$.

Constants. With $c_1:=\frac{(\log 5)^{2/3}}{\Gamma(5/3)}\approx 1.52131$, we obtain $\kappa^*=c_1\|\psi\|^{1/3}\approx 0.27350$ and $(q-1)\kappa^{*\,q-2}=3\kappa^{*2}\approx 0.22441$. Hence

$$\Lambda_0=c_1^3\|\psi\|\approx 0.02046,\qquad c_\beta:=(1-\beta)+\tfrac{\beta(T-a)^\beta}{\Gamma(\beta+1)}\approx 2.55814,\qquad \Lambda_*=|\lambda|+c_\beta\Lambda_0\approx 2.05234,$$
$$\Lambda_1=2\|\vartheta\|\Lambda_*\approx 0.28461,\quad \Lambda_2\approx 0.17070,\quad \Lambda_3=2\beta\,(3\kappa^{*2})\,{}^{RL}I^{\beta}\,{}^{H}I^{\alpha}(1)(5)\approx 1.19933,$$
$$\Lambda_4\approx 0.08015,\qquad \mathcal{K}\mathcal{M}\approx 0.02805.$$

Therefore, setting $\Lambda:=\Lambda_1+\mathcal{K}\mathcal{M}(\Lambda_2+\Lambda_3)+\Lambda_4$,

$$\Lambda=\Lambda_1+\mathcal{K}\mathcal{M}(\Lambda_2+\Lambda_3)+\Lambda_4\approx 0.285+0.038+0.080=0.403<1,$$

so, by Theorem 3.2, Problem (5.1) has a unique solution on $[1,5]$. Since (H_1)–(H_4) and condition (4.2) hold and, $q=4\geq 2$, Theorem 4.1 applies and problem (5.1) is Ulam–Hyers stable; with $\kappa^\circ=c_1$ and $\varepsilon\in(0,1]$, the constant (4.3) is $\mathfrak{q}_1\leq\mathcal{M}c_\beta\,3(\kappa^*+c_1)^2c_1\approx 47.5$, giving

$$C_{\mathrm{UH}}=\frac{\mathfrak{q}_1}{1-0.40319}\approx 80.$$

Figure 1 illustrates this numerically: the successive approximations generated by the operator (3.8) from $\zeta^{(0)}\equiv 0$ converge to the unique solution ζ^*, and the consecutive differences decay geometrically, with the observed contraction ratio (≈ 0.10) being well below the theoretical bound $\Lambda\approx 0.40$.

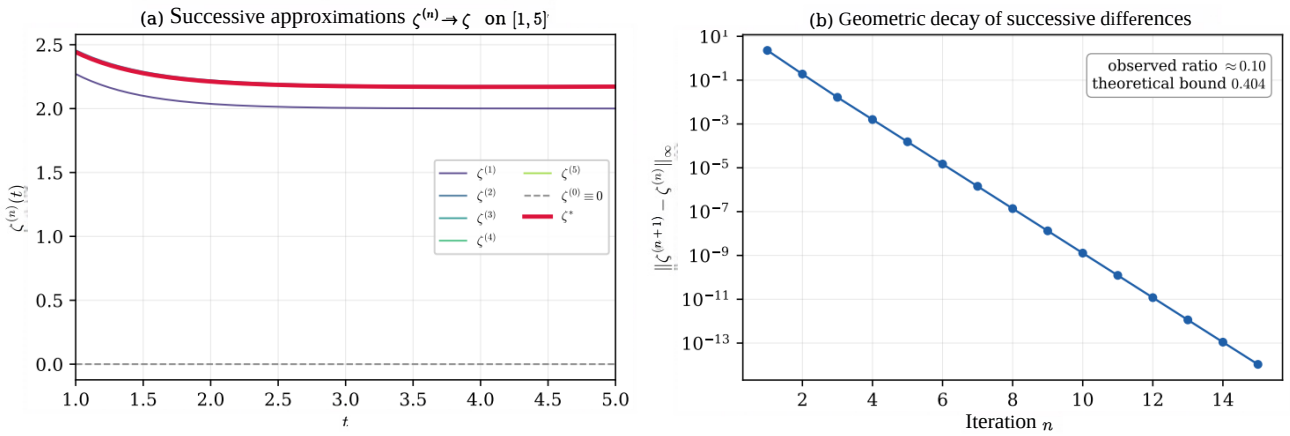

Figure 1. Numerical illustration for Example 1. Panel (a) displays selected successive iterates generated from $\zeta^{(0)} \equiv 0$ by the fixed-point operator (3.8). Panel (b) displays the corresponding discrete successive-difference norms $\|\zeta^{(n+1)} - \zeta^{(n)}\|_{\infty,h}$.

Figure 1 provides a numerical illustration of the iteration. Panel (a) displays selected successive iterates generated from $\zeta^{(0)} \equiv 0$, while panel (b) displays the corresponding discrete successive-difference norms. Since the contraction condition of Theorem 3.2 holds for this example, the iteration is consistent with the unique integral solution established by that theorem.

Remark 5.1. *In panel (a), the successive approximations $\zeta^{(n)}$, generated by the fixed-point operator* (3.8) *from $\zeta^{(0)} \equiv 0$, converge to the unique solution ζ^* (shown in bold red).*

In panel (b), the successive differences $\|\zeta^{(n+1)} - \zeta^{(n)}\|_\infty$ decay geometrically, as evidenced by the approximately linear behavior on the semi-logarithmic scale. The observed decay ratio of successive differences is approximately 0.096*, which is substantially smaller than the theoretical contraction constant $\Lambda \approx 0.40$ established in Theorem 3.2. This behavior is consistent with the predicted convergence of the fixed-point iteration to the unique integral solution.*

Example 2 (illustrating Theorem 3.3).

$$\begin{cases} {}^{CH}D_1^{\frac{1}{2}}\left[\varphi_2\left({}^{ABC}D_1^{\frac{2}{3}}\left(\dfrac{\zeta(t) - \sum_{i=1}^{2} {}^{AB}I_1^{\frac{1}{4}}\mathscr{F}_i(t,\zeta(t),\zeta(\frac{t}{5}))}{\mathscr{G}(t,\zeta(t),\zeta(\frac{t}{3}))}\right)\right)\right] - \mathscr{H}(t,\zeta(t),\zeta(\frac{t}{2})) = 0, \\ t \in \Omega = [1,4], \quad \dfrac{\zeta(1)}{\mathscr{G}(1,\zeta(1),\zeta(\frac{1}{3}))} = 1, \quad \mathscr{F}_i(1,x,y) = 0, \ \ \forall (x,y) \in \mathbb{R}^2, \end{cases} \tag{5.2}$$

with $\alpha = \frac{1}{2}$, $\beta = \frac{2}{3}$, $\delta = \frac{1}{5}$, $\rho = \frac{1}{3}$, $\gamma = \frac{1}{2}$, $\sigma_1 = \sigma_2 = \frac{1}{4}$, $p = 2$ (so $\varphi_2(r) = r$, $q = 2$). We take

$$\begin{aligned} \mathscr{H}(t,x,y) &= \tfrac{t}{10}\sin x + \tfrac{1}{5}\arctan y + \tfrac{e^{-t}}{2}, \\ \mathscr{F}_i(t,x,y) &= \frac{t^2-1}{200\,i}\left(\frac{x}{1+x^2} + \frac{y}{1+|y|}\right), \quad i = 1,2, \\ \mathscr{G}(t,x,y) &= 2 + \tfrac{1}{30}\cos x + \frac{1}{30}\cdot\frac{y}{1+y^2}. \end{aligned}$$

Again, the factor (t^2-1) gives $\mathscr{F}_i(1,x,y)=0$ for all (x,y), and, since $|\cos x|\le 1$ and $\left|\frac{y}{1+y^2}\right|\le\frac{1}{2}$, one has $\frac{39}{20}\le\mathscr{G}\le\frac{41}{20}$, so $\mathscr{G}$ is bounded and valued in $\mathbb{R}\setminus\{0\}$.

Verification of the hypotheses.

(H1) $|\mathscr{H}(t,x_1,y_1)-\mathscr{H}(t,x_2,y_2)|\le\frac{2}{5}(|x_1-x_2|+|y_1-y_2|)$, i.e., $\mathcal{K}=\frac{2}{5}$ (using $\frac{t}{10}\le\frac{2}{5}$ on $[1,4]$); and, since $u\mapsto\frac{u}{1+u^2}$ and $u\mapsto\frac{u}{1+|u|}$ are 1-Lipschitz, $|\mathscr{F}_i(t,x_1,y_1)-\mathscr{F}_i(t,x_2,y_2)|\le\widehat{\mathcal{K}_i}(|x_1-x_2|+|y_1-y_2|)$ with $\widehat{\mathcal{K}_1}=\frac{15}{200}=\frac{3}{40}$, $\widehat{\mathcal{K}_2}=\frac{15}{400}=\frac{3}{80}$ (suprema of $\frac{t^2-1}{200i}$ over $[1,4]$).

(H2)–(H3) $|\mathscr{G}(t,x_1,y_1)-\mathscr{G}(t,x_2,y_2)|\le\frac{1}{30}(|x_1-x_2|+|y_1-y_2|)$, so $\vartheta(t)=\frac{1}{30}$, $\|\vartheta\|=\frac{1}{30}$; $\mathcal{M}=\frac{41}{20}$; and $\mathfrak{m}_0=\sup_t|\mathscr{G}(t,0,0)|=2+\frac{1}{30}=\frac{61}{30}$.

(H4) $|\mathscr{H}(t,x,y)|\le\frac{t}{10}+\frac{\pi}{10}+\frac{1}{2}\le\frac{t}{5}+1=\varphi_2(\psi(t))$ with $\psi(t)=\frac{t}{5}+1>0$, so $\|\psi\|=\frac{9}{5}$.

Constants. Since $q=2$, $\varphi_q=\mathrm{Id}$ and $(q-1)\kappa^{q-2}=1$, so no constant κ is needed. With $c_1=\frac{\sqrt{\log 4}}{\Gamma(3/2)}\approx 1.32857$, we get $\Lambda_0=c_1\|\psi\|\approx 2.39142$, $c_\beta=(1-\beta)+\frac{\beta(T-a)^\beta}{\Gamma(\beta+1)}\approx 1.86945$, and hence $\Lambda_*=|\lambda|+c_\beta\Lambda_0\approx 5.47064$ and $\Lambda_4\approx 0.25042$, $\Lambda_5=0$ (as $\mathfrak{m}_i=0$). The smallness condition (3.15) reads

$$2(\|\vartheta\|\Lambda_*+\Lambda_4)\approx 2\,(0.182+0.250)=0.866<1,$$

so Theorem 3.3 applies: Problem (5.2) has at least one solution in $[1,4]$, and any radius

$$r\;\ge\;\frac{\mathfrak{m}_0\Lambda_*+\Lambda_5}{1-2(\|\vartheta\|\Lambda_*+\Lambda_4)}\approx\frac{2.03333\times 5.47064}{0.13444}\approx 83$$

is admissible; e.g., $r=83$.

Remark 5.2. *Here, Banach's principle (Theorem 3.2) does* not *apply: with the above data, one computes* $\Lambda_1=2\|\vartheta\|\Lambda_*\approx 0.36471$, $\Lambda_2\approx 0.88571$, $\Lambda_3\approx 3.31037$, *and* $\mathcal{K}\mathcal{M}=0.82$, *where* $\Lambda_1+\mathcal{K}\mathcal{M}(\Lambda_2+\Lambda_3)+\Lambda_4\approx 4.06>1$. *This is precisely why Example 2 is treated by Dhage's hybrid fixed-point theorem rather than by the contraction principle.*

For illustration, we generate a finite sequence of iterates associated with the fixed-point operator (3.8), starting from $\zeta_0=0$. The resulting numerical profiles and the corresponding successive differences are displayed in Figure 2.

Figure 2 displays the computed iterates associated with the operator (3.8) and the corresponding successive-difference norms. For the selected initial function and numerical discretization, the successive differences decrease over the displayed iterations. This is a numerical observation only; it is not used to prove convergence of the iteration and does not identify the computed profile with the solution whose existence is guaranteed by Theorem 3.3.

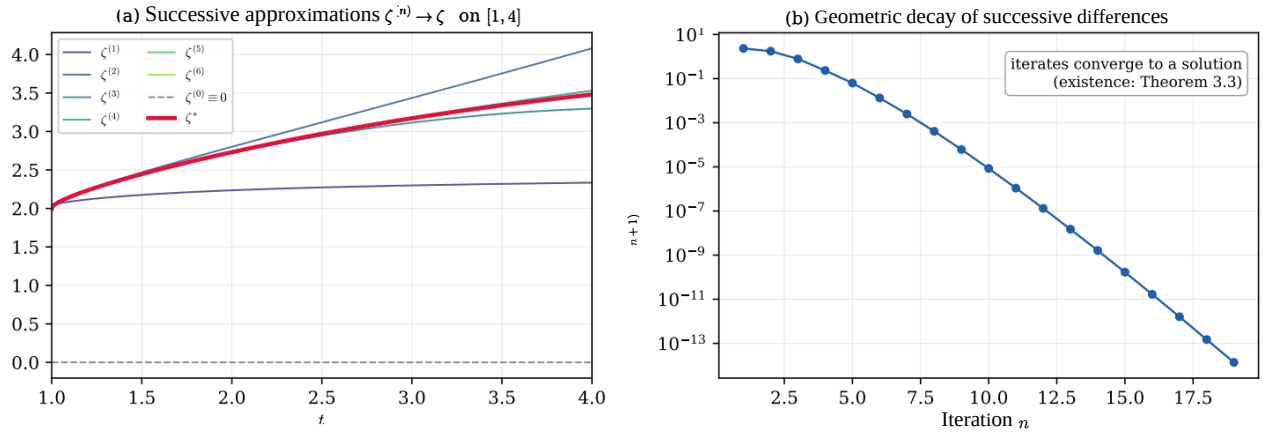

Figure 2. Numerical illustration for Example 2. Panel (a) displays selected iterates generated by the fixed-point operator (3.8); panel (b) displays the corresponding discrete successive-difference norms $\|\zeta^{(n+1)} - \zeta^{(n)}\|_{\infty,h}$. The observed decrease is a numerical observation only and is not used as a proof of convergence of the iteration.

Figure 2 is included solely as a numerical illustration. The computed decrease of the discrete successive-difference norms does not constitute a proof that the full fixed-point iteration converges. In particular, Theorem 3.3 establishes the existence of at least one integral solution, but it does not imply convergence of the displayed iteration.

6. Conclusions

This study has established the existence and uniqueness of solutions for a class of sequential pantograph fractional differential equations coupling the Caputo–Hadamard and the Atangana–Baleanu–Caputo fractional derivatives through a p-Laplacian operator, with hybrid nonlocal terms and proportional delays. After deriving an integral representation associated with the problem, Banach's contraction principle was employed to establish the uniqueness of solutions under suitable Lipschitz conditions, whereas Dhage's Krasnoselskii-type hybrid fixed-point theorem in Banach algebras provided existence results under less–restrictive assumptions; the Ulam–Hyers stability of the problem was established as well.

The theoretical results were illustrated through two examples that satisfy explicitly all the required hypotheses, including the Lipschitz continuity and growth conditions imposed on the nonlinear terms. These examples demonstrate the applicability of the obtained theoretical results to nonlinear fractional problems involving the Atangana–Baleanu fractional operator and the p-Laplacian structure.

Future research directions could include extensions to systems of coupled fractional equations, as well as the development of reliable structure-preserving numerical schemes for the considered fractional model. Particular emphasis will be placed on designing numerical methods that preserve relevant physical and mathematical properties under suitable conditions. Recent advances, such as the work of Yang and Zhang [24], provide a promising direction for this development.

Author contributions

Hammou Benmehidi: Conceptualization, methodology, investigation, writing–original draft, writing–review and editing. Tayeb Mahrouz: Conceptualization, methodology, investigation, writing–original draft, writing–review and editing. Zoubir Dahmani: Conceptualization, methodology, investigation, writing–original draft, writing–review and editing. Abdelkader Moumen: Writing–original draft, writing–review and editing. Hussien Albala: Visualization, writing–original draft, writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors have no conflicts of interest to declare.

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